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Review

Unimodular Approaches to the Cosmological Constant Problem

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Abstract: We review selected aspects of unimodular gravity and we discuss its viability as a solution of the old cosmological constant problem. In unimodular gravity, the cosmological constant is promoted to a global degree of freedom. We highlight the importance of correctly setting up its initial data in order to achieve a resolution of the cosmological constant problem on a semi-classical level. We review recent path integral analysis of quantum aspects of unimodular gravity to note that the semi-classical findings carry over to the quantum level as well. We point out that a resolution of the problem inherently relies on a global constraint on the spacetime four-volume. This makes the theory closely related to the vacuum energy sequester, which operates in a similar way. We discuss possible avenues of extending unimodular gravity that preserve the resolution of the cosmological constant problem.

Keywords: unimodular gravity; cosmological constant problem; modified gravity; Weyl invariance

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1. Introduction

Prior to the measurement of the accelerated nature of the expansion of the Universe [1,2], the cosmology community predominantly expected that the effective value of the cosmological constant (CC) would be zero. However, even before we had been burdened with reconciling the puzzling minuscule value of the CC, that we observe today [3], it was recognized that there is an underlying problem with the vanishing of the vacuum energy [4–6]. This problem stems from the observation that the energy of the vacuum state of quantum fields behaves exactly as an effective cosmological constant. In the semi-classical approximation, where gravity behaves classically, while matter fields are quantized, these vacuum energies appear to be able to drive an accelerated expansion of the universe. However, any attempt at estimating these contributions has produced values of such magnitude that their effect on cosmology would be impossible to miss. It is needless to say that such effects have not been observed and while the vacuum energy has ultimately been measured to be non-zero it is still orders and orders of magnitude smaller than any estimation obtained from quantum field theory. The question then arises: Why do we not observe these large vacuum energies or rather what mechanism causes them to cancel out or vanish? For more details see, e.g., [7–11]. Due to its origin, the above problem is often referred to as the 'old' cosmological constant problem. Note that more recent questions regarding the value of the vacuum energy like the coincidence problem often assume that the old cosmological constant problem is somehow solved and usually do not address it in any way.

The old cosmological constant problem is commonly considered to be a fine tuning problem as one can carefully tune the bare cosmological constant of general relativity (GR) in such a way that it cancels the quantum contributions up to the tiny residual amount that we observe. However, this view is grossly oversimplified. As has been pointed out [12–14], the cosmological constant also receives large contributions from higher loop correction of matter fields, which do not diminish with higher loop orders. One is then forced to tune the cosmological constant at every step of loop expansion to a very high degree of precision, which entails an infinite amount of fine tunings, each as bad as the previous one. This signals that the running of such a *renormalized* cosmological constant is ultrasensitive to the UV completion of the matter theory, which we have no knowledge of. This is of course a disaster since it would imply that our understanding of gravity at the lowest energies depends significantly on the microscopic physics of large energies. Hence, we need to protect the effective cosmological constant from such effects.

In this paper we are going to discuss a popular theory commonly used to address this problem—*unimodular gravity* (UG) [15–17]. The origins of this theory go back to Einstein himself who used the unimodular condition $\sqrt{-g} = 1$ as a partial gauge fixing of diffeomorphism invariance to simplify calculations in GR [18]. Only later was it realized that assuming such a gauge fixing prior to variation of the Einstein–Hilbert action yields a modification of GR, where the trace of the Einstein equations is directly subtracted[1] [15,16,19–21],

$$G_{\mu\nu} - \frac{1}{4} G g_{\mu\nu} = T_{\mu\nu} - \frac{1}{4} T g_{\mu\nu} \,. \tag{1}$$

Surprisingly, these equations turn out to be classically equivalent to those of GR, however, with an unspecified cosmological constant. The key property of Equation (1) is that they appear to be blind to any cosmological constant term. This has raised hopes that within UG the quantum corrections to the vacuum energy would fail to affect the spacetime geometry. This would clearly solve the old cosmological constant problem. Alas, this claim is often merely stated or it is considered to be an obvious fact, without providing any detail or references on the argument. Furthermore, it has been recently argued that the quantum corrections to vacuum energy in fact do not decouple in UG and that the old CC problem remains present in UG as well [8,22,23]. As far as we are aware, the discussion on this topic is not settled and a consensus has not been reached. It is one of the aims of this work to highlight the origin of this difference in views.

In Section 2, we discuss several popular formulations of UG and the status of the old CC problem within them. We show that the resolution of whether UG solves the old CC problem or not hinges on the way we provide initial data that determine the effective cosmological constant. In particular, in formulations that rely on the use of a Lagrange multiplier to fix the unimodular condition for metric determinant, i.e.,

$$\lambda(\sqrt{-g} - 1) \,, \tag{2}$$

directly specifying the initial value for the Lagrange multiplier spoils the decoupling mechanism. Conversely, in formulations where the same restriction is achieved via a composite structure of the minimally coupled metric, for example:

$$g^{phys}_{\mu\nu} = \frac{g_{\mu\nu}}{\sqrt[4]{-g}} \,, \tag{3}$$

such initial conditions cannot be given and the old CC problem appears to be resolved. We first discuss the ambiguity in the initial data on the level of equations of motion, then we provide a discussion for transverse diffeomorphism invariant formulations in Section 2.1. We review the fully diffeomorphism invariant theories à la Henneaux and Teitelboim [24] in Section 2.2 and its Weyl invariant extensions [25,26] in Section 2.3. We further comment on the usefulness of these extensions for further study of unimodular gravity. Finally, we comment on the appearance of a global constraint on four-volume of spacetime in Section 2.4.

The problem with specifying the initial value of the Lagrange multiplier carries over to the quantum regime. We review a partial path integration procedure of the unimodular degrees of freedom that are extra in comparison to GR, for the Henneaux and Teitelboim formulation [24]. Quantum aspects of UG have been studied using path integral techniques in multiple recent works, i.e., [22,27–31]. The integration can either be carried out, while keeping the initial value of the Lagrange multiplier fixed as in [22,28,31] or by keeping it free. The former reduces to GR with a directly specified cosmological constant. Consequently, the old CC problem persists. The latter calculation results in a theory that corresponds to GR with a CC that is selected by a global constraint on the spacetime four-volume. Such fixing is inherently invariant under quantum corrections to vacuum energy, as has been pointed out in [14]. Hence, the old cosmological constant problem seems to be alleviated. We also briefly discuss the appearance of quantum fluctuations of the cosmological constant that naively appear due to the promotion of CC to a degree of freedom [32].

Lastly, we review the proposal of vacuum energy sequestering [13,14], which also achieves the decoupling of the quantum corrections to CC. This mechanism relies on a similar blindness of the equations of motion to the vacuum energies as we find in UG. Unlike UG, this approach forcibly introduces a pair of global constraints, which completely removes the ambiguity in providing initial data that exists in UG. Vacuum energy sequester can be formulated as a local theory [33] that is very similar to UG. We point out that the local approach again introduces the ambiguity in providing initial data, which affected the solution of the old cosmological constant problem. Finally, we discuss the relation between the local and global version and propose how such relation can be used to formulate a global version of UG.

2. Classical Formulations of Unimodular Gravity

As we alluded to in the introduction, the motivation for UG stems from the observation that the trace-free Einstein equations[2] [15,16,19–21]

$$G_{\mu\nu} - \frac{1}{4} G g_{\mu\nu} = T_{\mu\nu} - \frac{1}{4} T g_{\mu\nu} \,, \tag{4}$$

contain no information about the cosmological constant term in the Einstein–Hilbert action. Despite this, these equations are almost equivalent to standard Einstein equations. We can see this by taking the covariant divergence of both sides, which gives

$$\partial_\mu (G - T) = 0 \,. \tag{5}$$

This is a differential constraint, which can be easily solved as

$$G - T = 4\Lambda \,, \tag{6}$$

where Λ is an integration constant. Plugging this into the original traceless equations (4) yields the Einstein equations with a cosmological constant Λ

$$G_{\mu\nu} = T_{\mu\nu} + \Lambda g_{\mu\nu} \,, \tag{7}$$

A crucial difference in comparison to GR is that any value for Λ is admissible here. In other words, any solution of Einstein equations with an *arbitrary cosmological constant* is a solution of the traceless equations (4). The key property responsible for this arbitrariness is that Equation (4) are invariant under constant shifts of vacuum energy

$$T_{\mu\nu} \to T_{\mu\nu} + \rho_{vac}\, g_{\mu\nu} \,, \tag{8}$$

where ρ_{vac} is a constant. Crucially, the shift in the energy-momentum tensor, which is generated by accounting for the quantum corrections to vacuum energy, has exactly the form (8) and therefore the original Equation (4) are indeed blind to such corrections. The symmetry (8) holds even in Equation (5), but it is finally broken once we specify the cosmological constant Λ. If we wish to evolve some initial conditions using trace-free equations (4), we would soon find that we need to specify the effective cosmological constant (6) in order to obtain a unique solution. However, as it has been pointed out [22,23], specifying this parameter immediately yields Equation (7), which are just Einstein equations with a cosmological constant. This is where the original cosmological constant problem arose in the first place. This seems to imply that we have not succeeded in solving the old CC problem; rather, we have just shifted it one step away.

However, the situation is not completely hopeless, as the way we have chosen Λ is far from unique. If we first consider a splitting of the energy-momentum tensor into two pieces

$$T_{\mu\nu} = \tilde{T}_{\mu\nu} + T^{vac}_{\mu\nu} \,. \tag{9}$$

where the second piece on the right hand side accounts for the constant vacuum energy

$$T^{vac}_{\mu\nu} = \frac{1}{4} T^{vac} g_{\mu\nu} \,, \tag{10}$$

where T^{vac} is a spacetime constant. Furthermore, we assign any additional quantum corrections to the vacuum energy to $T^{vac}_{\mu\nu}$. Hence, $\tilde{T}_{\mu\nu}$ receives no such contributions. Plugging this splitting into (4), we will find that $T^{vac}_{\mu\nu}$ completely drops out of the equations and therefore we can write

$$G_{\mu\nu} - \frac{1}{4} G g_{\mu\nu} = \tilde{T}_{\mu\nu} - \frac{1}{4} \tilde{T} g_{\mu\nu} \,. \tag{11}$$

Repeating the argument above, we obtain a differential constraint

$$\partial_\mu (G - \tilde{T}) = 0 \,, \tag{12}$$

which we integrate as

$$G - \tilde{T} = 4\tilde{\Lambda} \,. \tag{13}$$

This yields the Einstein equations

$$G_{\mu\nu} = \tilde{T}_{\mu\nu} + \tilde{\Lambda} g_{\mu\nu} \,. \tag{14}$$

We can see that now only $\tilde{T}_{\mu\nu}$ appears in the above equations, and since $\tilde{T}_{\mu\nu}$ does not, by definition, receive any quantum correction to the vacuum energy it follows that a choice of $\tilde{\Lambda}$ is also stable. Hence, if we perform a measurement of the cosmological constant and interpret it as a parameter of these equations rather than (7), we will obtain a constant stable under quantum corrections.

It seems that the cosmological constant problem in Equation (4) is not solved automatically but allows us a leeway in how we interpret the measurement of the cosmological constant. Some ways are stable, while others are not. This is in a stark contrast with GR where the cosmological constant can only be interpreted as the bare coupling constant of the Einstein–Hilbert action. Note that this does not imply that GR and UG are physically inequivalent on a classical level. Rather, the change in description in UG allows us to understand the measurement of a cosmological constant in a different manner, where the old cosmological constant problem does not arise. Going forward, we will see that this is the case in other formulations of unimodular gravity as well.

Before we move on to other formulations of UG, it is important to point out that in order to arrive at the conclusion (6) and (13) we have assumed that the energy-momentum tensor is covariantly conserved. While this is required in GR, the Einstein traceless equations (4) allow for a certain type of violation of energy conservation. This allows us to reconcile gravity with situations and models where energy is not conserved. This can occur in the context of a wave function collapse during a quantum measurement or in some models of quantum gravity [35–37]. The traceless equations allow for the following violation of conservation of the energy-momentum tensor

$$\nabla^\mu T_{\mu\nu} = \partial_\mu Q \,, \tag{15}$$

where Q is an unspecified scalar field. Indeed, considering (15) and taking the divergence of the traceless Einstein equations yields

$$\partial_\mu (G - T) = 4\partial_\mu Q \,, \tag{16}$$

which we can easily integrate to obtain

$$G - T = 4\Lambda + 4Q \,. \tag{17}$$

Plugging this into the traceless equations (4), we find

$$G_{\mu\nu} = T_{\mu\nu} + (\Lambda + Q) g_{\mu\nu} \,. \tag{18}$$

The right hand side of this equation is now covariantly conserved due to the appearance of the field Q, which exactly compensates the non-conservation of the energy-momentum tensor. In this sense, the unaccounted for dissipating energy of $T_{\mu\nu}$ reappears in the form of a change of the effective cosmological 'constant', which is now spacetime-dependent due to this energy transfer

$$\Lambda_{eff} = \Lambda + Q \,. \tag{19}$$

Note that the violations of the energy conservation are assumed to be small; nevertheless, they can add up over the course of the history of the Universe, resulting in non-negligible changes in the effective cosmological constant [35]. It has been suggested that such effects may be responsible for the discrepancy between the late and early time measurements of the Hubble constant [38–40] and that they may provide a possible origin of the cosmological constant itself [41,42]. It should be stressed that the change in Λ_{eff} is primarily governed by the details of the energy diffusion processes in the matter sector and not by UG itself. It is also important to note that the compatibility with the non-conservation (15) is not a general feature of UG. Indeed, in diffeomorphism covariant formulations of UG, i.e., [24], the constancy of the cosmological 'constant' arises as an equation of motion rather than due to Bianchi identity. In these formulations, further modifications are needed to allow for the present behavior. We will briefly comment on this later in Section 2.2.

Finally, it is crucial that the mechanism that allowed us to screen away the quantum corrections to vacuum energy remain intact even when we consider energy diffusion in the matter sector. Indeed, the splitting (9) can be introduced to eliminate $T^{vac}_{\mu\nu}$ prior to introducing Λ in Equation (17). Consequently, repeating the above analysis, we arrive at

$$G_{\mu\nu} = \tilde{T}_{\mu\nu} + (\tilde{\Lambda} + Q) g_{\mu\nu} \,, \tag{20}$$

where $\tilde{T}_{\mu\nu}$ is free from quantum corrections as in (14). Hence, we can still achieve a resolution of the old cosmological constant problem.

2.1. The Unimodular Constraint

Maybe the most common formulation of UG in the literature is based on the so-called unimodular condition, from which UG gets its name

$$\sqrt{-g} = 1 \,. \tag{21}$$

In order to actually modify the dynamics of GR, this condition is enforced prior to variation of the Einstein–Hilbert (EH) action. This can be achieved in multiple ways; however, the most common one is to enforce it via a Lagrange multiplier directly in the action [15,16]

$$S[g, \lambda, \Psi] = \int_{\mathcal{M}} d^4x \left[-\frac{1}{2}\sqrt{-g}R + \lambda\left(\sqrt{-g} - 1\right) \right] + S_{matter}[g, \Psi] \,. \tag{22}$$

Here, S_{matter} accounts for any matter fields Ψ that we consider along the gravitational sector and $\mathcal{M}$ is the spacetime region under consideration. A downside of this formulation is that the action clearly breaks the diffeomorphism invariance of GR. This is because the metric density $\sqrt{-g}$ is set to be equal to a scalar quantity, in this case a unity. Hence, the action is invariant only under transverse diffeomorphisms generated by ξ^μ satisfying

$$\nabla_\mu \xi^\mu = 0\,. \tag{23}$$

Such diffeomorphisms indeed preserve the metric density since

$$\delta_\xi \sqrt{-g} = \mathcal{L}_\xi \sqrt{-g} = \frac{1}{2}\sqrt{-g}\nabla_\mu \xi^\mu = 0\,. \tag{24}$$

Hence, the symmetry group of this theory is substantially different from GR. Crucially, the only diffeomorphism breaking term depends only on λ, while the rest of the action is still diffeomorphism invariant. Consequently, all the matter and gravity equations of motion remain covariant. In particular, the Einstein equations implied by action (22) are

$$G_{\mu\nu} + \lambda\, g_{\mu\nu} = T_{\mu\nu}\,. \tag{25}$$

Clearly the Lagrange multiplier λ plays the role of the cosmological 'constant', which, at this point, is a general scalar field. However, since the matter sector of the action is assumed to be diffeomorphism invariant, it follows that the right hand side is covariantly conserved on-shell. By taking the covariant divergence of both sides, we find

$$\partial_\mu \lambda = 0\,. \tag{26}$$

Therefore, consistency requires that λ is indeed a constant. The unimodular constraint enforced by λ can be viewed locally as a mere gauge choice. Hence, it naively seems that any solution of GR with the cosmological constant λ in *any* coordinates can be considered as a solution of the above unimodular equations. Indeed, any such solution can be locally transformed into coordinates such that (21) is satisfied.

Note that the reduction of the symmetry to transverse diffeomorphisms relaxes the requirement of the covariant conservation of the energy-momentum tensor to (15) for an unspecified Q. This in general leads to

$$\partial_\mu \lambda = \partial_\mu Q\,, \tag{27}$$

as opposed to (26), which ultimately yields Equation (18). Nevertheless, as long as the matter action is fully diffeomorphism covariant the energy-momentum tensor in (25) is still conserved even though the symmetry does not hold for the entire action. Thus, any violations of energy conservation must be introduced by hand or, alternatively, one has to break the full diffeomorphism invariance in the matter action.

Let us now discuss the fate of the quantum corrections to the vacuum energy in this formulation. As opposed to the previous trace-free equations (4), the cosmological constant in (25) is an independent field and enters the Einstein equations directly. Consequently, such variable appears to have a privileged status in the theory and it seems only natural to provide initial conditions for it in order to solve Equation (26). This, however, directly leads to the old cosmological constant problem. To demonstrate this, we proceed with the only consistent initial condition. That is, when λ is a spatial constant Λ

$$\lambda(t_1) = \Lambda\,. \tag{28}$$

Equation (26) then immediately fixes $\lambda(t) = \Lambda$ for the rest of the time evolution. Consequently, we are left with standard GR with a cosmological constant Λ and the old cosmological constant problem appears again. A common counter-argument to this conclusion is that any quantum correction ρ_{vac} to the cosmological constant appears in the action coupled to $\sqrt{-g}$. Surely, we can decouple such terms from gravity by using the constraint to eliminate the $\sqrt{-g}$ dependence:

$$\int_{\mathcal{M}} d^4x \left[\lambda\left(\sqrt{-g}-1\right)+\rho_{vac}\sqrt{-g}\right] \to \int_{\mathcal{M}} d^4x \left[\lambda\left(\sqrt{-g}-1\right)+\rho_{vac}\right]. \tag{29}$$

Doing so prior to the derivation of equations of motion eliminates any information about the quantum corrections in the Einstein equations. However, while this seems like a straightforward step, we must realize that using the constraint within the action necessarily entails a redefinition of the Lagrange multiplier λ. In this case, this redefinition is the shift

$$\lambda \to \lambda - \rho_{vac}. \tag{30}$$

Consequently, any initial condition for λ (28) posed prior to the use of the constraint within the action is shifted exactly by the same amount ρ_{vac} in the opposite direction

$$\Lambda \to \Lambda + \rho_{vac}. \tag{31}$$

Hence, the value of the cosmological constant Λ clearly receives the quantum contributions. One could be tempted to argue that we should therefore set up the initial conditions for λ only after we calculate the quantum corrections to vacuum energy and use the constraint to decouple them. However, this is no different from fine-tuning the cosmological constant at each level of the loop expansion, because we would need to set up the right initial value for each loop order separately. Hence, this 'solution' amounts to an infinite amount of fine tunings.

Interestingly, we can use the constraint to modify the action in a more substantial way so that the quantum vacuum energy contributions decouple automatically. Indeed, consider the following substitution

$$g_{\mu\nu} \to \hat{g}_{\mu\nu} = g_{\mu\nu}\,|g|^{-1/4}, \tag{32}$$

which is carried out everywhere in the action outside of the constraint itself. This results in a theory with the following form

$$S[g,\lambda,\Psi] = \int_{\mathcal{M}} d^4x \left[-\frac{1}{2}\hat{R} + \lambda\left(\sqrt{-g}-1\right)\right] + S_{matter}[\hat{g},\Psi]. \tag{33}$$

Here, $\hat{R}$ is the scalar curvature evaluated using $\hat{g}_{\mu\nu}$. The variation of (33) with respect to $g_{\mu\nu}$ yields the equations of motion

$$\hat{G}_{\mu\nu} - \frac{1}{4}\hat{G}\hat{g}_{\mu\nu} + \lambda g_{\mu\nu} = \hat{T}_{\mu\nu} - \frac{1}{4}\hat{T}\hat{g}_{\mu\nu}, \tag{34}$$

where the hat above $G_{\mu\nu}$ and $T_{\mu\nu}$ signifies that the tensors are evaluated using the metric $\hat{g}_{\mu\nu}$. However, upon the constraint (21) we have $\hat{g}_{\mu\nu} = g_{\mu\nu}$ and thus we can drop the hats in the above equation. Note that, even though the matter action (33) is not fully diffeomorphism covariant, the energy-momentum tensor $\hat{T}_{\mu\nu}$ is covariantly conserved with respect to the covariant derivative associated with $\hat{g}_{\mu\nu}$. Consequently, this conservation extends to $g_{\mu\nu}$ on-shell. Crucially, taking the trace of Equation (34) immediately implies $\lambda = 0$. We get this conclusion without ever specifying its initial conditions. In fact, specifying non-zero initial conditions for λ is clearly inconsistent. Substituting $\lambda = 0$ back into (34) yields the traceless Einstein equations. We see that λ no longer plays the role of the cosmological constant. Furthermore, since the entire matter and gravitational Lagrangian depend strictly on $\hat{g}_{\mu\nu}$, any quantum correction will contribute as $\rho_{vac}\sqrt{-\hat{g}}$ to action (33). Since the novel composite metric $\hat{g}_{\mu\nu}$ has a unit determinant by construction

$$\sqrt{-\hat{g}} = 1, \tag{35}$$

these contributions decouple trivially, without the need to use the constraint or, equivalently, redefine λ. Therefore, the energy-momentum tensor that appears in the resulting trace-free equations is automatically *free of any quantum corrections to vacuum energy*. A downside of this is that the trace free equations lack *all information* about the cosmological constant and its effective value must be put in by hand as it is seemingly not tied to initial conditions of any fields.

Considering the action (33), it is clear that the constraint is rendered unnecessary (at least on the classical level); thus, we can remove it from the action to obtain yet another formulation of UG

$$S[g,\Psi] = -\frac{1}{2}\int_{\mathcal{M}} d^4x\,\hat{R} + S_{matter}[\hat{g},\Psi]\,. \tag{36}$$

Since all terms in the action now depend purely on $\hat{g}_{\mu\nu}$, the action has a manifest Weyl invariance under transformations of the metric

$$g_{\mu\nu} \to \omega^2 g_{\mu\nu}\,, \tag{37}$$

where ω is an arbitrary non-zero function. Consequently, the resulting equations of motion associated with $g_{\mu\nu}$ are necessarily traceless. The equations of motion associated with this action are indeed the Einstein traceless equations (11) taken together with the unimodularity condition (21). Since the Lagrange multiplier is no longer present in this formulation, choosing its initial value is clearly impossible here.

It is interesting to note that the Weyl symmetry (37) arises only after we eliminate the 'cosmological constant' term in the action

$$\lambda\sqrt{-g}\,. \tag{38}$$

Hence, the symmetry of the action is increased by having $\lambda = 0$. This bears a striking resemblance to the technical naturalness [43] for the cosmological constant; however, in this case the extra symmetry is a gauge symmetry rather than a regular symmetry and consequently the conclusion is not that Λ is protected against quantum corrections but instead it is necessarily vanishing. It is important to note that the Weyl symmetry in UG does not become anomalous in the quantum regime as has been shown in [44,45].

Note that the main difference between the original action (22) and the theory (36) is that in the former, the unimodular condition (21) is enforced via a Lagrange multiplier, while the latter achieves the same by universally coupling to a *composite metric* (32). It is not immediately clear why this should make a difference as standard intuition dictates that these theories should be entirely equivalent. Yet as we have seen, they behave differently. The difference stems from the fact that in the theory (22) we are tempted to introduce initial condition for the constant part of λ. This implies that the zero mode is not varied and thus the integral conclusion of the unimodular condition

$$\int_{\mathcal{M}} d^4x\sqrt{-g} = \int_{\mathcal{M}} d^4x\,, \tag{39}$$

is not meant to be enforced. Note that this is the only diffeomorphism invariant information in (21) and hence it represents a physical constraint [30,46,47]. It is thus not surprising that abandoning it leaves the theory unmodified—equivalent to GR. Leaving the initial value for λ unspecified allows us to use the constraint *freely* and hence we can use (29) without limits to decouple any contribution to vacuum energy. Consequently, the cosmological constant problem is alleviated. The condition (39) should then provide the missing information in the traceless Einstein equations (11) and consequently allows us to determine the effective cosmological constant. In the current setting, this point is difficult to demonstrate; however, we will revisit it for the HT formulation in Section 2.4, where an analogous situation occurs.

2.2. Henneaux and Teitelboim UG

The introduction of the unimodularity condition (21) in the previous formulation has the unfortunate consequence of reducing the gauge group of the theory from diffeomorphisms to transverse diffeomorphisms. However, the full diffeomorphism invariance can be restored by introducing a novel vector density V^μ. This construction has been described in [24] and the resulting theory is given by the following action,[3] which is usually referred to as Henneaux and Teitelboim (HT) unimodular gravity

$$S_{HT}[g,\lambda,V] = \int_{\mathcal{M}} d^4x \Big[-\frac{1}{2}\sqrt{-g}R - \lambda\big(\partial_\mu V^\mu - \sqrt{-g}\big)\Big] + S_{matter}[g,\Psi]\,. \tag{40}$$

Note that the divergence of a vector density is a scalar density and therefore the above action is fully diffeomorphism invariant. We can clearly see that this action reduces to (22), when we fix $\partial_\mu V^\mu = 1$, which can be achieved locally by a partial gauge fixing of diffeomorphisms. A gauge fixing prior to variation is not a generally admissible step and thus this does not guarantee the equivalence of the two theories. Nevertheless, the classical equivalence can be immediately demonstrated from the equations of motion. The variation with respect to $g_{\mu\nu}$ yields

$$G_{\mu\nu} + \lambda\, g_{\mu\nu} = T_{\mu\nu}\,, \tag{41}$$

and by varying V^μ we obtain

$$\partial_\mu \lambda = 0\,. \tag{42}$$

These equations are clearly the same as (25) and (26). The difference is that the second equation now arose as an equation of motion rather then due to Bianchi identity. Consequently, one cannot introduce violations of the conservation of the energy-momentum tensor (15) in a straightforward manner.[4] Finally, the constraint associated with λ forces

$$\partial_\mu V^\mu = \sqrt{-g}\,. \tag{43}$$

The status of the cosmological constant problem in this formulation is very similar to the version discussed in Section 2.1. We can see that the Lagrange multiplier λ enters the Einstein equations in the same way as in (25). It is thus not surprising that determining the cosmological constant by fixing λ directly through an initial condition will lead to the CC problem. Similarly, trying to use the constraint within the action to decouple any terms of the form

$$\rho_{vac}\sqrt{-g}\,, \tag{44}$$

will yield shifts of the initial value set up for λ exactly like in (31). On the other hand, the same solution that worked in Section 2.1 works here as well. If we give up the initial condition on λ, we may find a form of the action where λ can be determined uniquely. The steps are similar as well. We use the constraint (43) in the action to substitute

$$g_{\mu\nu} \to \hat{g}_{\mu\nu} = g_{\mu\nu}\left(\frac{\partial_\mu V^\mu}{\sqrt{-g}}\right)^{1/2}, \tag{45}$$

which yields the action

$$S[g,\lambda,V,\Psi] = \int_{\mathcal{M}} d^4x \left[-\frac{1}{2}\hat{R}\partial_\mu V^\mu - \lambda\big(\partial_\mu V^\mu - \sqrt{-g}\big)\right] + S_{matter}[\hat{g},\Psi]\,. \tag{46}$$

Upon variation, this gives the following tensor equation of motion

$$\hat{G}_{\mu\nu} - \frac{1}{4}\hat{G}\hat{g}_{\mu\nu} + \lambda g_{\mu\nu} = \hat{T}_{\mu\nu} - \frac{1}{4}\hat{T}\hat{g}_{\mu\nu}\,. \tag{47}$$

Taking the trace of this equation immediately gives us $\lambda = 0$ and thus λ is determined uniquely. Crucially, any quantum correction to the vacuum energy from the matter sector or the gravitational sector couples directly to $\sqrt{-\hat{g}}$, which immediately reduces it to a boundary term since the metric $\hat{g}_{\mu\nu}$ by construction satisfies

$$\sqrt{-\hat{g}} = \partial_\mu V^\mu \,. \tag{48}$$

Hence, such terms decouple trivially, which renders the cosmological constant stable under quantum corrections of the vacuum energy. Note that unlike (32) the metric $\hat{g}_{\mu\nu}$ is a metric in a true sense—a rank 2 tensor with a density weight of 0. Furthermore, the constraint (43) reduces it to

$$g_{\mu\nu} = \hat{g}_{\mu\nu} \,. \tag{49}$$

so we can drop the hats in tensor equation of motion (47).

2.3. Diffeomorphism Covariant, Weyl Invariant UG

Finally, similarly to (33) the constraint in the action (46) can be omitted to give

$$S[g,V,\Psi] = S_{GR}[\hat{g}(g,V),\Psi] = -\frac{1}{2}\int_{\mathcal{M}} d^4x \hat{R}\partial_\mu V^\mu + S_{matter}[\hat{g},\Psi] \,. \tag{50}$$

This theory amounts to ordinary GR, whose metric is transformed using the *manisfestly Weyl invariant* redefinition (45). The Weyl symmetry of the ansatz is then inherited by the entire action (50) and thus we obtain a Weyl invariant, fully covariant theory of UG. This theory was first suggested in [48] and was later found as a generalization of mimetic gravity in [25], where the classical equivalence to HT formulation of UG (40) was pointed out.

In comparison to (36), the presence of the derivative of the vector field V^μ in the redefinition (45) implies that (50) is a higher derivative vector-tensor theory. This can be easily seen by expanding the scalar curvature in the gravitational part of (50) to obtain

$$S_{grav} = -\frac{1}{2}\int d^4x\,\sqrt{-g}\left[\sqrt{D}R + \frac{3}{8}\frac{g^{\mu\nu}\partial_\mu D\partial_\nu D}{D^{3/2}}\right], \tag{51}$$

where we denote

$$D = \frac{\partial_\mu V^\mu}{\sqrt{-g}} \,. \tag{52}$$

We can directly see that the action contains up to second time-derivatives of the time component of V^μ. Nevertheless, due to the structure, in which these terms enter the action, and crucially, due to the universal coupling of V^μ to the matter sector, the equations of motion for V^μ reduce to a very simple form when written using $\hat{g}_{\mu\nu}$. Specifically, they become

$$\partial_\mu(\hat{G} - \hat{T}) = 0 \,. \tag{53}$$

The dynamics of the vector field V^μ, as perceived in the spacetime that is given by $\hat{g}_{\mu\nu}$, is determined by the built-in constraint (48). Finally, the tensor equations of motion for $g_{\mu\nu}$ are the traceless equations (11) evaluated using the metric $\hat{g}_{\mu\nu}$. Note that these equations are understood as equations for the field $\hat{g}_{\mu\nu}$ rather then for $g_{\mu\nu}$. Hence, despite the higher derivative structure of this theory, the classical dynamics are equivalent to UG (40) and the theory does not suffer from any ghost instability.[5] Crucially, in contrast to (36) the reliance of this mechanism on the extra vector field V^μ allows us to propose deviations from the basic theory, without affecting the decoupling mechanism for vacuum energies. For example, we can introduce novel terms in the action, which alter the dynamics of the vector field. To preserve the decoupling mechanism, these terms must depend strictly on $\hat{g}_{\mu\nu}$ and on the composite vector field

$$W^\mu = \frac{V^\mu}{\partial_\nu V^\nu} \,. \tag{54}$$

By adding such terms, the theory no longer describes UG, as the original cosmological constant can acquire non-trivial dynamics and thus could become a more complicated model of dark energy, which not only models late time acceleration of the universe but also solves the old cosmological constant problem.

The form of the redefinition (45) is not unique in its ability to facilitate the decoupling of vacuum energies from gravity. An alternative ansatz was proposed in [26], where the metric $\hat{g}_{\mu\nu}$ is given through the following relation

$$g_{\mu\nu} \to \hat{g}_{\mu\nu} = \frac{g_{\mu\nu}}{\sqrt[4]{-g}} \sqrt{\mathcal{P}} \tag{55}$$

where $\mathcal{P}$ is the Pontryagin density

$$\mathcal{P} = \mathrm{Tr}\, \tilde{F}^{\mu\nu} F_{\mu\nu} \,, \tag{56}$$

constructed from the Yang–Mills gauge field strength

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu \,. \tag{57}$$

The derivative D_μ is the covariant derivative associated with A_μ. Finally, $\tilde{F}^{\mu\nu}$ is the density dual of $F_{\mu\nu}$

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \,, \tag{58}$$

and $\epsilon^{\mu\nu\rho\sigma}$ is the Levi–Civita symbol. Crucially, the mechanism by which the vacuum energies decouple is intact as the metric determinant is again constrained to[6]

$$\sqrt{-\hat{g}} = \mathcal{P} \,, \tag{59}$$

which is a total derivative of the Chern–Simons current density C^μ

$$\mathcal{P} = \partial_\mu C^\mu \,, \qquad \text{where} \qquad C^\mu = \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr} \left(F_{\nu\rho} A_\sigma - \frac{2if}{3} A_\nu A_\rho A_\sigma \right) . \tag{60}$$

Here, f is the coupling constant of the associated gauge theory. Hence, any corrections to vacuum energy are by construction decoupled as they equate to a total derivative in the Lagrangian

$$\int_{\mathcal{M}} d^4x\, \rho_{vac} \sqrt{-\hat{g}} = \int_{\mathcal{M}} d^4x\, \rho_{vac} \mathcal{P} \,. \tag{61}$$

Applying the redefinition (55) to GR results in the following theory

$$S[g, A, \Psi] = S_{GR}[\hat{g}(g, A), \Psi] = -\frac{1}{2} \int_{\mathcal{M}} d^4x \hat{R} \mathcal{P} + S_{matter}[\hat{g}, \Psi] \,. \tag{62}$$

Due to the Weyl invariance of (55) the tensor equations of motion are again the traceless Einstein equations (11) evaluated using the metric $\hat{g}_{\mu\nu}$. Furthermore, the equations of motion for A_μ give

$$\tilde{F}^{\mu\nu} \partial_\nu (\hat{G} - \hat{T}) = 0 \,. \tag{63}$$

For the U(1) case, the non-vanishing value of $\mathcal{P}$ implies that $\tilde{F}^{\mu\nu}$ is an invertible matrix, so we still recover Equation (53). For SU(N), $\tilde{F}^{\mu\nu}$ is a Lie algebra valued object, but we find that at least one of its components (Lie algebra component) is again an invertible matrix and thus we arrive at an equivalent conclusion (53). Hence, the gravitational dynamics are clearly equivalent to the HT formulation of UG. The dynamics of A_μ are determined by the built-in constraint (59). Note that solutions for this equation exist for arbitrary globally hyperbolic spacetime as long as the gauge group contains an SU(2) subgroup [50]. For the U(1) case, a general proof of existence has not been found yet, but for the standard FRW spacetimes a solution has been constructed explicitly.

There are multiple advantages of using (55) in contrast to (45) and other formulations of UG. The gauge fields A_μ are clearly more natural objects in the context of the Standard Model of particle physics. Hence, the present formulation is advantageous for the study of modifications via possible couplings to ordinary matter. Such extensions could be very interesting to explore as they could provide additional dynamics that might allow us to study selection mechanisms for the otherwise unspecified effective CC in UG. Furthermore, while the quantum corrections to vacuum energy get automatically converted into a total derivative, the resulting boundary term is not necessarily physically irrelevant. In this particular case, it plays the role of the theta term [51,52] for the corresponding Yang–Mills theory, under the assumption that we also include appropriate kinetic term. This then allows us to naively connect the old cosmological constant problem with the strong CP problem of quantum chromodynamics. From Equation (63) it is clear that upon the addition of a kinetic term for A_μ, the unimodular dynamics are altered, and the theory no longer describes GR with a cosmological constant [53]. Nevertheless, the decoupling mechanism for corrections to vacuum energy (61) is still applicable.

Finally, the theory can be written using a Lagrange multiplier in a form analogous to (40). That is

$$S[g,\lambda,A] = \int_{\mathcal{M}} d^4x \Big[-\frac{1}{2}\sqrt{-g}R + \lambda\big(\mathrm{Tr}\,\tilde{F}^{\mu\nu}F_{\mu\nu} - \sqrt{-g}\big)\Big] + S_{matter}[g,\Psi]\,. \tag{64}$$

We can see that in this form the Lagrange multiplier couples to the gauge fields like an axion field. Since λ is constrained to be a constant through the A_μ equation of motion, we can even add a kinetic term $\partial_\mu\lambda\partial^\mu\lambda$ to the action to increase this resemblance further.[7] We obtain

$$S[g,\lambda,A] = \int_{\mathcal{M}} d^4x \Big[-\frac{1}{2}\sqrt{-g}R + \frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\lambda\partial_\nu\lambda + \lambda\big(\mathrm{Tr}\,\tilde{F}^{\mu\nu}F_{\mu\nu} - \sqrt{-g}\big)\Big] + S_{matter}[g,\Psi]\,. \tag{65}$$

Note that this action cannot be straightforwardly traced back to a formulation without Lagrange multiplier. Nevertheless, since the kinetic term is shift symmetric, we are still able to decouple any *constant* corrections to vacuum energy as long as we let the initial conditions on zero mode of λ be free. Again adding a kinetic term for A_μ changes the dynamics significantly; however, it still does not affect the decoupling mechanism. It is an attractive speculation whether unimodular gravity can arise as a dynamical regime of axion where the dynamics of the gauge field become frozen.

2.4. Degrees of Freedom of UG

The field content in the Henneaux and Teitelboim theory is clearly larger than in GR; however, it is accompanied by a large symmetry group of divergenceless shifts of the vector field

$$V^\mu \to V^\mu + \zeta^\mu\,, \qquad \text{where} \qquad \partial_\mu\zeta^\mu = 0\,. \tag{66}$$

Consequently, much of the field content is a pure gauge and the theory can be shown to only contain a single extra global degree of freedom in comparison with its GR counterpart [24,49]. The global degree of freedom here is given as the overall charge associated with the current V^μ, which is defined on a given foliation Σ_t,

$$\mathcal{T} = \int_{\Sigma_t} d\Sigma_\mu V^\mu\,. \tag{67}$$

Here, $d\Sigma_\mu = n_\mu\, d^3y$, where d^3y is the associated coordinate volume element on Σ_t and n_μ is perpendicular to Σ_t. Note that if Σ_t is infinite the definition of $\mathcal{T}$ might result in an infinity as well. In such cases, one has to consider some appropriate regularization in order to make sense of this quantity. $\mathcal{T}$ is often referred to as the 'cosmic time'. The symmetry (66) shifts $\mathcal{T}$ by a constant value

$$\mathcal{T} \to \mathcal{T} + const.\,, \tag{68}$$

as long as ζ^μ vanishes at infinity or on the boundary of Σ_t. Such a shift corresponds to a symmetry of the theory, which naturally results in a conservation law, in this case, the conservation of the conjugate momentum λ. Transformations that leave $\mathcal{T}$ intact form a gauge group of the theory. It follows that $\mathcal{T}$ is the only gauge-invariant information contained within the vector field V^μ and the gauge symmetry can be taken to fix the non-zero mode of $n_\mu V^\mu$ arbitrarily.

Integrating the expression (43), while assuming appropriate conditions for V^μ on the boundary of Σ_t, enables us to calculate the change in $\mathcal{T}$ as

$$\mathcal{T}(t_2) - \mathcal{T}(t_1) = \mathrm{Vol}_{\mathcal{M}}[g] \,, \tag{69}$$

where $\partial\mathcal{M} = \Sigma_{t_2} \cup \Sigma_{t_1}$. The two variables λ and $\mathcal{T}$ are conjugates of each other in the sense that their Dirac bracket is

$$\{\lambda, \mathcal{T}\}_D = 1 \,. \tag{70}$$

Since the constraint part of the action (40) is linear in time derivatives, we can glance this commutation directly from the action using the Fadeev–Jackiw procedure [54,55]. Considering only the spatially constant part of λ, we find that the only term containing time derivatives has the form

$$\int d^4x \lambda \partial_\mu V^\mu \approx \int dt \lambda \frac{d}{dt} \int_{\Sigma_t} d\Sigma_\mu V^\mu = \int dt \lambda \dot{\mathcal{T}} \,. \tag{71}$$

Hence, we see that the momentum associated with $\mathcal{T}$ is indeed λ and relation (70) immediately follows. This can be also confirmed directly using Dirac analysis. Note that while this is true in the original HT formulation (40), in classically equivalent action (46) and (50) this no longer holds as λ is constrained to vanish in the former and is not present in the latter.

In order to obtain a unique evolution in UG, one has to provide both the initial cosmic time $\mathcal{T}_i$ as well as the value for the cosmological constant λ_i. In practise, $\mathcal{T}_i$ can be easily omitted as the evolution of the cosmic time does not affect the dynamics of gravity and other fields. It is often considered unphysical [22,56]. However, we would like to point out that, while $\mathcal{T}$ is indeed unphysical, the difference of the cosmic time between two hypersurfaces Σ_{t_1} and Σ_{t_2} is a physical quantity, namely the total volume [30,46]. This encodes the information about the effective cosmological constant, which can be reconstructed from the knowledge of such difference. This can be seen by fixing $\mathcal{T}(t_2) = \mathcal{T}_2$ and $\mathcal{T}(t_1) = \mathcal{T}_1$ in Equation (69), which yields a global constraint on the four-volume. We can then determine the value of the effective cosmological constant by solving the Einstein equations with an unspecified (positive) cosmological constant Λ and label their solutions by its value. Hence, we obtain a one parameter family of solutions

$$g_{\mu\nu}(\Lambda) \,. \tag{72}$$

Plugging such solutions into Equation (69) with fixed values of the cosmic time yields a single equation that is in general able to determine Λ as a function of $\Delta\mathcal{T} = \mathcal{T}_2 - \mathcal{T}_1$. Let us demonstrate this explicitly on a very simple example. We consider a flat FRW universe, which is void of matter and energy, up to the unspecified cosmological constant. Hence, the solutions of Friedmann equations are

$$a(t) = a_0 e^{\sqrt{\Lambda/3}(t-t_1)} \,. \tag{73}$$

Plugging this into Equation (69) yields the following relation

$$\Delta\mathcal{T} = \frac{a_0^3}{\sqrt{3\Lambda}} \left(e^{\sqrt{\Lambda/3}(t_2-t_1)} - 1 \right) , \tag{74}$$

where we have rescaled the values of the cosmic time to factor out the infinite coordinate volume $\mathrm{Vol}_3 = \int d^3x$. The above equation is an algebraic equation for Λ, which can be solved explicitly using the Lambert W function as

$$\Lambda(\Delta\mathcal{T}) = \frac{1}{3\Delta t^2}\left[\frac{a_0^3\Delta t}{\Delta\mathcal{T}} + W_{-1}\left(-\frac{a_0^3\Delta t}{\Delta\mathcal{T}}\, e^{-\frac{a_0^3\Delta t}{\Delta\mathcal{T}}}\right)\right]^2 , \tag{75}$$

where $\Delta t = t_2 - t_1$ and the fraction $a_0^3\Delta t/\Delta\mathcal{T}$ is assumed to be smaller then 1.[8] Hence, the solution for the scale factor can be written as

$$a(t) = a_0 e^{\sqrt{\Lambda(\Delta\mathcal{T})/3}(t-t_1)} . \tag{76}$$

Note that any correction to the value $\Lambda \to \Lambda + \rho_{vac}$ is irrelevant here. We will reproduce the above solution (76) for any value of ρ_{vac} we account for. Hence, specifying the effective cosmological constant by providing the value $\mathcal{T}$ for two given times t_2 and t_1 is stable under quantum corrections. As has been noted in [14], in this case "it is the spacetime volume that remains fixed, forcing Λ to adjust". Note that the global constraint (69) is qualitatively equivalent to the diffeomorphism invariant constraint (39), with the difference that instead of $\mathcal{T}_{1,2}$ we are given a coordinate volume of the spacetime region $\mathcal{M}$. The constraint (39) is automatically present in formulations, which do not rely on the use of the Lagrange multiplier. This implies that resolution of the CC problem in UG inherently involves the existence of such a global constraint. A similar procedure, where the cosmological constant is determined from a global constraint was utilized in the theory of vacuum energy sequestering [13], which we discuss later in Section 4, as well as in the context of normalized General Relativity [57,58].

Note that these global constraints do not need to entail 'knowledge of the future' as both hypersurfaces Σ_{t_1,t_2} can be located in the past. This still fixes the cosmological constant, which can then be taken to determine the solutions for arbitrary future times. Note, that the hypersurfaces must not be too close to each other as the infinitesimal change in $\mathcal{T}$ becomes insensitive to the cosmological constant. Indeed, in the current setting, taking the limit $t_2 \to t_1$ would give us

$$\dot{\mathcal{T}} = \int_{\Sigma_t} d^3y\sqrt{-g} . \tag{77}$$

which for the FRW solution (73) gives us

$$\dot{\mathcal{T}}_1 \propto a_1^3 = a_0^3 . \tag{78}$$

which does not depend on Λ.

3. Quantum Aspects of UG

As we have seen in the previous section, the way we specify the initial conditions for our variables substantially affects the behavior of the theory with respect to the quantum correction of the effective cosmological constant. We have demonstrated this behavior on a semi-classical level, where the quantum corrections of the vacuum energy have been accounted for only as unspecified shifts of the energy-momentum tensor (8). The entire gravitational sector has been considered only on a classical level. In order to resolve the old CC problem to full satisfaction, one should address it in a quantum setting. Since the structure of UG is so similar to GR the problem of finding its fully quantum formulation is as problematic as that in Einstein's theory. Hence, the full quantum treatment is not within our technical reach. However, the extra degrees of freedom that are present in UG have a very simple structure and can be quantized separately from the degrees of freedom of the metric and matter.

In this section, we review a procedure where these degrees of freedom are integrated out in the path integral sense and by doing so they introduce a minor or no modification of the ordinary GR dynamics [22,27,28,31,56]. The distinction hinges on the way we carry out such integration. In particular we will explore two ways. Either we fix the initial condition for λ, or we keep λ free. The effect of such a procedure is in line with our argument from Section 2. That is, the former way reconstructs GR, while the latter offers a resolution to the old CC problem. Finally, it has been observed that the canonical structure of the unimodular degrees of freedom implies non-trivial commutation relations for the cosmological constant and the spacetime volume. This naively implies the presence of quantum fluctuations of these quantities, which could present a possible distinction between GR and UG on a quantum level.

3.1. Path Integral

The full formal expression for the generating function in UG for the action (40) can be written *formally* as a path integral

$$Z[J] = \int [Dg][D\Psi][D\lambda][DV] \exp(iS[g,\Psi,\lambda,V] + iS_{ext}[g,\Psi,J]) \,. \tag{79}$$

Note that we couple the external current only to the metric and matter degrees of freedom and not the fields λ or V^μ. The extra degrees of freedom of unimodular gravity are not deeply intertwined with the rest of the gravitational dynamic as they are neatly isolated within the constraint part of the action. Hence, we can integrate them out separately prior to integration of the metric degrees of freedom or matter degrees of freedom.[9] We may thus define this partial integration as

$$\mathcal{I} \equiv \int [D\lambda][DV] \exp(iS[g,\Psi,\lambda,V]) \,. \tag{80}$$

The generating functional (79) can then be calculated by integrating $\mathcal{I}$ over the metric and matter degrees of freedom along with external sources. The integration (80) can be carried out in more than one way depending on how we fix the initial and final condition for our fields λ and V^μ or rather for the associated degree of freedom λ and $\mathcal{T}$. We are going to be mainly looking at two ways: First, we fix the initial and final value of the cosmic time $\mathcal{T}$ and second we fix the initial and final value of λ. Note that the latter case has been worked out in [22]. Technically, it is possible to fix both as has been done in [31]; however, we would like to point out that the knowledge of both values of λ and $\mathcal{T}$ is prohibited due to the commutation relation (70). Hence, the physically relevant calculation fixes only one of these variables on a given spatial slice.

We first consider the following path integral where the endpoint values for $\mathcal{T}$ are fixed

$$\mathcal{I}_{\mathcal{T}} \equiv \int_{\mathcal{T}_i}^{\mathcal{T}_f} [D\lambda][DV] \exp(iS[g,\Psi,\lambda,V]) \,. \tag{81}$$

The action in the exponent is taken to be the HT action for UG (40). The integration over the field V^μ in Equation (81) is taken only over configurations that satisfy the following conditions

$$\int_{\Sigma_i} d\Sigma_\mu V^\mu = \mathcal{T}_i \,, \qquad \text{and} \qquad \int_{\Sigma_f} d\Sigma_\mu V^\mu = \mathcal{T}_f \,. \tag{82}$$

Note that the cosmic time $\mathcal{T}$ is the only gauge invariant information in V^μ and thus when we take the appropriate fixing of the symmetry (66) into account the above conditions determine the initial and final configurations of V^μ completely. The integration over λ is carried out freely without fixing the endpoints. As a first step, we divide the action into the GR component and the constraint

$$S_{HT}[g,\Psi,\lambda,V] = S_{EH}[g,\Psi] + \int_{\mathcal{M}} d^4x\, \lambda\left(\partial_\mu V^\mu - \sqrt{-g}\right) . \tag{83}$$

Here, S_{EH} corresponds to the Einstein–Hilbert action together with an unspecified action for matter fields Ψ in the theory. This part of the action is unaffected by the integration and thus we may focus on the constraint itself. In order to isolate the initial and final condition on the cosmic time, we first integrate by parts to obtain

$$S_{HT}[g,\Psi,\lambda,V] = S_{EH}[g,\Psi] + \int_{\mathcal{M}} d^4x \left(-V^\mu\partial_\mu\lambda - \lambda\sqrt{-g}\right) + \int_{\partial\mathcal{M}} d\Sigma_\mu \lambda\, V^\mu \,. \tag{84}$$

Since the last term is evaluated on the boundary $\partial\mathcal{M} = \Sigma_f \cup \Sigma_i$ where V^μ is fixed, this term is unaffected by the integration over V^μ. Hence, the integration over V^μ gives us a delta function

$$I_{\mathcal{T}} = \int_{\mathcal{T}_i}^{\mathcal{T}_f} [D\lambda]\delta(\partial_\mu\lambda) \exp\left(iS_{EH}[g,\Psi] - i\int_{\mathcal{M}} d^4x\, \sqrt{-g}\lambda + i\int_{\partial\mathcal{M}} d\Sigma_\mu\lambda\, V^\mu\right) . \tag{85}$$

The integration over the delta function fixes λ to a constant and thus it can be taken in front of the integral in the action. This allows us to express V^μ completely as the cosmic time $\mathcal{T}$

$$I_{\mathcal{T}} = \int_{-\infty}^{\infty} d\lambda \exp\left(iS_{EH}[g,\Psi] - i\lambda\left(\mathcal{T}_f - \mathcal{T}_i - \int_{\mathcal{M}} d^4x\sqrt{-g}\right)\right) . \tag{86}$$

Note that the delta function did not fix λ completely and thus we are meant to integrate over the residual constant part. This gives us an ordinary delta function fixing a global constraint

$$\mathcal{I}_{\mathcal{T}} = \delta(\mathcal{T}_f - \mathcal{T}_i - \mathrm{Vol}_{\mathcal{M}}[g]) \exp(iS_{EH}[g,\Psi]) \,. \tag{87}$$

We can see that the integration over the Lagrange multiplier λ introduces an extra global constraint on the metric volume of the considered spacetime region $\mathcal{M}$

$$\mathrm{Vol}_{\mathcal{M}}[g] = \mathcal{T}_f - \mathcal{T}_i \,. \tag{88}$$

From Equation (86), we can see that the Einstein–Hilbert action thus obtains an unspecified cosmological constant term, which is, however, classically fixed by the global volume as we have explained in Section 2.4. Note that any shift of the vacuum energy, which we may obtain by integrating out some heavy modes of matter fields $\int[D\Psi]$ only acts to rescale the entire expression

$$\mathcal{I}_{\mathcal{T}} \to \exp\left(i\rho_{vac}(\mathcal{T}_f - \mathcal{T}_i)\right) I_{\mathcal{T}} \,, \qquad \text{as} \qquad S_{EH}[g,\Psi] \to S_{EH}[g,\Psi] + \rho_{vac}\mathrm{Vol}_{\mathcal{M}}[g] \tag{89}$$

Hence, correlation functions of any kind remain unaffected by such shifts and consequently local measurements are unaffected as well.

Now we consider the situation where we fix the endpoint values of λ. This gives an expression nearly identical to (81)

$$\mathcal{I}_\lambda = \int_{\lambda_i}^{\lambda_f} [D\lambda][DV] \exp(iS[g,\Psi,\lambda,V]) \,. \tag{90}$$

However, in order to obtain a consistent result, we must alter the action that we use. Note that in (40) one must pose a vanishing boundary condition for the field V^μ. This corresponds to a fixing of initial and final configuration of V^μ and consequently of $\mathcal{T}$. Such an action is thus suitable for calculating path integrals with fixed initial and final $\mathcal{T}$. The appropriate action for calculating the integral with fixed initial and final value of λ is instead (84) with the boundary term being dropped [56]. Hence,

$$S[g,\Psi,\lambda,V] = S_{EH}[g,\Psi] + \int_{\mathcal{M}} d^4x \left(-V^\mu\partial_\mu\lambda - \lambda\sqrt{-g}\right) . \tag{91}$$

In this action, we must pose the vanishing of variation of λ in order to obtain equations of motion. The integration over V^μ in (90) is completely free, while the integration over λ has fixed endpoints

$$\lambda(t_i) = \lambda_i\,, \qquad \text{and} \qquad \lambda(t_f) = \lambda_f\,. \tag{92}$$

We can carry out the integration over V^μ directly to obtain

$$\mathcal{I}_\lambda = \int_{\lambda_i}^{\lambda_f} [D\lambda]\delta(\partial_\mu \lambda) \exp(iS_{GR}[g,\Psi] - i\lambda \mathrm{Vol}_{\mathcal{M}}[g])\,. \tag{93}$$

The integration over the delta function now gives

$$\int_{\lambda_i}^{\lambda_f} [D\lambda]\delta(\partial_\mu \lambda) = \delta(\lambda_f - \lambda_i)\,. \tag{94}$$

Hence, we obtain

$$\mathcal{I}_\lambda = \delta(\lambda_f - \lambda_i) \exp(iS_{GR}[g,\Psi] - i\lambda_i \mathrm{Vol}_{\mathcal{M}}[g])\,. \tag{95}$$

In this case, the cosmological constant is specified directly by λ_i. Consequently, the effective cosmological constant receives all shifts of vacuum energy ρ_{vac} from the matter sector.

We can see that the result (87) is insensitive to quantum corrections of the vacuum energy while the latter (95) is affected. Crucially, the difference in considerations that lead to these results is exactly in line with the semi-classical case that we have discussed in Section 2. In particular, choosing the initial value for the Lagrange multiplier λ spoils the solution of the old cosmological constant problem. Instead, allowing λ to be free yields a formulation where the effective cosmological constant is stable against radiative corrections. In the semi-classical case, this corresponds to solving for λ algebraically, without initial conditions, while in the present setting it corresponds to integration over the Lagrange multiplier including its zero mode. This distinction is consistent with other results on various aspects of quantum UG. For example, refs. [22,28,31,56] fix λ by hand and the results point toward the conclusion that the status of the cosmological constant in UG is not any different from GR. On the other hand, refs. [27,30,62–65] base their calculations on formulations that do not rely on a Lagrange multiplier to enforce constraint (21) and their conclusions are consistent with the old CC problem being solved in UG.

3.2. *Quantum Fluctuations*

As we have seen in Section 2.4, the two global quantities $\mathcal{T}$ and λ form a conjugate pair, with the following Dirac bracket relation

$$\{\mathcal{T}, \lambda\} = 1\,. \tag{96}$$

Upon standard canonical quantization, such a relation becomes a commutator of operators due to the correspondence principle

$$[\hat{\mathcal{T}}, \hat{\lambda}] = 1\,. \tag{97}$$

Consequently, the two associated observables are not simultaneously measurable and the corresponding quantities are subjected to quantum fluctuations, whose size is constrained by the Heisenberg uncertainty relations

$$\delta\mathcal{T} \times \delta\lambda \geq \frac{1}{2}\,. \tag{98}$$

Since the measurement of cosmic time between two hypersurfaces corresponds to the spacetime four-volume, any uncertainty in the measurement of $\mathcal{T}$ is translated to an uncertainty of the four-volume itself. Hence, we can write [32,66]

$$\delta\lambda \times \delta\mathrm{Vol}_{\mathcal{M}}[g] \geq \frac{1}{2} \,. \tag{99}$$

Such fluctuations are mostly harmless as the four-volume is typically very large and any fluctuations in it can be localized very far from a local observer, potentially even in a causally disconnected region. Hence, we may usually measure λ with an arbitrary precision. It follows that such fluctuations are unlikely to have any effect in our Universe; however, they present a conceptual difference between quantum GR and UG.

Nevertheless, we can imagine situations where such fluctuations can have significant effects. Consider a closed, radiation-dominated Friedman universe. The associated scale factor hence evolves as

$$a(\eta) = a_m \sin(\eta) \,, \tag{100}$$

where a_m is the scale factor at the turning point and η is the conformal time. It is reasonable to assume that the fluctuations of the cosmic time are smaller than the total four-volume. Hence, we obtain [66]

$$\delta\mathcal{T} < \mathrm{Vol}_4[g] = \frac{3\pi^3}{4} a_m^4 \,. \tag{101}$$

Using the uncertainty relation, we find a lower bound on the fluctuations of the cosmological constant

$$\delta\lambda > \frac{2}{3\pi^3} a_m^{-4} \,. \tag{102}$$

Clearly, this is negligible in a large universe, but it renders small universes inconsistent as large fluctuations of λ violate the assumption of radiation domination. It would be interesting to see whether such a small universe, which would quickly collapse in the ordinary GR setting, could grow large due to such a fluctuation in λ. It would also be of interest if such fluctuations could be recovered through the path integral techniques, which have been explored in greater detail. Finally, we would like to note that just a slight modification of the Henneaux and Teitelboim construction (40) can be used to promote any physical constant to a degree of freedom by promoting the constant α to a scalar field and introducing the term

$$V^\mu \partial_\mu \alpha \,. \tag{103}$$

By extension, we can obtain Heisenberg uncertainty relations for various constants with their associated global conjugates. This has been performed for the Planck mass and the Planck constant [32] as well as various others [67–69].

4. Vacuum Energy Sequestering

Another notable theory that aims to address the old cosmological constant problem is vacuum energy sequestering [13,14]. This proposal shares multiple similarities with UG, in particular in its local formulation [33] and hence it is useful to compare the two here. The original idea relies on an introduction of global mechanics, which enforce a pair of global constraints. These constraints then determine the effective cosmological constant in a manner that is stable against quantum corrections of vacuum energy. The global dynamics are introduced by considering a pair of *global variables* θ and Λ. The former is input by hand as a rescaling of the physical metric in the gravitational sector

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \theta^{-2} g_{\mu\nu} \,. \tag{104}$$

The second variable is the cosmological constant of GR promoted to an independent variable with no spacetime dependence. Hence, the *local* part of the action is modified as

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2\theta^2}R + \Lambda + \mathcal{L}(g_{\mu\nu}, \Psi)\right] . \quad (105)$$

Furthermore, this action is supplemented by a *global* term

$$\sigma\left(\frac{\Lambda}{\mu^4}\right) . \quad (106)$$

where σ is an arbitrary monotonous function and μ is an unspecified dimensionful parameter, which is meant to be measured. The total action is

$$S[g, \Psi, \Lambda, \theta] = \int d^4x\sqrt{-g}\left[-\frac{1}{2\theta^2}R + \Lambda + \mathcal{L}(g_{\mu\nu}, \Psi)\right] + \sigma\left(\frac{\Lambda}{\mu^4}\right) . \quad (107)$$

Crucially, the novel variables θ and Λ are not fields and have no spacetime dependence, yet they are subjected to the variation principle. Consequently, their equations of motion yield two global constraints

$$\int d^4x\sqrt{-g}R = 0 , \qquad \frac{\sigma'}{\mu^4} = \int d^4x\,\sqrt{-g} . \quad (108)$$

It is useful to introduce a spacetime average of a scalar quantity as $\langle\phi\rangle \equiv \int d^4x\sqrt{-g}\phi / \int d^4x\sqrt{-g}$. Using this, we can rewrite the first constraint as

$$\langle R\rangle = 0 . \quad (109)$$

The equations of motion for the metric $g_{\mu\nu}$ are given as

$$\theta^{-2}G_{\mu\nu} = T_{\mu\nu} - \Lambda g_{\mu\nu} . \quad (110)$$

We can clearly see that these are just Einstein equations with an unspecified cosmological constant and unspecified rescaling of the Planck mass. The key property of the sequestering mechanism is that the global constraints (108) allows us to find an explicit expression for Λ that does not reduce the Einstein equations to traceless equations (4). This is achieved by taking the trace and a spacetime average of (110). By doing so, we obtain

$$-\langle R\rangle = \langle T\rangle - 4\Lambda . \quad (111)$$

We can use the first of the two global constraints (108) to eliminate the average curvature to find[10]

$$\Lambda = \frac{1}{4}\langle T\rangle . \quad (112)$$

This can be plugged back into Einstein equations (110), which now read

$$\theta^{-2}G_{\mu\nu} = T_{\mu\nu} - \frac{1}{4}\langle T\rangle g_{\mu\nu} . \quad (113)$$

This equation clearly possesses the same symmetry (8) as UG. However, unlike UG the covariant divergence of this equation vanishes identically. Hence, there is no differential constraint, which would give rise to an additional component of the cosmological constant. Instead, we obtain a full set of 10 equations. The form of (113) is rather unusual as it contains a term that is non-local in time. Hence, it would seem it might be difficult to interpret it as an evolution equation for a set of initial data. Nevertheless, finding solutions of these equations is rather straightforward. The method is exactly the same as we discussed in

Section 2.4. We consider the Einstein equations with an *unspecified* cosmological constant λ and the parameter θ

$$\theta^{-2} G_{\mu\nu} = T_{\mu\nu} - \lambda g_{\mu\nu}\,, \tag{114}$$

and find a family of solutions labeled by their values: $g_{\mu\nu}(\theta,\lambda)$. For such solutions, we evaluate the energy-momentum tensor $T_{\mu\nu}(\theta,\lambda)$ and calculate its associated spacetime average. Plugging such an expression into Equation (112) and setting $\Lambda=\lambda$, we find a consistency equation

$$\Lambda = \frac{1}{4}\left\langle T\right\rangle(\theta,\Lambda)\,. \tag{115}$$

The actual value of Λ is then selected as a solution of this equation in terms of θ. Note that the existence of a solution is not in general guaranteed. If it does not exist, then the entire family of metrics $g_{\mu\nu}(\theta,\Lambda)$ are not solutions of Equation (113). Once we find Λ, we can plug it back into the second global constraint in (108) in order to determine θ. Crucially, unlike UG, the vacuum energy sequester does not allow for an arbitrary value of the cosmological constant but a specific one which is determined by the above procedure.

Local Formulation

The main disadvantage of the sequestering proposal is that its formulation requires an unusual global term and variables. To remedy this, a local formulation of the theory has been proposed in [33], which can be obtained by a 'localization' of the global dynamics of (107). The strategy to localize action (107) is rather simple. The global variables θ and Λ are promoted to local variables—scalar fields

$$\Lambda \to \Lambda(x)\,, \qquad \theta \to \theta(x)\,. \tag{116}$$

However, their non-constant part is immediately constrained to vanish using vector Lagrange multipliers

$$V^{\nu}\,\partial_{\nu}\sigma\left(\frac{\Lambda}{\mu^4}\right), \qquad W^{\nu}\,\partial_{\nu}\tilde{\sigma}\left(\frac{1}{\theta}\right), \tag{117}$$

where σ and $\tilde{\sigma}$ are taken to be monotonous functions and V^{μ} and W^{μ} are vector densities. Hence, the equations of motion for V^{μ} and W^{μ} yield

$$\partial_{\mu}\Lambda = \partial_{\mu}\theta = 0\,. \tag{118}$$

Λ and θ thus become global variables only on equations of motions rather than a priori. Including the constraint terms in the action yields

$$\begin{aligned} S[g,\Psi,\Lambda,\theta] = \int d^4x\sqrt{-g}\left[-\frac{1}{2\theta^2}R - \Lambda + \mathcal{L}(g_{\mu\nu},\Psi)\right] \\ + \int d^4x\left[\partial_{\mu}V^{\mu}\,\sigma\left(\frac{\Lambda}{\mu^4}\right) + \partial_{\mu}W^{\mu}\,\tilde{\sigma}\left(\frac{1}{\theta}\right)\right]. \end{aligned} \tag{119}$$

The extra constraint terms are metric independent and thus they do not affect the gravitational equations; however, we obtain new relations that govern the dynamics of the extra fields. In total, we obtain the following set of equations

$$\theta^{-2}G_{\mu\nu} = T_{\mu\nu} - \Lambda g_{\mu\nu}\,, \tag{120}$$

$$\frac{\sigma'}{\mu^4}\partial_{\mu}V^{\mu} = \sqrt{-g}\,, \qquad \frac{\tilde{\sigma}'}{\theta}\partial_{\mu}W^{\mu} = \sqrt{-g}R\,, \tag{121}$$

$$\partial_{\mu}\Lambda = 0\,, \qquad \partial_{\mu}\theta = 0\,. \tag{122}$$

We can see that the action (119), as well as the associated equations of motion, bear a strong resemblance to unimodular gravity. The tensor equation of motion remains unaffected by the change in description and retains its form (110). The vector field V^μ has the same role as it had in the HT formulation of UG—to force constancy of Λ. Analogously, the vector density W^μ is used to force the constancy of θ. In essence, the effective Planck mass and the cosmological constant are promoted to integration constants from the original bare coupling constants. The physical interpretation and pitfalls of this theory are consequently very similar to UG. Indeed, if we provide initial conditions for Λ and θ directly in order to solve Equation (122), we obtain an ordinary Einstein equations (120) with the chosen constants. Such an approach is clearly no different from choosing the cosmological constant directly in UG, and hence it will be unstable against radiative corrections. However, similar to UG, the cosmological constant can be prescribed in a stable manner. To demonstrate this, we first analyze the equations of motion for θ and Λ (121). These are local analogues of the global constraints (108). While being local, these equations describe the evolution of two global quantities, namely the 'cosmic times' $\mathcal{T}$ and $\tilde{\mathcal{T}}$ associated with V^μ and W^μ, which can be introduced as in (67). Such quantities are sourced by the spacetime volume and the integrated curvature, respectively,

$$\mathcal{T}_{t_2} - \mathcal{T}_{t_1} = \frac{\mu^4}{\sigma'}\mathrm{Vol}_{\mathcal{M}}[g]\,, \qquad \tilde{\mathcal{T}}_{t_2} - \tilde{\mathcal{T}}_{t_1} = \frac{\theta}{\tilde{\sigma}'}\int_{\mathcal{M}} d^4x\sqrt{-g}R\,. \tag{123}$$

These are global equations, which can be used in the same manner as the original global constraints (108). In particular, consider taking the trace and a spacetime average of Equation (120). We again find (111). By taking the ratio of Equation (123), we can express the averaged curvature

$$\langle R\rangle = \frac{\sigma'}{\tilde{\sigma}'}\frac{1}{\theta\mu^4}\frac{\tilde{\mathcal{T}}_{t_2} - \tilde{\mathcal{T}}_{t_1}}{\mathcal{T}_{t_2} - \mathcal{T}_{t_1}}\,. \tag{124}$$

Hence, using (111), we find

$$\Lambda = \frac{1}{4}\langle T\rangle + \Delta\Lambda\,, \tag{125}$$

where

$$\Delta\Lambda = \frac{\sigma'}{\tilde{\sigma}'}\frac{1}{\theta^3\mu^4}\frac{\tilde{\mathcal{T}}_{t_2} - \tilde{\mathcal{T}}_{t_1}}{\mathcal{T}_{t_2} - \mathcal{T}_{t_1}}\,. \tag{126}$$

Note that (125) is not in general an explicit solution for Λ as σ is a function of Λ. Nevertheless, plugging this expression into the Einstein equations (120), we find the sequestered equations (110) up to an extra vacuum energy piece $\Delta\Lambda$

$$\theta^{-2}G_{\mu\nu} = T_{\mu\nu} - \frac{1}{4}\langle T\rangle g_{\mu\nu} - \Delta\Lambda g_{\mu\nu}\,. \tag{127}$$

These equations again have the shift symmetry (8), which cancels out the quantum corrections of vacuum energy on the right hand side. Furthermore, the extra piece of the effective cosmological constant, $\Delta\Lambda$, does not depend on the energy and momentum of matter at all. It depends only on gravitational quantities such as the integral of V^μ and W^μ, which are sourced by the four-volume and the scalar curvature. Thus, $\Delta\Lambda$ does not directly carry the information about the energy and momentum of matter. Consequently, since gravitational degrees of freedom are protected via the symmetry (8) in the tensor equation, $\Delta\Lambda$ does not receive any such corrections. Note that unlike in the original global version of the sequester, we do not have a direct expression for the effective cosmological constant. Instead, such a constant must be determined through measurement. The main point of the above discussion is to demonstrate that such a value is then stable under the radiative corrections.

We would like to comment here that the equations of the local version of the sequester can be reduced to the original (113) if we allow ourselves to prescribe initial and final values of the cosmic times $\mathcal{T}$ and $\tilde{\mathcal{T}}$. In particular, the choice $\tilde{\mathcal{T}}_{t_2} = \tilde{\mathcal{T}}_{t_1}$ reduces the second Equation (123) to the constraint (109). Consequently, we obtain $\Delta\Lambda = 0$, so Equation (127) reduces exactly to (113). Finally, the localization procedure described in this section can be reversed and applied to unimodular gravity (40) to find a unimodular analogue of the sequestering mechanism. Doing so implies that λ becomes a global variable

$$\lambda(x) \to \lambda \,. \tag{128}$$

This allows us to integrate the divergence of the vector in the constraint part of the action (40) to obtain

$$S_{const} = \lambda\Big(\mathcal{T}_f - \mathcal{T}_i - \mathrm{Vol}_{\mathcal{M}}[g]\Big) \,. \tag{129}$$

The variation of the global λ now implies Equation (69), where the cosmic time at the endpoints must be a priori specified. Upon variation, this yields Einstein equations with a cosmological constant that is determined through the global constraint (69). Note that the crucial difference in comparison to sequestering is that the freedom in choosing $\mathcal{T}_f - \mathcal{T}_i$ allows us to reconstruct *any* value of the cosmological constant. In sequestering the global constraint (109) that determines Λ does not present any such choice. On the other hand, the solution for θ is affected by the choice of the function σ.

5. Conclusions

In this work, we discussed whether unimodular gravity is or is not able to reconcile the old cosmological constant problem. In Section 2, we pointed out that the answer hinges on a rather minute technicality—on how one provides the data that determine the effective cosmological constant. This point is completely mute on a classical level; however, it becomes crucial on a semi-classical level, when we introduce quantum corrections to vacuum energy. The distinction arises in theories that use Lagrange multiplier (22) and (40) in order to enforce their respective constraints (21) and (43). In such formulations, one often encounters that the initial condition is set up for the Lagrange multiplier directly. Such fixing implies that the zero mode of the multiplier is not varied in the action and hence the associated constraint is enforced only locally. The local versions of the constraints are, however, nearly 'empty' as GR possesses enough gauge symmetry to satisfy them without any effect on the dynamics. Consequently, setting up the cosmological constant in this manner amounts to little to no change in the dynamics in comparison to GR with a chosen CC. Hence, we found that the cosmological constant problem is still present when CC is chosen in this way. Leaving the initial conditions of the Lagrange multiplier be free implies that the constraints are enforced fully. This introduces a global constraint on the four-volume (39) and (69), which is able to fix the effective cosmological constant in a manner that is stable against quantum corrections (76). Hence, such a route offers a resolution of the old cosmological constant problem. The above-mentioned issues are not encountered in theories, where the metric is endowed with a composite structure, which enforces the appropriate constraints (33) and (50) automatically. When there are no Lagrange multipliers, we cannot assign initial values to them. For this reason such formulations can be considered to have an advantage over the Lagrange multiplier ones and indeed the cosmological constant problem has been reported to be solved in these versions of UG [62,63].

We discussed a recently proposed pair of formulations of UG (50) and (62) [25,26] in Section 2.3. These proposals combine many desired properties as they are fully diffeomorphism covariant, Weyl invariant theories that do not rely on a Lagrange multiplier. Hence, the cosmological constant problem is unambiguously solved within them. We discuss possible extensions of these theories beyond unimodular gravity and how they can fit within the Standard Model of particle physics, while making sure that the decoupling mechanism for quantum corrections of vacuum energy is still functional. We point out a striking similarity of the proposal (64) to the axion dynamics of SU(3) Yang–Mills theory.

In Section 3, we reviewed the path integral quantization of the unimodular degree of freedom, the cosmological constant, in the HT formulation (40) with Lagrange multiplier. The structure of the additional degree of freedom is very simple and can be integrated out easily separately from the metric and matter degrees of freedom. Such an integration can be carried out in two ways: by either fixing the initial and final value of the cosmological constant itself or by doing the same for its conjugate quantity—the cosmic time (67). This is a direct analogue of the initial value ambiguity in the semi-classical case discussed in Section 2 and leads to the same conclusion. That is, directly choosing the initial value of the cosmological constant spoils the solution of the cosmological constant problem. Conversely, the second route, fixing the cosmic time, leads to a reconciliation of the CC problem by introducing a global constraint. We further discuss that the promotion of the cosmological constant to a degree of freedom naively leads to an appearance of global fluctuations of the cosmological constant. Such fluctuations have likely no effect in our Universe due to its large size. Nevertheless, the existence of these fluctuations presents a conceptual difference of UG from GR.

Finally, we discussed the vacuum energy sequestering [13] in Section 4. We reviewed its basic formulation and compared its working with UG. In our view, the mechanism that allows UG to alleviate the CC problem is surprisingly similar to the mechanism of vacuum energy sequestering in that the two theories can be both viewed as relying on a global constraint to achieve the resolution of the old cosmological constant problem. In contrast to UG, the original sequestering proposal does not allow us to stray away from this global structure and thus it is guaranteed to provide a reconciliation of the old cosmological constant problem. Furthermore, in comparison to UG, the constraint that determines the cosmological constant in sequestering is uniquely fixed. In this sense, sequestering is more constrained than UG. The similarities between sequestering and UG are even more pronounced in the local formulation (119), which unfortunately introduces the same ambiguity in providing the initial value for the cosmological constant. The relation between the local and global formulation of sequestering can be extrapolated to allow us to write down a formulation of UG analogous to the global vacuum energy sequestering (129).

In our view, unimodular gravity indeed offers a resolution of the old cosmological constant problem. However, only as long as one is careful in setting up the initial value for the cosmological constant in a correct way. This particular distinction goes beyond the classical consideration, which leads to conflicting reports on the viability of UG with regards to the old CC problem. However, these findings are consistent when the above distinction is understood.

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Abbreviations

The following abbreviations are used in this manuscript:

CC	cosmological constant
GR	general relativity
UG	unimodular gravity
HT	Henneaux and Teitelboim
EH	Einstein–Hilbert

Notes

1 We use the reduced Planck units $8\pi G_N = \hbar = 1$ and signature convention $(+,-,-,-)$

2 A similar equation was written down originally by Einstein himself [34], however, only for a priori traceless energy-momentum tensor (of radiation). Only later was it realized that these equations describe UG.

3 This theory can be very easily rewritten in several other forms that are immediately equivalent. The only difference is that the fields V^μ and λ can be redefined in such a way that the constraint part of the action becomes

$$\sqrt{-g}\lambda(\nabla_\mu W^\mu - 1)\,, \tag{130}$$

where W^μ is an ordinary vector field. Another popular choice is

$$\lambda(\frac{1}{4!}\epsilon^{\mu\nu\sigma\rho}F_{\mu\nu\sigma\rho} - \sqrt{-g})\,, \tag{131}$$

where $F_{\mu\nu\sigma\rho}$ is the field strength of a 3-form gauge field $A_{\mu\nu\sigma}$ given as $F_{\mu\nu\sigma\rho} = 4\partial_{[\mu}A_{\nu\sigma\rho]}$.

4 A possible way to circumvent this limitation is to introduce additional terms dependent on the vector density V^μ, which modify Equation (42). Such terms provide the necessary couplings to the 'non-conserved' matter sector, which allow for spacetime dependence of λ. However, it is not clear whether such couplings can reproduce the diffusion processes considered in [35,38–40].

5 Interestingly, the corresponding Hamiltonian is linear in the momentum λ and thus it is unbounded from below. Nevertheless, since λ becomes a constant on-shell, the system is perfectly stable [49].

6 In the Abelian case $\mathcal{P}$ represents the Pfaffian of the matrix $F_{\mu\nu}$.

7 This preserves the original solutions while also adding a novel branch, where λ becomes dynamical. This addition can be also applied to (40), where the constraint on constancy of λ is stricter and no new branch appears.

8 Note that the Lambert W function $W_{-1}(xe^x) \neq x$ for $x > -1$.

9 Note that a more rigorous approach requires first going to the ADM formalism to work out the canonical structure of the theory and then calculating the associated path integral in the Hamiltonian formalism along with any necessary fixing of gauge symmetries and associated Faddeev–Popov determinants. Such procedure has been carried out in [31], while the ghost sector has been discussed in [28]. Furthermore, the structure of the ghost sector was analyzed in the BRST formalism in [47,59–61]. Nevertheless, such considerations do not meaningfully affect the result in comparison to a more naive approach we consider here.

10 Interestingly, a similar constraint for Λ was considered in [19,27].

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