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Lorentz Symmetry Violation Effects Caused by the Coupling between the Field $f^{\mu}\gamma^{5}$ and the Derivative of the Fermionic Field on One-Dimensional Potentials

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Abstract: In search of physics beyond the standard model, new phenomena can be relevant in low energies. In view of the Standard Model Extension is an effective field theory, in this study we explore the fermionic sector by showing that the properties of nonrelativistic quantum systems can be modified. We study one-dimensional nonrelativistic quantum systems under Lorentz symmetry violation effects caused by the coupling between the fixed vector field $f^{\mu}\gamma^{5}$ and the derivative of the fermionic field. We deal with the quantum bouncer, the attractive inverse-square potential, a modified attractive inverse-square potential, and a scalar exponential potential inside this scenario of the Lorentz symmetry violation. Then, we show that the spectra of energy are influenced by the Lorentz symmetry violation effects.

Keywords: violation of the Lorentz symmetry; derivative of the fermionic field; quantum bouncer; attractive inverse-square potential; scalar exponential potential

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1. Introduction

General Relativity (GR) succeeded in describing the gravitational interaction as a manifestation of the curvature of spacetime. So far, GR has passed all the tests to which it was submitted. On the other hand, the effects of dark matter and the accelerated expansion of the universe (dark energy) pose a challenge to understand which local theory is adequate to understand these phenomena. The quantum description of gravitation is the major motivation in search of a more fundamental theory. In this way, an extension of the Standard Model of Particle Physics (SM) was proposed in Ref. [1]. The search for an extended theory has promoted the idea that the Lorentz symmetry can be violated at some energy scale. In recent years, two forms of investigation of this type of violation have been explored: the Lorentz Symmetry Violation (LSV) by noncommutative theories [2] and by spontaneously generated LSV. The spontaneous violation of the Lorentz symmetry allows to investigate this option at low energies by exploring contexts in which laboratory measurements can be analyzed, and then, traces of this break can appear. Kostelecký and Samuel proposed the idea of the extension of the Higgs mechanism by the presence of nonzero vacuum expectation values of tensor-valued fields in the context of String Theory [3]. The immediate consequence is the spontaneous break in the Lorentz symmetry [4–6]. This proposal takes into account the renormalizability, and it is known as the Standard Model Extension (SME) [7,8]. If we relax the renormalizability condition, then we can explore several possibilities out of SME (nonminimal SME) [9–39].

Our research perspective is to establish low energy contexts that can show spontaneous symmetry violation by a background field. By observing that SME presents explicitly LSV in the fermionic sector, we thus explore this scenario in $(1+1)$-dimensions. Therefore, our focus is to investigate how background fields can modify the properties of nonrelativistic quantum systems. However, why should we investigate this possibility of detection LSV since LSV is expected to occur at high energies? We should observe that SME is an effective field theory, and nowadays we can perform very precise measurements in low-energy physics. Thereby, we might hope to detect minute modifications of such physics arising from new phenomena in low energies.

In this work, we discuss the Lorentz symmetry violation effects caused by the coupling between a fixed vector field $B^{\mu} = \gamma^5 f^{\mu}$ and the derivative of the fermionic field proposed by Kostelecký and Lane [6] on one-dimensional potentials at low-energy regime. Our analysis of the Lorentz symmetry violation effects is made from the eigenvalues of energy. We present a different perspective on the search for Lorentz symmetry violation effects in the low-energy regime.

This paper is organized as follows: in Section 2, we introduce the coupling between the vector field $B^{\mu} = f^{\mu}\gamma^5$ and the derivative of the fermionic field, which gives rise to the background of the Lorentz symmetry violation; in Section 3, we analyze the Lorentz symmetry violation effects on the quantum bouncer [40–43]; in Section 4, we deal with the Lorentz symmetry violation effects on the attractive inverse-square potential [44]; in Section 5; we discuss the Lorentz symmetry violation effects on the one-dimensional harmonic oscillator and the attractive inverse-square potential; in Section 6 we study the Lorentz symmetry violation effects on the scalar exponential potential [41,45]; in Section 7, we present our conclusions.

2. Non-Relativistic Wave Equation in a Background of the Lorentz Symmetry Violation

Based on Ref. [6], a model of describing a Lorentz symmetry violation background is made by introducing a coupling between the vector field $f^{\mu}\gamma^5$ and the derivative of the fermionic field. The Dirac equation is written in the following form (with the units $\hbar = 1$ and $c = 1$):

$$i\gamma^{\mu}\partial_{\mu}\Psi + i\, f^{\mu}\,\gamma^5\, i\,\partial_{\mu}\,\Psi = m\,\Psi. \tag{1}$$

The term $f^{\mu}\gamma^5$ is a fixed vector field that yields a privileged direction in the spacetime, where f^{μ} establishes the extent of the Lorentz symmetry violation [6]. Moreover, γ^{μ} corresponds to the Dirac γ matrices, where $\gamma^0 = \hat{\beta}$ and $\gamma^k = \hat{\beta}\,\hat{\alpha}^k$ [46].

Let us consider $(1+1)$-dimensions $\left(ds^2 = -dt^2 + dx^2\right)$. In this case, the Dirac matrices are defined as [47]: $\hat{\beta} = \sigma^3$, $\hat{\alpha}^1 = \sigma^1$ and $\hat{\beta}\,\gamma^5 = \sigma^2$, where the matrices $\sigma^i = \left(\sigma^1, \sigma^2, \sigma^3\right)$ are the standard Pauli matrices [42]. Henceforth, we assume that the extent of the Lorentz symmetry violation is determined by the space-like vector [48]:

$$f^{\mu} = (0,\, \varsigma). \tag{2}$$

Thereby, the Dirac Equation (1) becomes [48,49]

$$i\,\frac{\partial\Psi}{\partial t} = m\,\sigma^3\Psi - i\sigma^1\,\frac{\partial\Psi}{\partial x} + \varsigma\,\sigma^2\,\frac{\partial\Psi}{\partial x}. \tag{3}$$

By following Ref. [46] in order to achieve the nonrelativistic limit of the Dirac Equation (3), the four-component spinor $\Psi(t,x)$ can be written in terms of two-component spinors: $\psi(t,x)$ and $\chi(t,x)$. This is made by $\Psi(t,\,x) = e^{-i\,m\,t}(\psi\;\chi)^{\mathrm{T}}$, where m is the rest mass of the particle. Note that $\psi(t,x)$ corresponds to the large components of $\Psi(t,\,x)$, while $\chi(t,x)$ corresponds to the small components of $\Psi(t,\,x)$ [46]. Then, after some calculations, we obtain the Schrödinger equation (with the units $\hbar = 1$ and $c = 1$):

$$i\,\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\left(1 - \varsigma^2\right)\frac{\partial^2\psi}{\partial x^2}. \tag{4}$$

In the following sections, we bring a discussion about the nonrelativistic effects of the Lorentz symmetry violation caused by the coupling between the vector field $f^{\mu}\gamma^{5}$ and the derivative of the fermionic field on one-dimensional potentials. By considering a quantum particle subject to a scalar potential $V(x)$, the Schrödinger Equation (4) becomes [48,49]

$$i\,\frac{\partial\psi}{\partial t}=-\frac{1}{2m}\left(1-\varsigma^{2}\right)\frac{\partial^{2}\psi}{\partial x^{2}}+V(x)\psi. \tag{5}$$

We thus search for Lorentz symmetry violation effects on a particle subject to a quantum bouncer [40–42], the attractive inverse-square potential [44], a modified attractive inverse-square potential [50], and the scalar exponential potential [41,45]. Despite working with $(1+1)$-dimensions, we should observe that these potentials can be dealt with by using the spherical and cylindrical symmetries. Examples of them are given in Refs. [38,39,50–53].

3. Quantum Bouncer

We start our search for nonrelativistic effects of the Lorentz symmetry violation by considering a particle subject to a constant force, for instance, a particle of mass m in a constant gravitational field or an electric charge in a uniform electric field [40–43]. This quantum system is called quantum bouncer. The name "quantum bouncer" was given by Gibbs [40] and it has been studied in several contexts [41,54–57]. From a semiclassical analysis, it is worth citing the studies of the quantum bouncer made in Refs. [42,58] and the analogues of it in Refs. [59,60]. The quantum bouncer is described by the potential energy [40–43]:

$$V(x)=\begin{cases} m\,g\,x & (x>0);\\ \infty & (x<0),\end{cases} \tag{6}$$

where m is the mass of the particle, g is the gravity acceleration, and x is the height over Earth's surface.

In this section, we consider the extent of the Lorentz symmetry violation to be determined by the space-like vector (2), hence, with the potential energy (6), the Schrödinger Equation (5) for $x>0$ is given by

$$i\,\frac{\partial\psi}{\partial t}=-\frac{1}{2m}\left(1-\varsigma^{2}\right)\frac{\partial^{2}\psi}{\partial x^{2}}+m\,g\,x\,\psi. \tag{7}$$

Observe that the Hamiltonian operator given in the right-hand side of Equation (7) does not depend on t, therefore, the solution to Equation (7) can be given by $\psi(t,\,x)=e^{-i\,E\,t}\,u(x)$. By substituting this solution into Equation (7), we find

$$\frac{d^{2}u}{dx^{2}}-\frac{2m^{2}g}{(1-\varsigma^{2})}\,x\,u+\frac{2mE}{(1-\varsigma^{2})}\,u=0. \tag{8}$$

Let us go further by defining the parameter:

$$y=\left(\frac{2m^{2}g}{1-\varsigma^{2}}\right)^{1/3}\times\left[x-\frac{E}{mg}\right], \tag{9}$$

thus, Equation (8) becomes

$$\frac{d^{2}u}{dy^{2}}-y\,u=0. \tag{10}$$

Hence, the second order differential Equation (10) is known as the Airy equation [40,42,61]. Its solution is given by

$$u(y)=b_{1}\,A_{i}(y)+b_{2}\,B_{i}(y), \tag{11}$$

where b_1 and b_2 are constants and the functions $A_i(y)$ and $B_i(y)$ are the Airy functions [40–42,61]. Note that, when $x \to \infty$, then, $y \to \infty$. Therefore, we have the boundary condition:

$$u(y \to \infty) = 0. \tag{12}$$

By substituting the wave function (11) into Equation (12), we have that the boundary condition (12) requires that $b_2 = 0$, because $B_i(y) \to \infty$ when $y \to \infty$ [40–42,61]. Thereby, the wave function (11) becomes

$$u(y) = b_1\, A_i(y). \tag{13}$$

On the other hand, when $x = 0$, then, $y \to y_0 = -\frac{E}{mg} \times \left(\frac{2m^2 g}{1-\varsigma^2}\right)^{1/3}$. In this way, we have the second boundary condition:

$$u(y_0) = 0. \tag{14}$$

By substituting (13) into Equation (14), we obtain

$$u(y_0) = b_1\, A_i(y_0) = 0. \tag{15}$$

With the purpose of obtaining the energy levels explicitly, let us consider the case where $y_0 \ll 0$. By following Refs. [41,42,61], for $y_0 \ll 0$, the function $A_i(y_0)$ can be written in the form:

$$A_i(y_0) \approx \frac{1}{\sqrt{\pi}\,(-y_0)^{1/4}}\,\sin\left(\frac{2}{3}(-y_0)^{3/2} + \frac{\pi}{4}\right). \tag{16}$$

We should observe that $y_0 < 0$, thereby, after substituting (16) into Equation (15) we obtain:

$$\frac{2}{3}\,(-y_0)^{3/2} + \frac{\pi}{4} = n\pi, \tag{17}$$

where $n = 1, 2, 3, \ldots$. Since $y_0 = -\frac{E}{mg} \times \left(\frac{2m^2 g}{1-\varsigma^2}\right)^{1/3}$, we obtain from Equation (17):

$$E_n \approx \left(\frac{mg^2\,(1-\varsigma^2)}{2}\right)^{1/3} \times \left[\frac{3\pi}{2}\left(n - \frac{1}{4}\right)\right]^{2/3}. \tag{18}$$

Hence, the energy eigenvalues of the quantum bouncer given in Equation (18) is influenced by the Lorentz symmetry violation effects caused by the coupling between the vector field $f^{\mu}\gamma^{5}$ and the derivative of the fermionic field. The nonrelativistic effects of the Lorentz symmetry violation on the energy levels are determined by the presence of the parameter ς, which arises from the extent of the Lorentz symmetry violation defined by the space-like vector (2). In the limit $\varsigma \to 0$, we recover the energy levels of the quantum bouncer in the absence of the Lorentz symmetry violation background [41].

From another point of view, the linear confining potential given in Equation (6) can be obtained when an electron is confined to a triangular well [43,62–67], which has a great interest in semiconductor devices [62–67]. Therefore, nanostructures such as the triangular well give us a good hint about searching for Lorentz symmetry breaking effects.

4. Attractive Inverse-Square Potential

Let us consider a particle subject to the attractive inverse-square potential. For the one-dimensional case, we follow Ref. [44] and write the attractive inverse-square potential as the "regularized" potential:

$$V(x) = \begin{cases} \infty & (x \leq \epsilon); \\ -\frac{a}{x^2} & (x > \epsilon), \end{cases} \tag{19}$$

where $\epsilon > 0$ and $a > 0$. In recent years, analogues of the attractive inverse-square potential have been studied in Lorentz symmetry violation backgrounds [38,39].

Therefore, by considering the potential energy (19), the Schrödinger Equation (5) in the region $x > \epsilon$ is given in the form:

$$i\,\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\left(1-\varsigma^2\right)\frac{\partial^2\psi}{\partial x^2} - \frac{a}{x^2}\,\psi. \tag{20}$$

In the same way of the previous section, the solution to Equation (20) can be written in the form: $\psi(t,\,x) = e^{-i\,E\,t}\,u(x)$. Thereby, after substituting $\psi(t,\,x) = e^{-i\,E\,t}\,u(x)$ in Equation (20), we have

$$\frac{d^2u}{dx^2} + \frac{2ma}{(1-\varsigma^2)}\frac{u}{x^2} + \frac{2mE}{(1-\varsigma^2)}\,u = 0. \tag{21}$$

In search of bound states, we assume that $E < 0$ from now on. Then, we define the parameter:

$$\lambda = \sqrt{\frac{-2mE}{(1-\varsigma^2)}}, \tag{22}$$

and perform the change of variables: $\rho = \lambda\,x$. In this way, Equation (21) becomes

$$\frac{d^2u}{d\rho^2} + \frac{2ma}{(1-\varsigma^2)}\frac{u}{\rho^2} - u = 0. \tag{23}$$

Let us write the solution to Equation (23) in the following form:

$$u(\rho) = \sqrt{\rho}\,W(\rho), \tag{24}$$

where $W(\rho)$ is an unknown function. After substituting the function (24) into Equation (23), we obtain the following equation for the function $W(\rho)$:

$$\frac{d^2W}{d\rho^2} + \frac{1}{\rho}\frac{dW}{d\rho} + \frac{\nu^2}{\rho^2}\,W - W = 0. \tag{25}$$

The second order differential Equation (25) is known as the Bessel equation [61,68] and the parameter ν is defined as

$$\nu = \sqrt{\frac{2ma}{(1-\varsigma^2)} - \frac{1}{4}}. \tag{26}$$

Let us impose that $W(\rho) \to 0$ when $\rho \to \infty$. Thereby, a solution to Equation (25) is given in terms of the modified Bessel function of the third kind of imaginary order [44,69–73]:

$$W(\rho) = d_1\,K_{i\nu}(\rho), \tag{27}$$

where d_1 is a constant.

Let us write $\rho_n = \lambda_n \epsilon$ when $x = \epsilon$. From the potential energy (19), when $x = \epsilon$ we have the boundary condition:

$$W(\rho_n) = d_1\, K_{i\nu}(\rho_n) = 0. \tag{28}$$

With the aim of obtaining the energy levels explicitly, we consider $\rho_n \ll 1$. For $\rho_n \ll 1$, the function $K_{i\nu}(\rho_n)$ can be written in the form [69,72–74]:

$$K_{i\nu}(\rho_n) \propto \sin(\nu\, \ln(\rho_n/2) + \delta), \tag{29}$$

where δ is a constant [69,72–74]. By substituting (29) into Equation (28), we obtain

$$\rho_n = \frac{2}{e^{\delta/\nu}}\, e^{\alpha\pi/\nu}, \tag{30}$$

where $\alpha = 0, \pm1, \pm2, \pm3, \ldots$. However, since $\rho_n \ll 1$, we have that the possible values of the parameter α must be given by $\alpha = -n$, where $n = 1, 2, 3, 4, \ldots$ [75]. With $\rho_n = \lambda_n \epsilon$, where λ_n is given in Equation (22), we obtain from Equation (30):

$$E_n = -\frac{2(1-\varsigma^2)}{m\,\epsilon^2\, e^{2\delta/\sqrt{\frac{2ma}{(1-\varsigma^2)}-\frac{1}{4}}}}\, \exp\left(-\frac{2n\pi}{\sqrt{\frac{2ma}{(1-\varsigma^2)}-\frac{1}{4}}}\right). \tag{31}$$

The energy levels given in Equation (31) show us that there exists the influence the Lorentz symmetry violation effects caused by the coupling between the vector field $f^\mu \gamma^5$ and the derivative of the fermionic field on them. The energy levels (31) decrease exponentially with the quantum number n. When $n \to \infty$ we have that $E_{n\to\infty} = 0$. Therefore, there is an accumulation point of energy levels in the energy level $E_n = 0$ [73]. The effects of the Lorentz symmetry violation is also determined by the parameter ς. It is worth emphasizing that ς stems from the extent of the Lorentz symmetry violation defined by the space-like vector (2). Besides, we recover the energy levels of the attractive inverse-square potential in the absence of Lorentz symmetry breaking effects [44] by using the limit $\varsigma \to 0$ in Equation (31).

5. Modified Attractive Inverse-Square Potential

Let us extend the discussion about the attractive inverse-square potential of Ref. [44] by including a contribution to the "regularized" potential (19) given by the one-dimensional harmonic oscillator. Thereby, the "regularized" potential (19) becomes

$$V(x) = \begin{cases} \infty & (x \leq \epsilon); \\ -\frac{a}{x^2} + \frac{1}{2}\, m\,\omega^2\, x^2 & (x > \epsilon), \end{cases} \tag{32}$$

where we also have $\epsilon > 0$ and $a > 0$. Recently, we studied an analogue of the potential (32) in a system of a neutral particle with an induced electric dipole moment [50].

Therefore, by considering the potential energy (32), the Schrödinger Equation (5) for $x > \epsilon$ is given by

$$i\,\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\left(1-\varsigma^2\right)\frac{\partial^2\psi}{\partial x^2} - \frac{a}{x^2}\,\psi + \frac{1}{2}\, m\,\omega^2\, x^2\,\psi. \tag{33}$$

Let us also write $\psi(t,\, x) = e^{-i\,E\,t}\, u(x)$, and thus, Equation (33) becomes

$$\frac{d^2u}{dx^2} + \frac{2ma}{(1-\varsigma^2)}\frac{u}{x^2} - \frac{m^2\omega^2}{(1-\varsigma^2)}\, x^2\, u + \frac{2mE}{(1-\varsigma^2)}\, u = 0. \tag{34}$$

We go further by performing the change of variables: $r = m\,\omega\, x^2/\sqrt{1-\varsigma^2}$. Thereby, Equation (34) becomes

$$\frac{d^2u}{dr^2} + \frac{1}{2r}\frac{du}{dr} + \frac{ma}{2(1-\varsigma^2)}\frac{u}{r^2} - \frac{1}{4}u + \frac{E}{2\omega\sqrt{1-\varsigma^2}}\frac{u}{r} = 0. \tag{35}$$

The solution to Equation (35) is given in the form:

$$u(r) = \frac{c_1}{r^{1/4}}\, M_{\kappa,\, i\mu}(r) + \frac{c_2}{r^{1/4}}\, W_{\kappa,\, i\mu}(r), \tag{36}$$

where c_1 and c_2 are constants. The functions $M_{\kappa,\, i\mu}(r)$ and $W_{\kappa,\, i\mu}(r)$ are the Whittaker functions of first and second kinds of imaginary order, respectively [50,76]. Besides, the parameters κ and μ are defined as

$$\begin{aligned} \kappa &= \frac{E}{2\omega\sqrt{1-\varsigma^2}}; \\ \mu &= \sqrt{\frac{ma}{2(1-\varsigma^2)} - \frac{1}{16}}. \end{aligned} \tag{37}$$

Let us impose that $u(r) \to 0$ when $r \to \infty$. Thereby, let us take $c_1 = 0$ in Equation (36), and thus, the solution to Equation (35) can be given in terms of the Whittaker functions of second kind of imaginary order [50,76]:

$$u(r) = \frac{c_2}{r^{1/4}}\, W_{\kappa,\, i\mu}(r). \tag{38}$$

Note that, when $x = \epsilon$ we can write $r_0 = m\,\omega\,\epsilon^2/\sqrt{1-\varsigma^2}$. In addition, when $x = \epsilon$, we have the boundary condition:

$$u(r_0) = 0. \tag{39}$$

Then, by substituting Equation (38) into Equation (39), we have

$$u(r_0) = \frac{c_2}{r_0^{1/4}}\, W_{\kappa,\, i\mu}(r_0) = 0. \tag{40}$$

In order to obtain the energy levels explicitly, we focus on the particular case where $r \ll 1$. As shown in Ref. [50], for $r \ll 1$, we can write

$$W_{\kappa,\, i\mu}(r) \approx 2A\,\sqrt{r}\,\cos\left(2\mu + \mu\,\ln\left(\frac{\beta\, r}{4\mu^2}\right) + \frac{\pi}{4}\right), \tag{41}$$

where $\beta = \frac{1}{2} - \kappa$ and

$$A = \frac{e^{-\mu\pi+\beta}}{\sqrt{2\mu}\;\beta^{\beta-1/2}}. \tag{42}$$

Therefore, after substituting (41) into the boundary condition (40), we obtain

$$r_0 = \frac{4\mu^2}{\beta}\, e^{\left(\frac{\pi}{4\mu}+2\right)}\, e^{\pi\alpha/\mu}, \tag{43}$$

where $\alpha = 0, \pm1, \pm2, \pm3, \ldots$. Observe that the possible values of α in which ww satisfy the condition $r_0 \ll 1$ are $\alpha = -n$, where $n = 1, 2, 3, \ldots$. In this way, with $\beta = \frac{1}{2} - \kappa$ and by substituting Equation (37) into Equation (43), we find

$$E_n = \omega\sqrt{1-\varsigma^2} - \frac{8\mu^2\left(1-\varsigma^2\right)e^{\left(\frac{\pi}{4\mu}-2\right)}}{m\,\epsilon^2}\,\exp\left(-\frac{n\pi}{\sqrt{\frac{ma}{2(1-\varsigma^2)}-\frac{1}{16}}}\right). \tag{44}$$

Hence, Equation (44) yields the energy levels of the modified attractive inverse-square potential (32) in the background of the Lorentz symmetry violation effects caused by the coupling between the vector field $f^{\mu}\gamma^5$ and the derivative of the fermionic field. The influence of the Lorentz symmetry violation background is given by the presence of the parameter ς in the energy levels which, in turn, arises from the extent of the Lorentz symmetry violation defined by the space-like vector (2). Note that the Lorentz symmetry violation effects modify the angular frequency of the harmonic oscillator by yielding an effective angular frequency $\omega' = \omega\,\sqrt{1-\varsigma^2}$. By taking the limit $\varsigma \to 0$ in Equation (44), we obtain the energy levels of the modified attractive inverse-square potential (32) in the absence of Lorentz symmetry breaking effects.

6. Scalar Exponential Potential

In this section, we analyze a scalar exponential potential [41,45] in the background of the Lorentz symmetry violation yielded by the coupling between the vector field $f^{\mu}\gamma^5$ and the derivative of the fermionic field. The extent of the Lorentz symmetry violation is also defined by the space-like vector (2). Therefore, the scalar exponential potential is given by [41,45]:

$$V(x) = -V_0\,e^{-x/a}, \tag{45}$$

where V_0 and a are constants and x is defined in the range $0 \leq x < \infty$. Despite being a spherically symmetric potential and possessing exact solutions only for s-waves [41], we can study this system in the Lorentz symmetry violation background by dealing with the nonrelativistic limit of the Dirac equation in $(1+1)$-dimensions. Indeed, the nonrelativistic limit of the Dirac equation given by Equation (5) allow us to analyze the s-waves in a quantum system with spherical symmetry. In this way, by considering the potential energy (45), the Schrödinger Equation (5) becomes

$$i\,\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\left(1-\varsigma^2\right)\frac{\partial^2\psi}{\partial x^2} - V_0\,e^{-x/a}\,\psi. \tag{46}$$

We proceed with the solution to Equation (46) by being $\psi(t,\,x) = e^{-i\,E\,t}\,u(x)$, then, Equation (46) becomes

$$\frac{d^2u}{dx^2} + \frac{2mV_0}{(1-\varsigma^2)}e^{-x/a}\,u + \frac{2mE}{(1-\varsigma^2)}\,u = 0. \tag{47}$$

We thus perform the change of variables:

$$y = e^{-x/2a}, \tag{48}$$

and rewrite Equation (47) as follows:

$$\frac{d^2u}{dy^2} + \frac{1}{y}\frac{du}{dy} + \frac{8mE\,a^2}{(1-\varsigma^2)}\frac{u}{y^2} + \frac{8mV_0\,a^2}{(1-\varsigma^2)}\,u = 0. \tag{49}$$

Our aim is to achieve bound states, hence, we assume that $E < 0$. Next, we define

$$\begin{aligned} \bar{\nu} &= \sqrt{-\frac{8mE\,a^2}{(1-\varsigma^2)}}; \\ \bar{\beta} &= \sqrt{\frac{8m\,V_0\,a^2}{(1-\varsigma^2)}}, \end{aligned} \tag{50}$$

and write Equation (49) in the form:

$$\frac{d^2u}{dy^2} + \frac{1}{y}\frac{du}{dy} - \frac{\bar{\nu}^2}{y^2}\,u + \bar{\beta}^2\,u = 0. \tag{51}$$

Note that Equation (51) is also the Bessel differential equation [61,68]. Let us consider a regular solution to Equation (51) at $y = 0$ $(x \to \infty)$:

$$u(y) = c_1\,J_{\bar{\nu}}(\bar{\beta}\,y), \tag{52}$$

where c_1 is a constant and $J_{\bar{\nu}}(\bar{\beta}\,y)$ is the Bessel function of first kind [61,68]. Next, let us impose that $u(y) \to 0$ when $y = 1$ $(x = 0)$. This yields the following boundary condition:

$$u(y = 1) = c_1\,J_{\bar{\nu}}(\bar{\beta}) = 0. \tag{53}$$

From now on, we focus on the particular case where $\bar{\beta}\,y \gg 1$. In short, in the case where $z_0 \gg 1$, the Bessel function can be written in the form [61]:

$$J_{\nu}(z_0) \to \sqrt{\frac{2}{\pi\,z_0}}\cos\left(z_0 - \frac{\bar{\nu}\,\pi}{2} - \frac{\pi}{4}\right). \tag{54}$$

Thereby, by substituting Equation (54) into Equation (53), with $z_0 = \bar{\beta}$ $(y = 1)$, we find

$$\bar{\nu} = \frac{2\bar{\beta}}{\pi} - 2\left(n + \frac{3}{4}\right), \tag{55}$$

where $n = 0, 1, 2, 3, \ldots$. Then, after substituting Equation (50) into Equation (55) we obtain

$$E_n = -\frac{4V_0}{\pi}\left[1 - \frac{\pi}{a}\sqrt{\frac{1-\varsigma^2}{2mV_0}}\left(n + \frac{3}{4}\right)\right]^2. \tag{56}$$

Hence, Equation (56) yields the spectrum of energy of the scalar exponential potential (45) under the effects of the Lorentz symmetry violation effects caused by the coupling between the vector field $f^{\mu}\gamma^5$ and the derivative of the fermionic field. The effects of the Lorentz symmetry violation is viewed through the parameter ς, where ς stems from the extent of the Lorentz symmetry violation defined by the space-like vector (2). Furthermore, Equation (55) shows that bound states exist only if $\frac{2\bar{\beta}}{\pi} - 2\left(n + \frac{3}{4}\right) > 0$, because $\bar{\nu} > 0$. In this way, we obtain an upper limit to the quantum number n given by

$$n_{\max} < \frac{a}{\pi}\sqrt{\frac{2m\,V_0}{(1-\varsigma^2)}} - \frac{3}{4}, \tag{57}$$

otherwise, no bound states exist. Therefore, the quantum number n takes values from zero to the upper limit $(n_{\max})$ given in Equation (57). In addition, this upper limit is influenced by the background of the Lorentz symmetry violation. By comparing with the discussion about the Morse potential made in Ref. [77], the existence of this upper limit means that the number of energy levels is limited.

Finally, by taking the limit $\varsigma \to 0$ in Equation (57), we obtain the spectrum of energy of the scalar exponential potential in the absence of Lorentz symmetry breaking effects.

We should observe that the scalar exponential potential (45) has been used to describe spherical quantum dots [51,52]. Therefore, we have another example of nanostructures that provide us with insights into the search for Lorentz symmetry breaking effects in low energies [6].

7. Conclusions

The search for physics beyond the standard model is a topic of fundamental interest and justified the construction of the LHC. In this work, we have investigated in a low-energy scenario how the model could be extended if a spontaneous breaking of the Lorentz symmetry is detected. We have dealt with one-dimensional quantum systems in the Lorentz symmetry violation background yielded by the coupling between a fixed vector field $f^{\mu}\gamma^{5}$ and the derivative of the fermionic field. Our starting point was the Dirac equation in $(1+1)$-dimensions, which has ensured a general effective theory at low energies. Moreover, the Dirac equation in $(1+1)$-dimensions permit us to deal with the s-waves in a quantum system with spherical symmetry. As an example, we have studied the scalar exponential potential (45). Besides the influence of the Lorentz symmetry breaking effects on the spectrum of energy of the scalar exponential potential, we have seen that the quantum number n has an upper limit $(n_{\max})$, in turn, it is influenced by the background of the Lorentz symmetry violation. With regard to the influence of the Lorentz symmetry violation on the quantum bouncer, the attractive inverse-square potential and the modified attractive inverse-square potential, we have shown in all these cases that the energy levels are influenced by Lorentz symmetry breaking effects.

It is worth observing that the study of the Lorentz symmetry violation yielded by the coupling between a fixed vector field $f^{\mu}\gamma^{5}$ and the derivative of the fermionic field can be extended to $(2+1)$-dimensions and $(3+1)$-dimensions. This gives us a perspective of searching for effects of the Lorentz symmetry violation through the magnetization [78–81] and the thermodynamic properties [82–87] of quantum systems, quantum Hall effect [88], Boson–Fermi systems [89], and the scattering of Dirac particles [90].

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