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Curvature Dynamics in General Relativity

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Abstract: The equations of general relativity are recast in the form of a wave equation for the Weyl tensor. This allows reformulation of gravitational wave theory in terms of curvature waves, rather than metric waves. The existence of two transverse polarization states for curvature waves is proven and in the linearized approximation the quadrupole formula is rederived. A perturbative scheme to extend the linearized result to the non-linear regime is outlined.

Keywords: general relativity; Weyl tensor; gravitational waves

1. Introduction

The essential content of general relativity (GR) resides in the identification of gravity with the geometric property of a space–time curvature [1]. All observer frames in free fall can be identified with local inertial frames in which gravity is absent, as the local space–time geometry is flat. However, the relative acceleration between local inertial frames at different points in space at different times is encoded in the space–time curvature and cannot be eliminated by any choice of reference frame.

Therefore, an essential description of gravitation is to be cast in terms of the dynamics of curvature. It is the aim of this paper to provide such a description and to show how some familiar results of GR describing observed gravitational phenomena can be rederived in such a framework.

2. Space–Time Curvature

In this section, we summarize the properties of space–time curvature and establish our notation.

In geometry, curvature is measured by the extent to which parallel displacements of vectors and higher rank tensors in two independent directions commute. In differential form, this is expressed by the Ricci identity, which when implemented on a covariant vector field $V_\mu(x)$ takes the form

$$[\nabla_\mu, \nabla_\nu]\, V_\kappa = -R_{\mu\nu\kappa}^{\;\;\;\;\;\;\lambda}\, V_\lambda, \tag{1}$$

where the coefficients $R_{\mu\nu\lambda}^{\;\;\;\;\;\;\kappa}$ are the components of the Riemann curvature tensor. This identity relates the Riemann curvature tensor to the covariant derivative ∇_μ and the associated connection with components $\Gamma_{\mu\nu}^{\;\;\;\lambda}$:

$$R_{\mu\nu\kappa}^{\;\;\;\;\;\;\lambda} = \left(\partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu - [\Gamma_\mu, \Gamma_\nu]\right)_\kappa^{\;\;\lambda}. \tag{2}$$

The standard choice of connection is the one which transports the metric parallel to itself:

$$\nabla_\lambda g_{\mu\nu} = 0. \tag{3}$$

This condition is sometimes known as the metric postulate; it results in the Riemann–Chistoffel connection

$$\Gamma_{\mu\nu}^{\;\;\;\lambda} = \frac{1}{2}\, g^{\lambda\kappa}\left(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}\right). \tag{4}$$

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By Definition (1) and applying the Ricci identity to the metric postulate, one establishes the symmetry properties of the Riemann tensor in the fully covariant representation

$$R_{\mu\nu\kappa\lambda} = -R_{\nu\mu\kappa\lambda} = -R_{\mu\nu\lambda\kappa} = R_{\kappa\lambda\mu\nu}, \tag{5}$$

and the cyclic property

$$R_{\mu\nu\kappa\lambda} + R_{\nu\kappa\mu\lambda} + R_{\kappa\mu\nu\lambda} = 0. \tag{6}$$

In addition to these algebraic identities, the cyclic Jacobi identity for three covariant derivatives guarantees the Bianchi identity for the Riemann tensor:

$$\nabla_\sigma R_{\mu\nu\kappa\lambda} + \nabla_\mu R_{\nu\sigma\kappa\lambda} + \nabla_\nu R_{\sigma\mu\kappa\lambda} = 0. \tag{7}$$

By contraction with the inverse metric $g^{\sigma\lambda}$, this identity implies another one for the divergence of the Riemann tensor:

$$\nabla^\lambda R_{\lambda\kappa\mu\nu} = \nabla_\mu R_{\nu\kappa} - \nabla_\nu R_{\mu\kappa}. \tag{8}$$

In view of its symmetry properties, the Riemann tensor in four space–time dimensions has 20 independent components. Of these, 10 are contained in the trace of the Riemann tensor, the symmetric Ricci tensor

$$R_{\mu\nu} = R_{\nu\mu} = R_{\mu\lambda\nu}{}^{\lambda}. \tag{9}$$

The trace of the Ricci tensor is the Riemann curvature scalar $R = R_\mu{}^\mu$. The other 10 components of the Riemann tensor are contained in its traceless part, known as the Weyl tensor [2], with components

$$W_{\mu\nu\kappa\lambda} = R_{\mu\nu\kappa\lambda} - \frac{1}{2}\left(g_{\mu\kappa}R_{\nu\lambda} - g_{\mu\lambda}R_{\nu\kappa} - g_{\nu\kappa}R_{\mu\lambda} + g_{\nu\lambda}R_{\mu\kappa}\right) + \frac{1}{6}\left(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}\right)R. \tag{10}$$

It is straightforward to check that the Weyl tensor has the same algebraic symmetry properties (5), (6) as the Riemann tensor, and that, in addition, its trace vanishes:

$$W_{\mu\nu\kappa}{}^{\nu} = 0. \tag{11}$$

Indeed, this condition eliminates 10 of the original 20 components of the Riemann tensor, leaving 10 other components as claimed.

Taking traces of the Bianchi identity (7) and its divergence (8), it follows that the Ricci tensor has a divergence-free extension, the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \qquad \nabla^\mu G_{\mu\nu} = 0. \tag{12}$$

The Riemann tensor can now be decomposed into the Weyl tensor, the Einstein tensor and the Riemann scalar, by inverting (10):

$$R_{\mu\nu\kappa\lambda} = W_{\mu\nu\kappa\lambda} + \frac{1}{2}\left(g_{\mu\kappa}G_{\nu\lambda} - g_{\mu\lambda}G_{\nu\kappa} - g_{\nu\kappa}G_{\mu\lambda} + g_{\nu\lambda}G_{\mu\kappa}\right) + \frac{1}{3}\left(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}\right)R. \tag{13}$$

The part of curvature dynamics determined by the physical content of the universe formed by matter and radiation is described by the Einstein equations

$$G_{\mu\nu} = -8\pi G T_{\mu\nu}, \tag{14}$$

with $T_{\mu\nu}$ the energy-momentum tensor of matter and radiation and G the gravitational constant; note that we use units in which the speed of light $c = 1$. The condition (12) of vanishing divergence of the Einstein tensor thereby becomes a consistency requirement for the local conservation of energy and momentum.

Taking the trace of the Einstein tensor, it follows that

$$G_\mu^{\ \mu} = -R = -8\pi G T, \tag{15}$$

using the notation $T = T_\mu^{\ \mu}$ for the trace. The general expression for the Riemann curvature in regions with energy-momentum density $T_{\mu\nu}$, therefore, is

$$R_{\mu\nu\kappa\lambda} = W_{\mu\nu\kappa\lambda} - 8\pi G\Big(\tfrac{1}{2}\big(g_{\mu\kappa}T_{\nu\lambda} - g_{\mu\lambda}T_{\nu\kappa} - g_{\nu\kappa}T_{\mu\lambda} + g_{\nu\lambda}T_{\mu\kappa}\big) - \tfrac{1}{3}\big(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}\big)T\Big), \tag{16}$$

which in the vacuum (empty space-time regions where $T_{\mu\nu} = 0$) reduces to the Weyl tensor. The Bianchi identity (7) can be rewritten as an identity for the Weyl tensor; it then takes the form

$$\nabla_{[\sigma} W_{\mu\nu]\kappa\lambda} = -g_{\kappa[\mu}\nabla_\sigma G_{\nu]\lambda} + g_{\lambda[\mu}\nabla_\sigma G_{\nu]\kappa} - \frac{2}{3}\, g_{\kappa[\mu}\nabla_\sigma R\, g_{\nu]\lambda}, \tag{17}$$

where the square brackets denote complete anti-symmetrization of the enclosed indices $[\mu\sigma\nu]$ with total weight equal to one. By the same substitution (14), (15), it follows that the right-hand side vanishes in the vacuum. Finally, by contraction with $g^{\sigma\lambda}$, we find a result analogous to (8) for the divergence of the Weyl tensor [3,4]:

$$\nabla^\lambda W_{\lambda\kappa\mu\nu} = \frac{1}{2}\big(\nabla_\mu G_{\nu\kappa} - \nabla_\nu G_{\mu\kappa}\big) + \frac{1}{6}\big(g_{\kappa\nu}\nabla_\mu R - g_{\kappa\mu}\nabla_\nu R\big). \tag{18}$$

3. Curvature Dynamics

In the previous section, we established results for the mathematical and physical properties of space–time curvature, with the Riemann tensor expressing the overall curvature in the presence of energy densities due to matter and radiation, whilst the Weyl tensor expresses the purely gravitational contribution to the curvature as it exists in a vacuum. In this section, we discuss the dynamics of these curvature tensors themselves, as follow from these properties.

In the case of gravity, it is simpler to derive the equation for the overall curvature in the presence of energy densities (the Riemann curvature) than for pure gravity (the Weyl curvature). The derivation starts from the Bianchi identity (7), taking a divergence:

$$\nabla^2 R_{\mu\nu\kappa\lambda} - \nabla^\sigma\nabla_\mu R_{\sigma\nu\kappa\lambda} + \nabla^\sigma\nabla_\nu R_{\sigma\mu\kappa\lambda} = 0, \tag{19}$$

and using the Ricci identity and the result (8) for the divergence of the Riemann tensor, to end up with

$$\begin{aligned} &\nabla^2 R_{\mu\nu\kappa\lambda} - 2R_{\mu\sigma\kappa}^{\ \ \ \ \rho} R_{\lambda\rho\nu}^{\ \ \ \ \sigma} + 2R_{\mu\sigma\lambda}^{\ \ \ \ \rho} R_{\kappa\rho\nu}^{\ \ \ \ \sigma} + R_{\mu\nu\rho}^{\ \ \ \ \sigma} R_{\kappa\lambda\sigma}^{\ \ \ \ \rho} = \\ &\quad = -\frac{1}{2}\Big(R_{\mu\nu\kappa\rho}R^\rho_{\ \lambda} - R_{\mu\nu\lambda\rho}R^\rho_{\ \kappa} + R_{\kappa\lambda\mu\rho}R^\rho_{\ \nu} - R_{\kappa\lambda\nu\rho}R^\rho_{\ \mu}\Big) \\ &\qquad + \frac{1}{2}\big(\{\nabla_\mu, \nabla_\kappa\}R_{\nu\lambda} - \{\nabla_\mu, \nabla_\lambda\}R_{\nu\kappa} - \{\nabla_\nu, \nabla_\kappa\}R_{\mu\lambda} + \{\nabla_\nu, \nabla_\lambda\}R_{\mu\kappa}\big), \end{aligned} \tag{20}$$

where the curly braces denote the symmetric anti-commutator of the covariant derivatives enclosed. Note that the right-hand side vanishes in a vacuum environment, in which case the Riemann tensor can be replaced by the Weyl tensor. Indeed, written in terms of the Weyl tensor, the full equation becomes

$$
\begin{aligned}
&\nabla^2 W_{\mu\nu\kappa\lambda} - 2W_{\mu\sigma\kappa}{}^{\rho} W_{\lambda\rho\nu}{}^{\sigma} + 2W_{\mu\sigma\lambda}{}^{\rho} W_{\kappa\rho\nu}{}^{\sigma} - W_{\mu\nu\rho}{}^{\sigma} W_{\kappa\lambda\sigma}{}^{\rho} = \\
&= -\frac{1}{2}\Big(W_{\mu\nu\kappa\rho}G^{\rho}{}_{\lambda} - W_{\mu\nu\lambda\rho}G^{\rho}{}_{\kappa} + W_{\kappa\lambda\mu\rho}G^{\rho}{}_{\nu} - W_{\kappa\lambda\nu\rho}G^{\rho}{}_{\mu}\Big) - RW_{\mu\nu\kappa\lambda} \\
&\quad + \Big(g_{\mu\kappa}W_{\nu\rho\lambda\sigma} - g_{\mu\lambda}W_{\nu\rho\kappa\sigma} - g_{\nu\kappa}W_{\mu\rho\lambda\sigma} + g_{\nu\lambda}W_{\mu\rho\kappa\sigma}\Big)G^{\rho\sigma} \\
&\quad - \Big(g_{\mu\kappa}[G^2]_{\nu\lambda} - g_{\mu\lambda}[G^2]_{\nu\kappa} - g_{\nu\kappa}[G^2]_{\mu\lambda} + g_{\nu\lambda}[G^2]_{\mu\kappa}\Big) + \frac{1}{2}\Big(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}\Big)\operatorname{Tr} G^2 \\
&\quad - \frac{2}{3}R\Big(g_{\mu\kappa}G_{\nu\lambda} - g_{\mu\lambda}G_{\nu\kappa} - g_{\nu\kappa}G_{\mu\lambda} + g_{\nu\lambda}G_{\mu\kappa}\Big) - \frac{1}{3}R^2\Big(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}\Big) \\
&\quad + \frac{1}{2}\Big(\{\nabla_\mu, \nabla_\kappa\}G_{\nu\lambda} - \{\nabla_\mu, \nabla_\lambda\}G_{\nu\kappa} - \{\nabla_\nu, \nabla_\kappa\}G_{\mu\lambda} + \{\nabla_\nu, \nabla_\lambda\}G_{\mu\kappa}\Big) \\
&\quad - \frac{1}{2}\Big(g_{\mu\kappa}\nabla^2 G_{\nu\lambda} - g_{\mu\lambda}\nabla^2 G_{\nu\kappa} - g_{\nu\kappa}\nabla^2 G_{\mu\lambda} + g_{\nu\lambda}\nabla^2 G_{\mu\kappa}\Big) \\
&\quad + \frac{1}{2}\big(g_{\nu\mu}\nabla_\kappa\nabla_\lambda R - g_{\nu\lambda}\nabla_\kappa\nabla_\mu R - g_{\kappa\mu}\nabla_\nu\nabla_\lambda R + g_{\kappa\lambda}\nabla_\nu\nabla_\mu R\big) \\
&\quad - \frac{1}{3}\Big(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}\Big)\nabla^2 R.
\end{aligned}
\tag{21}
$$

As the wave equations for the curvature tensors are derived from the Ricci and Bianchi identities, their general solution is given by Equations (2) and (4). Specific expressions for the metric and connection are implied by imposing the Einstein Equation (14) as a constraint on $G_{\mu\nu}$. Therefore, Equations (20) and (21) are automatically satisfied by all solutions of the Einstein equations.

Equation (21) is a non-linear wave equation for the Weyl curvature tensor. Given a particular vacuum metric—making the right-hand side of this equation to vanish—it defines an extremum, under free variations in the tensor $W_{\mu\nu\kappa\lambda}$, of the action functional

$$
\begin{aligned}
S[W; g] \;=\; & \int d^4x \sqrt{-g}\Big[-\frac{1}{2}\nabla^{\rho}W^{\mu\nu\kappa\lambda}\nabla_{\rho}W_{\mu\nu\kappa\lambda} - \frac{1}{3}W_{\mu\nu}{}^{\kappa\lambda}W_{\kappa\lambda}{}^{\rho\sigma}W_{\rho\sigma}{}^{\mu\nu} \\
& - \frac{4}{3}W^{\mu}{}_{\nu}{}^{\kappa}{}_{\lambda}W_{\mu}{}^{\sigma}{}_{\kappa}{}^{\rho}W_{\sigma}{}^{\nu}{}_{\rho}{}^{\lambda}\Big].
\end{aligned}
\tag{22}
$$

A brief discussion and generalization of this action is presented in Appendix A.

4. Curvature Polarization Modes

In a vacuum, the dynamical solutions of the curvature equations

$$
\nabla^2 W_{\mu\nu\kappa\lambda} - 2W_{\mu\sigma\kappa}{}^{\rho} W_{\lambda\rho\nu}{}^{\sigma} + 2W_{\mu\sigma\lambda}{}^{\rho} W_{\kappa\rho\nu}{}^{\sigma} - W_{\mu\nu\rho}{}^{\sigma} W_{\kappa\lambda\sigma}{}^{\rho} = 0, \tag{23}
$$

describe gravitational waves as waves of curvature. These equations follow from the Bianchi and Ricci identities, in particular (17) and (18), which in vacuum reduce to

$$
\nabla_\sigma W_{\mu\nu\kappa\lambda} + \nabla_\mu W_{\nu\sigma\kappa\lambda} + \nabla_\nu W_{\sigma\mu\kappa\lambda} = 0, \qquad \nabla^{\lambda} W_{\lambda\kappa\mu\nu} = 0. \tag{24}
$$

Recalling also the cyclic property of the curvature tensor

$$
W_{\mu\nu\kappa\lambda} + W_{\nu\kappa\mu\lambda} + W_{\kappa\mu\nu\lambda} = 0,
$$

we can now establish that curvature waves have precisely two polarization modes. To show this it is convenient to make a 3 + 1 space–time split, and define

$$E_{ij} = W_{0i0j}, \qquad B_i^{\ j} = \frac{1}{2\sqrt{-g}}\,\varepsilon^{0jmn}W_{0imn}, \qquad P^{ij} = -\frac{1}{4g}\,\varepsilon^{0imn}\varepsilon^{0jkl}W_{mnkl}, \tag{25}$$

where the latin indices $k, l, m, n = (1, 2, 3)$ denote components of three spatial tensors. All three spatial tensors are traceless:

$$\begin{aligned} &E_j^{\ j} = g^{ij}W_{0i0j} = g^{\mu\nu}W_{0\mu0\nu} = 0, \quad B_j^{\ j} = \frac{1}{2\sqrt{-g}}\,\varepsilon^{0\mu\nu\lambda}W_{0\mu\nu\lambda} = 0, \\ &P_j^{\ j} = -\frac{1}{4g}\,g_{\rho\sigma}\varepsilon^{0\rho\mu\nu}\varepsilon^{0\sigma\kappa\lambda}W_{\mu\nu\kappa\lambda} = 0, \end{aligned} \tag{26}$$

as a result of the four-dimensional tracelessness and cyclic property of the Weyl tensor. Next, in empty space–time, the three divergence of these tensors vanishes:

$$\begin{aligned} &\nabla^i E_{ij} = \nabla^\mu W_{0\mu0j} = 0, \quad \nabla^i B_i^{\ j} = \frac{1}{2\sqrt{-g}}\,\varepsilon^{0j\kappa\lambda}\nabla^\mu W_{0\mu\kappa\lambda} = 0, \\ &\nabla_i P^{ij} = -\frac{1}{4g}\,\varepsilon^{0j\mu\nu}\varepsilon^{0\sigma\kappa\lambda}\nabla_\sigma W_{\kappa\lambda\mu\nu} = 0. \end{aligned} \tag{27}$$

Therefore, the three tensors E_{ij} and P^{ij} are symmetric, traceless and divergence-free, implying that they have only two independent components. Finally, these three tensors are related in terms of time derivatives

$$\nabla_0 P^{ij} = -\frac{1}{4g}\,\varepsilon^{0imn}\varepsilon^{0jkl}\nabla_0 W_{mnkl} = -\frac{1}{2g}\,\varepsilon^{0imn}\varepsilon^{0jkl}\nabla_m W_{n0kl} = \frac{1}{\sqrt{-g}}\,\varepsilon^{0imn}\nabla_m B_n^{\ j}, \tag{28}$$

and

$$\nabla^0 B_i^{\ j} = \frac{1}{2\sqrt{-g}}\,\varepsilon^{0jmn}\nabla^0 W_{0imn} = -\frac{1}{2\sqrt{-g}}\,\varepsilon^{0jmn}\nabla^k W_{kimn} = \sqrt{-g}\,\varepsilon_{0ikl}\nabla^k P^{lj}. \tag{29}$$

Similarly

$$\nabla_0 B_i^{\ j} = \frac{1}{2\sqrt{-g}}\,\varepsilon^{0j\mu\nu}\nabla_0 W_{\mu\nu0i} = \frac{1}{\sqrt{-g}}\,\varepsilon^{0j\mu\nu}\nabla_\nu W_{0\mu0i} = \frac{1}{\sqrt{-g}}\,\varepsilon^{0jkl}\nabla_l E_{ki}, \tag{30}$$

and finally

$$\nabla^0 E_{ij} = \nabla^0 W_{0i0j} = -\nabla^k W_{ki0j} = \frac{1}{2}\,\varepsilon_{0lki}\varepsilon^{0lmn}\nabla^k W_{0jmn} = \sqrt{-g}\,\varepsilon_{0lki}\nabla^k B_j^{\ l}. \tag{31}$$

Thus, $B_i^{\ j}$ and E_{ij} describe magnetic and electric components of the curvature, encoding the time evolution of the two physical degrees of freedom contained in P^{ij}. These two degrees of freedom in P^{ij} represent the independent physical spatial polarization components of the Weyl tensor.

5. Weak Gravity: The Linear Approximation

The curvature dynamics in general relativity simplifies considerably in the weak gravity limit, in which metric and curvature fluctuations can be treated in linearized approximation on a flat background space–time. The starting point of this linear theory is to split the metric $g_{\mu\nu}$ into a constant flat Minkowski background plus metric fluctuations:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + 2h_{\mu\nu}(x). \tag{32}$$

It is possible that such a single split can be made only in a restricted part of space–time, because the local co-ordinates x^μ cannot be extended to all of space–time. If in a neighboring part of space–time, overlapping only partly with the previous one, a different local set of co-ordinates x'^μ is necessary, such that in the domain of the overlap of the co-ordinate systems

$$x'^\mu - x^\mu = -2\xi^\mu(x), \tag{33}$$

then to first order in ξ the split of the corresponding new metric $g'_{\mu\nu}(x')$ in the domain of overlap is

$$g'_{\mu\nu}(x) = \eta_{\mu\nu} + 2h'_{\mu\nu}(x), \qquad h'_{\mu\nu} = h_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \tag{34}$$

The linear approximation restricts all expressions for geometric quantities to the part linear in the fluctuations $h_{\mu\nu}$; thus, for the inverse metric

$$g^{\mu\nu} = \eta^{\mu\nu} - 2h^{\mu\nu} + \mathcal{O}[h^2], \qquad h^{\mu\nu} \equiv \eta^{\mu\kappa} h_{\kappa\lambda} \eta^{\lambda\nu}, \tag{35}$$

for the connection

$$\Gamma_{\mu\nu}^{\;\;\lambda} = \eta^{\lambda\kappa}\left(\partial_\mu h_{\nu\kappa} + \partial_\nu h_{\mu\kappa} - \partial_\kappa h_{\mu\nu}\right) + \mathcal{O}[h^2], \tag{36}$$

and for Riemann curvature

$$R_{\mu\nu\kappa\lambda} = \partial_\nu \partial_\lambda h_{\mu\kappa} - \partial_\nu \partial_\kappa h_{\mu\lambda} - \partial_\mu \partial_\lambda h_{\nu\kappa} + \partial_\mu \partial_\kappa h_{\nu\lambda} + \mathcal{O}[h^2]. \tag{37}$$

Note that in this approximation the co- and contravariant components of vectors and tensors are always related by contraction with the Minkowski metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$. Furthermore, the co-ordinate transformations (33) now reduce to gauge transformations

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \qquad \Gamma'^{\;\;\lambda}_{\mu\nu} = \Gamma_{\mu\nu}^{\;\;\lambda} + 2\partial_\mu \partial_\nu \xi^\lambda, \tag{38}$$

which leave the Riemann tensor invariant: $R'_{\mu\nu\kappa\lambda} = R_{\mu\nu\kappa\lambda}$.

The expressions for the Ricci tensor and Riemann scalar reduce in the linear approximation to

$$\begin{aligned} R_{\mu\nu} &= \Box h_{\mu\nu} - \partial_\mu \partial^\lambda h_{\lambda\nu} - \partial_\nu \partial^\lambda h_{\lambda\mu} + \partial_\mu \partial_\nu h_\lambda^{\;\lambda}, \\ R &= 2\Big(\Box h_\mu^{\;\mu} - \partial^\mu \partial^\nu h_{\mu\nu}\Big), \end{aligned} \tag{39}$$

where $\Box$ is the d'Alembertian in Minkowski space. Therefore, the Einstein equation becomes

$$G_{\mu\nu} = \Box h_{\mu\nu} - \partial_\mu \partial^\lambda h_{\lambda\nu} - \partial_\nu \partial^\lambda h_{\lambda\mu} + \partial_\mu \partial^\nu h_\lambda^{\;\lambda} - \eta_{\mu\nu}\Big(\Box h_\lambda^{\;\lambda} - \partial^\kappa \partial^\lambda h_{\kappa\lambda}\Big) = -8\pi G T_{\mu\nu}, \tag{40}$$

where the linearized Einstein tensor satisfies the conservation law

$$\partial^\mu G_{\mu\nu} = -8\pi G\, \partial^\mu T_{\mu\nu} = 0. \tag{41}$$

The differential identities (7) and (8) satisfied by the Riemann tensor simplify to

$$\partial_\sigma R_{\mu\nu\kappa\lambda} + \partial_\mu R_{\nu\sigma\kappa\lambda} + \partial_\nu R_{\sigma\mu\kappa\lambda} = 0, \qquad \partial^\mu R_{\mu\nu\kappa\lambda} = -8\pi G\left(\partial_\kappa \hat{T}_{\nu\lambda} - \partial_\lambda T_{\nu\kappa}\right), \tag{42}$$

where

$$\hat{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T. \tag{43}$$

These equations now imply the inhomogeneous linear wave equation[1]

$$\Box R_{\mu\nu\kappa\lambda} = -8\pi G\left(\partial_\mu \partial_\kappa \hat{T}_{\nu\lambda} - \partial_\mu \partial_\lambda \hat{T}_{\nu\kappa} - \partial_\nu \partial_\kappa \hat{T}_{\mu\lambda} + \partial_\nu \partial_\kappa \hat{T}_{\mu\lambda}\right), \tag{44}$$

Following the decomposition of the Weyl tensor in Section 4, we introduce a $3+1$ decomposition of the linearized Riemann tensor:

$$E_{ij} = R_{0i0j}, \quad B_{ij} = -\frac{1}{2}\varepsilon_{jmn}R_{0imn}, \quad P_{ij} = \frac{1}{4}\varepsilon_{ikl}\varepsilon_{jmn}R_{klmn}, \tag{45}$$

By the Bianchi identity, the cyclic property of the Riemann tensor and the Einstein equations, it follows that

$$\begin{aligned} E_{jj} &= -8\pi G\,\hat{T}_{00} = -4\pi G\big(T_{jj} + T_{00}\big), \quad B_{jj} = 0, \\ P_{jj} &= -4\pi G\big(\hat{T}_{jj} + \hat{T}_{00}\big) = -8\pi G\,T_{00}. \end{aligned} \tag{46}$$

Also

$$\begin{aligned} \partial_i E_{ij} &= -8\pi G\big(\partial_j \hat{T}_{00} - \partial_0 \hat{T}_{0j}\big), \quad \partial_j B_{ij} = 0, \\ \partial_i B_{ij} &= -8\pi G\,\varepsilon_{jkl}\partial_k \hat{T}_{l0}, \qquad\qquad \partial_i P_{ij} = 0, \end{aligned} \tag{47}$$

and

$$\begin{aligned} \varepsilon_{ikl}\partial_k E_{lj} &= -\partial_t B_{ij}, \quad \varepsilon_{ikl}\partial_k B_{lj} = \partial_t E_{ij} + 8\pi G\big(\partial_j \hat{T}_{i0} - \partial_0 \hat{T}_{ij}\big), \\ \varepsilon_{jkl}\partial_k B_{il} &= -\partial_t P_{ij}, \quad \varepsilon_{ikl}\partial_k P_{lj} = \partial_t B_{ij} + 8\pi G\,\varepsilon_{jkl}\partial_k \hat{T}_{li}. \end{aligned} \tag{48}$$

Thus, in a vacuum, all three-tensor fields $\mathbf{F} = (\mathbf{E}, \mathbf{B}, \mathbf{P})$ are traceless and transverse, and they satisfy the homogeneous wave equation $\Box\mathbf{F} = 0$.

6. Static Curvature Geometries

As is well-known, the gauge transformations (38) can be used to fix $h_{\mu\nu}(x_0) = \Gamma_{\lambda\nu}{}^{\mu}(x_0) = 0$ at a given point with co-ordinates x_0^μ; this is achieved by taking

$$\xi_\mu = \frac{1}{2}h_{\mu\nu}(x_0)(x^\nu - x_0^\nu) - \frac{1}{4}\Gamma_{\lambda\nu\mu}(x_0)\left(x^\lambda - x_0^\lambda\right)(x^\nu - x_0^\nu) + \mathcal{O}[(x - x_0)^3]. \tag{49}$$

In this way, a locally flat co-ordinate system in the neighborhood of x_0 is contructed. Of course, the curvature as encoded by the Riemann tensor cannot be transformed away, as the Riemann tensor is gauge invariant. In a locally flat co-ordinate system with x_0 as the origin, such a geometry with *constant* Riemann tensor in the neighborhood of the origin is described by the tensor field

$$h_{\mu\nu}(x) = \frac{1}{6}R_{\mu\kappa\nu\lambda}x^\kappa x^\lambda, \qquad \partial_\sigma R_{\mu\kappa\nu\lambda} = 0. \tag{50}$$

The corresponding Riemann–Christoffel connection is

$$\Gamma_{\lambda\nu\mu} = -\frac{1}{3}(R_{\lambda\mu\nu\kappa} + R_{\nu\mu\lambda\kappa})x^\kappa, \tag{51}$$

and, indeed, $h_{\mu\nu}(0) = \Gamma_{\lambda\nu\mu}(0) = 0$. In a vacuum, such a geometry is possible if $R_{\mu\nu\kappa\lambda} = W_{\mu\nu\kappa\lambda}$, i.e., if $R_{\mu\nu} = 0$. In the presence of matter, it obviously requires a constant energy-momentum density: $\partial_\lambda T_{\mu\nu} = 0$. Clearly, a constant curvature automatically satisfies the Bianchi identity and the wave equatio, in agreement with Equations (42) and (44).

A less trivial example is the asymptotic curvature of a spherically symmetric point-like test mass. In the linear approximation, such a mass may be modeled by a δ-function localizing the test mass on a world line $X^\mu(\tau)$; the corresponding energy-momentum tensor takes the form

$$T_{\mu\nu}(x) = m\int d\tau\, u_\mu u_\nu\, \delta^4[x - X(\tau)], \tag{52}$$

with $u^\mu = dX^\mu/d\tau$ the four velocity of the test mass. For a test mass at rest in the origin $u^\mu = (1,0,0,0)$, the energy-momentum tensor reduces to a single component

$$T_{00}(x) = m\,\delta^3(\mathbf{x}), \qquad \partial_0 T_{00} = 0; \qquad T_{ij} = T_{i0} = 0. \tag{53}$$

Equivalently

$$\hat{T}_{ij} = \delta_{ij}\hat{T}_{00} = \frac{1}{2}\,\delta_{ij}T_{00}, \qquad \hat{T}_{i0} = 0. \tag{54}$$

Using the linearized Einstein equations, it follows that

$$R_{ij} = \delta_{ij}R_{00}, \qquad R_{00} = -4\pi G\,T_{00} = -4\pi G m\,\delta^3(\mathbf{x}). \tag{55}$$

The full Riemann curvature tensor can be constructed by integrating Equation (44); using the retarded Green's function, the solution of this equation for arbitrary $\hat{T}_{\mu\nu}$ localized inside some finite volume S becomes (37):

$$R_{\mu\nu\kappa\lambda} = \partial_\mu\partial_\kappa\hat{h}_{\nu\lambda} - \partial_\mu\partial_\lambda\hat{h}_{\nu\kappa} - \partial_\nu\partial_\kappa\hat{h}_{\mu\lambda} + \partial_\nu\partial_\lambda\hat{h}_{\mu\kappa},$$

where $\hat{h}_{\mu\nu}$ is *defined* by

$$\hat{h}_{\mu\nu}(\mathbf{x},t) = 2G\int_S d^3x'\,\frac{\hat{T}_{\mu\nu}(\mathbf{x}',t_{ret})}{|\mathbf{x}-\mathbf{x}'|}, \tag{56}$$

with the integrand being evaluated at the retarded time $t_{ret} = t - |\mathbf{x}-\mathbf{x}'|$. Notice that in view of Equation (37), $\hat{h}_{\mu\nu}$ can be identified with a gauge-fixed expression for the metric fluctuation $h_{\mu\nu}$; however, as no gauge transformation applied to $h_{\mu\nu}$ will affect the curvature, it also will not affect the result (56) for $\hat{h}_{\mu\nu}$, for that matter.

As the partial derivatives commute with $\Box^{-1}$ in Minkowski space, it follows that it is possible to replace the terms in the expression for the Riemann curvature by the equivalent

$$\partial_\mu\partial_\kappa\hat{h}_{\nu\lambda}(\mathbf{x},t) = 2G\int_S d^3x'\,\frac{\left[\partial'_\mu\partial'_\kappa\hat{T}_{\nu\lambda}\right](\mathbf{x}',t_{ret})}{|\mathbf{x}-\mathbf{x}'|}. \tag{57}$$

Substitution of (53) and (54) into (56) or (57) now gives a direct expression for the curvature components, without requiring to fix a gauge for the tensor field $h_{\mu\nu}$.

The final result for the curvature field of the test mass m now is the quadrupole field

$$E_{ij} = -P_{ij} = \frac{3Gm}{r^5}\left(r_i r_i - \frac{1}{3}\,\delta_{ij}r^2\right), \qquad B_{ij} = 0, \tag{58}$$

where $\mathbf{r} = \mathbf{x} - \mathbf{x}' \to \mathbf{x}$ if the test mass is located in the origin. Equivalently,

$$R_{ikjl} = \frac{3Gm}{r^5}\left[\delta_{ij}r_k r_l - \delta_{il}r_k r_j - \delta_{jk}r_i r_l + \delta_{kl}r_i r_j - \frac{2}{3}\,r^2\left(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{kj}\right)\right]. \tag{59}$$

This is in full agreement with the asymptotic form for large r of the Schwarzschild geometry.

7. Gravitational Radiation

We now turn to the more general asymptotic dynamical curvature solution of localized sources, the most important example of which is gravitational radiation[2]. The general expression for the curvature in the weak-gravity limit is again (37) for general time-dependent energy-momentum distributions $\hat{T}_{\mu\nu}$ localized inside a finite volume S, which may for practical purposes be taken to be a sphere surrounding the source region; for non-relativistic sources, it is convenient to take the center of mass of the source as the center of this sphere.

As the linearized expression for $E_{ij} = R_{0i0j}$ reads

$$E_{ij} = 2G\left(\partial_0^2 h_{ij} - \partial_0\partial_i h_{0j} - \partial_0\partial_j h_{0i} + \partial_i\partial_j h_{00}\right), \tag{60}$$

in the long-range limit $|\mathbf{x} - \mathbf{x}'| \simeq |\mathbf{x}| = r$, the leading terms in $1/r$ are

$$\begin{aligned} E_{ij}(\mathbf{x}, t) &\simeq \frac{2G}{r}\, \partial_0^2 \int_S d^3x' \left(\hat{T}_{ij} + \hat{r}_i \hat{T}_{0j} + \hat{r}_j \hat{T}_{0i} + \hat{r}_i \hat{r}_j \hat{T}_{00} \right)(\mathbf{x}', t_{ret}) \\ &= \frac{2G}{r}\, \partial_0^2 \int_S d^3x' \left[T_{ij} - \frac{1}{2}(\delta_{ij} - \hat{r}_i\hat{r}_j) T_{kk} + \hat{r}_i T_{0j} + \hat{r}_j T_{0i} \right. \\ &\qquad \left. + \frac{1}{2}(\delta_{ij} + \hat{r}_i \hat{r}_j) T_{00} \right](\mathbf{x}', t_{ret}). \end{aligned} \tag{61}$$

where $\hat{r}_i = x_i/r$ are the components of the unit vector in the direction of $\mathbf{x}$. Using a well-known identity for localized sources, this can be written alternatively as

$$\begin{aligned} E_{ij}(\mathbf{x}, t) &= \frac{2G}{r} \left[\frac{1}{2}\, \partial_0^4 \int_S d^3x'\, x_i' x_j' T_{00}(\mathbf{x}', t_{ret}) - \frac{1}{4}\left(\delta_{ij} - \hat{r}_i \hat{r}_j\right) \partial_0^4 \int_S d^3x'\, \mathbf{x}'^2 T_{00}(\mathbf{x}', t_{ret}) \right. \\ &\quad + \hat{r}_i\, \partial_0^3 \int_S d^3x'\, x_j'\, T_{00}(\mathbf{x}', t_{ret}) + \hat{r}_j\, \partial_0^3 \int_S d^3x'\, x_i'\, T_{00}(\mathbf{x}', t_{ret}) \\ &\quad \left. + \frac{1}{2}\left(\delta_{ij} + \hat{r}_i \hat{r}_j\right) \partial_0^2 \int_S d^3x' T_{00}(\mathbf{x}', t_{ret}) \right]. \end{aligned} \tag{62}$$

Now in the non-relativistic limit all terms except the first two can be discarded, as they represent the time changes in the mass dipole and the mass monopole moment, which vanish in the CM frame. Thus, we are left with

$$E_{ij}(\mathbf{x}, t) = \frac{G}{r}\, \partial_0^4 \int d^3x' \left[\left(x_i' x_j' - \frac{1}{3}\, \mathbf{x}'^2 \delta_{ij} \right) - \frac{1}{2} \left(\hat{r}_i \hat{r}_j - \frac{1}{3}\, \delta_{ij} \right) \mathbf{x}'^2 \right] T_{00}(\mathbf{x}', t_{ret}). \tag{63}$$

Note that the integral in the second term is equivalent to

$$\partial_0^4 \int_S d^3x'\, \mathbf{x}'^2 T_{00} = 2\, \partial_0^2 \int_S d^3x'\, T_{kk},$$

which for non-relativistic sources is generally very small compared to the first term representing the mass quadrupole. Therefore, in most practical situations, the spatial trace term can be ignored.

Although the expression (63) is manifestly traceless, as a result of the truncations made it is not manifestly transverse. This can be repared by considering the transverse projection of E_{ij}, which leads to the result:

$$R_{0i0j}(\mathbf{x}, t) \simeq E_{ij}^{TT} = (\delta_{ik} - \hat{r}_i \hat{r}_k) \left(\delta_{jl} - \hat{r}_j \hat{r}_l \right) \left(E_{kl} + \frac{1}{2}\, \delta_{kl}\, \hat{r} \cdot E \cdot \hat{r} \right). \tag{64}$$

Note that the TT label here is not the result of gauge fixing; it is simply a manifestly transverse and traceless set of components of the linearized Riemann—or, equivalently, a linearized Weyl– curvature tensor.

This expression (64) is all that is needed to interpret the data of gravitational-wave detectors; in general, these monitor the space–time intervals between several test masses. If the space–time interval between two test masses moving on geodesics is n^μ, the change in the interval due to space–time curvature is given by the geodesic deviation equation[3]

$$D_\tau^2 n^\mu = R_{\kappa\nu\lambda}^{\quad\ \mu} u^\kappa u^\lambda n^\nu. \tag{65}$$

where D_τ is a covariant proper-time derivative along the geodesic world line of one of the masses, u^μ is its four-velocity and n^μ measures the space–time interval of a second mass with respect to the first one. In the frame in which the first test mass is originally at rest in a local Minkowski frame: $u^\mu = (1,0,0,0)$, the equations simplify to

$$\frac{d^2 n^i}{d\tau^2} = E^{TT}_{ij} n^j, \qquad \frac{d^2 n^0}{d\tau^2} = 0. \tag{66}$$

Thus, the relative motion of the test masses, determined by the curvature components E_{ij}, give direct access to the variations in the mass quadrupole moment of the source.

8. Beyond Leading Order

In the previous sections, we derived expressions for the curvature induced by gravitational fields at leading order. However, the expressions obtained there can be used as the starting point for an improved treatment of gravitational-wave propagation including terms beyond leading order. The general procedure consists of expanding the Weyl tensor into the first-order contribution (37) plus the next-order term:

$$W_{\mu\nu\kappa\lambda} = W^{(1)}_{\mu\nu\kappa\lambda} + W^{(2)}_{\mu\nu\kappa\lambda}, \tag{67}$$

with $W^{(1)}_{\mu\nu\kappa\lambda}$ given by the linear approximation:

$$W^{(1)}_{\mu\nu\kappa\lambda} = R^{(1)}_{\mu\nu\kappa\lambda} = \partial_\mu\partial_\kappa h^{(1)}_{\nu\lambda} - \partial_\mu\partial_\lambda h^{(1)}_{\nu\kappa} - \partial_\nu\partial_\kappa h^{(1)}_{\mu\lambda} + \partial_\nu\partial_\lambda h^{(1)}_{\mu\kappa},$$

while at the same time expanding the metric and the connection according to (32):

$$g_{\mu\nu} = \eta_{\mu\nu} + 2h^{(1)}_{\mu\nu}, \qquad \Gamma^{(1)}_{\lambda\nu\mu} = \Gamma^{(1)\,\rho}_{\lambda\nu}\,\eta_{\rho\mu} = \partial_\lambda h^{(1)}_{\nu\mu} + \partial_\nu h^{(1)}_{\lambda\mu} - \partial_\mu h^{(1)}_{\lambda\nu},$$

where $h^{(1)}_{\mu\nu} = \hat{h}_{\mu\nu}$, as given by Equation (56).

Armed with these results, one now expands Equation (21) as

$$\begin{aligned}\Box W^{(2)}_{\mu\nu\kappa\lambda} = \nabla^2_{(1)} W^{(1)}_{\mu\nu\kappa\lambda} + \eta^{\rho\eta}\eta^{\sigma\tau}\left(2W^{(1)}_{\mu\sigma\kappa\rho}W^{(1)}_{\lambda\eta\nu\tau} - W^{(1)}_{\mu\sigma\lambda\rho}W^{(1)}_{\kappa\eta\nu\tau} + W^{(1)}_{\mu\nu\rho\sigma}W^{(1)}_{\kappa\lambda\eta\tau}\right)\\ + (\textit{source terms}),\end{aligned} \tag{68}$$

where, obviously, the source terms are absent in the source-free far region. The first term on the right-hand side becomes, in this approximation,

$$\begin{aligned}\nabla^2_{(1)} W^{(1)}_{\mu\nu\kappa\lambda} = \Box W^{(1)}_{\mu\nu\kappa\lambda} + \eta^{\rho\eta}\eta^{\sigma\tau}\Big[-2h^{(1)}_{\rho\sigma}\partial_\eta\partial_\tau W^{(1)}_{\mu\nu\kappa\lambda} - \Gamma^{(1)}_{\tau\sigma\rho}\,\partial_\eta W^{(1)}_{\mu\nu\kappa\lambda}\\ + 2\left(\Gamma^{(1)}_{\mu\sigma\rho}\,\partial_\tau W^{(1)}_{\eta\nu\kappa\lambda} + \Gamma^{(1)}_{\nu\sigma\rho}\,\partial_\tau W^{(1)}_{\mu\eta\kappa\lambda} + \Gamma^{(1)}_{\kappa\sigma\rho}\,\partial_\tau W^{(1)}_{\mu\nu\eta\lambda} + \Gamma^{(1)}_{\lambda\sigma\rho}\,\partial_\tau W^{(1)}_{\mu\nu\kappa\eta}\right)\\ + (\partial_\tau\Gamma^{(1)}_{\mu\sigma\rho})W^{(1)}_{\eta\nu\kappa\lambda} + (\partial_\tau\Gamma^{(1)}_{\nu\sigma\rho})W^{(1)}_{\mu\eta\kappa\lambda} + (\partial_\tau\Gamma^{(1)}_{\kappa\sigma\rho})W^{(1)}_{\mu\nu\eta\lambda} + (\partial_\tau\Gamma^{(1)}_{\lambda\sigma\rho})W^{(1)}_{\mu\nu\kappa\eta}\Big].\end{aligned} \tag{69}$$

Of course the first term on the right-hand side here vanishes outside the source region, as by construction

$$\Box W^{(1)}_{\mu\nu\kappa\lambda} = 0$$

there. Clearly, this perturbative scheme can be extended to still higher orders.

9. Discussion

General relativity is presently the best theory of gravity available, in agreement with almost all observations and experiments [8,9], with the possible exception of those phenomena which are usually interpreted as evidence for dark matter. Although all available tests are in the classical regime and we still do not have a complete workable quantum theory of gravity, the observation of gravitational waves shows without a doubt that in GR, space–time is a genuine dynamical system. These waves in the space–time geometry are observed to propagate at the speed of light and have two transverse polarization modes. In view of the geometry of space–time being encoded in an observer-independent way by the curvature, it was the aim of this investigation to recast GR in such a way as to obtain a direct description of curvature dynamics.

As the Ricci tensor and Riemann scalar are directly given in terms of the local energy-momentum distribution, the actual problem to be solved was to derive a wave equation for the purely gravitational degrees of freedom contained in the Weyl tensor. This was achieved in Equation (21). It was also shown that in vacuum, the propagating degrees of freedom of the Weyl tensor correspond to two independent transverse polarization modes, as was to be expected, and that in the non-relativistic linear approximation the curvature waves are sourced by the quadrupole modes of the energy-momentum distribution.

Corrections to these results are of two kinds. First, the self-interaction of the gravitational field corrects the propagation of gravitational waves even on a flat background. As discussed in sect. 8, the linearized curvature creates a non-flat background affecting the propagation of the non-linear second-order contributions to the gravitational waves. This will change the dispersion relation for the waves beyond first order.

Another type of correction occurs if the gravitational fluctuations propagate in a non-vacuum environment. In the context of Equation (21), this means not only that there are non-vanishing source terms on the right-hand side, but also that the 4-D laplacean ∇^2 is modified. This may manifest itself in additional propagating degrees of freedom, as $\nabla^i E_{ij}$ and $\nabla^i B_i^{\;j}$ receive contributions from the right-hand side of Equation (18). These corrections require additional study; they could show up, e.g., in gravitational memory effects in non-flat background geometries [5,10].

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Appendix A

This appendix addresses action principles for the Riemann and Weyl tensors. First, observe that Equation (20) can be rewritten using the Einstein tensor in the form

$$\begin{aligned}\nabla^2 R_{\mu\nu\kappa\lambda} &- 2R_{\mu\sigma\kappa}^{\quad\;\rho} R_{\lambda\rho\nu}^{\quad\;\sigma} + 2R_{\mu\sigma\lambda}^{\quad\;\rho} R_{\kappa\rho\nu}^{\quad\;\sigma} + R_{\mu\nu\rho}^{\quad\;\sigma} R_{\kappa\lambda\sigma}^{\quad\;\rho} \\ &+ \frac{1}{2}\Big(G_\lambda^{\;\rho} R_{\mu\nu\kappa\rho} - G_\kappa^{\;\rho} R_{\mu\nu\lambda\rho} + G_\nu^{\;\rho} R_{\kappa\lambda\mu\rho} - R_\mu^{\;\rho} R_{\kappa\lambda\nu\rho}\Big) - 2\,R\,R^{\mu\nu\kappa\lambda} R_{\mu\nu\kappa\lambda} \\ &= \frac{1}{2}\big(\{\nabla_\mu, \nabla_\kappa\} G_{\nu\lambda} - \{\nabla_\mu, \nabla_\lambda\} G_{\nu\kappa} - \{\nabla_\nu, \nabla_\kappa\} G_{\mu\lambda} + \{\nabla_\nu, \nabla_\lambda\} G_{\mu\kappa}\big)\end{aligned} \tag{A1}$$

For a fixed metric and connection, taking the tensor $R_{\mu\nu\kappa\lambda}$ as an independent set of variables, this equation defines an extremal point of the action functional

$$\begin{aligned} S[R; g, H] = \int d^4x \sqrt{-g} \Big[&-\frac{1}{2} \nabla^\sigma R^{\mu\nu\kappa\lambda} \nabla_\sigma R_{\mu\nu\kappa\lambda} - \frac{1}{3} R_{\mu\nu}^{\quad\kappa\lambda} R_{\kappa\lambda}^{\quad\rho\sigma} R_{\rho\sigma}^{\quad\mu\nu} \\ &+ \frac{4}{3}\, R_{\mu\nu}^{\quad\kappa\lambda} R_{\kappa\rho}^{\quad\mu\sigma} R^{\nu\;\;\rho}_{\;\;\sigma\;\;\lambda} - 2\,G^\rho_{\;\lambda} R^{\mu\nu\kappa\lambda} R_{\mu\nu\kappa\rho} - R\,R^{\mu\nu\kappa\lambda} R_{\mu\nu\kappa\lambda} - H_{\mu\nu\kappa\lambda} R^{\mu\nu\kappa\lambda} \Big], \end{aligned} \tag{A2}$$

where, using the Einstein equations, the source term can be replaced by

$$H_{\mu\nu\kappa\lambda} = -4\pi G\big(\{\nabla_\mu, \nabla_\kappa\} T_{\nu\lambda} - \{\nabla_\mu, \nabla_\lambda\} T_{\nu\kappa} - \{\nabla_\nu, \nabla_\kappa\} T_{\mu\lambda} + \{\nabla_\nu, \nabla_\lambda\} T_{\mu\kappa}\big). \tag{A3}$$

For the case of vacuum solutions with $G_{\mu\nu} = R = H_{\mu\nu\kappa\lambda} = 0$, the action $S[R; g, h]$ straightforwardly reduces to the action $S[W; g]$ in Equation (22) for the corresponding traceless part $W_{\mu\nu\kappa\lambda}$ of the tensor $R_{\mu\nu\kappa\lambda}$. Obviously, the given metric and connection solutions of (A1) for the tensor $R_{\mu\nu\kappa\lambda}$ are given by Equation (2). However, as the example of deriving curvature fluctuations directly in the linearized theory shows, the alternative approach presented here is feasible for certain applications, which could, e.g., include the related problem in non-vacuum space–times with additional matter fields.

The observation that this equation defines an extremum of the functional (A2) can be useful in such cases; for example, to derive WKB-type approximations. More generally, for given background $(G^{\mu}{}_{\nu}, R)$, as defined by the energy-momentum tensor, Equations (A1) and (A2) relate fluctuations in the curved-space d'Alembertian ∇^2 to curvature fluctuations, thereby connecting two different approaches—spectral analysis and differential geometry—to the description and analysis of gravitational fluctuations.

Notes

1. This linear form of the equation was derived before in ref. [5]
2. For a detailed and thorough introduction with references see [6].
3. For a derivation and discussion with references see [7].

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