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Unruh Effect for Mixed Neutrinos and the KMS Condition

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Abstract: The quantization of mixed (neutrino) fields in an accelerated background reveals a non-thermal nature for Unruh radiation, which can be fitted by a Tsallis-like distribution function. However, for relativistic flavor neutrinos, which are represented by the standard Pontecorvo states, such a correction turns out to be negligible and thermality is restored. We show that the usage of Pontecorvo states for the calculation of the decay rate of an accelerated proton in the laboratory and comoving frames leads to consistent results and correctly implements the KMS thermal condition. Thus, the employment of these states in the above framework is not at odds with the principle of general covariance, in contrast to recent claims in the literature.

Keywords: neutrino mixing; weak interactions; Unruh effect; non-extensive Tsallis statistics; quantum gravity

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1. Introduction

In spite of many decades of enormous efforts, quantum gravity (QG) is a goal yet to be reached. Several lines of research originating from expertise in one or another area of theoretical physics have provided tentative solutions to this problem. Among these, the most popular approaches are string (and the dual gauge field) theory [1–3], where the quantum nature of gravity emerges as a manifestation of vibrational states of strings emitting gravitons, and loop quantum gravity, which postulates that the spacetime is fundamentally composed of finite loops nested into spin networks at approximately the Planck scale [4]. In parallel, other models such as non-commutative geometry and asymptotic safe gravity rely on either the introduction of non-commutative metric spaces based on the quantum paradigm of non-commuting observables [5–8], or the concept of asymptotic safety, whose main ingredient is a fixed point of the theory's renormalization group flow that rules the behavior of the gravitational coupling in the UV regime [9–13]. In the absence of a complete understanding of the early quantum universe, preliminary aspects of QG might also be figured out by looking at the emergence of our observed classical universe from quantum Planck domain through late-inflation decoherence [14–20] or wave-function collapse [21,22]. On more phenomenological grounds, attempts to study low-energy effects of QG have been carried out by suitably modifying the Heisenberg Uncertainty Principle [23–37] and/or relativity laws [38,39] at Planck scale, or looking for peculiar implications of QG in tabletop analogue gravity experiments [40–52] (see Ref. [53] for a recent review on QG phenomenology). At a different conceptual level, it is believed that unification of quantum and gravity should actually require a rethinking of quantum gravity as a dissipative deterministic system [54], which would be in tune with the deterministic view of quantum mechanics proposed in [55–57].

While a fully consistent theory of quantum gravity is currently lacking, a semiclassical approach is highly successful in explaining a variety of phenomena on the border between general relativity and quantum theory is the quantum field theory (QFT) in curved spacetime [58,59]. Among the most distinctive predictions obtained in this framework, some results as the discovery of Hawking radiation for black holes [60] and the related Unruh effect [61] for accelerated observers turn out to be solid and indeed constitute important steps in the direction of a quantized theory of gravity. Moreover, they provide a unique arena for the study of general features of QG [62–84] and the potential extension of the equivalence principle [85,86] to quantum reference frames [87]. In particular, it has been argued that the Unruh effect appears as a necessary consequence of QFT, in a similar way as Coriolis force naturally arises in classical mechanics [88], accounting for the non-inertiality of the reference frame.

In this respect, a "theoretical proof" for the existence of Unruh effect has been obtained by considering particle decay in accelerated frames [89]. In particular, protons—which are notoriously stable in the Standard Model—can decay via inverse beta decay when sufficiently accelerated [90]. The mean proper lifetime of a uniformly accelerated proton has been explicitly computed in [89] in both the laboratory frame, where the particle is accelerated, and the comoving frame, where it is supposed to be at rest. By invoking the scalar nature of the proper lifetime, it has been shown that the equality between the two expressions is achieved (as dictated by the general covariance of QFT), provided that the Unruh effect is taken into account [61].

In the above approach, the emitted neutrino has been treated as a massless [89] or definite-mass particle [91]. Only recently, the analysis has been refined by including neutrino mixing and oscillations [92–97]. Surprisingly, conflicting conclusions on the maintenance of general covariance have been reached in this case. Indeed, in [92] it has been shown that the proton mean lifetimes as calculated by inertial and accelerated observers would disagree when considering neutrino mixing. In particular, this occurs if one assumes asymptotic neutrinos to be in flavor eigenstates in the laboratory frame, while in mass eigenstates in the comoving system. On the other hand, it has been claimed that consistency is recovered if the flavor representation is adopted in the accelerated frame as well [92,93,95–97], the price to be paid would be the violation of the Kubo–Martin–Schwinger (KMS) thermality condition for the accelerated vacuum [92]. The question thus arises on how to find a way out of this apparent contradiction.

To approach this problem, it should be noted that the study of flavor mixing in the context of QFT has shown that a consistent definition of states with definite flavor is associated with a redefinition of the vacuum state, which does not coincide with the one for the free fields with definite masses [98]. Indeed, a Bogoliubov transformation is hidden in the rotation for the neutrino fields, and this is at the origin of the unitary inequivalence of mass and flavor vacua. Recently, such formalism has been extended to the case of Rindler (i.e., uniformly accelerated) metric. Particularly, it has been proven that for a Rindler observer, the Unruh condensate for mixed (neutrino) fields exhibits deviations from thermality due to the interplay between the Bogoliubov transformation contained in field mixing and the one arising from the Rindler spacetime structure [99].

The non-thermal character of Unruh radiation for mixed neutrinos has been later identified [100,101] as the one associated with a generalized thermostatistics arising from nonextensive Tsallis entropy [102]. In this respect, a thermal field theory based on Tsallis distribution has been investigated in [103], obtaining a generalized KMS thermal state condition [103,104]. Consequently, the concerns raised in Ref. [92] about the use of flavor neutrino states for the calculation of the proton decay rate in the comoving frame are actually not fully justified. However, when the usual relativistic approximation is adopted for neutrinos, the exact flavor states reduce to the well-known Pontecorvo states: in this approximation, the thermal character of the Unruh radiation for mixed neutrinos is restored. Here, we show by explicit calculations that general covariance is guaranteed when Pontecorvo states are considered as being representative of neutrinos both in the decay products of the proton (in the laboratory frame) and in the Unruh radiation (in the comoving frame).

The work is organized as follows. In Section 2, we review some aspects of the quantization for mixed (Dirac) fields both in Minkowski and in Rindler frames and the non-thermal nature of the associated Unruh radiation. In Section 3, we show the equivalence of the proton decay rates calculated in both frames, by means of flavor states. Conclusions and outlook are finally summarized in Section 4. Throughout the whole manuscript, we use natural units $\hbar = c = k_B = 1$.

2. Nonextensive Tsallis Statistics in Unruh Effect for Mixed Neutrino Fields

The field theoretical analysis of flavor mixing [98] has revealed the shortcomings of the original Pontecorvo treatment [105]. Particularly, it has highlighted the *unitary inequivalence* between the Fock space for fields with definite flavors and that for fields with definite masses. Mathematically speaking, this feature emerges from the action of a Bogoliubov transformation inherent to the superposition of fields with different masses. One of the most striking consequences of this transformation is the non-trivial nature acquired by the vacuum for flavor fields, which turns out to be a condensate of massive particle-antiparticle pairs [98].

To better explore the main features of QFT mixing, let us rewrite Pontecorvo transformations for mixed fields in terms of the algebraic generator $G_\theta(t)$ as in [98],

$$\Psi_{\nu_\alpha}(t,\mathbf{x}) \;=\; G_\theta^{-1}(t)\,\Psi_{\nu_i}(t,\mathbf{x})\,G_\theta(t)\,, \qquad (\alpha,i) = \{(e,1),(\mu,2)\}\,, \tag{1}$$

where Ψ_{ν_α} and Ψ_{ν_i} denote the fields with definite flavors and masses, respectively. (For simplicity, we consider a minimalistic model with only two flavors labeled by $\alpha = e,\mu$. The extension to three generations does not affect the overall validity of our considerations). The generator $G_\theta(t)$ is given by

$$G_\theta(t) \;=\; \exp\left\{\theta \int d^3x \left[\Psi_1^\dagger(t,\mathbf{x})\Psi_2(t,\mathbf{x}) - \Psi_2^\dagger(t,\mathbf{x})\Psi_1(t,\mathbf{x})\right]\right\}. \tag{2}$$

Using the standard free-field expansion for (Dirac) fields Ψ_{ν_i}, insertion of Equation (2) into (1) yields

$$\Psi_{\nu_\alpha}(t,\mathbf{x}) = \sum_{r=1,2}\int d^3k\; N\left[a^r_{\mathbf{k},\nu_\alpha}(\theta,t)\, u^r_{\mathbf{k},\nu_i}(t)e^{i\mathbf{k}\cdot\mathbf{x}} + b^{r\dagger}_{\mathbf{k},\nu_\alpha}(\theta,t)\, v^r_{\mathbf{k},\nu_i}(t)e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{3}$$

where the (time-dependent) annihilators in the flavor basis read

$$a^r_{\mathbf{k},\nu_\alpha}(\theta,t) = G_\theta^{-1}(t)a^r_{\mathbf{k},\nu_i}G_\theta(t), \qquad (\nu_\alpha,\nu_i) = \{(e,1),(\mu,2)\}, \tag{4}$$

and similarly for $b^r_{\mathbf{k},\nu_i}(\theta,t)$. For instance, we explicitly obtain for $a^r_{\mathbf{k},\nu_e}(\theta,t)$ [98]

$$a^r_{\mathbf{k},\nu_e} = \cos\theta\, a^r_{\mathbf{k},\nu_1} + \sin\theta \sum_{s=1,2}\left[\left(u^{r\dagger}_{\mathbf{k},\nu_1}(t), u^s_{\mathbf{k},\nu_2}(t)\right) a^s_{\mathbf{k},\nu_2} + \left(u^{r\dagger}_{\mathbf{k},\nu_1}(t), v^s_{-\mathbf{k},\nu_2}(t)\right) b^{s\dagger}_{-\mathbf{k},\nu_2}\right], \tag{5}$$

and similarly for $b^r_{\mathbf{k},\nu_e}$. This equation shows that flavor and mass ladder operators are connected to each other by the combination of the standard Pontecorvo rotation (encoded by the $\cos\theta$- and $\sin\theta$-factors) and a Bogoliubov transformation arising from the products of (anti-)neutrino Dirac spinors with different masses (the terms in the square brackets).

The above formalism holds in Minkowski spacetime. In [99], it has been extended to Rindler metric, which is the background pertaining to a uniformly accelerated observer. It is well-known that the field quantization within this framework is accompanied by a Bogoliubov transformation between the Fock spaces for Minkowski (inertial) and Rindler (accelerated) observers, even for the simplest case of a free (i.e., unmixed) field. This transformation is indeed at the root of the Unruh effect as derived in its original formulation [61]. It is then natural to expect that the study of field mixing in Rindler spacetime can somehow involve an interplay of the two Bogoliubov transformations at stake. This has been formalized in [106], showing that the single Bogoliubov transformation responsible for the Unruh effect for non-interacting fields

$$r_{i,\kappa} = \frac{c_{i,\kappa}\, e^{\frac{\pi\omega}{2}} + d^{\dagger}_{i,\kappa}\, e^{-\frac{\pi\omega}{2}}}{\sqrt{2\cosh(\pi\omega)}}\,, \tag{6}$$

must be generalized to the doubled-Bogoliubov transformation structure. (We are assuming the acceleration to be along the z-axis.)

$$\begin{aligned} r_{i,\kappa,\nu_\alpha}(\theta,t) \;=\; & \frac{1}{\sqrt{2\cosh(\pi\omega)}} \sum_{r=1,2} \int dk_z \Bigg\{ e^{\frac{\pi\omega}{2}} F_{i,r,\nu_i}(k_z,\omega) \\ & \times \Bigg\{ \cos\theta\, a^{r}_{\mathbf{k},1} + \sin\theta \sum_{s=1,2} \Big[(u^{r\dagger}_{\mathbf{k},1}(t), u^{s}_{\mathbf{k},2}(t))\, a^{s}_{\mathbf{k},2} + (u^{r\dagger}_{\mathbf{k},1}(t), v^{s}_{-\mathbf{k},2}(t))\, b^{s\dagger}_{-\mathbf{k},2} \Big] \Bigg\} \\ & + e^{-\frac{\pi\omega}{2}} G_{i,r,\nu_i}(k_z,\omega) \\ & \times \Bigg\{ \cos\theta\, b^{r\,\dagger}_{\mathbf{k},1} + \sin\theta \sum_{s=1,2} \Big[(v^{s\dagger}_{\mathbf{k},2}(t), v^{r}_{\mathbf{k},1}(t))^{*}\, b^{s\dagger}_{\mathbf{k},2} + (u^{s\dagger}_{-\mathbf{k},2}(t), v^{r}_{\mathbf{k},1}(t))^{*}\, a^{s}_{-\mathbf{k},2} \Big] \Bigg\} \Bigg\}, \end{aligned} \tag{7}$$

when including flavor mixing as well. Here, we have denoted by $c_{i,\kappa}$ ($d^{\dagger}_{i,\kappa}$) suitable integral combinations of Minkowski annihilation (creation) operators through the convolution functions $F_{i,r,\nu_i}(k_z,\omega)$ and $G_{i,r,\nu_i}(k_z,\omega)$. These functions are explicitly given in [100].

In turn, the generalized transformation (7) spoils the thermal nature of Unruh vacuum distribution

$${}_{\mathrm{M}}\langle 0| r^{\dagger}_{i,\kappa} r_{i,\kappa'} |0\rangle_{\mathrm{M}} = \frac{1}{e^{\beta_{\mathrm{U}}\, a\omega} + 1}\, \delta^{3}(\kappa-\kappa')\,, \qquad \beta_{\mathrm{U}} = 1/T_{\mathrm{U}} = 2\pi/a\,, \tag{8}$$

generated by Equation (6), giving rise to the modified spectrum

$${}_{\mathrm{M}}\langle 0| r^{\dagger}_{i,\kappa,e}(\theta)\, r_{i,\kappa,e}(\theta) |0\rangle_{\mathrm{M}} \simeq \frac{1}{e^{2\pi\omega}+1} + \frac{|\Delta m^2| \sin^2\theta\, \mathrm{Re}\{\mathcal{H}(\mu_{k,i})\}}{\mu_{k,i}^2} \frac{e^{-\pi\omega}}{\cosh(\pi\omega)}\,, \tag{9}$$

where $\mu_{k,i} = \sqrt{m_i^2 + k_x^2 + k_y^2}$ is the reduced Minkowski frequency and $|0\rangle_{\mathrm{M}}$ the (Minkowski) vacuum state for definite mass fields. T_{U} is the well-known Unruh temperature, with a being the proper acceleration of Rindler observer. The function $\mathcal{H}(\mu_{k,i})$ is defined in [100]. Notice that the above result has been derived in the realistic approximation of small mass-difference between neutrinos. It is straightforward to see that in the absence of mixing (i.e., for θ and/or $|\Delta m|$ going to zero), the correction over the thermal profile vanishes, consistently with the recovery of the traditional Unruh radiation in this limit. The same behavior occurs in the ultrarelativistic approximation $\Delta m/\mu_k \to 0$, where the effects of the mixing Bogoliubov transformation in (7) become trivial and the quantum mechanical (Pontecorvo) formalism is restored [98]. This shows that deviations from thermality are actually peculiar to the QFT treatment of mixing.

Now, in [100,101] it has been argued that the non-thermal distribution (9) can still be mapped into a thermal-like profile, provided that one assumes Tsallis statistics for the particle-antiparticle pairs in the vacuum condensate [102]. Tsallis thermostatistics is a non-extensive generalization of Boltzmann–Gibbs theory, based on the following non-additive definition of entropy

$$S_q = \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1} = \sum_{i=1}^{W} p_i \log_q \frac{1}{p_i}\,, \qquad q \in \mathbb{R}^{+}\,, \tag{10}$$

where

$$\log_q z \equiv \frac{z^{1-q}-1}{1-q}, \quad (\log_1 z = \log z)\,. \tag{11}$$

Departure from extensivity is quantified by the entropic parameter q. Clearly, Boltzmann-Gibbs entropy is a special case of S_q for $q \to 1$.

Tsallis entropy (10) has provided encouraging results in the description of many complex (strongly correlated) systems [107–112]. In the case of QFT flavor mixing, its usage is justified by the non-trivial entangled structure exhibited by the quantum vacuum for mixed fields [113,114], which is a (time-dependent) $SU(2)$ coherent state.

Core thermodynamic relations in Tsallis statistics are significantly affected and the basic tools of Tsallis-based thermodynamics are modified accordingly [115]. For instance, in [116] it has been shown that the entropy (10) leads to the following generalized Fermi-Dirac distribution

$$N_q(\omega) = \frac{1}{[1+(q-1)\beta\omega]^{1/(q-1)}+1}\,, \tag{12}$$

which is obtained by maximizing Tsallis entropy under the constraints of keeping the average internal energy and number of particles constant. Since deviations from extensivity are in general expected to be relatively small, for practical purposes it is convenient to expand $N_q(\omega)$ for $q \ll 1$. To the leading order, we obtain

$$N_q(\omega) \simeq \frac{1}{e^{\beta\omega}+1} + \frac{1}{8}(\beta\omega)^2\,\mathrm{sech}^2\left(\frac{\beta\omega}{2}\right)(q-1)\,. \tag{13}$$

Following [100], we now speculate on the possibility that mixing corrections to the Unruh condensate (9) can be framed in Tsallis picture. By demanding consistency between Equations (9) and (13), we can fix the value of q that validates this scenario, obtaining

$$q \simeq 1 + \frac{\mathcal{F}_\theta(\Delta m^2, \mu_{k,i})}{\Omega^2}\,, \tag{14}$$

where

$$\mathcal{F}_\theta(\Delta m^2, \mu_{k,i}) = \frac{|\Delta m^2|\ \mathrm{Re}\{\mathcal{H}(\mu_{k,i})\}}{\pi^2 \mu_{k,i}^2}\sin^2\theta\,. \tag{15}$$

It is worth noting that, since $\mathrm{Re}\{\mathcal{H}(\mu_{\vec{k},1})\} > 0$ [100], we have $q > 1$, which is indicative of the sub-additivity of Tsallis entropy [102]. In contrast, one has $q < 1$ [101] for boson mixing. A possible explanation for this behavior has been provided in [100] in compliance with the Pauli exclusion principle. Furthermore, we have found a running (i.e., energy dependent) expression for q. Although not contemplated in the original formalism by Tsallis, this is expected for field theoretical systems, as discussed in [117].

Therefore, the above framework shows that non-thermal properties of Unruh effect for mixed fields can effectively be described in terms of a Tsallis-induced deviation from extensivity, the q-exponent satisfying the condition (14). Clearly, the dependence of q from the mixing parameters $\sin\theta$ and Δm is such that $q \to 1$ for $\theta, \Delta m \to 0$, consistently with the vanishing of mixing in both cases. The same happens in the Pontecorvo ultrarelativistic limit, since $\mathrm{Re}\{\mathcal{H}(\mu_{\vec{k},1})\}$ keeps finite values for large momenta [100]. This is in line with expectations, since the entangled structure of flavor vacuum which motivates the usage of Tsallis statistics is a richness of the QFT mixing only [98,114], while it is missing in the quantum mechanical formalism.

3. Unruh Effect in Accelerated Proton Decay

In what follows we review the general formalism for the evaluation of the proton proper lifetime. We sketch the main steps of calculations in both the laboratory and comoving frames. For a complete treatment, see [93,95,96].

3.1. Laboratory Frame

In the laboratory frame the uniformly accelerated proton decays according to the weak process $p \to n + e^+ + \nu_e$, where p, n denote the proton and neutron, while e^+, ν_e the positron and (electron) neutrino, respectively. Following [89], we consider p and n as unexcited and excited states of a single quantum system, the nucleon, whose spacetime trajectory is assumed to be well-defined.

The interaction ruling the proton decay can be described by the semiclassical Fermi coupling $\hat{J}^\mu_\ell \hat{J}^{(cl)}_{h,\mu}$, where $\hat{J}^{(cl)}_{h,\mu} = \hat{q}(\tau) u_\mu \delta(x)\delta(y)\delta(u - a^{-1})$ is the classical hadronic current, while $\hat{J}^\lambda_\ell = \sum_{\alpha=e,\mu} \left(\hat{\overline{\Psi}}_{\nu_\alpha} \gamma^\lambda \hat{\Psi}_\alpha + \hat{\overline{\Psi}}_\alpha \gamma^\lambda \hat{\Psi}_{\nu_\alpha} \right)$ the quantum lepton current. Here, $\tau = v/a$ is the nucleon proper time (v denotes the Rindler time coordinate), a is the proper acceleration and $u = a^{-1} = const.$ the spatial Rindler coordinate that defines the nucleon worldline. The four-velocity is given by $u^\mu = (a, 0, 0, 0)$ and $u^\mu = (\sqrt{a^2 t^2 + 1}, 0, 0, at)$ in Rindler and Minkowski coordinates, respectively, where $t = u \sinh v$ and $z = u \cosh v$. The monopole $\hat{q}(\tau)$ takes the form $\hat{q}(\tau) \equiv e^{i\hat{H}\tau} \hat{q}_0 e^{-i\hat{H}\tau}$, where $\hat{H}$ is the nucleon Hamiltonian of eigenvalues $\hat{H}|p\rangle = m_p|p\rangle$, $\hat{H}|n\rangle = m_n|n\rangle$, $m_{p(n)}$ being the proton (neutron) mass. The operator $\hat{q}_0$ spits out the Fermi constant via the matrix element $G_F \equiv \langle p|\hat{q}_0|n\rangle$ [58]. Concerning the lepton current, we have used the standard notation $\hat{\Psi}_\alpha$, $\hat{\Psi}_{\nu_\alpha}$ for the α-charged lepton and neutrino fields. More details on their explicit expansions are given in [93,95,96].

In the above setting, the Fermi action for the proton decay reads

$$\hat{S}_I \equiv \sum_{\alpha=e,\mu} \int d^4x \sqrt{-g}\, \hat{J}^{(cl)}_{h,\lambda} \left(\hat{\overline{\Psi}}_{\nu_\alpha} \gamma^\lambda \hat{\Psi}_\alpha + \hat{\overline{\Psi}}_\alpha \gamma^\lambda \hat{\Psi}_{\nu_\alpha} \right). \tag{16}$$

To account for neutrinos being in superposition of mass states, we use the Pontecorvo transformation [105]

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} \\ U_{\mu 1} & U_{\mu 2} \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}, \tag{17}$$

and similarly for related fields. Here $U_{e1} = U_{\mu 2} = \cos\theta$ and $U_{e2} = -U_{\mu 1} = \sin\theta$, θ being the mixing angle.

At tree level, the transition amplitude in the laboratory frame is given by [93,95,96]

$$\begin{aligned} \mathcal{A}^{(lab)} &\equiv \langle n| \otimes \langle e^+, \nu_e| \hat{S}_I |0\rangle \otimes |p\rangle \\ &= \frac{G_F}{2^4 \pi^3} \left[\cos^2\theta\, \mathcal{I}_{\sigma_\nu \sigma_e}(\omega_{\nu_1}, \omega_e) + \sin^2\theta\, \mathcal{I}_{\sigma_\nu \sigma_e}(\omega_{\nu_2}, \omega_e) \right], \end{aligned} \tag{18}$$

where we have used the flavor representation for asymptotic neutrino states, consistently with the discussion at the beginning of this Section. The functions $\mathcal{I}_{\sigma_\nu \sigma_e}(\omega_{\nu_i}, \omega_e)$, $i = 1, 2$, are given by Dirac products of positron and neutrino wave-functions, integrated over the proper time τ. They are explicitly exhibited in [93]. Here $\omega_{e(\nu_e)}$ is the usual Minkowski frequency for the positron and neutrino modes.

In turn, the proper decay rate (i.e., the inverse of the mean proper lifetime) is $\Gamma^{(lab)} \equiv \mathcal{P}^{(lab)}/T$, where $d^6\mathcal{P}^{(lab)}/d^3k_\nu\, d^3k_e \equiv \sum_{\sigma_e, \sigma_\nu} \left| \mathcal{A}^{(lab)} \right|^2$ is the differential transition probability, while $T \equiv \int_{-\infty}^{+\infty} d\tau$ denotes the total nucleon proper time. By direct calculations, it can be proved that

$$\Gamma^{(lab)} = \cos^4\theta\, \Gamma_1 + \sin^4\theta\, \Gamma_2 + \cos^2\theta \sin^2\theta\, \Gamma_{12}. \tag{19}$$

Again, the reader can refer to [93] for the explicit form of Γ_i, $i = 1, 2$, and Γ_{12}.

Some comments are in order here: first, we notice that, while the diagonal terms Γ_i, $i = 1, 2$, correspond to the decay rates for the process with an outgoing neutrino of definite mass m_i [89], the presence of Γ_{12} is peculiar to the flavor-basis description, as it arises from the interference (i.e., coherent superposition) between different mass states. Furthermore, we have not considered neutrino oscillations in the above treatment. This, however, does not affect the overall validity of our results. A comprehensive study including oscillations is given in [95,96].

3.2. Comoving Frame

Let us now describe the process from the point of view of an observer comoving with the proton. In this case, the particle "at rest" would be unable to decay, unless some providential mechanism comes into play. In [90], this mechanism is identified with the Unruh effect. Specifically, the proton is supposed to experience a thermal bath of electrons and antineutrinos in the inertial vacuum, giving rise to the following processes at tree level:

$$a)\ p^+ + e^- \to n + \nu_e\,, \qquad b)\ p^+ + \overline{\nu}_e \to n + e^+\,, \qquad c)\ p^+ + e^- + \overline{\nu}_e \to n\,. \tag{20}$$

Here, there is the subtle point of the proton-decay analysis. Following the recipe of [92], we initially assume that the proton interacts with neutrinos in Rindler states [118,119], which display an effective thermal weight and are mass eigenstates. Then, for each channel in Equation (20) the decay amplitude is evaluated by implementing the Rindler–Fulling quantization for fermion fields [89]. After some algebra, we obtain for the first process

$$\begin{aligned} \mathcal{A}^{(com)}_{a,i} &\equiv \langle n| \otimes \langle \nu_i|\hat{S}_I|e^-\rangle \otimes |p\rangle \qquad (21) \\ &= \frac{G_F}{(2\pi)^2}\, U_{ei}\, \mathcal{J}^{(i)}_{\sigma_\nu\sigma_e}(\omega_\nu,\omega_e)\,, \qquad i = 1,2\,, \end{aligned}$$

where $\mathcal{J}^{(i)}_{\sigma_\nu\sigma_e}(\omega_\nu,\omega_e)$ is given by a product of Dirac spinors in Rindler metric [93] and we have used Pontecorvo transformation at level of neutrino field only. Similar expression can be derived for the other two decay channels.

Since the three processes (20) are mutually exclusive, the total proper decay rate is equal to the incoherent sum of the three decay rates taken separately. A somewhat laborious calculation yields

$$\Gamma^{(com)}_m = \cos^2\theta\,\widetilde{\Gamma}_1 + \sin^2\theta\,\widetilde{\Gamma}_2\,, \tag{22}$$

where we have additionally summed over neutrino mass eigenstates, as indicated in [92] (the subscript m on the l.h.s. is a reminder for mass representation). The functions $\widetilde{\Gamma}_i$, $i = 1,2$, are given in [93], where it is shown that $\widetilde{\Gamma}_i = \Gamma_i$.

By comparison with Equation (19), it follows that $\Gamma^{(lab)} \neq \Gamma^{(com)}_m$. Thus, the usage of neutrino mass representation in the comoving frame spoils the general covariance of QFT. In [92], this is motivated by the requirement of maintaining the KMS thermality for the Unruh effect, which is posed at the foundations of the whole analysis. However, as explicitly shown in Equation (9), the non–thermal contribution induced by neutrino mixing is only due to the QFT corrections, which are negligible for relativistic neutrinos. Thus, if one adopts (as we do here) the Pontecorvo flavor states, no violation of KMS condition is to be expected.

Based on the above considerations, we can then repeat the computation of $\mathcal{A}^{(com)}_a$ by considering flavor (Pontecorvo) states for asymptotic neutrinos. Following the same steps as above, we are led to [93]

$$\begin{aligned} \mathcal{A}^{(com)}_{a,e} &\equiv \langle n| \otimes \langle \nu_e|\hat{S}_I|e^-\rangle \otimes |p\rangle \qquad (23) \\ &= \frac{G_F}{(2\pi)^2}\Big[\cos^2\theta\, \mathcal{J}^{(1)}_{\sigma_\nu\sigma_e}(\omega_\nu,\omega_e) + \sin^2\theta\, \mathcal{J}^{(2)}_{\sigma_\nu\sigma_e}(\omega_\nu,\omega_e)\Big]. \end{aligned}$$

Combined with the corresponding expressions for the processes (b) and (c), this gives for the total proper decay rate

$$\Gamma^{(com)}_f = \cos^4\theta\,\widetilde{\Gamma}_1 + \sin^4\theta\,\widetilde{\Gamma}_2 + \cos^2\theta\sin^2\theta\,\widetilde{\Gamma}_{12}\,, \tag{24}$$

(the index f now stands for flavor representation). Again, the use of flavor states leads to the appearance of the extra interference term $\widetilde{\Gamma}_{12}$. In [93] this has been calculated explicitly, showing that $\widetilde{\Gamma}_{12} = \Gamma_{12}$, at least to the leading order in the approximation of small-mass difference between mixed neutrinos. Therefore, the flavor-based picture allows us to restore the expected equality $\Gamma^{(lab)} = \Gamma^{(com)}$, consistently with the general covariance of the theory.

4. Discussion and Conclusions

We have discussed the issue of thermality of the Unruh radiation for mixed neutrino fields. By reviewing recent results, we have shown that corrections to the Fermi–Dirac distribution arise due to the non-trivial nature of the flavor vacuum for neutrinos. Such corrections can be cast, for sufficiently small mixing angles and/or small mass differences, in the form of a Tsallis distribution, corresponding to a non–extensive thermostatistics. However, these corrections disappear for relativistic neutrinos, which are well described by the usual Pontecorvo flavor states.

Thus, the concerns expressed in Ref. [92] that the non-implementability of KMS condition for flavor neutrinos in the Unruh radiation would lead to a difference in the accelerated proton decay rates (laboratory and comoving), do not apply and general covariance is safe. This is true at least for Pontecorvo flavor neutrino states, on which the discussion of Refs. [93,95,96] is based and which we have reviewed here.

Of course, it remains to be checked the consistency of the calculations in the two reference frames when one would use the exact flavor neutrino states [120], which are of course mandatory in the non–relativistic regime. We expect that in such a case, the calculation in the proton's comoving frame would require the use of Tsallis distribution functions as statistical weights for the flavor neutrinos absorbed/emitted by the proton from the Unruh thermal vacuum.

Finally, we would like to comment on the approach of Ref. [94], where the massive neutrino states have been used in the above calculations for both frames. In Ref. [96], we have extended the treatment to include three flavor mixing and CP violation, and general covariance has been checked to be valid also in this case if flavor neutrino states are used. On the other hand, the decay rate calculated by means of massive neutrino states does not exhibit any dependence of the (physical) CP phase and this casts some doubts on this approach.

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Abbreviations

The following abbreviations are used in this manuscript:

QFT	Quantum Field Theory
QG	Quantum Gravity
KMS	Kubo–Martin–Schwinger

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