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Highly Dispersive Optical Solitons with Four Forms of Self-Phase Modulation

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Abstract: This paper implements the enhanced Kudryashov approach to retrieve highly dispersive optical solitons and study it with four nonlinear forms. These are the power law, generalized quadratic-cubic law, triple-power law, and the generalized non-local law. This approach reveals bright and singular optical solitons along with the respective parameter constraints.

Keywords: solitons; bright; singular; Kudryashov; dispersive; constraints

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1. Introduction

A few years ago, the idea of highly dispersive (HD) solitons was proposed. It came out of extreme necessity for sustaining the careful balance between chromatic dispersion (CD) and the self-phase modulation (SPM) for these solitons to travel across inter-continental distances without any loss or pulse collapse. Occasionally, the source of dispersion, namely the CD, could become depleted during such trans-continental and trans-oceanic distances. Therefore, it is deemed necessary to replenish this loss of dispersion by introducing additional dispersion terms. This led to the inclusion of inter-modal dispersion (IMD), third-order dispersion (3OD), fourth-order dispersion (4OD), fifth-order dispersion (5OD) and sixth-order dispersion (6OD). Together, these dispersion terms constitute HD optical solitons. Thus, these six dispersion terms, together with the self–phase modulation (SPM) effect, lead to the structure of HD nonlinear Schrödinger's Equation (NLSE). One issue that immediately pops up is the slowdown of solitons with the inclusion of these dispersion terms and the other issue is the soliton radiation. While soliton radiation is ignored, the slowdown of solitons is inevitable.

This study will concentrate on four distinct types of SPM. They are power law, generalized quadratic-cubic law, triple-power law and the generalized non-local law. The enhanced Kudryashov's approach will be the integration scheme that will be employed to recover the soliton solutions. It is only the bright and singular soliton solutions that are recoverable by the application of this integration architecture. The parameter constraints that naturally emerge from the soliton solutions guarantee the existence of such solitons. The details of the integration algorithm are discussed and the solutions, recovered for these four forms of nonlinear refractive index structure, are enumerated and exhibited. The details follow through next.

Governing Model

Through an optical waveguide, the dimensionless form of HD-NLSE with a generic non-Kerr law of nonlinear refractive index is given as:

$$iq_t + ia_1 q_x + a_2 q_{xx} + ia_3 q_{xxx} + a_4 q_{xxxx} + ia_5 q_{xxxxx} + a_6 q_{xxxxxx} + F[|q|^2]q = 0. \tag{1}$$

Here, in (1), the complex-valued function $q(x,t)$ depicts the wave profile that passes through a monomode optical fiber across intercontinental distances. The two independent variables are x and t, that account for spatial and temporal variables, respectively. The first term is with linear temporal evolution, whose coefficient is $i = \sqrt{-1}$, while the six dispersion terms with coefficients a_j for $j = 1,2,3,4,5,6$ represent the IMD, CD, 4OD, 4OD, 5OD and 6OD. The generalized version of the refractive index that is dependent on intensity is denoted by the functional F. This model, which is a partial differential equation, will now be addressed by means of Kudryashov's improved technique, after it is reduced to an ordinary differential Equation (ODE).

The subsequent section revisits the enhanced Kudryashov's integration scheme. Section 3 applies this scheme to HD-NLSE to recover the bright and singular 1–soliton solutions to the model for the four forms of SPM. Finally, in Section 4, some conclusive statements are given.

2. Enhanced Kudryashov's Method

We will consider the following nonlinear evolution equation (NLEE):

$$P(u, u_x, u_t, u_{xt}, u_{xx}, \ldots) = 0. \tag{2}$$

where P is a polynomial function of u and several of its derivatives, and u is the dependent variable of both t and x. Furthermore, u is an unknown function which was investigated using this technique.

The key algorithmic framework of the improved Kudryashov technique is as follows [1–5]:

Step–1: By applying the following traveling wave hypothesis:

$$u(x,t) = U(\xi), \quad \xi = k(x - vt), \tag{3}$$

where k and v are undetermined constants. Then, Equation (2) is simplified to the form of a nonlinear ordinary differential equation, in general, given by

$$P(U, -kvs.U', kU', k^2U'', \ldots) = 0. \tag{4}$$

Step–2: Presuming that the solution to Equation (4) may be written as

$$U(\xi) = \sum_{l=0}^{N} \lambda_i R^i(\xi), \tag{5}$$

where λ_i are constants to be determined for every $i = 0, 1, \ldots, N$ and the function $R(\xi)$ satisfies the following nonlinear ODE:

$$R'(\xi)^2 = R(\xi)^2(1 - \chi R(\xi)^2), \tag{6}$$

The solutions of (6) is

$$R(\xi) = \frac{4a}{4a^2 e^{\xi} + \chi e^{-\xi}}, \tag{7}$$

where a and χ turn out to be arbitrary constants.

Step–3: To determine N, the number of summands in Equation (5), we must compute the balance between the nonlinear term and the higher-order derivative in Equation (4).

Step–4: Utilizing (5) in (4) in addition to (6) and (5). When we do this substitution, we obtain a polynomial in the bases $R(\xi)$ and $R'(\xi)$. By collecting all terms of the same power in this polynomial and setting them equal to zero, we have a system of overdetermined algebraic equations which can be solved using software such as Maple or Mathematica to find the values of k, v, a, χ, and λ_i for every $i = 0, 1, \ldots, N$. Finally, as a consequence of all of the above, we will obtain several exact solution families of (2).

3. Mathematical Analysis

For this system to be solved, the following solution structure is proposed.

$$q(x,t) = U(\xi)e^{i\phi(x,t)}, \tag{8}$$

where ξ is the variable of the wave, given by

$$\xi = k(x - vt). \tag{9}$$

$U(\xi)$ represents the amplitude component of the soliton solution in this case, the speed of the soliton by v, and the phase component which, denoted by $\phi(x,t)$, is defined as

$$\phi(x,t) = -\kappa x + \omega t + \theta. \tag{10}$$

κ is the soliton frequency, ω is the wave number, and θ is a phase constant in Equation (10). Substituting (8) in (1) and then splitting into real and imaginary components yields

$$\begin{aligned}
&a_6 k^6 U^{(6)} + F\left(U^2\right)U + 5a_5\kappa k^4 U^{(4)} + a_4 k^4 U^{(4)} + 15a_6\kappa^4 k^2 U'' - 10a_5\kappa^3 k^2 U'' \\
&-6a_4\kappa^2 k^2 U'' + 3a_3\kappa k^2 U'' + a_2 k^2 U'' + U\left(-a_6\kappa^6 + a_5\kappa^5 + a_4\kappa^4 - a_3\kappa^3 - a_2\kappa^2 + a_1\kappa - \omega\right) \\
&-15a_6\kappa^2 k^4 U^{(4)} = 0,
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&k^5(a_5 - 6a_6\kappa)U^{(5)} - k\left(6a_6\kappa^5 - 5a_5\kappa^4 - 4a_4\kappa^3 + 3a_3\kappa^2 + 2a_2\kappa - a_1 + v\right)U' + \\
&k^3 U^{(3)}\left(20a_6\kappa^3 - 10a_5\kappa^2 - 4a_4\kappa + a_3\right)U^{(3)} = 0.
\end{aligned} \tag{12}$$

Equation (12) yields the speed of the soliton

$$v = -6a_6\kappa^5 + 5a_5\kappa^4 + 4a_4\kappa^3 - 3a_3\kappa^2 - 2a_2\kappa + a_1, \tag{13}$$

and the parameter constraints

$$20a_6\kappa^3 - 10a_5\kappa^2 - 4a_4\kappa + a_3 = 0, \quad a_5 - 6a_6\kappa = 0. \tag{14}$$

3.1. Power Law

For power-law nonlinearity, the model stands as:

$$iq_t + ia_1 q_x + a_2 q_{xx} + ia_3 q_{xxx} + a_4 q_{xxxx} + ia_5 q_{xxxxx} + a_6 q_{xxxxxx} + b|q|^{2n} q = 0. \tag{15}$$

In this case, Equation (11) can be written as:

$$k^2U^{(6)} + H_1U + H_2U'' + H_3U^{(4)} + H_4U^{2n+1} = 0, \tag{16}$$

where

$$\begin{cases} H_1 = \frac{-a_6\kappa^6+a_5\kappa^5+a_4\kappa^4-a_3\kappa^3-a_2\kappa^2+a_1\kappa-\omega}{a_6k^4}, \\ H_2 = \frac{15a_6\kappa^4-10a_5\kappa^3-6a_4\kappa^2+3a_3\kappa+a_2}{a_6k^2}, \\ H_3 = \frac{-15a_6\kappa^2+5a_5\kappa+a_4}{a_6}, \\ H_4 = \frac{b}{a_6k^4}. \end{cases} \tag{17}$$

Applying the transformation

$$U = V^{\frac{3}{n}}$$

Equation (16) is transformed into

$$\begin{aligned}
&V^5\left(3H_3n^5V^{(4)} + 3H_2n^5V'' + 3k^2n^5V^{(6)}\right) + V^4(-9H_3(n-3)n^4V''^2 \\
&-3H_2(n-3)n^4V'^2 - 12H_3(n-3)n^4V^{(3)}V' - 30k^2(n-3)n^4\left(V^{(3)}\right)^2 \\
&-18k^2(n-3)n^4V^{(5)}V' - 45k^2(n-3)n^4V^{(4)}V'') + V^2(-9H_3n^2\left(2n^3 - 11n^2 + 18n - 9\right)(V')^4 \\
&-405k^2(n-1)n^2\left(2n^2 - 9n + 9\right)(V')^2(V'')^2 - 180k^2n^2\left(2n^3 - 11n^2 + 18n - 9\right)V^{(3)}(V')^3) \\
&+V(\xi)^3(V''(18H_3\left(2n^2 - 9n + 9\right)n^3(V')^2 + 180k^2(n-3)(2n-3)n^3V^{(3)}V') \\
&+45k^2\left(2n^2 - 9n + 9\right)n^3(V'')^3 + 45k^2\left(2n^2 - 9n + 9\right)n^3V^{(4)}(V')^2) + \\
&H_4n^6V^{12} + H_1n^6V^6 + 135k^2n\left(8n^4 - 50n^3 + 105n^2 - 90n + 27\right)V(V')^4V'' - \\
&9k^2(40n^5 - 274n^4 + 675n^3 - 765n^2 + 405n - 81)(V')^6 = 0.
\end{aligned} \tag{18}$$

Balancing $V^5V^{(6)}$ with V^{12} in Equation (18), it turns out that $N = 1$. Consequently, we arrive at

$$V(\xi) = \lambda_0 + \lambda_1R(\xi), \tag{19}$$

Substituting (19) into (18) along with (6). When we do this substitution, we obtain a polynomial in the bases $R(\xi)$ and $R'(\xi)$. By collecting all terms of the same power in this polynomial and setting them equal to zero, we have a system of overdetermined algebraic equations which can be solved using software such as Maple or Mathematica to obtain the following result:

$$\begin{aligned}
&\lambda_0 = 0, \quad \lambda_1 = \pm\sqrt[6]{\frac{9H_2(40n^5 + 274n^4 + 675n^3 + 765n^2 + 405n + 81)\chi^3}{H_4n^2(64n^4 + 288n^3 + 648n^2 + 648n + 243)}}, \\
&H_1 = -\frac{9H_2\left(8n^2 + 18n + 9\right)^2}{n^2(64n^4 + 288n^3 + 648n^2 + 648n + 243)}, \\
&H_3 = -\frac{H_2n^2\left(20n^2 + 36n + 27\right)}{64n^4 + 288n^3 + 648n^2 + 648n + 243}, \\
&k = \pm\sqrt{\frac{H_2n^4}{64n^4 + 288n^3 + 648n^2 + 648n + 243}}.
\end{aligned} \tag{20}$$

Inserting (20) together with (7) into (19) leads to a solution of the form

$$q(x,t) = \left\{ \pm \sqrt[6]{\frac{9H_2(40n^5+274n^4+675n^3+765n^2+405n+81)\chi^3}{H_4 n^2(64n^4+288n^3+648n^2+648n+243)}} \right. \\ \left. \times \left(\frac{4a \exp\left[\pm\sqrt{\frac{H_2 n^4}{64n^4+288n^3+648n^2+648n+243}}(x-vt)\right]}{4a^2 \exp\left[\pm\sqrt{\frac{4H_2 n^4}{64n^4+288n^3+648n^2+648n+243}}(x-vt)\right]+\chi} \right) \right\}^{\frac{3}{n}} e^{i(-\kappa x+\omega t+\theta)}. \tag{21}$$

Setting $\chi = \pm 4a^2$, and $H_2 > 0$. The bright soliton can be found by considering for Equation (21):

$$q(x,t) = \left\{ \pm \sqrt[6]{\frac{9H_2(40n^5+274n^4+675n^3+765n^2+405n+81)}{H_4 n^2(64n^4+288n^3+648n^2+648n+243)}} \right. \\ \left. \times \operatorname{sech}\left[\sqrt{\frac{H_2 n^4}{64n^4+288n^3+648n^2+648n+243}}(x-vt)\right] \right\}^{\frac{3}{n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{22}$$

as well as the singular soliton:

$$q(x,t) = \left\{ \pm \sqrt[6]{\frac{9H_2(40n^5+274n^4+675n^3+765n^2+405n+81)}{H_4 n^2(64n^4+288n^3+648n^2+648n+243)}} \right. \\ \left. \times \operatorname{csch}\left[\sqrt{\frac{H_2 n^4}{64n^4+288n^3+648n^2+648n+243}}(x-vt)\right] \right\}^{\frac{3}{n}} e^{i(-\kappa x+\omega t+\theta)}. \tag{23}$$

3.2. Generalized Quadratic-Cubic Law

For generalized quadratic-cubic law nonlinearity, the model stands as:

$$iq_t + ia_1 q_x + a_2 q_{xx} + ia_3 q_{xxx} + a_4 q_{xxxx} + ia_5 q_{xxxxx} + a_6 q_{xxxxxx} + \left(b_1 |q|^n + b_2 |q|^{2n}\right) q = 0. \tag{24}$$

In this case, Equation (11) can be written as:

$$k^2 U^{(6)} + H_1 U + H_2 U'' + H_3 U^{(4)} + H_4 U^{n+1} + H_5 U^{2n+1} = 0, \tag{25}$$

where

$$\begin{cases} H_1 = \frac{-a_6\kappa^6+a_5\kappa^5+a_4\kappa^4-a_3\kappa^3-a_2\kappa^2+a_1\kappa-\omega}{a_6k^4}, \\ H_2 = \frac{15a_6\kappa^4-10a_5\kappa^3-6a_4\kappa^2+3a_3\kappa+a_2}{a_6k^2}, \\ H_3 = \frac{-15a_6\kappa^2+5a_5\kappa+a_4}{a_6}, \\ H_4 = \frac{b_1}{a_6k^4}, \\ H_5 = \frac{b_2}{a_6k^4}. \end{cases} \tag{26}$$

Applying the transformation

$$U(\xi) = V(\xi)^{\frac{3}{n}}$$

Equation (25) transformed to

$$
\begin{aligned}
&H_4 n^6 V^9 + H_1 n^6 V^6 + 135k^2 n\left(8n^4 - 50n^3 + 105n^2 - 90n + 27\right) V V'^4 V'' \\
&+H_5 n^6 V^{12} - 9k^2\left(40n^5 - 274n^4 + 675n^3 - 765n^2 + 405n - 81\right) V'^6 \\
&+V(^2(9H_3 n^2\left(-2n^3 + 11n^2 - 18n + 9\right) V'^4 - 180k^2 n^2\left(2n^3 - 11n^2 + 18n - 9\right) V^{(3)} V'^3 \\
&-405k^2 n^2\left(2n^3 - 11n^2 + 18n - 9\right) V'^2 V''^2) + V^5\left(3H_3 n^5 V^{(4)} + 3H_2 n^5 V'' + 3k^2 n^5 V^{(6)}\right) \\
&+V^3(V''\left(18H_3\left(2n^2 - 9n + 9\right) n^3 V'^2 + 180k^2\left(2n^2 - 9n + 9\right) n^3 V^{(3)} V'\right) \\
&+45k^2\left(2n^2 - 9n + 9\right) n^3 V''^3 + 45k^2\left(2n^2 - 9n + 9\right) n^3 V^{(4)} V'^2) \\
&+V(\xi)^4(-9H_3(n-3)n^4 V''^2 - 3H_2(n-3)n^4 V'^2 - 12H_3(n-3)n^4 V^{(3)} V' \\
&-30k^2(n-3)n^4 {V^{(3)}}^2 - 18k^2(n-3)n^4 V^{(5)} V' - 45k^2(n-3)n^4 V^{(4)} V'') = 0.
\end{aligned} \tag{27}
$$

Balancing $V^5 V^{(6)}$ with V^{12} in Equation (27) gives $N = 1$. Consequently, we arrive at

$$
V(\xi) = \lambda_0 + \lambda_1 R(\xi), \tag{28}
$$

Substituting (28) into (27) along with (6), we obtain a polynomial in the bases $R(\xi)$ and $R'(\xi)$. By collecting all terms of the same power in this polynomial and setting them equal to zero, we have a system of overdetermined algebraic equations which can be solved using software such as Maple or Mathematica to obtain the following result:

$$
\begin{aligned}
&\lambda_0 = 0, \quad \lambda_1 = \pm\sqrt[6]{\frac{9H_2(40n^5 + 274n^4 + 675n^3 + 765n^2 + 405n + 81)\chi^3}{H_5 n^2(64n^4 + 288n^3 + 648n^2 + 648n + 243)}}, \quad H_4 = 0, \\
&H_1 = -\frac{9H_2\left(8n^2 + 18n + 9\right)^2}{n^2(64n^4 + 288n^3 + 648n^2 + 648n + 243)}, \\
&H_3 = -\frac{H_2 n^2\left(20n^2 + 36n + 27\right)}{64n^4 + 288n^3 + 648n^2 + 648n + 243}, \\
&k = \pm\sqrt{\frac{H_2 n^4}{64n^4 + 288n^3 + 648n^2 + 648n + 243}}.
\end{aligned} \tag{29}
$$

Inserting (29) together with (7) into (28) leads to a solution of the form

$$
\begin{aligned}
q(x,t) = &\left\{ \pm\sqrt[6]{\frac{9H_2(40n^5 + 274n^4 + 675n^3 + 765n^2 + 405n + 81)\chi^3}{H_5 n^2(64n^4 + 288n^3 + 648n^2 + 648n + 243)}} \right. \\
&\left. \times \left(\frac{4a \exp\left[\pm\sqrt{\frac{H_2 n^4}{64n^4+288n^3+648n^2+648n+243}}\,(x - vt)\right]}{4a^2 \exp\left[\pm\sqrt{\frac{4H_2 n^4}{64n^4+288n^3+648n^2+648n+243}}\,(x - vt)\right] + \chi} \right) \right\}^{\frac{3}{n}} e^{i(-\kappa x + \omega t + \theta)}.
\end{aligned} \tag{30}
$$

Setting $\chi = \pm 4a^2$, and $H_2 > 0$, Equation (30) reduces to the bright soliton:

$$
\begin{aligned}
q(x,t) = &\left\{ \pm\sqrt[6]{\frac{9H_2(40n^5 + 274n^4 + 675n^3 + 765n^2 + 405n + 81)}{H_5 n^2(64n^4 + 288n^3 + 648n^2 + 648n + 243)}} \times \right. \\
&\left. \operatorname{sech}\left[\sqrt{\frac{H_2 n^4}{64n^4 + 288n^3 + 648n^2 + 648n + 243}}\,(x - vt)\right] \right\}^{\frac{3}{n}} e^{i(-\kappa x + \omega t + \theta)},
\end{aligned} \tag{31}
$$

and the singular soliton:

$$q(x,t) = \left\{ \pm \sqrt[6]{\frac{9H_2(40n^5+274n^4+675n^3+765n^2+405n+81)}{H_5n^2(64n^4+288n^3+648n^2+648n+243)}} \times \right. \\ \left. \operatorname{csch}\left[\sqrt{\frac{H_2n^4}{64n^4+288n^3+648n^2+648n+243}}\,(x-vt)\right] \right\}^{\frac{3}{n}} e^{i(-\kappa x+\omega t+\theta)}. \tag{32}$$

3.3. Triple-Power Law

For triple-power law nonlinearity, the model stands as:

$$iq_t + ia_1q_x + a_2q_{xx} + ia_3q_{xxx} + a_4q_{xxxx} + ia_5q_{xxxxx} + a_6q_{xxxxxx} + \left(b_1|q|^{2n} + b_2|q|^{4n} + b_3|q|^{6n}\right)q = 0. \tag{33}$$

In this case, Equation (11) can be written as:

$$H_1U + H_2U'' + H_3U^{(4)} + H_4U^{2n+1} + H_5U^{4n+1} + H_6U^{6n+1} + k^2U^{(6)} = 0, \tag{34}$$

where

$$\begin{cases} H_1 = \frac{-a_6\kappa^6+a_5\kappa^5+a_4\kappa^4-a_3\kappa^3-a_2\kappa^2+a_1\kappa-\omega}{a_6k^4}, \\ H_2 = \frac{15a_6\kappa^4-10a_5\kappa^3-6a_4\kappa^2+3a_3\kappa+a_2}{a_6k^2}, \\ H_3 = \frac{-15a_6\kappa^2+5a_5\kappa+a_4}{a_6}, \\ H_4 = \frac{b_1}{a_6k^4},\ H_5 = \frac{b_2}{a_6k^4},\ H_6 = \frac{b_3}{a_6k^4}. \end{cases} \tag{35}$$

Applying the transformation

$$U = V^{\frac{1}{n}}$$

Equation (34) is transformed into

$$\begin{aligned} &H_1n^6V^6 + \left(-120k^2n^5+274k^2n^4-225k^2n^3+85k^2n^2-15k^2n+k^2\right)V'^6 + H_6n^6V^{12} \\ &+H_5n^6V^{10} + H_4n^6V^8 + \left(360k^2n^5-750k^2n^4+525k^2n^3-150k^2n^2+15k^2n\right)VV'^4V'' \\ &V^2(\left(-6H_3n^5+11H_3n^4-6H_3n^3+H_3n^2\right)V'^4 + (-270k^2n^5+495k^2n^4-270k^2n^3 \\ &+45k^2n^2)V'^2V''^2 + \left(-120k^2n^5+220k^2n^4-120k^2n^3+20k^2n^2\right)V^{(3)}V'^3) \\ &+V^3(\left(30k^2n^5-45k^2n^4+15k^2n^3\right)V''^3 + \left(120k^2n^5-180k^2n^4+60k^2n^3\right)V^{(3)}V'V'' \\ &+V'^2\left(\left(12H_3n^5-18H_3n^4+6H_3n^3\right)V'' + \left(30k^2n^5-45k^2n^4+15k^2n^3\right)V^{(4)}\right)) \\ &+V^4(\left(3H_3n^4-3H_3n^5\right)V''^2 + \left(H_2n^4-H_2n^5\right)V'^2 + \left(10k^2n^4-10k^2n^5\right){V^{(3)}}^2 \\ &V'\left(\left(4H_3n^4-4H_3n^5\right)V^{(3)} + \left(6k^2n^4-6k^2n^5\right)V^{(5)}\right) + +\left(15k^2n^4-15k^2n^5\right)V^{(4)}(\xi)V''(\xi)) \\ &+V^5\left(H_3n^5V^{(4)} + H_2n^5V'' + k^2n^5V^{(6)}\right) = 0. \end{aligned} \tag{36}$$

Balancing $V^5V^{(6)}$ with V^{12} in Equation (36) gives $N = 1$. Consequently, we arrive at

$$V(\xi) = \lambda_0 + \lambda_1R(\xi), \tag{37}$$

Substituting (37) into (36) along with (6). When we do this substitution, we obtain a polynomial in the bases $R(\xi)$ and $R'(\xi)$. By collecting all terms of the same power in this polynomial and setting them equal to zero, we have a system of overdetermined algebraic equations which can be solved using software such as Maple or Mathematica to obtain the following result:

$$\lambda_0 = 0, \quad \lambda_1 = \pm\sqrt{\frac{2(n+1)\chi(H_2 n^4 \tau_1 + H_3 n^2 \tau_3)}{H_4 n^6 \tau_2}}, \quad k = \pm\sqrt{\frac{H_2 n^4}{\tau_2}},$$
$$H_1 = -\frac{H_3}{n^4} - \frac{4H_2\left(16n^4 + 72n^3 + 162n^2 + 162n + 61\right)}{n^2 \tau_2},$$
$$H_5 = -\frac{H_4^2 n^4\left(6n^2 + 5n + 1\right)\tau_2\left(H_2\left(20n^2 + 12n + 3\right)n^2 + H_3\tau_2\right)}{4(n+1)(H_2 n^2 \tau_1 + H_3 \tau_3)^2},$$
$$H_6 = \frac{H_2 H_4^3 n^{10}\left(120n^4 + 154n^3 + 71n^2 + 14n + 1\right)\tau_2^2}{8(n+1)^2(H_2 n^2 \tau_1 + H_3 \tau_3)^3}. \tag{38}$$

where

$$\tau_1 = 40n^4 + 160n^3 + 338n^2 + 330n + 123,$$
$$\tau_2 = 64n^4 + 288n^3 + 648n^2 + 648n + 243,$$
$$\tau_3 = 128n^6 + 704n^5 + 1936n^4 + 2880n^3 + 2430n^2 + 1134n + 243. \tag{39}$$

Inserting (38) together with (7) into (37) leads to a solution of the form

$$q(x,t) = \left\{ \pm\sqrt{\frac{2(n+1)\chi(H_2 n^4 \tau_1 + H_3 n^2 \tau_3)}{H_4 n^6 \tau_2}} \left(\frac{4a \exp\left[\pm\sqrt{\frac{H_2 n^4}{\tau_2}}(x - vt)\right]}{4a^2 \exp\left[\pm\sqrt{\frac{4H_2 n^4}{\tau_2}}(x - vs.t)\right] + \chi} \right) \right\}^{\frac{1}{n}} \times e^{i(-\kappa x + \omega t + \theta)}. \tag{40}$$

Setting $\chi = \pm 4a^2$, and $H_2 > 0$, The bright soliton can be found by considering for Equation (30):

$$q(x,t) = \left\{ \pm\sqrt{\frac{2(n+1)(H_2 n^4 \tau_1 + H_3 n^2 \tau_3)}{H_4 n^6 \tau_2}} \operatorname{sech}\left[\sqrt{\frac{H_2 n^4}{\tau_2}}(x - vt)\right] \right\}^{\frac{1}{n}} e^{i(-\kappa x + \omega t + \theta)}. \tag{41}$$

as well as the singular soliton:

$$q(x,t) = \left\{ \pm\sqrt{\frac{2(n+1)(H_2 n^4 \tau_1 + H_3 n^2 \tau_3)}{H_4 n^6 \tau_2}} \operatorname{csch}\left[\sqrt{\frac{H_2 n^4}{\tau_2}}(x - vt)\right] \right\}^{\frac{1}{n}} e^{i(-\kappa x + \omega t + \theta)}. \tag{42}$$

3.4. Generalized Non-Local Law

For generalized non-local law nonlinearity, the model stands as:

$$iq_t + ia_1 q_x + a_2 q_{xx} + ia_3 q_{xxx} + a_4 q_{xxxx} + ia_5 q_{xxxxx} + a_6 q_{xxxxxx} + b\left(|q|^{2n}\right)_{xx} q = 0. \tag{43}$$

In this case, Equation (11) can be written as:

$$H_1 U + H_2 U'' + H_3 U^{(4)} + H_4 U^{2n-1}(U')^2 + H_5 U^{2n} U'' + k^2 U^{(6)} = 0, \tag{44}$$

where

$$\begin{cases} H_1 = \frac{-a_6\kappa^6 + a_5\kappa^5 + a_4\kappa^4 - a_3\kappa^3 - a_2\kappa^2 + a_1\kappa - \omega}{a_6 k^4}, \\ H_2 = \frac{15a_6\kappa^4 - 10a_5\kappa^3 - 6a_4\kappa^2 + 3a_3\kappa + a_2}{a_6 k^2}, \\ H_3 = \frac{-15a_6\kappa^2 + 5a_5\kappa + a_4}{a_6}, \\ H_4 = \frac{b}{a_6 k^4}, \\ H_5 = \frac{2bn}{a_6 k^2}. \end{cases} \tag{45}$$

Applying the transformation to

$$U = V^{\frac{2}{n}}$$

transforms Equation (44) into

$$\begin{aligned}
&H_1 n^6 V^6 + \left(-2H_5 n^5 + 4H_4 n^4 + 4H_5 n^4\right) V^8 V'^2 + 2H_5 n^5 V^9 V'' \\
&+\left(-240k^2n^5 + 1096k^2n^4 - 1800k^2n^3 + 1360k^2n^2 - 480k^2n + 64k^2\right) V'^6 \\
&+\left(720k^2n^5 - 3000k^2n^4 + 4200k^2n^3 - 2400k^2n^2 + 480k^2n\right) V V'^4 V'' \\
&+V^2\left(\left(-12H_3n^5 + 44H_3n^4 - 48H_3n^3 + 16H_3n^2\right) V'^4\right. \\
&+\left(-540k^2n^5 + 1980k^2n^4 - 2160k^2n^3 + 720k^2n^2\right) V'^2 V''^2 \\
&+\left.\left(-240k^2n^5 + 880k^2n^4 - 960k^2n^3 + 320k^2n^2\right) V^{(3)} V'^3\right) \\
&+V^3(\left(60k^2n^5 - 180k^2n^4 + 120k^2n^3\right) V''^3 + \left(240k^2n^5 - 720k^2n^4 + 480k^2n^3\right) V^{(3)} V' V'' \\
&+V'^2\left(\left(24H_3n^5 - 72H_3n^4 + 48H_3n^3\right) V'' + \left(60k^2n^5 - 180k^2n^4 + 120k^2n^3\right) V^{(4)}\right)) \\
&+V^4(2H_2(2-n)n^4 V'^2 + \left(12H_3n^4 - 6H_3n^5\right) V''^2 + \left(40k^2n^4 - 20k^2n^5\right) {V^{(3)}}^2 \\
&+V'\left(\left(16H_3n^4 - 8H_3n^5\right) V^{(3)} + \left(24k^2n^4 - 12k^2n^5\right) V^{(5)}\right) + +\left(60k^2n^4 - 30k^2n^5\right) V^{(4)} V'') \\
&+V^5\left(2H_3n^5 V^{(4)} + 2H_2 n^5 V'' + 2k^2 n^5 V^{(6)}\right) = 0.
\end{aligned} \tag{46}$$

Balancing $V^5 V^{(6)}$ with $V^9 V''$ in Equation (46) gives $N = 1$. Consequently, we arrive at

$$V(\xi) = \lambda_0 + \lambda_1 R(\xi), \tag{47}$$

Substituting (47) into (46) along with (6). When we do this substitution, we obtain a polynomial in the bases $R(\xi)$ and $R'(\xi)$. By collecting all terms of the same power in this polynomial and setting them equal to zero, we have a system of overdetermined algebraic equations which can be solved using software such Maple or Mathematica to obtain the following result:

Result-1:

$$\begin{aligned}
&\lambda_0 = 0, \\
&\lambda_1 = \pm\sqrt[4]{-\frac{(3n^3 + 11n^2 + 12n + 4)\chi^8 (H_3n^3 + 2H_3n^2 + 20k^2n^3 + 24k^2n^2 + 24k^2n + 16k^2)}{H_4 n^5}}, \\
&\quad k = \pm\sqrt{\frac{n^2(H_1n^4 - 16H_3n^2 - 32H_3n - 16H_3)}{64n^4 + 256n^3 + 448n^2 + 384n + 128}}, \\
&H_2 = -\frac{H_1\left(n^4 + 4n^3 + 7n^2 + 6n + 3\right)n^4 + 16H_3(n+1)^4}{4n^2(n+1)^2(n^2 + 2n + 2)}, \\
&H_5 = \frac{2H_4\left(H_3n^2 + 2k^2(3n+4)\right)}{\chi^6(H_3n^2(n+2) + 4k^2(5n^3 + 6n^2 + 6n + 4))}.
\end{aligned} \tag{48}$$

Inserting (48) together with (7) into (47) leads to a solution of the form

$$q(x,t)=\left\{\pm\sqrt[4]{-\frac{(3n^3+11n^2+12n+4)\chi^8(H_3n^3+2H_3n^2+20k^2n^3+24k^2n^2+24k^2n+16k^2)}{H_4n^5}}\right.$$
$$\left.\times\left(\frac{4a\exp\left(\pm\sqrt{\frac{n^2(H_1n^4-16H_3n^2-32H_3n-16H_3)}{64n^4+256n^3+448n^2+384n+128}}(x-vt)\right)}{4a^2\exp\left(\pm\sqrt{\frac{4n^2(H_1n^4-16H_3n^2-32H_3n-16H_3)}{64n^4+256n^3+448n^2+384n+128}}(x-vt)\right)+\chi}\right)\right\}^{\frac{2}{n}}e^{i(-\kappa x+\omega t+\theta)}. \tag{49}$$

Setting $\chi=\pm 4a^2$, and $H_1n^4-16H_3n^2-32H_3n-16H_3>0$, Equation (49) reduces to the bright soliton:

$$q(x,t)=\left\{\pm\sqrt[4]{-\frac{(3n^3+11n^2+12n+4)(H_3n^3+2H_3n^2+20k^2n^3+24k^2n^2+24k^2n+16k^2)}{H_4n^5}}\right.$$
$$\left.\times\operatorname{sech}\left[\sqrt{\frac{n^2(H_1n^4-16H_3n^2-32H_3n-16H_3)}{64n^4+256n^3+448n^2+384n+128}}(x-vt)\right]\right\}^{\frac{2}{n}}e^{i(-\kappa x+\omega t+\theta)}, \tag{50}$$

and the singular soliton:

$$q(x,t)=\left\{\pm\sqrt[4]{-\frac{(3n^3+11n^2+12n+4)(H_3n^3+2H_3n^2+20k^2n^3+24k^2n^2+24k^2n+16k^2)}{H_4n^5}}\right.$$
$$\left.\times\operatorname{csch}\left[\sqrt{\frac{n^2(H_1n^4-16H_3n^2-32H_3n-16H_3)}{64n^4+256n^3+448n^2+384n+128}}(x-vt)\right]\right\}^{\frac{2}{n}}e^{i(-\kappa x+\omega t+\theta)}. \tag{51}$$

Result-2:

$$\begin{gathered}\lambda_0=0,\ \lambda_1=\pm\sqrt[4]{-\frac{H_1n(15n^3+46n^2+36n+8)\chi^2}{16H_4(n+1)}},\\ k=\pm\sqrt{\frac{H_1n^5}{64n^3+160n^2+128n+32}},\\ H_2=-\frac{H_1n(2n^4+5n^3+4n^2-2n-2)}{4(n+1)^2(2n+1)},\quad H_3=-\frac{H_1n^3(3n+4)}{16(n+1)^2(2n+1)},\quad H_5=0.\end{gathered} \tag{52}$$

Inserting (48) together with (7) into (47) leads to the soliton solution

$$q(x,t)=\left\{\pm\sqrt[4]{-\frac{H_1n(15n^3+46n^2+36n+8)\chi^2}{16H_4(n+1)}}\right.$$
$$\left.\times\left(\frac{4a\exp\left(\pm\sqrt{\frac{H_1n^5}{64n^3+160n^2+128n+32}}(x-vs.t)\right)}{4a^2\exp\left(\pm\sqrt{\frac{4H_1n^5}{64n^3+160n^2+128n+32}}(x-vt)\right)+\chi}\right)\right\}^{\frac{2}{n}}e^{i(-\kappa x+\omega t+\theta)}. \tag{53}$$

Setting $\chi=\pm 4a^2$, and $H_1>0$. The bright soliton can be found by considering, for Equation (53):

$$q(x,t) = \left\{ \pm \sqrt[4]{-\frac{H_1 n(15n^3+46n^2+36n+8)}{16H_4(n+1)}} \right.$$
$$\left. \times \operatorname{sech}\left[\sqrt{\frac{H_1 n^5}{64n^3+160n^2+128n+32}}(x - vs.t) \right] \right\}^{\frac{2}{n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{54}$$

as well as the singular soliton:

$$q(x,t) = \left\{ \pm \sqrt[4]{-\frac{H_1 n(15n^3+46n^2+36n+8)}{16H_4(n+1)}} \right.$$
$$\left. \times \operatorname{csch}\left[\sqrt{\frac{H_1 n^5}{64n^3+160n^2+128n+32}}(x - vs.t) \right] \right\}^{\frac{2}{n}} e^{i(-\kappa x+\omega t+\theta)}. \tag{55}$$

4. Conclusions

This paper addressed HD–NLSE by the usage of enhanced Kudryashov's approach. There are four forms of SPM that were addressed. These are power-law, quadratic-cubic law, triple-power law and the generalized non-local nonlinear law. These were addressed sequentially starting from the power law and then gradually moving in the ascending order of SPM structures for the first three forms of nonlinear refractive index. The fourth form of SPM is, however, independent of the first three forms. While this paper studies the HD solitons with four forms of SPM, it must be noted that such solitons with no SPM were already studied earlier during 2022 and the corresponding conservation laws were reported there too [6]. The immediate physical applications of the results would be rendered in long distance fiber-optic communications across inter-continental distances. Additional forms of waveguides where the proposed model along with the results would be applicable are metamaterials, metasurfaces, PCF and so on.

The bright and singular soliton solutions to the model were recovered and presented. It must be noted that bright solitons are the ones that are visible on an oscilloscope when these are experimentally demonstrated. Dark solitons are not visible on an oscilloscope without the presence of a background wave. Singular solitons, on the other hand, are viable candidates for rogue waves. These solutions included parameter constraints that ensure the existence of these kind of solitons. These constraints naturally and effortlessly fell out from the solution structure. A visible shortcoming of this integration approach is that it fails to reveal the dark optical soliton solutions to the model. The results of the model nevertheless show exceptional promise. The bright soliton solutions will be applied to derive the conservation laws to the model that can be obtained by the application of the multiplier approach or even by the aid of the Lagrangian. Subsequently, the issue of quasi–monochromatic dynamics of these solitons will be handled. This would lead to the phenomena of optical soliton cooling. Another avenue to explore is the construction of the perturbed HD solitons when the perturbation terms would have their greatest intensity. When a perturbed model fails the Painleve test of integrability and is therefore not integrable, the semi-inverse variational principle comes to the rescue. Such promises will be sequentially honored with time and the results will be revealed and disseminated across the board. The results would be aligned along the works that have been recently reported [6–24].

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