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Unimodular Theory of Gravity in Light of the Latest Cosmological Data

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Abstract: The unimodular theory of gravity is an alternative perspective to the traditional general relativity of Einstein and opens new possibilities for exploring its implications in cosmology. In this paper, we investigated Unimodular Gravity (UG) with the cosmological data from the Pantheon sample of Type Ia Supernovae (SNs) (2018), Baryon Acoustic Oscillations (BAOs), and the observational H(z) data from the Differential Age method (DA). We also used the Cosmic Microwave Background (CMB) distance priors from the Planck 2018 results. We considered a model consisting of a generalized cosmological constant, radiation, and a dark matter component along with normal matter. The considered theory respects only unimodular coordinate transformations. We first fit our model with low-redshift data from SNs and DA and determined the value of the model parameters (ξ, H_0). We found the best-fit value of parameter $\xi = 6.03 \pm 0.40$, which deviates slightly from 6, for which the theory becomes the standard general theory of relativity. We observed a small deviation in the value of the Hubble constant ($H_0 = 72.6 \pm 3.5$ km s^{-1} Mpc^{-1}) in the UG model compared with the standard ΛCDM model ($H_0 = 72.2 \pm 1.2$ km s^{-1} Mpc^{-1}). Using the BAO + CMB constraint in the UG model, we obtained $H_0 = 68.45 \pm 0.66$ km s^{-1}Mpc^{-1}, and ξ is ∼6.029. For the combined datasets (SN + DA + BAO + CMB), the estimated $H_0 = 69.01 \pm 0.60$ km s^{-1} Mpc^{-1} with ξ ∼6.037, and in standard gravity, $H_0 = 68.25 \pm 0.40$ km s^{-1} Mpc^{-1}.

Keywords: unimodular gravity; cosmological constant; Hubble parameter; Hubble tension; dark energy

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1. Introduction

There are many competing models to fit cosmological observations, such as the modified theory of gravity, the scalar field theory, the ΛCDM model, etc. [1]. In these models, the ΛCDM model is a plausible model for explaining current cosmological observations [2,3]. However, the cosmological constant Λ has its own fine-tuning problem. Some of the proposals, such as supersymmetry, supergravity, anthropic considerations, the adjustment mechanism, changing gravity, etc., were discussed in [4] regarding the cosmological constant problem. In this review paper [4], Unimodular Gravity (UG) was discussed as a possible generalization of gravity. Unimodular gravity is an interesting model that was first proposed by Anderson and Finkelstein [5] following the closely related proposal given by Einstein [6]. It might solve the cosmological constant problem since $\sqrt{-g}$ is not a dynamical field in this theory. However, in [4], it was explained that the cosmological constant reappears as an integration constant, and the fine-tuning problem remains in the theory. This is because, within that unimodular model of gravity, the theory still maintains full General Coordinate Invariance (GCI). Several studies regarding the cosmological constant problem in unimodular gravity were performed in [7–11].

In [12–30], various other aspects of unimodular gravity were studied. In this work, we are interested in studying its implications in cosmology. The full metric can be broken into a scalar field and an unimodular metric [12,13]. In [15,16], the implication of unimodular gravity was studied thoroughly using such a decomposition. In these works and [12,13], the theory respects covariance under unimodular coordinate transformations instead of general coordinate invariance. Broken general coordinate invariance introduces a parameter ξ [15,16] with a value other than 6, where $\xi = 6$ corresponds to full general coordinate invariance, and the outcome of the theory is the same as that of general relativity. Other values of ξ correspond to unimodular gravity and provide covariance under only unimodular coordinate transformations. In [15,16], the authors discussed the expansion of the universe considering the generalized non-relativistic matter and generalized cosmological constant separately. A model where only radiation was assumed was also discussed. The motivation for these models is to describe the current expansion of the universe by only one component, either dark matter or a cosmological constant, to solve the coincidence problem.

One of the challenging tasks in modern cosmology is to determine the precise value of the Hubble constant H_0. Considering the standard ΛCDM model, the H_0 value determined by the Cosmic microwave Background (CMB) experiments, such as WMAP and Planck, differs from the value determined by the local distance ladder approach, such as Supernovae and H_0 for the Equation of State (SH0ES) project. The Planck 2018 results gave the value $H_0 = 67.4 \pm 0.5$ km s^{-1}Mpc^{-1} [31]. The constraint of H_0 in CMB measurement is model-dependent. The most-recent results of the SH0ES program gave $H_0 = 73.30 \pm 1.04$ km s^{-1}Mpc^{-1} [32], which differ by 5σ from the final result of Planck. Since the local measurement of H_0 does not rely on any cosmological assumptions, it can be considered model-independent. To alleviate this tension in the measurement of H_0 from low- and high-redshift probes, primarily two methods are suggested in the literature: early universe modification and late universe modification [33–37]. Modified gravity might also be a solution for the Hubble tension [38,39].

In this paper, we investigated how the unimodular theory differs from the general theory of relativity and to what extent the Hubble tension problem can be addressed within this framework. Here, we took into account a unimodular gravity model [16] with a generalized cosmological constant term. We extended this model by including the dark matter and radiation energy components. In general relativity, it is well known that the radiation term has a small contribution at the lower redshift values. However, it is necessary to test if the same is true in unimodular gravity. The radiation term is added to check its contribution to the Hubble constant in our model. In addition, the value of the comoving sound horizon, which is the calibration scale for the BAO and CMB data, is sensitive to the radiation term. This might result in a change in the value of the Hubble constant when we consider the BAO and CMB data. For supernovae, we used the Pantheon datasets (2018) to estimate the parameters of the model. We further studied the Hubble constant problem in the theory by including the BAO data and CMB distance priors.

The manuscript is structured as follows. The review of the unimodular gravity model and unimodular field equations, with broken general coordinate invariance, is given in Section 2 and Section 3, respectively. In Section 4, we describe the unimodular gravity model having a generalized cosmological constant. In Section 5, we describe our methodology and datasets used for the analysis. In Section 6, we discuss our results, and then, we conclude in Section 7.

2. Field Decomposition in Unimodular Gravity

To begin, we decomposed the standard metric $g_{\mu\nu}$ into a scalar field χ and unimodular metric $\bar{g}_{\mu\nu}$ as $g_{\mu\nu} = \chi^2 \bar{g}_{\mu\nu}$. We point out that, in the Cartesian coordinate system, the determinant of $\bar{g}_{\mu\nu}$ is unity. We can generalize it to any coordinate system, such that $\det(\bar{g}_{\mu\nu}) = f(x)$ [16], where $f(x)$ is some specified function of space–time coordinates while keeping $\det(\bar{g}_{\mu\nu})$ as non-dynamical. The theory is supposed to be invariant under

Unimodular General Coordinate Transformations (UGCTs), such that the Jacobian of the transformation is unity, i.e.,

$$x^\mu \to x'^\mu \tag{1}$$

$$\det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) = 1 \tag{2}$$

Under this transformation, the determinant of $g_{\mu\nu}$ and, hence, the field χ behaves as a scalar. Following the definition of $g_{\mu\nu}$, we have

$$g^{\mu\nu} = \frac{\bar{g}^{\mu\nu}}{\chi^2}, \;\; \Gamma^\mu_{\alpha\beta} = \bar{\Gamma}^\mu_{\alpha\beta} + \tilde{\Gamma}^\mu_{\alpha\beta}, \tag{3}$$

where $\Gamma^\mu_{\alpha\beta}$ is the full affine connection, and the connection $\bar{\Gamma}^\mu_{\alpha\beta}$ corresponds to the unimodular metric $\bar{g}_{\mu\nu}$. $\tilde{\Gamma}^\mu_{\alpha\beta}$ contains all the terms of the scalar field χ and is given by

$$\tilde{\Gamma}^\mu_{\alpha\beta} = \bar{g}^\mu_\beta \partial_\alpha \ln\chi + \bar{g}^\mu_\alpha \partial_\beta \ln\chi - \bar{g}_{\alpha\beta}\partial^\mu \ln\chi. \tag{4}$$

Similarly, under this definition, the Ricci curvature tensor $R_{\mu\nu}$ and Ricci scalar R are decomposed. The first part of $R_{\mu\nu}$ is made of the unimodular metric, and the second one contains the terms of the scalar field χ (see [16]). One can write the gravitational action as

$$S_E = \int d^4x \frac{\sqrt{-\bar{g}}}{16\pi G}\left[\chi^2\bar{R} - \xi\bar{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi\right], \tag{5}$$

where, if parameter $\xi \neq 6$, GCI is broken and the theory respects only unimodular coordinate invariance.

3. Field Equations in Unimodular Gravity

We considered the action as discussed in [15,16]. The action follows covariance under unimodular coordinate transformations. Considering both matter and the cosmological constant, the action is given as follows:

$$S = \int d^4x\sqrt{-\bar{g}}\left[\frac{\chi^2}{\kappa}\bar{R} - \frac{\xi}{\kappa}\bar{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi\right] + S_M + S_\Lambda, \tag{6}$$

here $\kappa = 16\pi G$, ξ is the parameter of the theory, and S_M and S_Λ are the actions corresponding to matter and the cosmological constant. For the general theory of relativity, the parameter $\xi = 6$. Under this theory, the Einstein field equation and equation of motion for field χ are given by

$$\begin{aligned}-\chi^2\left[\bar{R}_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}\bar{R}\right] - \left[(\chi^2)_{;\mu;\nu} - \frac{1}{4}\bar{g}_{\mu\nu}(\chi^2)^{;\lambda}_{;\lambda}\right] + \xi\left[\partial_\mu\chi\partial_\nu\chi - \frac{1}{4}\bar{g}_{\mu\nu}\partial^\lambda\chi\partial_\lambda\chi\right] \\ = \frac{\kappa}{2}\left[T_{\mu\nu} - \frac{1}{4}\bar{g}_{\mu\nu}T^\lambda_\lambda\right]\end{aligned} \tag{7}$$

and

$$2\chi\bar{R} + 2\xi\bar{g}^{\mu\nu}(\chi)_{;\mu;\nu} = \kappa T_\chi, \tag{8}$$

respectively.

In this theory, the cosmological constant or vacuum energy term does not contribute to Equation (7); however, the term T_χ includes all the contribution due to the coupling of χ with matter fields and the cosmological constant. For the spatially flat Friedmann–Robertson–Walker (FLRW) metric, we can write $\bar{g}_{\mu\nu} = \text{diagonal}[1, -1, -1, -1]$ and identify the scale factor $a(\eta)$ with the scalar field χ.

4. Generalized Cosmological Constant with Radiation

In [16], the authors investigated a model for the coincidence problem based on a generalized cosmological constant in unimodular gravity, where only ordinary matter and the cosmological constant play a role. The action for the generalized cosmological constant term is defined as

$$S_\Lambda = -\int d^4x\sqrt{-\bar{g}}\Big[\Lambda\chi^{\xi-2} + \Lambda_1\chi\Big], \tag{9}$$

In this model, the generalized cosmological constant term $\Lambda\chi^\delta$ with a parameter δ appears in the action instead of the standard general relativity term $\Lambda\chi^4$. For a consistent solution, it is found that a term $\Lambda_1\chi$ is required in the action and $\delta = \xi - 2$. We followed this model. We are interested in considering both the dark matter (CDM) and cosmological constant term, along with radiation. Thus, in place of ordinary matter, we took into account both cold dark matter and ordinary matter, and we also included radiation. We expect that minimally broken invariance may explain cosmological observations. The purpose of including radiation is that it might cause a deviation in the Hubble constant problem.

For radiation, the term $T^\lambda_\lambda = 0$ and, also, $T_\chi = 0$ since there is no direct coupling to the vector field with field χ [16]. The energy densities corresponding to matter and radiation still hold the same proportionality: $\rho_m \propto \chi^{-3}$ and $\rho_R \propto \chi^{-4}$, respectively, as in standard general relativity [16]. Using Equations (7) and (8), we obtain

$$(\xi-2)\left(\frac{d\chi}{d\eta}\right)^2 - 2\chi\frac{d^2\chi}{d\eta^2} = \frac{\kappa}{2}\chi^4\rho_m + \frac{2\kappa}{3}\chi^4\rho_R. \tag{10}$$

Here, we used $T^R_{00} = \chi^4\rho_R$. The equation of motion of χ gives

$$2\xi\frac{d^2\chi}{d\eta^2} = \kappa\chi^3\rho_m + \kappa\Big[(\xi-2)\Lambda\chi^{\xi-3} + \Lambda_1\Big]. \tag{11}$$

Integrating Equation (11), we obtain

$$\left(\frac{d\chi}{d\eta}\right)^2 = \frac{\kappa(\rho_{0m}+\Lambda_1)\chi}{\xi} + \frac{\kappa\Lambda}{\xi}\chi^{\xi-2} + C_0, \tag{12}$$

where C_0 is the integration constant. For a consistent solution, we obtain

$$\rho_{0m} = \frac{2(\xi-3)\Lambda_1}{(6-\xi)}, \tag{13}$$

and

$$C_0 = \frac{2\kappa\rho_{0R}}{3(\xi-2)}. \tag{14}$$

Now, we considered $\chi = a(t)$, where $a(t)$ is the scale factor of the universe. Changing the conformal time "η" to the cosmological time "t" using $\chi d\eta = a(t)d\eta = dt$ leads to

$$H(a)^2 = \frac{\kappa(\rho_{0m}+\Lambda_1)}{\xi}a^{-3} + \frac{\kappa\Lambda}{\xi}a^{\xi-6} + \frac{2\kappa\rho_{0R}}{3(\xi-2)}a^{-4}, \tag{15}$$

and simplifying, we obtain

$$\begin{aligned} H(a) &= \frac{H_0}{(1+(\rho_{0m}\xi)/(2\Lambda(\xi-3)) + (2\xi\rho_{0R})/(3\Lambda(\xi-2)))^{1/2}} \\ &\quad \times a^{\xi/2-3}\Big[1 + \frac{\rho_{0m}\xi}{2\Lambda(\xi-3)}a^{3-\xi} + \frac{2\xi\rho_{0R}}{(3\Lambda(\xi-2))}a^{2-\xi}\Big]^{1/2}. \end{aligned} \tag{16}$$

Here, ρ_{0m} represents the current energy density of ordinary matter and CDM together, ρ_{0R} represents the current energy density of photons and neutrinos together, and H_0 is given by

$$H_0 = \sqrt{\frac{\kappa\Lambda}{\xi}\Big[1 + (\rho_{0m}\xi)/(2\Lambda(\xi-3)) + (2\xi\rho_{0R})/(3\Lambda(\xi-2))\Big]}. \tag{17}$$

5. Methodology and Datasets

We performed our analyses with the Hubble relation given in Equation (16), which can be written in standard notation as

$$H(z, \Omega_m, H_0, \xi) = H_0(1+z)^{3-\xi/2}\sqrt{\left(\frac{3}{\xi-3}\Omega_m^0(1+z)^{\xi-3} + \frac{4}{\xi-2}\Omega_R^0(1+z)^{\xi-2} + \frac{6}{\xi}\Omega_\Lambda\right)} \tag{18}$$

where, for each component, we define $\Omega_i^0 = \rho_i^0/\rho_{cr}$ with ρ_{cr} the critical energy density of the universe. For ΛCDM model, we set $\xi = 6$, and in the unimodular gravity model, we set (H_0, Ω_m, ξ) as free parameters. We used the observational datasets including the Supernovae (SNs), measurements of the BAO, and observational H(z) data obtained from the Differential Age method (DA) to estimate the model parameters. We further constrained our parameters using the distance priors from the Planck 2018 data [40]. The MCMC exploration of the model parameter space was carried out using the Python ensemble sampling toolkit *emcee* [41].

We used the Pantheon dataset [42] for supernovae, which spans the redshift range $0.01 < z < 2.26$ and contains 1048 data points. These observations provide the apparent magnitude $m_o(z)$ of the supernovae at peak brightness. The resulting apparent magnitude $m(z)$ is related to the luminosity distance $d_L(z)$ as

$$m_{th}(z, \Omega_m, H_0, \xi) = 5\log_{10}\left[\frac{d_L(z)}{Mpc}\right] + 25 + M_B \tag{19}$$

where the luminosity distance is

$$d_L(z) = (1+z)\int_0^z \frac{cdz'}{H(z', \Omega_i, \xi)} \tag{20}$$

and M_B is the supernova absolute magnitude. So, the distance modulus is given by

$$\mu(z) = m_{th}(z, \Omega_m, H_0, \xi) - M_B \tag{21}$$

We also considered the 31 observed Hubble data points from the Differential Age (DA) method [43] to constrain the value of H_0. The quantity measured in the differential age method is related to the Hubble parameter:

$$H(z) = -\frac{1}{1+z}\frac{dz}{dt}. \tag{22}$$

This method can be used to find the Hubble constant H_0. Table 1 shows the 31 points of $H(z)$ data given by differential age method [44].

Table 1. The 31 observational data points of $H(z)$ obtained from the differential age method.

z	$H(z)$	σ_H	Ref.	z	$H(z)$	σ_H	Ref.
0.09	69	12	[45]	0.3802	83	13.5	[46]
0.07	69.0	19.6	[47]	0.4004	77	10.2	[46]
0.12	68.6	26.2	[47]	0.4247	87.1	11.2	[46]
0.20	72.9	29.6	[47]	0.4497	92.8	12.9	[46]
0.28	88.8	36.6	[47]	0.4783	80.9	9	[46]
0.17	83	8	[48]	0.47	89	23	[49]
0.27	77	14	[48]	0.48	97	62	[50]
0.4	95	17	[48]	0.88	90	40	[50]
0.9	117	23	[48]	1.3	168	17	[48]
0.1791	75	4	[51]	1.43	177	18	[48]
0.1993	75	5	[51]	1.53	140	14	[48]
0.3519	83	14	[51]	1.75	202	40	[48]
0.5929	104	13	[51]	1.037	154	20	[51]
0.6797	92	8	[51]	1.363	160	33.6	[52]
0.7812	105	12	[51]	1.965	186.5	50.4	[52]
0.8754	125	17	[51]				

In this work, we also wanted to test our model for the Hubble tension with the SH0ES observations. The determination of H_0 in the SH0ES observations [32,53] is based on the constraint on the absolute magnitude M_B, which depends on the astrophysical properties of the sources. Furthermore, in the SH0ES observations, to determine the value of the Hubble constant H_0, the data of Type Ia supernovae in the redshift range $0.023 \leq z \leq 0.15$ are used, and cosmography with $q_0 = -0.55$ and $j_0 = 1$ is considered. In [54], it was argued that the supernova absolute magnitude M_B that is used to determine the value of the local H_0 is not compatible with the supernova, BAO, and CMB data, and it was shown that how the statistical analysis with the prior on M_B is an alternative to the prior on H_0. Further, in [54], it was shown that the statistical analysis is biased when the prior on H_0 is used instead of the prior on M_B. A detailed analysis was performed in [54]. Therefore, we adopted this method of applying the prior on M_B to determine the value of H_0. The χ^2 function for M_B is given by

$$\chi^2_{M_B} = \frac{(M_B - M_B^{R21})^2}{\sigma^2_{M_B^{R21}}}, \tag{23}$$

where M_B^{R21} is the calibration that corresponds to the latest constraint on H_0 by the SH0ES observation. For the Pantheon dataset, $M_B^{R21} = -19.2435 \pm 0.0373$ mag.

We performed the joint analysis of the SN and DA datasets by minimizing the chi-squared function, defined as

$$\chi^2 = \chi^2_{DA} + \chi^2_{SN} + \chi^2_{M_B}, \tag{24}$$

where

$$\begin{aligned}
\chi^2_{DA} &= \sum_i \frac{(H(z_i) - H(z_i, \Omega_m, H_0, \xi))^2}{\sigma_H^2}, & (25)\\
\chi^2_{SN} &= \sum_{i,j} \{m_{th}(z_i) - m_o(z_i)\}^T C^{-1}_{i,j} \{m_{th}(z_j) - m_o(z_j)\}, & (26)\\
&= \sum_{i,j} \{\Delta m(z_i) - M_B\}^T C^{-1}_{i,j} \{\Delta m(z_j) - M_B\}, & (27)\\
\Delta m(z_i) &= m_o(z_i) - \mu(z_i). & (28)
\end{aligned}$$

We took the values of $m_0(z)$ and the full covariance matrix, C_{ij}, from the Pantheon catalog. To minimize the chi-squared function of supernovae, we adopted the methodology of marginalization over M_B, as discussed in [54]. The marginalized chi-squared function will become

$$\chi^2_{SN} + \chi^2_{M_B} = \chi^2_{SN,marg} + \chi^2_{loc}, \tag{29}$$

where

$$\chi^2_{SN,marg} = A - \frac{B^2}{C}, \tag{30}$$

$$\chi^2_{loc} = \frac{(B/C - M_B^{R21})^2}{1/C + \sigma^2_{M_B}} \tag{31}$$

and

$$A = \sum_{i,j} \Delta m(z_i)^T C^{-1}_{i,j} \Delta m(z_j), \tag{32}$$

$$B = \sum_{i,j} \Delta m(z_i)^T C^{-1}_{i,j}, \tag{33}$$

$$C = \sum_{i,j} C^{-1}_{ij}. \tag{34}$$

We further used the observational datasets of the BAO measurements including Galaxy BAO and $Ly\alpha$ BAO (eBOSS) to constrain the model parameters. BAO studies along the line of sight measure the combination $H(z)r_d$, whereas investigations of the BAO feature in the transverse direction offer a value of $D_M(z)/r_d$ and $D_V(z)/r_d$, where $r_d \equiv r_s(z_d)$ is the comoving size of the sound horizon at the baryon drag epoch (z_d) [55]. Table 2 contains a list of the datasets we used.

Table 2. BAO data measurements we used in our analysis. D_M, D_V, and r_d are in units of Mpc, while $H(z)$ is in units of km s^{-1} Mpc^{-1}.

Survey	z_{ef}	Measurement	Observation	σ	Reference
6dFGS	0.106	D_V/r_d	2.9762	0.1329	[56]
SDSS MGS	0.15	D_V/r_d	4.4657	0.1681	[57]
BOSS DR12	0.38	$D_M \times r_{d,fid}/r_d$	1518	20	[58]
BOSS DR12	0.38	$H \times r_d/r_{d,fid}$	81.5	1.7	[58]
BOSS DR12	0.51	$D_M \times r_{d,fid}/r_d$	1977	24	[58]
BOSS DR12	0.51	$H \times r_d/r_{d,fid}$	90.5	1.7	[58]
BOSS DR12	0.61	$D_M \times r_{d,fid}/r_d$	2283	28	[58]
BOSS DR12	0.61	$H \times r_d/r_{d,fid}$	97.3	1.8	[58]
BOSS DR14	0.72	D_V/r_d	16.08472	0.41278	[59]
eBOSS QSO	0.978	$D_A \times r_{d,fid}/r_d$	1586.18	284.93	[60]
eBOSS QSO	0.978	$H \times r_d/r_{d,fid}$	113.72	14.63	[60]
eBOSS QSO	1.23	$D_A \times r_{d,fid}/r_d$	1769.08	159.67	[60]
eBOSS QSO	1.23	$H \times r_d/r_{d,fid}$	131.44	12.42	[60]
eBOSS QSO	1.526	$D_A \times r_{d,fid}/r_d$	1768.77	96.59	[60]
eBOSS QSO	1.526	$H \times r_d/r_{d,fid}$	148.11	12.75	[60]
eBOSS QSO	1.944	$D_A \times r_{d,fid}/r_d$	1807.98	146.46	[60]
eBOSS QSO	1.944	$H \times r_d/r_{d,fid}$	172.63	14.79	[60]
eBOSS Ly	2.34	D_M/r_d	37.41	1.86	[61]
eBOSS Ly	2.34	H/r_d	8.86	0.29	[61]
eBOSS QSOxLy	2.35	D_M/r_d	36.3	1.8	[61]
eBOSS QSOxLy	2.35	D_H/r_d	9.20	0.36	[61]
eBOSS combined	2.34	D_M/r_d	37	1.3	[61]
eBOSS combined	2.34	D_H/r_d	9.00	0.22	[61]

The comoving angular diameter distance $D_M(z)$ and volume averaged scale $D_V(z)$ are related to $H(z)$ as

$$D_M(z) = c\int_0^{z'} \frac{dz}{H(z')}, \tag{35}$$

$$D_V(z) = \left(zD_H(z)D_M^2(z)\right)^{1/3}, \tag{36}$$

$$D_A(z) = \frac{D_M}{(1+z)}. \tag{37}$$

The comoving sound horizon is given by

$$r_d = \int_{z_d}^{\infty} \frac{c_s(z)}{H(z)}dz, \tag{38}$$

where $c_s(z) = c[3 + \frac{9}{4}\rho_b(z)/\rho_\gamma(z)]^{-1/2}$ is the speed of sound in the baryon–photon fluid, with $\rho_b(z)$ and $\rho_\gamma(z)$ being the baryon and photon densities, respectively, and z_d being the redshift at the drag epoch. The reference point used to calibrate the BAO observations is the sound horizon r_d, also known as the standard ruler of BAO observations. From the BAO data, we can only constrain the combination of H_0 and sound horizon r_d. In order to approximate the H_0 from BAO data, we used the analytic approximation of z_d from [62] and took $\Omega_b h^2$ from the Planck 2018 results ($\Omega_b h^2 = 0.02236$) [31].

Also, to constrain the model parameters with the physics of the early universe, we utilized the compressed likelihood of the CMB, which is based on distance priors [40]. Distance priors provide effective information on the CMB through the acoustic scale l_A, which characterizes the variations of the peak spacing, and the shift parameter R, which affects the heights of the peaks.

Distance priors are defined as

$$l_A = (1+z_*)\frac{\pi D_A(z_*)}{r_*(z_*)}, \tag{39}$$

$$R(z_*) \equiv \frac{(1+z_*)D_{\mathrm{A}}(z_*)\sqrt{\Omega_m H_0^2}}{c}, \tag{40}$$

where z_* is the redshift at the photon decoupling epoch. At this epoch, the comoving sound horizon will be

$$r_* = \int_{z_*}^{\infty} \frac{c_s(z)}{H(z)}dz, \tag{41}$$

here we used the analytic approximation of z_* from [62] to estimate the model parameters.

Finally, we combined the BAO data and distance priors from the CMB with the DA and SN datasets and performed the joint analysis to explore the parameter space. We minimized the χ^2 function, defined as

$$\chi^2_{\text{final}} = \chi^2_{DA} + \chi^2_{SN} + \chi^2_{M_B} + \chi^2_{BAO} + \chi^2_{CMB}, \tag{42}$$

$$= \chi^2_{DA} + \chi^2_{SN,marg} + \chi^2_{loc} + \chi^2_{BAO} + \chi^2_{CMB}, \tag{43}$$

where

$$\chi^2_{CMB} = \sum(y_i - d_i)C_{ij}^{-1}(y_j - d_j), \tag{44}$$

and $y_i = \{R(z_*), l_A(z_*)\}$ are values predicted in the UG model, $d_i = \{R^{Planck}, l_A^{Planck}\}$ are set to their mean values, and C_{ij} is their covariance matrix in the ΛCDM model [40].

6. Results and Discussion

In the unimodular gravity model, we set the uniform prior on all three parameters (Ω_m, H_0, ξ) in the range $\Omega_m \in [0, 0.5]$, $H_0 \in [60, 80]$, and $\xi \in [5, 8]$ and performed the MCMC analysis of the joint SN + DA dataset, marginalized over M_B. The mean values of the parameters we obtained were $\Omega_m = 0.294 \pm 0.061, \xi = 6.03 \pm 0.40$, and $H_0 = 72.6 \pm 3.5\,\mathrm{km\,s^{-1}Mpc^{-1}}$. For the standard ($\Lambda$CDM) model ($\xi = 6$), the mean values of the parameters obtained for low redshift data are $H_0 = 72.2 \pm 1.2$ km s^{-1} Mpc^{-1}, and $\Omega_m = 0.285 \pm 0.021$. In Figure 1, we show the results for the unimodular gravity and standard gravity models using the SN + DA datasets.

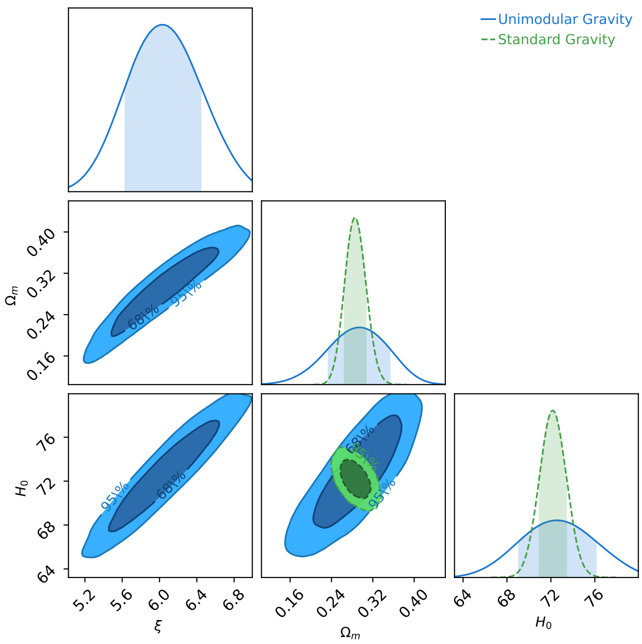

Figure 1. The 2D contour plot and 1D marginalized posterior distributions of model parameters ξ, Ω_m, and H_0 in the unimodular gravity model ($\xi \neq 6$) and the standard (ΛCDM) model ($\xi = 6$) using the SN + DA datasets.

We note that, while unimodular gravity theory can fit the low-redshift SN + DA data with a small deviation in the mean value of the parameters (ξ, Ω_m, H_0), the constraints on Ω_m and H_0 are not as tight as in the case of the standard ΛCDM model.

We next used the BAO dataset combined with the CMB distance priors and set the uniform prior on ξ, Ω_m, and H_0. We obtained the mean value of the parameters as $\Omega_m = 0.299 \pm 0.008, \xi = 6.029 \pm 0.022$, and $H_0 = 68.45 \pm 0.66\,\mathrm{km\,s^{-1}Mpc^{-1}}$. If we put these values in Equations (38) and (41), we find that the sound horizon at drag epoch (r_d) and at photon decoupling epoch (r_*) is 148.7 Mpc and 146.1 Mpc, respectively. The mean values of the parameters we found for the standard (ΛCDM) model ($\xi = 6$) are $\Omega_m = 0.306 \pm 0.006$, $H_0 = 67.80 \pm 0.42$ km s^{-1} Mpc^{-1}, and in this case, $r_d = 147.9$ Mpc and $r_* = 145.3$ Mpc.

In this case, we found that the mean values of Ω_m and H_0 are tightly constrained in the UG model and differ by $\sim 1\sigma$ from the standard ΛCDM model. Also, the value of r_d and r_* is consistent with the ΛCDM model. In Figure 2, we show the comparison of the unimodular gravity model with the standard gravity model using the BAO + CMB data.

Combining the SN + DA data with the BAO and CMB distance priors, we obtain $\xi = 6.037 \pm 0.020$, $\Omega_m = 0.292 \pm 0.007$, and $H_0 = 69.01 \pm 0.60\,\mathrm{km\,s^{-1}Mpc^{-1}}$ as the best-fit parameters for the unimodular gravity model and $\Omega_m = 0.30 \pm 0.005$ and $H_0 = 68.25 \pm 0.40\,\mathrm{km\,s^{-1}Mpc^{-1}}$ for the standard (ΛCDM) model. We found that the values of the model parameters in the unimodular gravity model using the SN + DA + BAO + CMB dataset are consistent with the Planck 2018 results. The results of both models are shown in

Figure 3. In Table 3, we list the best-fit value of the parameters inferred from the CMB, BAO, DA, and SN datasets in the standard (ΛCDM) model and the unimodular gravity model.

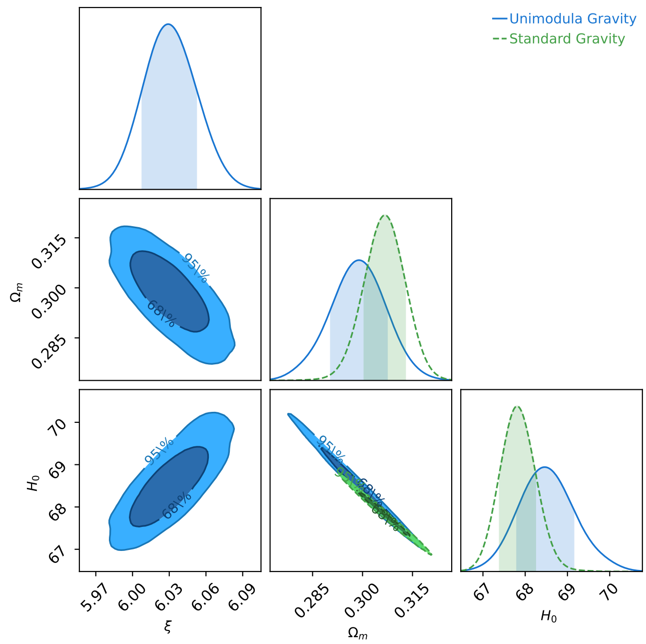

Figure 2. The 2D contour plot and 1D posterior distributions of model parameters ξ, Ω_m, and H_0 in the unimodular gravity model ($\xi \neq 6$) and the standard (ΛCDM) model ($\xi = 6$) using the BAO data and distance priors from the Planck 2018 data.

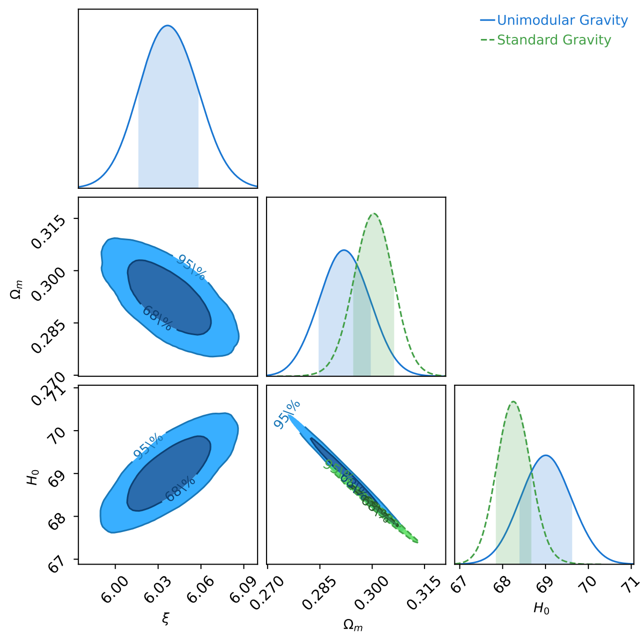

Figure 3. The 2D contour plot and 1D posterior distributions of model parameters ξ, Ω_m, and H_0 in the unimodular gravity model ($\xi \neq 6$) and the standard (ΛCDM) model ($\xi = 6$) using the BAO + SN + DA + CMB data.

Table 3. The results for the standard (ΛCDM) model and the unimodular gravity model using the BAO, DA, and SN datasets.

Data Sets	Parameter	Standard Gravity	Unimodular Gravity
SN + DA	H_0	72.2 ± 1.2	72.6 ± 3.5
	ξ	6	6.03 ± 0.40
	Ω_m	0.285 ± 0.021	0.294 ± 0.061
BAO + CMB	H_0	67.80 ± 0.42	68.45 ± 0.66
	ξ	6	6.029 ± 0.022
	Ω_m	0.306 ± 0.006	0.299 ± 0.008
	r_d	147.9	148.7
	r_*	145.3	146.1
SN + DA + BAO + CMB	H_0	68.25 ± 0.40	69.01 ± 0.60
	ξ	6	6.037 ± 0.020
	Ω_m	0.30 ± 0.005	0.292 ± 0.007
	r_d	148.2	149.2
	r_*	145.6	146.6

In order to study the impact of unimodular gravity on the Hubble tension problem, in Figure 4, we plot the 1D posterior distribution of H_0 and the corresponding median along with the 1σ error band plot for the unimodular gravity and standard (ΛCDM) models. Here, we also compared our results with the Planck 2018 and SH0ES 2022 estimates of H_0.

From these results, one can note that the value of H_0 is sensitive to the value of ξ. Here, we estimated $\xi \approx 6.03$ from all datasets. The mean value of H_0 obtained from SN + DA data changed significantly after including the BAO + CMB data, and the constraint became more stringent. Furthermore, the tension with the SH0ES 2022 results remained almost the same in our results.

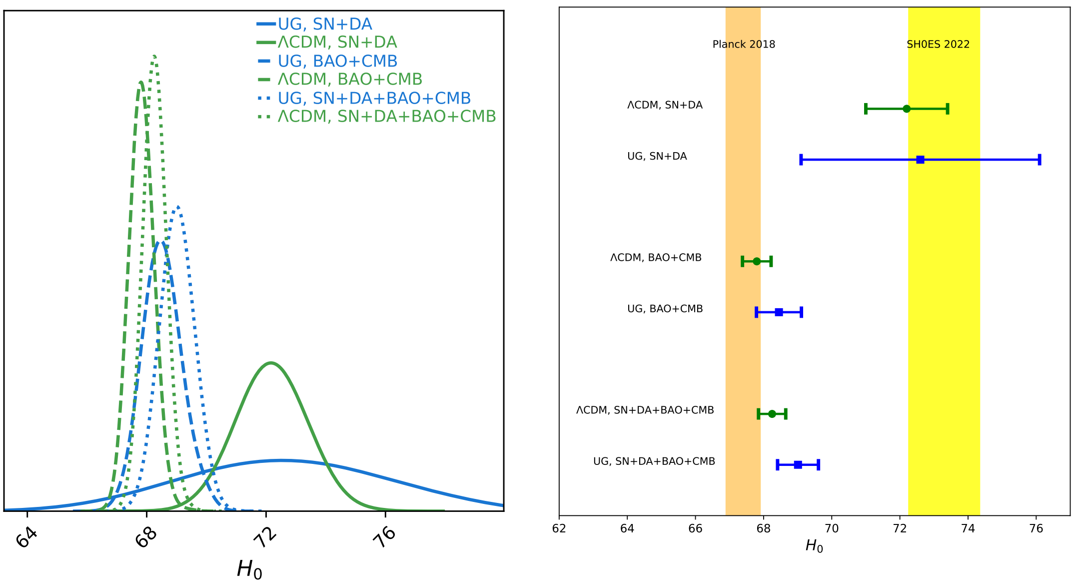

Figure 4. The 1D posterior distribution and corresponding median along with the 1σ error band plot of H_0 for the UG and standard (ΛCDM) models in comparison with the Planck 2018 [31] and SH0ES 2022 [32] results.

7. Conclusions

In this paper, we discussed the cosmological implication of the unimodular theory of gravity, which is an alternative approach to the general theory of relativity. The decomposition of the full metric in a scalar field and the unimodular metric led to the decomposition in the affine connection, Ricci tensor, and Ricci scalar. The general coordinate invariance was broken by introducing a parameter ξ. We considered a unimodular gravity model with broken general coordinate invariance and focused on estimating the cosmological parameters of the current universe in this model. We found that the UG model can fit the cosmological data with $\xi \approx 6.03$, which is very close to that of general relativity ($\xi = 6$). We also observed a small increase in the mean value of H_0 obtained in unimodular gravity compared with the standard (ΛCDM) model in all datasets.

We compared our theory of unimodular gravity with that studied in [28]. In this cosmological diffusion model, the unimodular constraint was applied to the full metric. In this way, a non-conservative current was established. Such a current can be due to the interaction of all matter. However, as the interaction of ordinary matter is well known, the authors explored the non-gravitational interaction of dark sector components. The cold dark matter interacts with dark energy, which appears as a time-dependent cosmological term $\Lambda(t)$. It was shown that, for some value of the diffusion parameter of the theory, it is possible to alleviate the Hubble tension by 2.4σ. On the other hand, in our theory, we applied the unimodular constraint only to the unimodular metric, and no such non-gravitational interacting term was considered. The introduced parameter ξ is effective only in the gravitational sector and generalized cosmological constant. As a result, the theory does not reduce the Hubble tension as much as in the diffusion model of unimodular gravity.

In the current work, we studied the cosmological parameters at the background level. A study of cosmological perturbation can further give more-precise values of the parameters of the theory. In [25], a study of linear perturbation was performed under such a construction of unimodular gravity where the decomposition of the metric and the introduction of ξ were performed in the same way. The power spectrum was derived, and the unimodular correction to the tensor-to-scalar ratio was obtained. It was shown that the unimodular correction can raise or lower the value of the tensor-to-scalar ratio. However, a detailed study is still required to know the shift in the peaks in the angular power spectrum in the CMB, which can further refine the value of the Hubble constant. One has to go for the complete study of cosmological perturbation assuming such a constraint of unimodular gravity.

A comparison with the model used in [16] is necessary. In that model, only the generalized cosmological constant or generalized non-relativistic matter was considered, whereas in our model, cold dark matter, the generalized cosmological constant, and radiation were all taken into account. Therefore, our model provides a more-satisfactory fit to the data. It could be a reliable model that consists of cold dark matter and the cosmological constant, similar to the ΛCDM of standard gravity.

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