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Point Charge Subject to an Attractive Inverse-Square-Type Potential and Anharmonic-Type Potentials

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Abstract: By applying the WKB (Wentzel, Kramers, Brillouin) approximation, we search for bound state solutions to the time-independent Schrödinger equation for an attractive inverse-square potential and anharmonic oscillators that stem from the interaction of a point charge with radial electric fields. We focus on the bound states associated with the *s*-waves. Further, we obtain the revival time associated with each case studied.

Keywords: attractive inverse-square potential; anharmonic oscillators; WKB approximation; semiclassical approximation; quantum revivals

1. Introduction

The semiclassical approximation proposed by Wentzel [1], Kramers [2], and Brillouin [3] showed a means of dealing with conservative systems that have one degree of freedom. This approximation is known in the literature as the WKB (Wentzel, Kramers, Brillouin) approximation [4,5]. Examples of the use of the WKB approximation are in anharmonic oscillators [5–8], electrons in a uniform magnetic field [9,10], and $\mathcal{PT}$-symmetric quantum mechanics [11–13]. However, by extending the WKB approximation to systems with spherical symmetry, the presence of the centrifugal term in the radial equation showed that the WKB approximation can fail. This problem with the centrifugal term and the singularity at the origin was analyzed by Langer [14]. There, Langer showed that the WKB approximation becomes valid in the spherical symmetry by applying a transformation known as the Langer transformation [15–17]. Since the proposal of the Langer transformation, several systems with spherical symmetry have been studied through the WKB approximation [14,17–20]. From the perspective of dealing with the WKB approximation in systems with cylindrical symmetry, Berry and co-workers [15,16] extended the Langer transformation to the cylindrical symmetry. Recently, the WKB approximation has been applied based on Berry and co-workers' proposal in neutral particle systems [21–23] and for a point charge in nonuniform electric fields under the influence of the disclination topology [24].

In this work, we study the interaction of a point charge with electric fields from a semiclassical point of view with the aim of achieving the bound states. We show that the interaction of a point charge with electric fields can give rise to an attractive inverse-square-type potential [4,25–29] and anharmonic-type oscillators [30–32]. The semiclassical analysis is based on the WKB (Wentzel, Kramers, Brillouin) approximation [4,5,17] by applying the Langer transformation for the cylindrical symmetry [15–17]. Therefore, we obtain the energy eigenvalues for a point charge subject to an attractive inverse-square-type potential and anharmonic-type potentials with regard to the *s*-waves. Later, we discuss the revival time [33–36] associated with each case studied.

The structure of this paper is as follows. In Section 2, we introduce the WKB approximation based on the Langer transformation for the cylindrical symmetry [15–17]; in Section 3, we show that the interaction of a point charge with an electric field can give rise

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to an attractive inverse-square-type potential [4,25–29], and, thus, we search for bound state solutions to the Schrödinger equation related to the s-waves; in Section 4, we search for the energy eigenvalues of s-waves for a quartic-type oscillator potential [30–32] that stems from the interaction of a point charge with a radial electric field; in Section 5, we also consider s-waves and discuss an analogue of r^6 potential; in Section 6, we focus on the revival time [33–36] related to the attractive inverse-square-type potential and the anharmonic-type potentials studied in the previous sections; in Section 7, we present our conclusions.

2. WKB Approximation

Let us consider the interaction of a point charge with radial electric fields. By dealing with the cylindrical symmetry, this interaction yields a potential energy that depends on the radial distance from the origin r, i.e., $V = V(r)$. Thereby, the time-independent Schrödinger equation that describes the the interaction of a point charge with a radial electric field is given by (we shall work with the units $\hbar = 1$ and $c = 1$ (natural units))

$$\mathcal{E}\psi = -\frac{1}{2m}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}\right]\psi + V(r)\,\psi, \tag{1}$$

where m is the mass of the point charge and $\mathcal{E}$ corresponds to the energy eigenvalues. The wave function is written in terms of the cylindrical coordinates $(r,\, \varphi,\, z)$, i.e., $\psi = \psi(r,\, \varphi,\, z)$.

From Equation (1), we can observe that $\hat{p}_z = -i\,\partial_z$ and $\hat{L}_z = -i\partial_\varphi$ commute with the Hamiltonian operator $\hat{H}$. Therefore, we can write the solution to Equation (1) in terms of the eigenvalues of the $\hat{p}_z$ and $\hat{L}_z$ as follows:

$$\psi(r,\, \varphi,\, z) = e^{i\ell\varphi + ikz}\,\frac{u(r)}{\sqrt{r}}. \tag{2}$$

Note that $u(r)$ is an unknown function of the radial coordinate, $\ell = 0, \pm1, \pm2, \ldots$ is the eigenvalue of $\hat{L}_z$, and k is a constant that corresponds to the eigenvalue of $\hat{p}_z$. Then, by substituting the wave function (2) into Equation (1), we obtain the radial equation:

$$\frac{d^2u}{dr^2} - \frac{(\ell^2 - 1/4)}{r^2}\,u + \left[2m\mathcal{E} - 2mV(r) - k^2\right]u = 0. \tag{3}$$

With the aim of working with the WKB approximation in cylindrical symmetry, we thus follow Berry and co-workers [15,16] and deal with the centrifugal term of the radial Equation (3) through the Langer transformation [15–17]. In the case of the cylindrical symmetry, Berry and co-workers [15,16] showed that the WKB approximation is valid by replacing the term $(\ell^2 - 1/4)$ with ℓ^2 in the centrifugal term of the radial equation [16,17]. In this way, the radial Equation (3) becomes

$$\frac{d^2u}{dr^2} - \frac{\ell^2}{r^2}\,u + \left[2m\mathcal{E} - 2mV(r) - k^2\right]u = 0. \tag{4}$$

Furthermore, let us consider $k = 0$ and define

$$Q(r) = \sqrt{2m\mathcal{E} - 2mV(r) - \frac{\ell^2}{r^2}}. \tag{5}$$

Hence, the radial Equation (4) can be rewritten in the form

$$u'' + Q^2(r)u = 0. \tag{6}$$

Hence, with the Langer transformation proposed by Berry and co-workers [15,16], the WKB approximation yields the radial wave function in the form [17]

$$u(r) \cong \frac{2}{\sqrt{Q(r)}} \cos\left(\int_{r_1}^{r} Q(r')\, dr' - \frac{\pi}{4}\right), \tag{7}$$

where r_1 is called the turning point.

In the following, we search for the energy eigenvalues of attractive inverse-square potential-type and anharmonic-type potentials that stem from the interaction of the point charge with radial electric fields. From these interactions, we can determine $Q(r)$ in Equation (5), and thus, introduce the Bohr–Sommerfeld quantization.

3. Attractive Inverse-Square-Type Potential

Let us consider the electric field produced by a nonuniform electric charge distribution inside a long nonconducting cylinder of inner radius r_a and outer radius r_0. The volume charge density is given by $\rho = \rho_0 r_a^4/r^4$, where $\rho_0 > 0$ is a constant. Note that $r_a \leq r \leq r_0$; thus, $\rho_0 \leq \rho \leq \rho_0 r_a^4/r_0^4$. Thereby, the electric field produced by this electric charge distribution in the region $r \geq r_a$ is

$$\vec{E}_1 = \left[-\frac{\rho_0\, r_a^4}{2r^3} + \frac{\rho_0\, r_a^2}{2r}\right] \hat{r}, \tag{8}$$

where $\hat{r}$ is a unit vector in the radial direction. In addition, we consider the electric field produced by a linear distribution of electric charges (λ) on the z-axis:

$$\vec{E}_2 = \frac{\lambda}{2\pi\, r}\, \hat{r}, \tag{9}$$

where $\lambda < 0$ is a constant. Thereby, in the region $r \geq r_a$, we have the presence of the electric fields (1) and (2). Our interest is in the particular case where $|\lambda| = \pi\,\rho_0\, r_a^2$. In this particular case, the electric field in the region $r \geq r_a$ is given by

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = -\frac{\rho_0\, r_a^4}{2r^3}\, \hat{r}. \tag{10}$$

Next, the potential energy that arises from the interaction of the point with the electric field (10) is

$$V(r) = -q \int_{r_a}^{r} \vec{E} \cdot d\vec{r} = -\frac{q\,\rho_0\, r_a^4}{4\, r^2} + \frac{q\,\rho_0\, r_a^2}{4}, \tag{11}$$

where $q > 0$ is the electric charge of the particle. Therefore, the first term of the potential energy (11) plays the role of an attractive inverse-square potential [4,25–29] for a point charge in the region $r \geq r_a$. In atomic systems, the inverse-square potential is of great interest, as shown in Refs. [22,37–40]. On the other hand, the second term of Equation (11) is a constant. Thereby, with the aim of studying the attractive inverse-square-type potential, as in Refs. [4,25–29], let us define the potential energy in the region $r \geq r_a$ (around the cylindrical cavity) as

$$\bar{V}(r) = V(r) - \frac{q\,\rho_0\, r_a^2}{4} = -\frac{q\,\rho_0\, r_a^4}{4\, r^2}. \tag{12}$$

Note that we recover the electric field given in Equation (10) by using the relation $\vec{E} = -\nabla\bar{V}$. In addition, we shall consider an infinity wall at $r = r_a$.

Henceforth, we deal with $\ell = 0$, which corresponds to the s-waves. From this perspective, by substituting Equation (12) into the radial Equation (4), we have

$$\frac{d^2u}{dr^2} + \frac{\nu^2}{r^2}\,u + 2m\mathcal{E}\,u = 0, \tag{13}$$

where the parameter ν is defined as

$$\nu^2 = \frac{m\,q\,\rho_0\,r_a^4}{2}. \tag{14}$$

In search of bound states, we assume that $\mathcal{E} < 0$ from now on. Thereby, for s-waves, the function $Q(r)$ given in Equation (5) becomes

$$Q(r) = \sqrt{\frac{\nu^2}{r^2} - (-2m\mathcal{E})}. \tag{15}$$

With the purpose of writing the Bohr–Sommerfeld quantization, we assume that the radial wave function vanishes at $r = r_a$, i.e., we assume that there is an infinity wall at $r = r_a$. It is worth observing that this boundary condition is analogous to the cut-off point used in Refs. [26–29]. According to Ref. [5], in the presence of the infinity wall at $r = r_a$, the Bohr–Sommerfeld quantization is given by

$$\int_{r_a}^{r_1} Q(r)\,dr = \left(n - \frac{1}{4}\right)\pi, \tag{16}$$

where $n = 1, 2, 3, \ldots$ is the radial quantum number and r_1 is the turning point. In the present case, the turning point is

$$r_1 = \frac{\nu}{\sqrt{-2m\mathcal{E}}}. \tag{17}$$

Next, we have that the left-hand side of Equation (16) is given by

$$\begin{aligned}
\int_{r_a}^{r_1} Q(r)\,dr &= \int_{r_a}^{r_1} \sqrt{\frac{\nu^2}{r^2} - (-2m\mathcal{E})}\;dr \\
&= \sqrt{-2m\mathcal{E}}\int_{r_a}^{r_1} \sqrt{r_1^2 - r^2}\,\frac{dr}{r} \\
&= \sqrt{-2m\mathcal{E}}\left[-r_1\sqrt{1 - \frac{r_a^2}{r_1^2}} + r_1\ln\left(\frac{r_1}{r_a}\right) + r_1\ln\left(1 + \sqrt{1 - \frac{r_a^2}{r_1^2}}\right)\right].
\end{aligned} \tag{18}$$

Let us assume that $r_1 \gg r_a$. Hence, we can apply the Taylor expansion [41] and neglect the terms of order $\mathcal{O}\left(\frac{r_a^2}{r_1^2}\right)$ [22]. Thereby, we obtain from Equation (18)

$$\begin{aligned}
\int_{r_a}^{r_1} Q(r)\,dr &\approx r_1\sqrt{-2m\mathcal{E}}\left[\ln\left(\frac{r_1}{r_a}\right) + c\right] \\
&\approx \nu\,\ln\left(\frac{\nu}{r_a\sqrt{-2m\mathcal{E}}}\right) + \nu\,c,
\end{aligned} \tag{19}$$

where c is a constant [22]. Hence, after substituting (19) into (16), we find the energy eigenvalues for s-waves:

$$\mathcal{E}_n \approx -\frac{\nu^2}{2m\,r_a^2\,e^{-2\delta/\nu}}\,e^{-\frac{2\pi n}{\nu}}, \tag{20}$$

where $\delta = \nu\,c + \frac{\pi}{4}$.

Hence, we have achieved in Equation (20) the spectrum of energy for s-waves that stems from a point charge subject to the attractive inverse-square-type potential (12). The energy levels (20) decrease exponentially with the radial quantum number n. Thus, when $n \to \infty$, we have that $\mathcal{E}_{n\to\infty} \to 0$. This means that there is a point of accumulation of energy levels in $\mathcal{E}_{n\to\infty} \to 0$ due to the fact that a large number of energy levels are closer to this energy level. On the other hand, the ground state is determined by $n = 1$. The energy of the ground state is $\mathcal{E}_1 = -\frac{\nu^2}{2m\, r_a^2\, e^{-2\delta/\nu}}\, e^{-\frac{2\pi}{\nu}}$. Therefore, the energy eigenvalues (19) are defined in the range

$$-\frac{\nu^2}{2m\, r_a^2\, e^{-2\delta/\nu}}\, e^{-\frac{2\pi}{\nu}} \leq \mathcal{E}_n \leq 0. \tag{21}$$

Besides, observe that if $r_a \to 0$, then $\mathcal{E}_n \to -\infty$. This characteristic of the energy levels corresponds to the fall of the particle to the center, discussed by Landau and Lifshitz [4]. Therefore, by assuming that r_a plays the role of the cut-off point used in Refs. [26–29], we can obtain the discrete spectrum of energy (20).

4. Quartic-Type Oscillator

Let us consider a nonuniform electric charge density given by $\rho = \frac{\rho_0}{r_0^2}\, r^2$ ($\rho_0 > 0$ is a constant) inside a long nonconducting cylinder of radius r_0. Since $0 \leq r \leq r_0$, then $0 \leq \rho \leq \rho_0$. Thus, the electric field inside the long nonconducting cylinder is

$$\vec{E}_3 = \frac{\rho_0}{4r_0^2}\, r^3\, \hat{r}. \tag{22}$$

The potential energy that arises from the interaction of the point charge with the electric field (22) is given by

$$V_3(r) = -q_1 \int_0^r \vec{E}_3 \cdot d\vec{r} = \frac{|q|\rho_0}{16\, r_0^2}\, r^4. \tag{23}$$

where $q_1 = -|q|$ is the electric charge of the particle. Note that the potential energy (23) gives rise to an anharmonic oscillator, which is an analogue of the pure quartic potential [30–32]. The quartic oscillator is of interest in molecular physics [42–44] and field theory [45].

Let us also deal with s-waves ($\ell = 0$). In this way, the radial Equation (4) becomes

$$\frac{d^2u}{dr^2} - 2m\alpha\, r^4\, u + 2m\mathcal{E}\, u = 0, \tag{24}$$

where $\alpha = \frac{|q|\rho_0}{16\, r_0^2}$. Then, the function $Q(r)$ in Equation (5) becomes

$$Q(r) = \sqrt{2m\mathcal{E} - 2m\alpha\, r^4}. \tag{25}$$

In this case, the Bohr–Sommerfeld quantization is given by [16,17]

$$\int_{r_1}^{r_2} Q(r)\, dr = \left(n - \frac{1}{2}\right)\pi, \tag{26}$$

where $n = 1, 2, 3, \ldots$ is also the radial quantum number and r_1 and r_2 are the turning points. As shown in Ref. [17], for s-waves, we have that $r_1 = 0$. Thereby, there is only one turning point, which is given by $\mathcal{E} = V(r_2)$. Hence, the turning point r_2 is

$$r_2 = \left[\frac{\mathcal{E}}{\alpha}\right]^{1/4}. \tag{27}$$

Therefore, by defining $x = \frac{\alpha}{\mathcal{E}}\, r^4$, then the left-hand side of Equation (26) yields

$$
\begin{aligned}
\int_0^{r_2} Q(r)\, dr &= \int_0^{r_2} \sqrt{2m\mathcal{E} - 2m\alpha\, r^4}\, dr \\
&= \sqrt{2m\mathcal{E}} \int_0^{r_2} \sqrt{1 - \frac{\alpha}{\mathcal{E}}\, r^4}\, dr \\
&= \frac{\sqrt{2m\mathcal{E}}}{4} \left(\frac{\mathcal{E}}{\alpha}\right)^{1/4} \int_0^1 x^{-3/4}\, \sqrt{1-x}\, dx \\
&= \frac{\sqrt{2m\pi}}{8}\, \frac{\mathcal{E}^{3/4}}{\alpha^{1/4}}\, \frac{\Gamma(5/4)}{\Gamma(7/4)},
\end{aligned}
\tag{28}
$$

where $\Gamma(5/4)$ and $\Gamma(7/4)$ are the gamma functions [5,41]. Then, by substituting Equation (28) into Equation (26), we obtain

$$
\mathcal{E}_n = \left[8\, \alpha^{1/4} \left(n - \frac{1}{2}\right) \times \sqrt{\frac{\pi}{2m}} \times \frac{\Gamma(7/4)}{\Gamma(5/4)}\right]^{4/3}. \tag{29}
$$

Hence, Equation (29) yields a discrete spectrum of energy for s-waves when a point charge interacts with the radial electric field (22). In this case, the potential energy (23) plays the role of a confining scalar potential, i.e., it yields an analogue of the pure quartic potential [30–32]. Observe that these bound states cannot be achieved for ℓ-waves ($\ell \neq 0$) through the WKB approximation. For $\ell \neq 0$, the turning points r_1 and r_2 are modified, and, thus, the energy eigenvalues (29) are no longer achieved.

5. Analogue of r^6 Potential

Let us consider a nonuniform electric charge density given by $\rho = \frac{\rho_0}{r_0^4}\, r^4$ ($\rho_0 > 0$ is a constant) inside a long nonconducting cylinder of radius r_0. Since $0 \leq r \leq r_0$, then $0 \leq \rho \leq \rho_0$. In this way, the electric field inside the long nonconducting cylinder is

$$
\vec{E}_4 = \frac{\rho_0}{5r_0^4}\, r^5\, \hat{r}, \tag{30}
$$

and the potential energy that arises from the interaction of the point charge with the electric field (30) is given by

$$
V_4(r) = -q_2 \int_0^r \vec{E}_4 \cdot d\vec{r} = \frac{|q|\rho_0}{30\, r_0^4}\, r^6. \tag{31}
$$

Note that $q_2 = -|q|$ is the electric charge of the particle. In this case, the potential energy (31) gives rise to another anharmonic oscillator, where we have an analogue of r^6 potential.

We also focus on the s-waves ($\ell = 0$); hence, the radial Equation (4) becomes

$$
\frac{d^2u}{dr^2} - 2m\beta\, r^6\, u + 2m\mathcal{E}\, u = 0, \tag{32}
$$

where $\beta = \frac{|q|\rho_0}{30\, r_0^4}$. Moreover, the function $Q(r)$ in Equation (5) becomes

$$
Q(r) = \sqrt{2m\mathcal{E} - 2m\beta\, r^6}. \tag{33}
$$

For s-waves, we have that the turning point $r_1 = 0$ [17]; thus, the Bohr–Sommerfeld quantization is also given by Equation (26). Furthermore, with $\mathcal{E} = V(r_2)$, the turning point r_2 is given by

$$r_2 = \left[\frac{\mathcal{E}}{\beta}\right]^{1/6}. \tag{34}$$

We can go further by defining $y = \frac{\beta}{\mathcal{E}}\, r^6$, and then the left-hand side of Equation (26) yields

$$\begin{aligned}
\int_0^{r_2} Q(r)\, dr &= \int_0^{r_2} \sqrt{2m\mathcal{E} - 2m\beta\, r^6}\, dr \\
&= \sqrt{2m\mathcal{E}} \int_0^{r_2} \sqrt{1 - \frac{\beta}{\mathcal{E}}\, r^6}\, dr \\
&= \frac{\sqrt{2m\mathcal{E}}}{6} \left(\frac{\mathcal{E}}{\beta}\right)^{1/6} \int_0^1 y^{-5/6}\, \sqrt{1-y}\, dy \\
&= \frac{\sqrt{2m\pi}}{12}\, \frac{\mathcal{E}^{2/3}}{\beta^{1/6}}\, \frac{\Gamma(7/6)}{\Gamma(5/3)},
\end{aligned} \tag{35}$$

where $\Gamma(7/6)$ and $\Gamma(5/3)$ are also the gamma functions [5,41]. Then, by substituting Equation (35) into Equation (26), we obtain

$$\mathcal{E}_n = \left[12\, \beta^{1/6} \left(n - \frac{1}{2}\right) \times \sqrt{\frac{\pi}{2m}} \times \frac{\Gamma(5/3)}{\Gamma(7/6)}\right]^{3/2}, \tag{36}$$

where $n = 1, 2, 3, \ldots$ is also the radial quantum number.

Thereby, we have obtained a discrete spectrum of energy for s-waves when a point charge interacts with the radial electric field (30). The potential energy (31) yields an anharmonic-type oscillator, which, in turn, also plays the role of a confining scalar potential. In this case, the potential energy yields an analogue of r^6 potential. We can also observe that the bound states cannot be achieved for ℓ-waves ($\ell \neq 0$) through the WKB approximation. The turning points r_1 and r_2 are also modified for $\ell \neq 0$; hence, the energy eigenvalues (36) are no longer achieved for $\ell \neq 0$.

6. Revival Time

An interesting point raised in Refs. [33–36] is that the wave function can recover its initial shape. This quantum effect occurs after a time called the revival time and it is called quantum revival. As shown in Refs. [33,34], the revival time is obtained from the energy eigenvalues. Due to the fact that we are dealing with s-waves, the revival time is determined by only the radial quantum number n. Therefore, with regard to the radial quantum number n, we need to expand the energy eigenvalues about the central value n_1 of this quantum number in Taylor series as follows:

$$\mathcal{E}_n \approx \mathcal{E}_{n_1} + \left(\frac{d\mathcal{E}_n}{dn}\right)_{n=n_1} \times (n - n_1) + \frac{1}{2}\left(\frac{d^2\mathcal{E}_n}{dn^2}\right)_{n=n_1} \times (n - n_1)^2 + \cdots \tag{37}$$

Therefore, the revival time is defined by (with $\hbar = 1$) [33,34]

$$\tau = \frac{4\pi}{\left|\left(\frac{d^2\mathcal{E}_n}{dn^2}\right)_{n=n_1}\right|}. \tag{38}$$

In recent years, the study of quantum revivals has been carried out in several contexts. Graphene [46,47] and Rydberg atoms [48–50] are interesting examples. Other recent studies are given in Refs. [51–55]. In the following, we discuss the revival time associated with the quantum systems studied in the previous sections.

6.1. Attractive Inverse-Square-Type Potential

From the energy levels (20), for a given value of n (i.e., for a fixed n), Equation (38) yields

$$\tau = \frac{2m\, r_a^2}{e^{2\delta/\nu}}\, e^{\frac{2\pi n}{\nu}}. \tag{39}$$

Hence, Equation (39) corresponds to the revival time for s-waves when a point charge is subject to the attractive inverse-square-type potential (12).

6.2. Quartic-Type Oscillator

For a given value of n, by substituting Equation (29) into Equation (38), we have

$$\tau = \frac{3\pi}{64}\, \frac{\left[8\,\alpha^{1/4}\left(n-\frac{1}{2}\right) \times \sqrt{\frac{\pi}{2m}} \times \frac{\Gamma(7/4)}{\Gamma(5/4)}\right]^{2/3}}{\left[\alpha^{1/4} \times \sqrt{\frac{\pi}{2m}} \times \frac{\Gamma(7/4)}{\Gamma(5/4)}\right]^{2}}. \tag{40}$$

Therefore, the revival time (40) is obtained for s-waves when a point charge is subject to the analogue of the pure quartic potential (23). As we have pointed out previously, due to the fact that the energy levels (29) are not achieved for ℓ-waves via WKB approximation, hence, the revival time (40) cannot be obtained for $\ell \neq 0$.

6.3. Analogue of r^6 Potential

From the energy levels (36), for a given value of n, the revival time is

$$\tau = \frac{\pi}{27}\, \frac{\left[12\,\beta^{1/6}\left(n-\frac{1}{2}\right) \times \sqrt{\frac{\pi}{2m}} \times \frac{\Gamma(5/3)}{\Gamma(7/6)}\right]^{1/2}}{\left[12\,\beta^{1/6} \times \sqrt{\frac{\pi}{2m}} \times \frac{\Gamma(5/3)}{\Gamma(7/6)}\right]^{2}}. \tag{41}$$

Hence, the revival time (41) is obtained for s-waves when a point charge is confined to the analogue of r^6 potential (31). Through the WKB approximation, this anharmonic-type oscillator possesses the discrete spectrum of energy (36) only for s-waves; therefore, the revival time (41) cannot be obtained for ℓ-waves.

7. Conclusions

We have studied the interaction of a point charge with nonuniform electric fields through the WKB approximation based on Berry and co-workers' proposal [15–17]. We have seen that analogues of the attractive inverse-square potential [4,25–29], pure quartic oscillator [30–32], and r^6 (anharmonic) oscillator can stem from the interaction of a point charge with electric fields. Therefore, through the WKB approximation, we have found and analyzed the eigenvalues of energy associated with the s-waves. Further, we have obtained the revival time with respect to each case studied. Since we have focused on the bound states associated with s-waves, the revival times have been defined in terms of the radial quantum number.

An interesting perspective is to search for bound states with regard to the ℓ-waves. In this case, there is the possibility of studying the the Aharonov–Bohm effect [56,57] and analogues effects associated with the topology of disclinations and dislocations in an elastic medium [58–60]. Moreover, with the ℓ-waves, the revival time becomes determined in terms of two of more quantum numbers [33,34,61–63].

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