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# Spherically Symmetric Exact Vacuum Solutions in Einstein-Aether Theory

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**Abstract:** We study spherically symmetric spacetimes in Einstein-aether theory in three different coordinate systems, the isotropic, Painlevé-Gullstrand, and Schwarzschild coordinates, in which the aether is always comoving, and present both time-dependent and time-independent exact vacuum solutions. In particular, in the isotropic coordinates we find a class of exact static solutions characterized by a single parameter  $c_{14}$  in closed forms, which satisfies all the current observational constraints of the theory, and reduces to the Schwarzschild vacuum black hole solution in the decoupling limit ( $c_{14} = 0$ ). However, as long as  $c_{14} \neq 0$ , a marginally trapped throat with a finite non-zero radius always exists, and on one side of it the spacetime is asymptotically flat, while on the other side the spacetime becomes singular within a finite proper distance from the throat, although the geometric area is infinitely large at the singularity. Moreover, the singularity is a strong and spacetime curvature singularity, at which both of the Ricci and Kretschmann scalars become infinitely large.

**Keywords:** Einstein-aether theory; spherical symmetry; exact solution; singularities; black holes; cosmological models

**PACS:** 04.50.Kd; 04.70.Bw; 04.40.Dg; 97.10.Kc; 97.60.Lf



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**Citation:** Oost, J.; Mukohyama, S.; Wang, A. Spherically Symmetric Exact Vacuum Solutions in Einstein-Aether Theory. *Universe* **2021**, *7*, 272. <https://doi.org/10.3390/universe7080272>

Academic Editor: Lorenzo Iorio

Received: 30 June 2021

Accepted: 22 July 2021

Published: 28 July 2021

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## 1. Introduction

Recently, there has been growing interest in exploring the possibility of violations of the Lorentz invariance (LI), and the development of new theories which feature LI-violating effects. Although divergences introduced by LI in quantum field theories help to motivate these explorations, LI-violations in the matter sector are tightly constrained by experiments [1–3]. On the other hand, in the gravitational sector, such experimental constraints are still rather weaker [4,5], and leave rooms for the development of theories that break LI, especially in the case where the breaking is at very high energies, such as in the very early Universe. In particular, if the quantization of spacetimes is a necessary feature of a full theory of gravity, then LI must be an emergent property of low energy physics rather than a fundamental symmetry, as it is a continuous symmetry and cannot exist in a discretized spacetime. Examples of theories which violate LI include Einstein-aether theory [6,7] and Hořava gravity [8–10].

Einstein-aether theory (sometimes shortened as æ-theory) is a vector-tensor theory that breaks LI by coupling a unit time-like vector field to the metric at every point in spacetime. It is the most general vector-tensor theory in the sense: (1) a metric theory, (2) generally covariant, (3) the aether field is unity and time-like, and (4) the field equations

are the second-order differential equations in terms of not only the metric but also the aether field. It was shown [11,12] that æ-theory can be considered as the low-energy limit of the non-projectable Hořava gravity [13,14]. The theory contains three different species of gravitons, scalar (spin-0), vector (spin-1), and tensor (spin-2), and each of them in principle travels at speeds not necessarily the same as the speed of light [15]. However, to avoid the existence of the vacuum gravi-Čerenkov radiation by matter, such as cosmic rays, each of them cannot be less than the speed of light [16]. Furthermore, the gravitational wave, GW170817, observed by the LIGO/Virgo collaboration [17], and the event of the gamma-ray burst GRB 170817A [18], provides a severe constraint on the speed of the spin-2 mode,  $-3 \times 10^{-15} < c_T - 1 < 7 \times 10^{-16}$ . Nevertheless, by properly choosing the coupling constants of the theory, it was shown that the theory is self-consistent (such as free of ghosts and instabilities) [7], and all the observational constraints carried out so far are satisfied [19].

In this paper, we study spherically symmetric *vacuum* solutions of Einstein-aether theory, both time-dependent and time-independent, by paying particular attention on exact solutions, solutions given analytically in closed and explicit forms. We shall study such solutions in three different sets of coordinate systems, namely, *the isotropic, Painlevé-Gullstrand, and Schwarzschild coordinates*, and present several exact solutions in closed forms. In all of these studies, we assume that the aether is at rest in the chosen coordinate system.

It should be noted that spherically symmetric *vacuum* spacetimes in æ-theory have been studied extensively in the past couple of years both analytically [20–34] and numerically [35–40]. In particular, it was shown that they can be also formed from gravitational collapse [41]. Unfortunately, in these studies, the parameter space of the coupling constants of the theory has all been ruled out by current observations [19]. (The only exception is the solutions obtained by taking the limit  $c_{13} \rightarrow 0$  from the ones with  $c_{14} = 0$  first found in [21] in the vacuum case, and later generalized to the charged cases [24–27], where  $c_{ij} \equiv c_i + c_j$ , and  $c_i$  ( $i = 1, 2, 3, 4$ ) are the four dimensionless coupling constants of æ-theory. It is remarkable to note that such obtained solutions are the charged Schwarzschild (Reissner–Nordstrom) solutions. Therefore, in these limiting cases, the aether field has no contributions to the spacetime geometry, and can be considered either as a test field [42], or a real time-like vector field but having no contributions to the spacetime curvature [43]. It is equally remarkable that the aether field remains time-like in the whole spacetime even inside the black holes [42,43].) Lately, spherically symmetric BH solutions that satisfy all the observational constraints were studied numerically in [43] and various black hole solutions were found. It was also shown that not only killing horizons but also a dynamical version of the universal horizons can be formed from the gravitational collapse of realistic matter even for the coupling constants of the theory satisfying all the observational constraints [44].

Note that, due to the fact that the speeds of the spin-0 and spin-1 gravitons can, in principle, be arbitrarily large, the boundaries of black holes in æ-theory are no longer the locations of the killing horizons, but the ones of the universal horizons, which are one-way membranes for particles moving with any speeds, including the speeds that are arbitrarily large. Universal horizons were first proposed in [38] (See also [39]), and recently have been extensively studied in [45–60] (For a recent review, see [10]).

The rest of the paper is organized as follows: In Section 2 we present a brief review of æ-theory, while in Sections 3–5, we consider both static and time-dependent spherically symmetric vacuum spacetimes of æ-theory in the isotropic, Painlevé-Gullstrand, and Schwarzschild coordinate systems, respectively, and find various exact solutions in closed forms, and some of which were found before but were written for the first time in closed forms. The paper is ended in Section 6, in which we summarize our main results and present some concluding remarks. There exists also an appendix, Appendix A, in which we present the Einstein-aether field equations in each of the three different sets of coordinate systems.

## 2. Einstein-Aether Theory

In this paper, we consider only vacuum solutions of the Einstein- aether theory [6,7],

$$S = \frac{1}{16\pi G} \int dx^4 \sqrt{-g}(\mathcal{R} + \mathcal{L}_{\mathfrak{a}}), \tag{1}$$

where  $\mathcal{R}$  is the Ricci scalar and the aether Lagrangian is given by,

$$\mathcal{L}_{\mathfrak{a}} = -M^{ab}{}_{mn} D_a u^m D_b u^n + \lambda(g_{ab} u^a u^b + 1), \tag{2}$$

where  $a, b = 0, 1, 2, 3$ ,  $D$  denotes the covariant derivative with respect to the metric  $g_{ab}$ , and  $\lambda$  is the Lagrangian multiplier, which insures that the aether is time-like and has a fixed norm over the whole spacetime. The tensor  $M^{ab}{}_{mn}$  is defined as

$$M^{ab}{}_{mn} = c_1 g^{ab} g_{mn} + c_2 \delta_m^a \delta_n^b + c_3 \delta_n^a \delta_m^b - c_4 u^a u^b g_{mn}, \tag{3}$$

where  $c_i$  ( $i = 1, 2, 3, 4$ ) are dimensionless coupling constants, as mentioned previously.

Note that the above theory was first studied by Gasperini using the tetrad formalism [61]. On the other hand, setting  $c_1 = -1/2$ ,  $c_{13} = c_2 = c_4 = 0$ , the theory reduces to the bumblebee model, first proposed by Kostelecký and Samuel (KS) [62], when the KS vector field is restricted to time-like and unity. Later, the bumblebee model was extended to the case  $c_1 = (\alpha - \beta)/2$ ,  $c_2 = \zeta$ ,  $c_3 = \zeta - \alpha/2$ ,  $c_4 = 0$ , where  $\alpha, \beta$  and  $\zeta$  are three independent coupling constants, and, in general, the BS vector field has a vacuum expectation value, which can be time-like, null or space-like [63].

Then, the variation of the above action with respect to  $g_{ab}$  yields

$$G_{ab} = T_{ab}^{\mathfrak{a}}, \tag{4}$$

where  $G_{ab} \equiv \mathcal{R}_{ab} - \frac{1}{2}g_{ab}\mathcal{R}$ , and

$$T_{ab}^{\mathfrak{a}} \equiv -\frac{1}{\sqrt{-g}} \frac{\lambda(\sqrt{-g}(\mathcal{L}_{\mathfrak{a}}))}{\lambda g^{ab}} = D_c [J^c{}_{(a} u_{b)} + J_{(ab)} u^c - u_{(b} J_a{}^c] + c_1 [(D_a u_c)(D_b u^c) - (D_c u_a)(D^c u_b)] + c_4 a_a a_b + \lambda u_a u_b - \frac{1}{2}g_{ab} J^d{}_c D_d u^c, \tag{5}$$

where  $J^a{}_b$  and  $a^a$  are defined by

$$J^a{}_b = M^{ac}{}_{bd} D_c u^d, \quad a^a = u^b D_b u^a. \tag{6}$$

In addition, the variation of the action with respect to  $u^a$  yields the aether field equations,

$$\mathfrak{A}_a = \frac{1}{\sqrt{-g}} \frac{\lambda(\sqrt{-g}\mathcal{L}_{\mathfrak{a}})}{\lambda u^a} = D_\alpha J^\alpha{}_\mu + c_4 a_\alpha D_\mu u^\alpha + \lambda u_\mu = 0, \tag{7}$$

while its variation with respect to  $\lambda$  gives,

$$u^a u_a = -1. \tag{8}$$

From Equations (7) and (8) we find that

$$\lambda = u_b D_a J^{ab} + c_4 a^2. \tag{9}$$

As mentioned above, the theory in general allows three different species of gravitons, spin-0, spin-1, and spin-2, and each of them move in principle with different speeds, given, respectively, by [15],

$$\begin{aligned} c_S^2 &= \frac{c_{123}(2-c_{14})}{c_{14}(1-c_{13})(2+c_{13}+3c_2)}, \\ c_V^2 &= \frac{2c_1-c_{13}(2c_1-c_{13})}{2c_{14}(1-c_{13})}, \\ c_T^2 &= \frac{1}{1-c_{13}}, \end{aligned} \tag{10}$$

where  $c_{ij} \equiv c_i + c_j$ ,  $c_{ijk} \equiv c_i + c_j + c_k$ , and  $c_{S,V,T}$  represent the speeds of the spin-0, spin-1, and spin-2 gravitons, respectively.

The most recent observational constraints on the coupling constants  $c_i$ , in light of the LIGO/Virgo gravitational wave detection GW170817 [17] and its concurrent gamma-ray burst GRB170817A [18], were found in [19], together with the self-consistent conditions, such as the absence of ghosts and instability [7]. Depending on the values of  $c_{14}$ , the constraints can be divided into three different bands, and are given, respectively, by [19],

$$(i) \quad 0 \leq c_{14} \leq 2 \times 10^{-7} : c_{14} \leq c_2 \leq 0.095, \tag{11}$$

$$(ii) \quad 2 \times 10^{-7} \leq c_{14} \leq 2 \times 10^{-6} : c_{14} \leq c_2 \leq 0.095, \\ -10^{-7} \leq \frac{c_{14}(c_{14} + 2c_2c_{14} - c_2)}{c_2(2 - c_{14})} \leq 10^{-7}, \tag{12}$$

$$(iii) \quad 2 \times 10^{-6} \leq c_{14} \leq 2.5 \times 10^{-5} : 0 \leq c_2 - c_{14} \leq c_2 \times 10^{-7}. \tag{13}$$

Therefore, each of the three parameters,  $c_2$ ,  $c_{14}$  and  $c_{13}$  are restricted, respectively, to the ranges,

$$0 < c_2 \lesssim 0.095, \tag{14}$$

$$0 < c_{14} \lesssim 2.5 \times 10^{-5}, \tag{15}$$

$$|c_{13}| \lesssim 10 \times 10^{-15}. \tag{16}$$

### 3. Spherically Symmetric Spacetimes in Isotropic Coordinates

#### 3.1. Spherically Symmetric Spacetimes

The general form for a spherically-symmetric metric can be written as,

$$ds^2 = g_{AB}dx^A dx^B + R^2 d\Omega^2 = -\mathcal{N}^2 dt^2 + \mathcal{B}^2 (dr + \mathcal{N}^r dt)^2 + R^2 d\Omega^2, \tag{17}$$

where  $\mathcal{N}$ ,  $\mathcal{B}$ ,  $\mathcal{N}^r$ , and  $R$  are functions of  $t$  and  $r$  only,  $x^\mu = (t, r, \theta, \phi)$ , and  $d\Omega^2 \equiv d\theta^2 + \sin^2 \theta d\phi^2$ . This metric clearly is invariant under the coordinate transformations,

$$t = f(\bar{t}, \bar{r}), \quad r = g(\bar{t}, \bar{r}), \tag{18}$$

where  $f$  and  $g$  are arbitrary functions of their indicated arguments. By properly choosing these functions, we are able to fix two of the four arbitrary functions  $\mathcal{N}$ ,  $\mathcal{B}$ ,  $\mathcal{N}^r$ , and  $R$ .

In this section, we shall use the gauge freedom (18) to set,

$$g_{rr} = R(t, r), \quad g_{tr} = 0, \tag{19}$$

so that the metric (17) takes the form,

$$ds^2 = -e^{2\mu(r,t)} dt^2 + e^{2\nu(t,r)} d\sigma^2, \tag{20}$$

where  $d\sigma^2$  is the spatial part of the metric, defined as,

$$d\sigma^2 \equiv dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \tag{21}$$

Then, the comoving aether<sup>1</sup> is given by

$$u^a = e^{-\mu} \delta_t^a. \tag{22}$$

To write down the field equations, we find convenient first to introduce the constant  $\alpha$  and the function  $\Sigma$  as,

$$\alpha^2 \equiv 3 \left( 1 + \frac{3c_2 + c_{13}}{2} \right), \tag{23}$$

$$\Sigma \equiv 3\dot{v}^2 + 2\ddot{v} - 2\dot{\mu}\dot{v}. \tag{24}$$

Then, the non-vanishing equation for the aether dynamics is,

$$0 = (3c_2 + c_{13} + c_{14})\mu'\dot{v} + c_{14}\dot{\mu}' - \beta\dot{v}', \tag{25}$$

where  $\beta \equiv 3c_2 + c_{13}$ . The non-vanishing components of  $G_{\mu\nu}$  and  $T_{\mu\nu}^{\text{aether}}$  are given by Equations (A4)–(A11), from which we find that currently there are four non-trivial equations, given, respectively, by the  $tt, tr, rr, \theta\theta$  components,

$$\alpha^2\dot{v}^2e^{2\nu} = e^{2\mu} \left[ c_{14} \left( \frac{\mu'^2}{2} + \mu'v' + \mu'' + 2\frac{\mu'}{r} \right) + v'^2 + 2v'' + 4\frac{v'}{r} \right], \tag{26}$$

$$c_{14}(\dot{\mu}' + \mu'\dot{v}) = 2(\mu'\dot{v} - \dot{v}'), \tag{27}$$

$$\frac{\alpha^2}{3}e^{2\nu}\Sigma = e^{2\mu} \left[ v'^2 + 2\mu'v' + \frac{2}{r}(\mu' + v') + c_{14}\frac{\mu'^2}{2} \right], \tag{28}$$

$$\frac{\alpha^2}{3}e^{2\nu}\Sigma = e^{2\mu} \left[ \mu'^2 + \mu'' + v'' + \frac{\mu' + v'}{r} - c_{14}\frac{\mu'^2}{2} \right]. \tag{29}$$

### 3.2. Time-Independent Solutions

With no time-dependence, the five equations are reduced to three. Then, from the  $tt$  and  $\theta\theta$  equations, we find

$$0 = f'' + f'^2 + \frac{3}{r}f', \tag{30}$$

where

$$f \equiv \mu + \nu. \tag{31}$$

To solve Equation (30) we first divide both sides of the equation by  $f'$  and then integrate it, leading to,

$$\ln(L_0f') = -f - 3\ln\left(\frac{r}{r_0}\right), \tag{32}$$

where  $L_0$  and  $r_0$  are the integration constants with dimensions of length. Equation (32) has the general solutions,

$$f = \ln\left(f_0\left(1 - \frac{r_0^2}{r^2}\right)\right), \tag{33}$$

where  $f_0 \equiv r_0/2L_0$  is a dimensionless constant. Next we subtract the  $rr$  equation from the  $tt$  one, leading to,

$$2v'' + 2\frac{v'}{r} = (2 - c_{14})\mu'v' + (2 - 2c_{14})\frac{\mu'}{r} - c_{14}\mu''. \tag{34}$$

Now, from the  $\theta\theta$  equation, we find,

$$2v'' + 2\frac{v'}{r} = -2\mu'^2 - 2\mu'' - 2\frac{\mu'}{r} + c_{14}\mu'^2. \tag{35}$$

The combination of Equations (34) and (35) yields,

$$0 = (c_{14} - 2) \left[ \mu'^2 + \mu'' - 2\frac{\mu'}{r} + \mu'v' \right]. \tag{36}$$

Note that the observational constraints lead to Equation (15), from which we can see that  $c_{14} - 2 \neq 0$  always holds. Therefore, the above equation yields,

$$0 = \mu'^2 + \mu'' + 2\frac{\mu'}{r} + \mu'v', \tag{37}$$

which has the solution,

$$f = -\ln\left(\frac{f_0 r_0 q}{\mu' r^2}\right), \tag{38}$$

where  $q$  is an arbitrary dimensionless constant. Then, combining Equations (33) and (38), we find,

$$\mu' = \frac{r_0 q}{r^2 - r_0^2}, \tag{39}$$

which yields,

$$\mu = \frac{q}{2} \ln\left(U_0 \frac{r - r_0}{r + r_0}\right), \tag{40}$$

where  $U_0$  is a dimensionless constant. We can now solve for  $v$  by using Equations (33) and (40), and find,

$$v = \ln\left[\frac{f_0}{U_0} \left(1 - \frac{r_0^2}{r^2}\right) \left(\frac{r + r_0}{r - r_0}\right)^{\frac{q}{2}}\right]. \tag{41}$$

These solutions for  $\mu$  and  $v$  solve the field equations exactly provided that  $q$  is given by,

$$q \equiv 2\sqrt{\frac{2}{2 - c_{14}}}. \tag{42}$$

Since  $0 \leq c_{14} \leq 2.5 \times 10^{-5}$ , we find that

$$2 \leq q \lesssim 2\left(1 + 1.25 \times 10^{-5}\right). \tag{43}$$

So, the spacetime is given by,

$$ds^2 = \frac{f_0^2}{U_0^2} \left\{ -\frac{U_0^{2+q}}{f_0^2} \left(\frac{r - r_0}{r + r_0}\right)^q dt^2 + \left(1 - \frac{r_0^2}{r^2}\right)^2 \left(\frac{r + r_0}{r - r_0}\right)^q d\sigma^2 \right\}. \tag{44}$$

Rescaling  $t$ , we can set the factor  $U_0^{2+q}/f_0^2 = 1$ , so the above metric takes the form  $ds^2 = (f_0^2/U_0^2)d\bar{s}^2$ . Since  $ds^2$  and  $d\bar{s}^2$  are conformally related by a constant, the spacetimes described by them have the same properties. Therefore, without loss of the generality, we can always set  $f_0 = U_0 = 1$ .

On the other hand, to see the meaning of  $r_0$ , let us consider the Schwarzschild metric in the isotropic coordinates, which is given by,

$$ds^2 = -\left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^2 dt^2 + \left(1 + \frac{m}{2r}\right)^4 d^2\sigma, \tag{45}$$

where  $d^2\sigma \equiv dr^2 + r^2 d^2\Omega$ , as noticed previously. In the  $c_{14} \rightarrow 0$  limit,  $q \rightarrow 2$ , so the spacetime given by Equation (44) does indeed reduce to the isotropic Schwarzschild solution given by Equation (45), provided that

$$f_0 = 1, \quad U_0 = 1, \quad r_0 = \frac{m}{2}, \tag{46}$$

for which the metric (44) takes the form,

$$ds^2 = -\left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^q dt^2 + \left(1 - \frac{m^2}{4r^2}\right)^2 \left(\frac{1 + \frac{m}{2r}}{1 - \frac{m}{2r}}\right)^q d^2\sigma, \tag{47}$$

where  $q$  is given by Equation (42) and  $r \geq m/2$  (which is also true for the Schwarzschild solution in the isotropic coordinates).

The spacetime given by Equation (47) has curvature singularities at  $r = \frac{m}{2}$  and at  $r = 0$ . Both are curvature singularities as can be seen by considering the Ricci scalar. However this is easier to see in a coordinate system similar to the Schwarzschild form. Consider the coordinate transformation:

$$\bar{r} = r\left(1 + \frac{m}{2r}\right)^2, \tag{48}$$

upon which the metric becomes

$$ds^2 = -\left(1 - \frac{2m}{\bar{r}}\right)^{q/2} dt^2 + \left(1 - \frac{2m}{\bar{r}}\right)^{-q/2} d\bar{r}^2 + \left(1 - \frac{2m}{\bar{r}}\right)^{1-q/2} \bar{r}^2 d\Omega^2. \tag{49}$$

Then, the Ricci scalar is given by,

$$\mathcal{R} = \frac{m^2(4 - q^2)}{2\bar{r}^4} \left(1 - \frac{2m}{\bar{r}}\right)^{\frac{q}{2}-2}, \tag{50}$$

and the Kretschmann scalar is given by

$$\mathcal{K} \equiv R_{abcd}R^{abcd} = \frac{m^2}{4\bar{r}^4} \left(1 - \frac{2m}{\bar{r}}\right)^{q-4} (a\bar{r}^2 + b\bar{r} + d), \tag{51}$$

where

$$a = 48q^2, \quad b = -32mq(q^2 + 3q + 2), \quad d = m^2(2 + q)^2(7q^2 + 4q + 12). \tag{52}$$

Obviously both the Ricci and Kretschmann scalars have curvature singularities at the origin, and upon carefully taking the limit when  $\bar{r}$  approaches  $2m$  we see that there are curvature singularities at  $\bar{r} = 2m$  as well. When  $c_{14}$  is set to zero they reduce to the correct values for the Schwarzschild solution's Ricci and Kretschmann scalars (expressed in the Schwarzschild coordinates). As can be seen from Equation (49) the area of a sphere centered on the origin is given by,

$$A = 4\pi\bar{r}^2 \left(\frac{\bar{r}}{\bar{r} - 2m}\right)^{q/2-1}. \tag{53}$$

When  $c_{14} = 0$  we have  $q = 2$ , and then  $\bar{r}$  becomes the areal radial coordinate and a sphere with coordinate radius  $\bar{r} = 2m$  has the area  $4\pi\bar{r}^2$  as expected. However, when  $c_{14} \neq 0$ , we have  $q \geq 2$ , and the area of a sphere is well defined only for  $\bar{r} \geq 2m$  [cf. Equation (43)], and becomes infinitely large at both  $\bar{r} = 2m$  and  $\bar{r} = \infty$ . This shows that while the spacetime of Equations (47) and (49) approach the Schwarzschild solution as  $c_{14}$  approaches zero, the approach is not completely continuous. In particular, as long as  $c_{14} \neq 0$ , the areal radius always reaches a minimum at,

$$\bar{r}_{\min} = 2m\left(\frac{2 + q}{4}\right) \geq 2m, \tag{54}$$

which serves as a throat and smoothly connects the two regions,  $\bar{r} \in (2m, \bar{r}_{\min}]$  and  $\bar{r} \in [\bar{r}_{\min}, \infty)$ , as shown schematically in Figure 1, where the point  $\bar{r} = \bar{r}_{\min}$  defined by

Equation (48) denotes the location of the throat. The spacetime is asymptotically flat as  $\bar{r} \rightarrow \infty$ , and the proper radial distance from the throat to  $\bar{r} = \infty$  is infinitely large (so is the geometric area  $A$ ). However, despite the fact that  $A$  also becomes infinitely large at  $\bar{r} = 2m$ , the proper radial distance between the throat and  $\bar{r} = 2m$  is finite,

$$L \equiv \left| \int_{\bar{r}_{\min}}^{\bar{r}} \left( \frac{\bar{r}}{\bar{r} - 2m} \right)^{q/4} d\bar{r} \right| = \begin{cases} \infty, & \bar{r} \rightarrow \infty, \\ \text{finite}, & \bar{r} \rightarrow 2m. \end{cases} \tag{55}$$

It should be noted that the proper radial distance between the throat and the singularity  $\bar{r} = 2m$  becomes infinite for  $3/2 < c_{14} < 2$  [20], for which we have  $q > 4$ . However, this violates the observational constraints given by Equation (43).

To see the nature of the curvature singularity located at  $\bar{r} = 2m$ , let us consider the expansions of null geodesics [64]. Let the metric of Equation (49) be written as,

$$ds^2 = -u_a u_b + s_a s_b + m_{ab}, \tag{56}$$

where  $u_a$  is given by Equation (22),  $s_a = e^v \delta_a^r$  and  $m_{ab}$  is the 2-dimensional metric induced on a closed, 2-dimensional, space-like hypersurface (which is conformal to the 2-sphere metric). Then, let the outgoing or ingoing null geodesics have the tangent vectors  $k_a^\pm$ , given by,

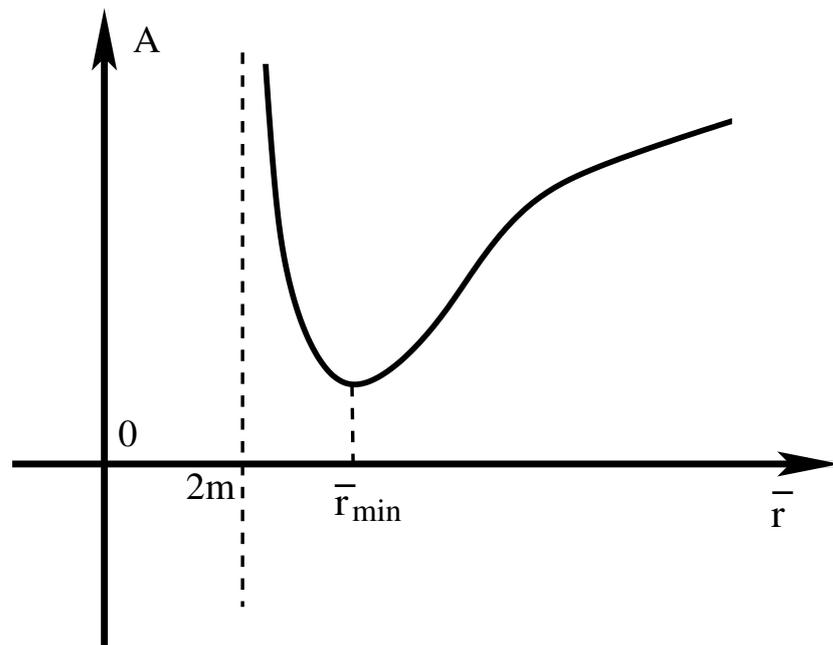
$$k_a^\pm = \frac{1}{\sqrt{2}}(u_a \pm s_a), \tag{57}$$

we find that the expansions of outgoing/ingoing null geodesics are given by

$$\Theta_\pm \equiv m^{ab} \nabla_a k_b^\pm = \pm \frac{\sqrt{2} \left(1 - \frac{2m}{\bar{r}}\right)^{q/4}}{\bar{r}(\bar{r} - 2m)} (\bar{r} - \bar{r}_{\min}), \tag{58}$$

which all vanish at  $\bar{r}_{\min}$ . However, across the throat the outgoing or ingoing null geodesics exchange their roles, so that  $\Theta_+ \Theta_- < 0$  holds in both sides of the throat. As an immediate result, no trapped regions exist, and the throat is only marginally trapped [65].

It should be noted that Eling and Jacobson studied the static aether case in the Schwarzschild coordinates, and found the general solutions (but given implicitly) [20], as to be shown explicitly in Section 5. Therefore, the above solutions must be the same ones. In addition, they also found that the minimal 2-sphere,  $\bar{r} = \bar{r}_{\min}$ , does not sit at a killing horizon, and that the singularity at  $\bar{r} = 2m$  is actually a null singularity [cf. [20], Figure 2]. In addition, these solutions with  $c_{123} \neq 0$  are stable against small spherically symmetric perturbations [7,66].



**Figure 1.** The geometric area  $A$  of the 2-dimensional spheres  $t, \bar{r} = \text{Constant}$  vs the radial coordinate  $\bar{r}$  for the spacetime described by the metric (49) with  $c_{14} \neq 0$ , where  $A$  is given by Equation (53). The point  $\bar{r} = \bar{r}_{\text{min}}$  defined by Equations (48) and (54) is the minimal surface for the area  $A$ —the throat. The area  $A$  becomes infinitely large at both  $\bar{r} = 2m$  and  $\bar{r} = \infty$ , and the spacetime is singular at  $\bar{r} = 2m$ , but asymptotically flat as  $\bar{r} \rightarrow \infty$ .

### 3.3. Time-Dependent Solutions

If we consider solutions such that  $e^\mu$  and  $e^\nu$  are separable in  $t$  and  $r$ , then we seek solutions of the form,

$$\mu(r, t) = \mu_0(r) + \mu_1(t), \tag{59}$$

$$\nu(r, t) = \nu_0(r) + \nu_1(t), \tag{60}$$

so that all mixed-partial derivatives of  $\mu$  and  $\nu$  are zero. However, redefining the time coordinate  $t$  by  $t'$ ,

$$t' \equiv \int e^{2\mu_1(t)} dt, \tag{61}$$

we can see that, without loss of generality, we can set  $\mu_1 = 0$ , and look for solutions of the form,

$$\mu(r, t) = \mu_0(r), \tag{62}$$

$$\nu(r, t) = \nu_0(r) + \nu_1(t). \tag{63}$$

If  $\dot{\nu} = 0$  then the equations of motion reduce to the static case, so we assume that  $\dot{\nu} \neq 0$ . In this case, the  $tr$  and aether equations reduce to,

$$c_{14}\mu'\dot{\nu} = 2\mu'\dot{\nu}, \tag{64}$$

$$c_{14}\mu'\dot{\nu} = -\beta\mu'\dot{\nu}. \tag{65}$$

Thus, there are three possibilities,

$$(i) \ c_{14} = 2 = -\beta; \quad (ii) \ \dot{\nu} = 0; \quad (iii) \ \mu' = 0, \tag{66}$$

where Case (i) is excluded by observations and Case (ii) is the static case, just studied in the last subsection. Therefore, in the following we shall need only to consider the last case,  $\mu' = 0$ .

When  $\mu' = 0$ , the three relevant equations are the ones of the  $tt$ ,  $rr$ , and  $\theta\theta$  components, given by,

$$\alpha^2 v_1^2 e^{2v_1} = e^{2\mu_0 - 2v_0} \left( v_0'^2 + 2v_0'' + 4\frac{v_0'}{r} \right), \tag{67}$$

$$\alpha^2 e^{2v_1} \left( v_1^2 + \frac{2}{3} \dot{v}_1 \right) = e^{2\mu_0 - 2v_0} \left( v_0'^2 + 2\frac{v_0'}{r} \right), \tag{68}$$

$$\alpha^2 e^{2v_1} \left( v_1^2 + \frac{2}{3} \dot{v}_1 \right) = e^{2\mu_0 - 2v_0} \left( v_0'' + \frac{v_0'}{r} \right). \tag{69}$$

Note that for each equation, the left-hand side is  $t$ -dependent and the right-hand side is  $r$ -dependent, thus both sides must be equal to the same constant. Setting

$$K_0^2 \equiv \alpha^2 e^{2v_1} v_1^2, \tag{70}$$

$$K_1^2 \equiv \alpha^2 e^{2v_1} \left( v_1^2 + \frac{2}{3} \dot{v}_1 \right), \tag{71}$$

from Equations (68) and (69) we have

$$v_0'' = v_0'^2 + \frac{v_0'}{r}, \tag{72}$$

and, thus, it can be shown that  $K_0^2 = K_1^2/3$ . Equation (72) has the general solution,

$$v_0(r) = \ln \left( \frac{r_1}{r^2 - r_0^2} \right), \tag{73}$$

where  $r_1$  and  $r_0$  are integration constants. Next we solve for  $v_1(t)$  using Equation (70),

$$e^{2v_1} \alpha^2 v_1^2 = K_0^2, \tag{74}$$

which has the solution,

$$v_1(t) = \ln \left[ \frac{K_0}{\alpha} (t - t_0) \right], \tag{75}$$

where  $t_0$  is an integration constant, and  $K_0 \neq 0$ . It is straightforward to show that the solutions given by Equations (73)–(75) solve the field Equations (67)–(69) provided that,

$$\frac{K_0^2 r_1^2}{\alpha} = 12 r_0^2 e^{2\mu_0}. \tag{76}$$

Then, the final solutions for  $\mu(t, r)$  and  $v(t, r)$  can be expressed as,

$$\mu(t, r) = \mu_0, \tag{77}$$

$$v(t, r) = \ln \left[ \sqrt{\frac{12}{\alpha}} \frac{r_0 (t - t_0)}{r^2 - r_0^2} \right] + \mu_0. \tag{78}$$

Then, using the gauge freedom  $\bar{t} = at + b$ , where  $a$  and  $b$  are constants, we can always set  $t_0 = \mu_0 = 0$ , so that the metric takes the form,

$$ds^2 = -d\bar{t}^2 + \frac{12 r_0^2 \bar{t}^2}{\alpha (r^2 - r_0^2)^2} \left( dr^2 + r^2 d^2\Omega \right). \tag{79}$$

It can be shown that the spacetime described by the above metric is conformally flat, that is, the Weyl tensor vanishes identically, and the spacetime is singular at  $t = 0$ , as can be seen from the Ricci and Kretschmann scalars, which are now given by,

$$R = -\frac{3\beta}{t^2}, \quad K = \frac{1}{t^4} \left( \frac{4}{3}\alpha^2 - 8\alpha + 12 \right), \tag{80}$$

where  $\beta = 3c_2 + c_{13}$ , as defined previously.

To study this solution further, let us consider the energy conditions. We define a time-like vector field  $t^\alpha$  in the  $(t, r)$ -plane,

$$t^\alpha = A\delta_t^\alpha + B\delta_r^\alpha, \quad A^2 = v^2 + B^2e^{2\nu}, \tag{81}$$

from which we find that  $t^\alpha t_\alpha = -v^2$ , where  $v$  is an arbitrary non-vanishing real function of  $x^\alpha$ . A stress-energy tensor that obeys the weak energy condition ensures that all observers following time-like trajectories will see only positive energy density [65], that is,

$$T_{\alpha\beta}^{\mathfrak{aether}} t^\alpha t^\beta \geq 0. \tag{82}$$

However, for the spacetime of Equation (79) we have,

$$T_{\alpha\beta}^{\mathfrak{aether}} t^\alpha t^\beta = -3\beta \left[ \frac{6B^2r_0^2}{\alpha(r^2 - r_0^2)^2} + \frac{v^2}{2t^2} \right], \tag{83}$$

which is always non-positive. Thus, the aether field in the current case always violates the weak energy condition.

A stress-energy tensor that obeys the strong energy condition ensures that gravity will always be attractive, which is equivalent to require [65],

$$T_{\alpha\beta}^{\mathfrak{aether}} t^\alpha t^\beta - \frac{1}{2} T_{\mathfrak{aether}}^{\mathfrak{aether}} t_\alpha t^\alpha \geq 0. \tag{84}$$

Again, in the current case, the above condition is violated, as now we have,

$$T_{\alpha\beta}^{\mathfrak{aether}} t^\alpha t^\beta - \frac{1}{2} T_{\mathfrak{aether}}^{\mathfrak{aether}} t_\alpha t^\alpha = -\frac{18B^2\beta r_0^2}{\alpha(r^2 - r_0^2)^2} < 0. \tag{85}$$

In addition, the above spacetime actually belongs to the Friedmann universe. To show this, we first introduce two new variables  $\eta$  and  $R$  via the relations,

$$t = e^{\gamma r_0(\eta - \eta_0)}, \quad \bar{r}^2 = \frac{r^2}{(r^2 - r_0^2)^2}, \tag{86}$$

where  $\eta_0$  is a constant and  $\gamma = \sqrt{12/\alpha}$ . It can be seen that  $R \equiv a\bar{r}$  is the areal radius. Then, in terms of  $\eta$  and  $\bar{r}$ , the above metric takes the form,

$$ds^2 = a(\eta)^2 \left( -d\eta^2 + \frac{d\bar{r}^2}{1 + 4r_0^2 R^2} + \bar{r}^2 d\Omega^2 \right), \tag{87}$$

where

$$a(\eta) = \gamma r_0 \exp(\gamma r_0(\eta - \eta_0)). \tag{88}$$

Remember that  $r_0$  was an integration constant, and from Equations (75) and (76) we see that we cannot set  $r_0$  to zero. If we set  $r_0^2 = 1/4$  then the metric of Equation (87) would be the traditional form for an FLRW metric of constant negative curvature ( $k = -1$ ). A solution equivalent to this was also found in [30].

#### 4. Spherically Symmetric Spacetimes in Painlevé-Gullstrand Coordinates

In this section, using the gauge freedom (18), we choose the gauge

$$g_{rr} = 1, \quad R(t, r) = r, \tag{89}$$

so the metric takes the Painlevé-Gullstrand (PG) form,

$$ds^2 = -e^{2\mu(t,r)} dt^2 + 2e^{\nu(t,r)} dr dt + dr^2 + r^2 d\Omega^2. \tag{90}$$

For this metric we only consider time-independent solutions, and assume that the aether is comoving,  $u^a = e^{-\mu} \delta_t^a$ . So, the aether is aligned with the time-like killing vector of the metric, which is itself hypersurface-orthogonal. To simplify the field equations, we first define the quantity  $\lambda$ ,

$$\lambda \equiv e^{2\mu} + e^{2\nu}. \tag{91}$$

Then, for static spacetimes,  $\mu = \mu(r)$ ,  $\nu = \nu(r)$ , we find that the non-vanishing components of  $G_{\mu\nu}$  and  $T_{\mu\nu}^{\text{ae}}$  are given by Equations (A14)–(A21). The aether dynamical equations are identically zero for any  $\mu$  and  $\nu$ , and the remaining field equations are the ones given by the  $(tt, rr, \theta\theta)$  components,

$$e^{4\mu-2\nu} [c_{14}(2r^2\mu'' + 4r\mu' + r^2\mu'^2)] + e^{2\mu} [c_{14}(2r^2\mu'' + 4r\mu' + r^2\mu'^2 + 2r^2\mu'(\mu' - \nu')) + 4r(\mu' - \nu')] - 2\lambda = 0, \tag{92}$$

$$e^{2\mu} [c_{14}(4r\mu' + 2r^2\mu'' - r^2\mu'^2) - 8r\mu'] e^{2\nu} [c_{14}(4r\mu' + 2r^2\mu'' + 2r^2\mu'(\mu' - \nu')) - 4r\nu'] - e^{4\mu-2\nu} [4r\mu' + c_{14}r^2\mu'^2] + 2\lambda = 0, \tag{93}$$

$$\lambda [(c_{14} - 2)r^2\mu'^2 - 2r^2\mu'' - 2r\mu'] + e^{2\nu} [2r\nu'(1 + r\mu') - 2r\mu' - 2r^2\mu'^2] = 0. \tag{94}$$

As can be shown from Equations (A14)–(A21), the Einstein-aether equations require that

$$\lambda \neq 0, \tag{95}$$

although this is not evident from the field equations. Therefore, as we proceed we must reject outright any solution that violates Equation (95).

Our strategy is to first solve the  $tt$  equation for  $\nu'$ . The result is

$$\begin{aligned} \nu' = & \frac{-2e^{4\nu} + c_{14}(4r\mu' + r^2\mu'^2 + 2r^2\mu'')e^{4\mu}}{2re^{2\mu+2\nu}(2 + c_{14}r\mu')} + \frac{e^{2\mu+2\nu}(c_{14}(r\mu'(4 + 3r\mu') + 2r^2\mu''))}{2re^{2\mu+2\nu}(2 + c_{14}r\mu')} \\ & + \frac{e^{2\mu+2\nu}(4r\mu' - 2)}{2re^{2\mu+2\nu}(2 + c_{14}r\mu')}. \end{aligned} \tag{96}$$

Note that in deriving the above expression, we assume that

$$r\mu' \neq -\frac{2}{c_{14}}. \tag{97}$$

When  $2 + c_{14}r\mu' = 0$ , the solutions are different. Therefore, let us pause here for a while, and first consider the case  $2 + c_{14}r\mu' = 0$ .

1.  $2 + c_{14}r\mu' = 0$

In this case from the  $tt$  equation we find,

$$\mu = \frac{2}{c_{14}} \ln \left( \frac{r_0}{r} \right), \tag{98}$$

where  $r_0$  is the integration constant. By substituting this into the  $rr$  equation we find,

$$(4 - c_{14})e^{4\nu} + 2\left(\frac{r_0}{r}\right)^{\frac{8}{c_{14}}} = (c_{14} - 6)e^{2\nu}\left(\frac{r_0}{r}\right)^{\frac{4}{c_{14}}}, \tag{99}$$

which has two solutions, but one does not satisfy Equation (95), so we must reject it. Then, we have,

$$e^{2\nu} = \frac{2}{c_{14} - 4}\left(\frac{r_0}{r}\right)^{\frac{4}{c_{14}}}. \tag{100}$$

Upon substituting Equations (98) and (100) into the  $tt$  and  $\theta\theta$  field equations, we find that

$$c_{14} = 2. \tag{101}$$

Unfortunately this leads the solution of Equation (100) to violate Equation (95). It is also unphysical, as it strongly violates the constraints (15), so in the case  $2 + c_{14}r\mu' = 0$  no physically acceptable solutions exist.

### 2. $2 + c_{14}r\mu' \neq 0$

This is the case in which Equation (96) holds. We substitute the value for  $\nu'$  from this equation into the  $rr$  equation and solve for  $e^{2\nu}$ . The result is,

$$e^{2\nu} = e^{2\mu}\left(2r\mu' + \frac{c_{14}}{2}r^2\mu'^2\right). \tag{102}$$

We can substitute this value into Equation (96), and then obtain the expressions for both  $\nu'$  and  $e^{2\nu}$  in terms of  $\mu$  and its derivatives. In particular, we find,

$$\nu' = \frac{c_{14}\mu''(2 + r\mu'(4 + c_{14}r\mu'))}{\mu'(2 + c_{14}r\mu')(4 + c_{14}r\mu')} + \frac{4(c_{14} - 1) + c_{14}r\mu'(2 + r\mu')(4 + c_{14}r\mu')}{r(2 + c_{14}r\mu')(4 + c_{14}r\mu')}. \tag{103}$$

Substituting it into either the  $tt$  or  $rr$  equation, we find,

$$(c_{14} - 2)\left(4\mu' + 4r\mu'^2 + c_{14}r^2\mu'^3 + 2r\mu''\right)\left[8 + 2(8 + c_{14})r\mu' + 8c_{14}r^2\mu'^2 + c_{14}^2r^3\mu'^3\right] = 0. \tag{104}$$

Therefore, there exist three possibilities,

$$0 = c_{14} - 2, \tag{105}$$

$$0 = 8 + 2(8 + c_{14})r\mu' + 8c_{14}r^2\mu'^2 + c_{14}^2r^3\mu'^3, \tag{106}$$

$$0 = 4\mu' + 4r\mu'^2 + c_{14}r^2\mu'^3 + 2r\mu''. \tag{107}$$

The case of Equation (105) is not only unphysical but also violates the constraint (96). Therefore, in the following we only need to consider the last two cases.

Let us first consider Equation (106), which can be written in the form,

$$0 = (f - \beta_0)(f - \beta_1)(f - \beta_2), \tag{108}$$

where now  $f \equiv r\mu'$ , and

$$\beta_0 = -\frac{4}{c_{14}}, \quad \beta_1 = -\frac{2 + \sqrt{4 - 2c_{14}}}{c_{14}}, \quad \beta_2 = \frac{-2 + \sqrt{4 - 2c_{14}}}{c_{14}}. \tag{109}$$

Generically, the solution to each case is of the form,

$$\mu = \ln\left(\frac{r}{r_0}\right)^{\beta_i}, \quad e^{2\mu} = \left(\frac{r}{r_0}\right)^{2\beta_i}, \tag{110}$$

where  $\beta_i$  is any of the ones given in Equation (109). When we insert Equation (110) into Equation (102) we find that,

$$e^{2\nu} = \frac{1}{2}\beta_i(4 + c_{14}\beta_i)\left(\frac{r}{r_0}\right)^{2\beta_i}. \tag{111}$$

Since any solution in which  $e^{2\nu} = 0$  is equivalent to the Minkowski metric, we ignore the case of  $\beta_0$ , as this would make  $e^{2\nu} = 0$ , as can be seen from Equation (111). If we insert the others  $\beta_i$  into Equation (111), then we have,

$$e^{2\nu} = -\left(\frac{r}{r_0}\right)^{2\beta_i}. \tag{112}$$

Unfortunately this violates the constraint of Equation (96), so we must reject it, and assume that Equation (106) does not hold. This brings us to Equation (107), which we rewrite it as,

$$0 = 4r\mu'4r^2\mu'^2 + c_{14}r^3\mu'^3 + 2r^2\mu''. \tag{113}$$

From  $f \equiv r\mu'$  we find,

$$r^2\mu'' = rf' - f, \tag{114}$$

and, thus, we can rewrite Equation (113) as

$$f' = -\frac{f}{r}\left(1 + 2f + \frac{c_{14}}{2}f^2\right). \tag{115}$$

However, this is precisely the same equation as for the static case in the Schwarzschild coordinates, given by Equation (26) of [20]. Then, we can find the corresponding solutions by proceeding exactly in the same way as performed in [20]. In particular, the solution for  $\mu$  is given by,

$$\mu(f) = \ln \left[ \left( f_0 \frac{1 - f/f_-}{1 - f/f_+} \right)^{\frac{f_+ - f_-}{f_+ - f_-}} \right], \tag{116}$$

where

$$f_{\pm} = \frac{-1 \pm \sqrt{1 - \alpha}}{\alpha}, \tag{117}$$

and  $f = f(r)$  is given implicitly via the relation,

$$\frac{r_0}{r} = \left(\frac{f}{f - f_-}\right) \left(\frac{f - f_-}{f - f_+}\right)^{\frac{1}{2(1+f_+)}}. \tag{118}$$

Considering the fact that the coordinate transformations,

$$\bar{t} = t + g(r), \tag{119}$$

can bring the PG metric to the Schwarzschild one by properly choosing the function  $g(r)$ , we find that in such coordinate systems we have  $g_{tt} = g_{\bar{t}\bar{t}}$ . So, the above solutions should be the ones found in [20], but written in the PG coordinates.

### 5. Spacetimes in the Schwarzschild Coordinates

The Schwarzschild coordinates correspond to the choice,

$$g_{tr} = 0, \quad R(t, r) = r, \tag{120}$$

for which the metric takes the form,

$$ds^2 = -e^{2\mu(t,r)} dt^2 + e^{2\nu(t,r)} dr^2 + r^2 d^2\Omega, \tag{121}$$

and the comoving aether vector field takes the form,  $u^a = e^{-\mu} \delta_t^a$ . For the sake of the simplicity, we first define the quantities,

$$Q \equiv \frac{\mu'^2}{2} - \mu'v' + \mu'', \tag{122}$$

$$H \equiv \frac{\dot{v}^2}{2} - \dot{\mu}\dot{v} + \ddot{v}. \tag{123}$$

Then, the non-vanishing equation for the aether dynamics is,

$$0 = (2c_{13} - (c_2 + c_{13} - c_{14})r\mu')\dot{v} + r(c_{123}\dot{v}' - c_{14}\dot{\mu}'). \tag{124}$$

The non-vanishing Einstein-aether vacuum equations  $G_{ab} = T_{ab}^{\text{ae}}$  are the  $(tt, tr, rr, \theta\theta)$  components, given, respectively, by,

$$0 = e^{2\mu} \left[ c_{14}Q + 2c_{14} \frac{\mu'}{r} - \frac{2v'}{r} + \frac{1}{r^2} \right] - e^{2v} \frac{c_{123}}{2} \dot{v}^2 - \frac{e^{2(\mu+v)}}{r^2}, \tag{125}$$

$$0 = c_{14}(\dot{\mu}' - \mu'\dot{v}) - \frac{2\dot{v}}{r}, \tag{126}$$

$$0 = e^{2v} [c_{123}H] + \frac{e^{2v+2\mu}}{r^2} - e^{2\mu} \left[ \frac{2\mu'}{r} + \frac{1}{r^2} + \frac{c_{14}}{2} \mu'^2 \right], \tag{127}$$

$$0 = e^{2\mu} \left[ \frac{\mu'^2}{2} (c_{14} - 1) + \frac{v' - \mu'}{r} - Q \right] + e^{2v} \left[ \frac{\dot{v}^2}{2} (1 - c_{13}) + (c_2 + 1)H \right]. \tag{128}$$

### 5.1. Time-Independent Solutions

The static solution was already found in [20] but with different (though equivalent) parameterizations of the metric. In the static case, all time-derivatives go to zero and the  $(tt, rr, \theta\theta)$  equations become,

$$\frac{e^{2v}}{r^2} = c_{14}Q + 2c_{14} \frac{\mu'}{r} - 2\frac{v'}{r} + \frac{1}{r^2}, \tag{129}$$

$$\frac{e^{2v}}{r^2} = c_{14} \frac{\mu'^2}{2} + 2\frac{\mu'}{r} + \frac{1}{r^2}, \tag{130}$$

$$0 = \frac{\mu'^2}{2} (c_{14} - 1) + c_{14}\mu'' - c_{14}\mu'v'. \tag{131}$$

Subtracting the  $rr$  equation from the  $tt$  one, we find,

$$2\frac{v'}{r} = c_{14}\mu'' - c_{14}\mu'v' + 2\frac{\mu'}{r} (c_{14} - 1). \tag{132}$$

On the other hand, from the  $\theta\theta$  equation we obtain,

$$2\frac{v'}{r} = 2\mu'' + \mu'^2(2 - c_{14}) + 2\frac{\mu'}{r} - 2\mu'v', \tag{133}$$

which, together with Equation (132) yields,

$$v' = \frac{\mu''}{\mu'} + \mu' + \frac{2}{r}. \tag{134}$$

We can rewrite the  $\theta\theta$  equation as,

$$v' \left( \frac{1}{r} + \mu' \right) = \frac{\beta}{2} \mu'^2 + \frac{\mu'}{r} + \mu''. \tag{135}$$

Inserting our expression (134) into this equation and after simplification, we find

$$r^2\mu'' + 2r\mu' + 2r^2\mu'^2 + \frac{c_{14}}{2}r^3\mu'^3 = 0, \tag{136}$$

which is equivalent to Equation (26) of [20], provided that we make the following substitutions,

$$c_{14} \rightarrow c_1, \quad \mu \rightarrow \frac{A}{2}. \tag{137}$$

We can solve this using an equivalent process as that given in [20]. In particular, setting  $f = r\mu'$ , we find that Equation (136) becomes

$$\frac{df}{dr} = -\frac{f}{r} (1 + 2f + \alpha f^2), \tag{138}$$

but now with  $\alpha \equiv c_{14}/2$ . From the chain role,  $\frac{d\mu}{dr} = \frac{d\mu}{df} \frac{df}{dr}$ , and the definition of  $f$  we find

$$\frac{d\mu}{df} = -\frac{1}{1 + 2f + \alpha f^2}. \tag{139}$$

Using the partial fraction decomposition we can solve the above equation, and find

$$\mu(f) = \ln \left[ \left( f_0 \frac{1 - f/f_-}{1 - f/f_+} \right)^{\frac{f_+ f_-}{f_+ - f_-}} \right], \tag{140}$$

where  $f_0$  is an integration constant whose square is unity. The equivalent equation in [20] is Equation (34), and this solution matches it exactly, bearing in mind that,

$$f_{\pm} = \frac{-1 \pm \sqrt{1 - \alpha}}{\alpha}. \tag{141}$$

Then we can solve Equation (138) and find,

$$\frac{r_0}{r} = \left( \frac{f}{f - f_-} \right) \left( \frac{f - f_-}{f - f_+} \right)^{\frac{1}{2(1+f_+)}} , \tag{142}$$

which is equivalent to Equation (35) of [20].

Note that, when  $c_{14} = 2$ , instead of Equation (140) now we have

$$\mu(f) = \ln \left[ \left( f_0 \frac{f + f_+}{f + f_-} \right)^{\frac{1}{f_- - f_+}} \right], \tag{143}$$

where now  $f_{\pm}$  are defined by

$$f_{\pm} = \frac{3}{4} \pm \frac{\sqrt{41}}{4}, \tag{144}$$

and instead of Equation (142) we have

$$\frac{r_0}{r} = f^{\frac{2}{f_+ f_-}} \left( \frac{(f - f_+)^{1/f_+}}{(f - f_-)^{1/f_-}} \right)^{\frac{2}{f_+ - f_-}} . \tag{145}$$

However, as shown above, this solution is physically not acceptable.

### 5.2. Time-Dependent Solutions

If we consider solutions, such that  $e^{\mu}$  and  $e^{\nu}$  are separable in  $t$  and  $r$ , then we seek solutions of the forms of Equations (62) and (63). In this case, the  $tr$  and aether equations reduce to,

$$\frac{2}{r} = -c_{14}\mu', \tag{146}$$

$$\frac{2c_{13}}{r} = \mu'(c_{14} - c_{123}). \tag{147}$$

We now consider separately the cases  $c_{13} = 0$  and  $c_{13} \neq 0$ .

1.  $c_{13} = 0$

By Equation (146) we must have

$$c_{14} \neq 0, \quad \mu' \neq 0. \tag{148}$$

Then, from Equation (147) we have,

$$c_2 = c_{14}, \tag{149}$$

for which Equation (146) yields,

$$\mu = \ln\left(\frac{U_0}{r^\alpha}\right), \quad \alpha \equiv \frac{2}{c_2}, \tag{150}$$

where  $U_0$  is an arbitrary constant. Then, the  $tt$ ,  $rr$ ,  $\theta\theta$  Equations (125), (127) and (128) become

$$\frac{U_0^2}{r^{2\alpha+2}}(\alpha - 1) = e^{2\nu} \left[ \frac{\dot{\nu}^2}{\alpha} + \frac{U_0^2}{r^{2\alpha+2}} \right], \tag{151}$$

$$\frac{U_0^2}{r^{2\alpha+2}}(\alpha - 1) = e^{2\nu} \left[ -\frac{2}{\alpha}\dot{\nu} - \frac{\dot{\nu}^2}{\alpha} - \frac{U_0^2}{r^{2\alpha+2}} \right], \tag{152}$$

$$\frac{U_0^2}{r^{2\alpha+2}}(\alpha - 1)(rv' + \alpha) = e^{2\nu} \left[ \frac{2}{\alpha}\dot{\nu} + \frac{\dot{\nu}^2}{\alpha} + \dot{\nu} + \dot{\nu}^2 \right]. \tag{153}$$

By combining the  $tt$  and  $rr$  equations we find

$$\frac{\dot{\nu}_1(t)^2 + \ddot{\nu}_1(t)}{\alpha} = -\frac{U_0^2}{r^{2\alpha+2}}, \tag{154}$$

where we have explicitly written the expressions for  $\dot{\nu}$  in terms of  $\nu_1(t)$  to emphasize the  $t$ -dependence. Since the left-hand side (LHS) is purely  $t$ -dependent, and the right-hand side (RHS) is purely  $r$ -dependent, then both sides must be equal to some constant. Since  $U_0 \neq 0$ , the only way to ensure that the RHS of Equation (154) is constant is to set,

$$\alpha = -1, \tag{155}$$

for which Equation (154) reduces to

$$\dot{\nu}^2 + \dot{\nu} = U_0^2. \tag{156}$$

By using Equation (156) with either the  $tt$  or  $rr$  equation, we arrive at

$$2U_0^2 = e^{2\nu_1} \left( \dot{\nu}_1^2 - U_0^2 \right), \tag{157}$$

which yields,

$$v_1(t) = \ln \left\{ \sqrt{2} \sinh[U_0(t_0 \pm t)] \right\}, \tag{158}$$

where  $t_0$  is an arbitrary constant. On the other hand, from Equation (153) we find

$$e^{2v} (\dot{v}^2 - U_0^2) = 2U_0^2(1 - rv'). \tag{159}$$

Comparing this to Equation (157), we find  $v_0(r) = \text{const.}$ , so that

$$v(t, r) = \ln \left\{ \sqrt{2} \sinh[U_0(t_0 \pm t)] + V_0 \right\}, \tag{160}$$

where  $V_0$  is a constant. Equations (150) and (160) satisfy all of the field equations, provided that  $V_0 = 0$ , with no other constraints on the remaining arbitrary constants. Thus, for the case  $c_{13} = 0$ , the solution is

$$\begin{aligned} \mu(r) &= \ln(U_0 r), \\ v(t, r) &= \ln \left[ \sqrt{2} \sinh(U_0(t_0 \pm t)) \right]. \end{aligned} \tag{161}$$

However, using the gauge freedom for the choice of  $t$ , we can always set  $U_0 = 1$  and  $t_0 = 0$ , so the metric finally takes the form,

$$ds^2 = -r^2 dt^2 + 2 \sinh^2(t) dr^2 + r^2 d^2\Omega. \tag{162}$$

Unfortunately, this solution is also excluded by the current observations, as Equations (149), (150), and (155) imply that

$$c_{14} = c_2 = -2. \tag{163}$$

### 2. $c_{13} \neq 0$

In this case we combine the  $tt$  and  $rr$  equations, and find

$$c_{123}(\dot{v}_1^2 + \ddot{v}_1) = -\frac{2U_0^2}{r^{2\alpha+2}}. \tag{164}$$

Substituting it into the  $tt$  equation, and then subtracting it from the  $\theta\theta$  equation, we find obtain,

$$2U_0^2 r v_0'(r) e^{-2v_0(r)} = e^{2v_1(t)} \left[ \frac{2c_{13}}{1 + c_{13}} U_0^2 \right]. \tag{165}$$

The right-hand is always different from zero, so the above equation holds only when  $v_1 = \text{const.}$  Then, from Equation (164) we find that the integration constant  $U_0$  must vanish. As a result, Equation (165) becomes an identity ( $0 = 0$ ). It can be shown that the rest of the Einstein-aether field equations will give the static solutions presented in the last subsection.

## 6. Conclusions

With the increasing interest of Einstein-aether theory in the recent years, in this paper we have studied spherically symmetric both static and time-dependent spacetimes in this theory, and found several exact solutions in closed forms. Such studies were carried out in three different coordinate systems: *the isotropic, Painlevé-Gullstrand, and Schwarzschild coordinates*, and in each of them exact solutions are found.

In particular, in the isotropic coordinates we have found a class of exact static solutions in closed forms, given by Equation (47), i.e.,

$$ds^2 = -\left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^q dt^2 + \frac{\left(1 + \frac{m}{2r}\right)^{q+2}}{\left(1 + \frac{m}{2r}\right)^{q-2}} \left[dr^2 + r^2(d^2\theta + \sin^2\theta d^2\phi)\right], \tag{166}$$

where

$$q \equiv 2\left(\frac{2}{2 - c_{14}}\right)^{1/2}. \tag{167}$$

Clearly, when  $c_{14} = 0$ , the above solution reduces to the Schwarzschild vacuum black hole solution but written in the isotropic coordinates, and the spacetime is free of spacetime curvature singularities at  $r = m/2$  [67].

However, as long as  $c_{14} \neq 0$  but satisfies the observational constraint (15), i.e.,

$$0 < c_{14} \leq 2.5 \times 10^{-5}, \tag{168}$$

the corresponding spacetime has several remarkable features:

- A minimal surface with non-zero area always exists at  $\bar{r} = \bar{r}_{\min}$ , given explicitly by Equation (54), the so-called throat of the spacetime. It smoothly connects two regions,  $\bar{r} \in (2m, \bar{r}_{\min}]$  and  $\bar{r} \in [\bar{r}_{\min}, \infty)$ , as schematically shown by Figure 1.
- The Kretschmann scalar always diverges at  $\bar{r} = 2m$  as long as  $c_{14} \neq 0$ , so a spacetime curvature singularity always appears. Despite the fact  $A(\bar{r} = 2m) = \infty$ , the proper radial distance between the throat and the singularity is always finite and non-zero [cf. Equation (55)].
- In the region  $\bar{r} \in [\bar{r}_{\min}, \infty)$ , the spacetime is asymptotically flat as  $\bar{r} \rightarrow \infty$ , and the proper radial distance between the throat and the spatial infinity  $\bar{r} = \infty$  is always infinitely large, so is the geometric area,  $A(\bar{r} = \infty) = \infty$ .
- The throat is only marginally trapped, as now  $\Theta_+ \Theta_-$  vanishes precisely only at the throat,  $\bar{r} = \bar{r}_{\min}$ , while away from it, we always have  $\Theta_+ \Theta_- < 0$ , as shown explicitly by Equation (58), where  $\Theta_{\pm}$  denote the expansions of the outgoing or ingoing null geodesic congruences.

With these remarkable features, it would be very interesting to consider other properties of the solution, including its stability against non-spherical perturbations and consistency with solar system tests [68] and the observations of the shadows of black holes [29]. In particular, in general relativity, in order to have a throat that connects two un-trapped regions, exotic matter (for example, the one that violates energy conditions) is often requested, which indicates some kind of instabilities [69]. In the present case, although such a kind of matter is provided by the aether field, it would be very interesting to show the stability of the throat against perturbations, especially against the non-spherical ones.

It would also be important to study the corresponding solutions in the context of the UV completion of the Einstein-aether theory, i.e., the non-projectable Hořava gravity. In particular it is intriguing to see whether and how the naked singularity present in the case of  $q \neq 2$  could be resolved or hidden behind a universal horizon.

In addition, to simplify mathematically the problems involved, in this paper we have considered only the cases in which the aether field is always comoving with the chosen coordinate systems. In general, the aether field can have radial motions, as long as it is time-like. It would be very interesting, if exact solutions with closed forms can be found in this case, too.

**Author Contributions:** The authors have equal contributions. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work was partially supported by the National Key Research and Development Program of China under the Grant No. 2020YFC2201503, the National Natural Science Foundation of China under the grant Nos. 11975203, the Japan Society for the Promotion of Science Grants-in-Aid

for Scientific Research No. 17H02890, No. 17H06359, and the World Premier International Research Center Initiative, MEXT, Japan.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Acknowledgments:** We would like to express our gratitude to Ted Jacobson for carefully reading our manuscript and valuable suggestions and comments.

**Conflicts of Interest:** The authors declare no conflict of interest.

### Appendix A. $G_{\mu\nu}$ and $T_{\mu\nu}^{\mathfrak{a}}$ in different Coordinate Systems

In this appendix, we shall present the non-vanishing components of the Einstein tensor  $G_{\mu\nu}$  and the effective energy-momentum tensor  $T_{\mu\nu}^{\mathfrak{a}}$  in three different coordinate systems. In all of them, the aether is assumed to be comoving with the coordinate systems, so we always have

$$u^\mu = \pm(-g_{tt})^{-1/2}\lambda_t^\mu, \tag{A1}$$

where the “+” sign means the aether field is moving along  $dt$  increasing direction, while the “−” sign corresponds to the case in which the aether field is moving along  $dt$  decreasing direction. From Equations (5) and (7), we can see that  $T_{\mu\nu}^{\mathfrak{a}}$  is independent of these choices, while  $\mathfrak{A}$  switches its sign. Clearly, these choices do not affect the field Equations (4) and (7). With these in mind, let us consider the three different coordinate systems.

#### Appendix A.1. Isotropic Coordinates

Choosing the gauge (19), the metric takes the form,

$$ds^2 = -e^{2\mu(r,t)}dt^2 + e^{2\nu(t,r)}(dr^2 + r^2d^2\Omega), \tag{A2}$$

where  $d^2\Omega \equiv d\theta^2 + \sin^2\theta d\phi^2$ . Introducing the quantity,

$$\Sigma = 3\dot{\nu}^2 + 2\ddot{\nu} - 2\dot{\mu}\dot{\nu}, \tag{A3}$$

we find that the non-zero and independent components of the Einstein tensor are,

$$G_{00} = 3\dot{\nu}^2 - e^{2\mu-2\nu}\left(\dot{\nu}^2 + 2\nu'' + 4\frac{\nu'}{r}\right), \tag{A4}$$

$$G_{01} = 2\mu'\dot{\nu} - 2\dot{\nu}', \tag{A5}$$

$$G_{11} = \dot{\nu}^2 + 2\mu'\nu' + \frac{2}{r}(\mu' + \nu') - \Sigma e^{-2\mu+2\nu}, \tag{A6}$$

$$G_{22} = r^2\left(\mu'^2 + \mu'' + \nu'' + \frac{\mu' + \nu'}{r} - \Sigma e^{-2\mu+2\nu}\right), \tag{A7}$$

and the non-zero and independent components of the aether stress-energy tensor are,

$$T_{00}^{\mathfrak{a}} = e^{2\mu-2\nu}\left[c_{14}\left(\frac{\mu'^2}{2} + \mu'\nu' + \mu'' + 2\frac{\mu'}{r}\right)\right] - \frac{3}{2}\beta\dot{\nu}^2, \tag{A8}$$

$$T_{01}^{\mathfrak{a}} = c_{14}(\dot{\mu}' + \mu'\dot{\nu}), \tag{A9}$$

$$T_{11}^{\mathfrak{a}} = \frac{\beta}{2}\Sigma e^{-2\mu+2\nu} - \frac{c_{14}}{2}\mu'^2, \tag{A10}$$

$$T_{22}^{\mathfrak{a}} = r^2\left(T_{11}^{\mathfrak{a}} + c_{14}\mu'^2\right), \tag{A11}$$

where  $\beta \equiv 3c_2 + c_{13}$ .

Appendix A.2. Painlevé-Gullstrand Coordinates

Choosing the gauge (89), and considering only the static spacetimes, we find that the metric takes the Painlevé-Gullstrand (PG) form,

$$ds^2 = -e^{2\mu(r)}dt^2 + 2e^{\nu(r)}drdt + dr^2 + r^2d\Omega^2. \tag{A12}$$

Setting

$$\lambda = e^{2\mu} + e^{2\nu}, \tag{A13}$$

we find that the non-zero components of the Einstein tensor are,

$$G_{00} = \frac{1}{\lambda^2 r^2} \left[ e^{4\mu+2\nu} (1 - 2r(\mu' + \nu')) + e^{2\mu+4\nu} \right], \tag{A14}$$

$$G_{01} = \frac{1}{\lambda^2 r^2} \left[ e^{2\mu+3\nu} (-1 + 2r(\mu' - \nu')) - e^{5\nu} \right], \tag{A15}$$

$$G_{11} = \frac{1}{\lambda^2 r^2} \left[ e^{2\mu+2\nu} (4r\mu' - 1) + 2e^{4\mu}r\mu' + e^{4\nu} (2r\nu' - 1) \right], \tag{A16}$$

$$G_{22} = \frac{1}{\lambda^2 r^2} \left[ e^{4\mu} (r\mu' + r^2\mu'^2 + r^2\mu'') + e^{2\mu+2\nu} (r(1 + r\mu') (2\mu' - \nu') + r^2\mu'') \right], \tag{A17}$$

while the non-zero components of the aether stress-energy tensor are,

$$T_{00}^{\text{ae}} = \frac{c_{14}}{2\lambda^2 r^2} \left[ e^{6\mu} (4\mu' + r\mu'^2 + 2r\mu'') + e^{4\mu+2\nu} (4\mu' + 3r\mu'^2 - 2r\mu'\nu' + 2r\mu'') \right], \tag{A18}$$

$$T_{01}^{\text{ae}} = \frac{c_{14}}{2\lambda^2 r^2} \left[ e^{2\mu+3\nu} (4\mu' + 3r\mu'^2 + 2r\mu'' - 2r\mu'\nu') - e^{4\mu+\nu} (4\mu' + r\mu'^2 + 2r\mu'\nu') \right], \tag{A19}$$

$$T_{11}^{\text{ae}} = \frac{c_{14}}{2\lambda^2 r^2} \left[ e^{4\nu} (2\mu' + r\mu'^2 + r\mu'') + e^{2\mu+2\nu} (4\mu' - r\mu'^2 + 2r\mu'\nu') - 4e^{4\mu}r\mu'^2 \right], \tag{A20}$$

$$T_{22}^{\text{ae}} = \frac{c_{14}e^{2\mu}r^2\mu'^2}{2\lambda}. \tag{A21}$$

Appendix A.3. Schwarzschild Coordinates

The Schwarzschild coordinates correspond to the choice (120), for which the metric takes the form,

$$ds^2 = -e^{2\mu(t,r)}dt^2 + e^{2\nu(t,r)}dr^2 + r^2d^2\Omega. \tag{A22}$$

We also define the quantities,

$$Q = \frac{\mu'^2}{2} - \mu'\nu' + \mu'', \tag{A23}$$

$$H = \frac{\dot{\nu}^2}{2} - \dot{\mu}\dot{\nu} + \ddot{\nu}. \tag{A24}$$

Then, the non-zero components of the Einstein tensor are,

$$G_{00} = \frac{1}{r^2} e^{2(\mu-\nu)} (e^{2\nu} + 2r\nu' - 1), \tag{A25}$$

$$G_{01} = \frac{2\dot{\nu}}{r}, \tag{A26}$$

$$G_{11} = \frac{1}{r^2} (1 - e^{2\nu} + 2r\mu'), \tag{A27}$$

$$G_{22} = r^2 \left\{ e^{-2\nu} \left[ Q + \left( \frac{\mu'^2}{2} - \frac{\nu' - \mu'}{r} \right) \right] - e^{-2\mu} \left( H + \frac{\dot{\nu}^2}{2} \right) \right\}, \tag{A28}$$

and the non-zero components of the aether stress-energy tensor are,

$$T_{00}^{\text{ae}} = e^{2\mu-2\nu} c_{14} \left( Q + \frac{2\mu'}{r} \right) - \frac{c_{123}}{2} \dot{\nu}^2, \quad (\text{A29})$$

$$T_{01}^{\text{ae}} = c_{14} (\dot{\mu}' - \mu' \dot{\nu}), \quad (\text{A30})$$

$$T_{11}^{\text{ae}} = c_{123} H e^{-2\mu+2\nu} - \frac{c_{14}}{2} \mu'^2, \quad (\text{A31})$$

$$T_{22}^{\text{ae}} = r^2 \left[ e^{-2\mu} \left( c_2 H - \frac{c_{13}}{2} \dot{\nu}^2 \right) - \frac{c_{14}}{2} e^{-2\nu} \mu'^2 \right]. \quad (\text{A32})$$

## Note

- <sup>1</sup> Here “comoving aether” means that the aether field is at rest in the chosen coordinates, so it has only the time-like component, while its spatial components vanish identically, i.e.,  $u^i = 0$  ( $i = 1, 2, 3$ ). When the spacetime is static, it aligns with the time-like killing vector,  $\xi^\mu = \delta_t^\mu$ .

## References

- Kostelecky, V.A.; Russell, N. Data Tables for Lorentz and CPT Violation. *Rev. Mod. Phys.* **2011**, *83*, 11. [\[CrossRef\]](#)
- Mattingly, D. Modern Tests of Lorentz Invariance. *Living Rev. Relat.* **2005**, *8*, 5. [\[CrossRef\]](#)
- Liberati, S. Tests of Lorentz invariance: A 2013 update. *Class. Quantum Grav.* **2013**, *30*, 133001. [\[CrossRef\]](#)
- Bourgoin, A.; Poncin-Lafitte, C.L.; Hees, A.; Bouquillon, S.; Francou, G.; Angonin, M.-C. Lorentz symmetry violations from matter-gravity couplings with lunar laser ranging. *Phys. Rev. Lett.* **2017**, *119*, 201102. [\[CrossRef\]](#)
- Flowers, N.A.; Goodge, C.; Tasson, J.D. Superconducting-gravimeter tests of local Lorentz invariance. *Phys. Rev. Lett.* **2017**, *119*, 201101. [\[CrossRef\]](#)
- Jacobson, T.; Mattingly, D. Gravity with a dynamical preferred frame. *Phys. Rev. D* **2001**, *64*, 024028. [\[CrossRef\]](#)
- Jacobson, T. Einstein-aether gravity: A status report. *arXiv* **2008**, arXiv:0801.1547.
- Hořava, P. Quantum gravity at a Lifshitz point. *Phys. Rev. D* **2009**, *79*, 084008. [\[CrossRef\]](#)
- Mukohyama, S. Hořava–Lifshitz cosmology: A review. *Class. Quantum Grav.* **2010**, *27*, 223101. [\[CrossRef\]](#)
- Wang, A. Hořava gravity at a Lifshitz point: A progress report. *Int. J. Mod. Phys. D* **2017**, *26*, 1730014. [\[CrossRef\]](#)
- Jacobson, T. Extended Hořava gravity and Einstein-aether theory. *Phys. Rev. D* **2017**, *81*, 101502. [\[CrossRef\]](#)
- Jacobson, T. Undoing the twist: The Hořava limit of Einstein-aether. *Phys. Rev. D* **2014**, *89*, 081501. [\[CrossRef\]](#)
- Blas, D.; Pujolas, O.; Sibiryakov, S. A healthy extension of Horava gravity. *Phys. Rev. Lett.* **2010**, *104*, 181302. [\[CrossRef\]](#)
- Blas, D.; Pujolas, O.; Sibiryakov, S. Models of non-relativistic quantum gravity, the good, the bad and the healthy. *J. High Energy Phys.* **2011**, *4*, 018. [\[CrossRef\]](#)
- Jacobson, T.; Mattingly, D. Einstein-aether waves. *Phys. Rev. D* **2004**, *70*, 024003. [\[CrossRef\]](#)
- Elliott, J.W.; Moore, G.D.; Stoica, H. Constraining the New Aether: Gravitational Cherenkov radiation. *J. High Energy Phys.* **2005**, *8*, 066. [\[CrossRef\]](#)
- Abbott, B. et al. [Virgo, LIGO Scientific Collaboration]. GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral. *Phys. Rev. Lett.* **2017**, *119*, 161101. [\[CrossRef\]](#)
- Abbott, B.P. et al. [Virgo, Fermi-GBM, INTEGRAL, LIGO Scientific Collaboration]. Gravitational Waves and Gamma-rays from a Binary Neutron Star Merger, GW170817 and GRB 170817A. *Astrophys. J.* **2017**, *848*, L13. [\[CrossRef\]](#)
- Oost, J.; Mukohyama, S.; Wang, A. Constraints on Einstein-aether theory after GW170817. *Phys. Rev. D* **2018**, *97*, 124023. [\[CrossRef\]](#)
- Eling, C.; Jacobson, T. Spherical Solutions in Einstein-Aether Theory: Static Aether and Stars. *Class. Quantum. Grav.* **2006**, *23*, 5625. [\[CrossRef\]](#)
- Berglund, P.; Bhattacharyya, J.; Mattingly, D. Mechanics of universal horizons. *Phys. Rev. D* **2012**, *85*, 124019. [\[CrossRef\]](#)
- Berglund, P.; Bhattacharyya, J.; Mattingly, D. Towards Thermodynamics of Universal Horizons in Einstein-aether Theory. *Phys. Rev. Lett.* **2013**, *110*, 071301. [\[CrossRef\]](#) [\[PubMed\]](#)
- Gao, C.; Shen, Y. Static spherically symmetric solution of the Einstein-aether theory. *Phys. Rev. D* **2013**, *88*, 103508. [\[CrossRef\]](#)
- Ding, C.; Wang, A.; Wang, X. Charged Einstein-aether black holes and Smarr formula. *Phys. Rev. D* **2015**, *92*, 084055. [\[CrossRef\]](#)
- Ding, C.; Liu, C.; Wang, A.; Jing, J. Three-dimensional charged Einstein-aether black holes and the Smarr formula. *Phys. Rev. D* **2016**, *94*, 124034. [\[CrossRef\]](#)
- Lin, K.; Ho, F.-H.; Qian, W.-L. Charged Einstein-aether black holes in n-dimensional spacetime. *Int. J. Mod. Phys. D* **2019**, *28*, 1950049. [\[CrossRef\]](#)
- Ding, C.; Wang, A. Thermodynamical study on universal horizons in higher D-dimensional spacetime and aether waves. *Phys. Rev. D* **2019**, *99*, 124011. [\[CrossRef\]](#)

28. Oost, J. Observational Constraints, Exact Plane Wave Solutions, and Exact Spherical Solutions in æ-Theory. Available online: [http://inspirehep.net/record/1778817/files/2104\\_10724.pdf](http://inspirehep.net/record/1778817/files/2104_10724.pdf) (accessed on 26 July 2021).
29. Azreg-Aïnou, M.; Chen, Z.-H.; Deng, B.-J.; Jamil, M.; Zhu, T.; Wu, Q.; Lim, Y.-K. Orbital mechanics for, and QPOs' resonances in, black holes of Einstein-æther theory. *Phys. Rev. D* **2020**, *102*, 044028. [[CrossRef](#)]
30. Chan, R.; da Silva, M.F.A.; Satheeshkumar, V.H. Existence of new singularities in Einstein-aether theory. *J. Cosmol. Astropart. Phys.* **2020**, *5*, 025. [[CrossRef](#)]
31. Chan, R.; da Silva, M.F.A.; Satheeshkumar, V.H. Spherically Symmetric Analytic Solutions and Naked Singularities in Einstein-Aether Theory. *arXiv* **2003**, arXiv:2003.00227.
32. Churilova, M.S. Black holes in Einstein-aether theory: Quasinormal modes and time-domain evolution. *Phys. Rev. D* **2020**, *102*, 024076. [[CrossRef](#)]
33. Khodadi, M.; Saridakis, E.N. Einstein-aether Gravity in the light of Event Horizon Telescope Observations of M87. *arXiv* **2012**, arXiv:2012.05186.
34. Rayimbaev, J.; Abdurjabbarov, A.; Jamil, M.; Han, W.-B. Dynamics of magnetized particles around Einstein-aether black hole with uniform magnetic field. *Nucl. Phys. B* **2021**, *966*, 115364. [[CrossRef](#)]
35. Eling, C.; Jacobson, T. Black holes in Einstein-aether theory. *Class. Quantum Grav.* **2006**, *23*, 5643. [[CrossRef](#)]
36. Eling, C.; Jacobson, T.; Miller, M.C. Neutron stars in Einstein-aether theory. *Phys. Rev. D* **2007**, *76*, 042003. [[CrossRef](#)]
37. Tamaki, T.; Miyamoto, U. Generic features of Einstein-Aether black holes. *Phys. Rev. D* **2008**, *77*, 024026. [[CrossRef](#)]
38. Blas, D.; Sibiryakov, S. Hořava gravity versus thermodynamics: The black hole case. *Phys. Rev. D* **2011**, *84*, 124043. [[CrossRef](#)]
39. Barausse, E.; Jacobson, T.; Sotiriou, T.P. Black holes in Einstein-aether and Hořava-Lifshitz gravity. *Phys. Rev. D* **2011**, *83*, 124043. [[CrossRef](#)]
40. Zhu, T.; Wu, Q.; Jamil, M.; Jusufi, K. Shadows and deflection angle of charged and slowly rotating black holes in Einstein-Æther theory. *Phys. Rev. D* **2019**, *100*, 044055. [[CrossRef](#)]
41. Garfinkle, D.; Eling, C.; Jacobson, T. Numerical simulations of gravitational collapse in Einstein-aether theory. *Phys. Rev. D* **2007**, *76*, 024003. [[CrossRef](#)]
42. Lin, K.; Goldoni, O.; da Silva, M.F.; Wang, A. New look at black holes: Existence of universal horizons. *Phys. Rev. D* **2015**, *91*, 024047. [[CrossRef](#)]
43. Zhang, C.; Zhao, X.; Lin, K.; Zhang, S.-J.; Zhao, W.; Wang, A. Spherically symmetric static black holes in Einstein-aether theory. *Phys. Rev. D* **2020**, *102*, 064043. [[CrossRef](#)]
44. Bhattacharjee, M.; Mukohyama, S.; Wan, M.-B.; Wang, A. Gravitational collapse and formation of universal horizons in Einstein-aether theory. *Phys. Rev. D* **2018**, *98*, 064010. [[CrossRef](#)]
45. Cropp, B.; Liberati, S.; Visser, M. Surface gravities for non-Killing horizons. *Class. Quantum Grav.* **2013**, *30*, 125001. [[CrossRef](#)]
46. Saravani, M.; Afshordi, N.; Mann, R.B. Dynamical emergence of universal horizons during the formation of black holes. *Phys. Rev. D* **2014**, *89*, 084029. [[CrossRef](#)]
47. Eling, C.; Oz, Y. Horava-Lifshitz black hole hydrodynamics. *J. High Energy Phys.* **2014**, *11*, 067. [[CrossRef](#)]
48. Bhattacharyya, J.; Mattingly, D. Universal horizons in maximally symmetric spaces. *Int. J. Mod. Phys. D* **2014**, *23*, 1443005. [[CrossRef](#)]
49. Lin, K.; Abdalla, E.; Cai, R.-G.; Wang, A. Universal horizons and black holes in gravitational theories with broken Lorentz symmetry. *Int. J. Mod. Phys. D* **2014**, *23*, 1443004. [[CrossRef](#)]
50. Shu, F.-W.; Lin, K.; Wang, A.; Wu, Q. Lifshitz spacetimes, solitons, and generalized BTZ black holes in quantum gravity at a Lifshitz point. *J. High Energy Phys.* **2014**, *4*, 056. [[CrossRef](#)]
51. Lin, K.; Shu, F.-W.; Wang, A.; Wu, Q. High-dimensional Lifshitz-type spacetimes, universal horizons, and black holes in Hořava-Lifshitz gravity. *Phys. Rev. D* **2015**, *91*, 044003. [[CrossRef](#)]
52. Michel, F.; Parentani, R. Black hole radiation in the presence of a universal horizon. *Phys. Rev. D* **2015**, *91*, 124049. [[CrossRef](#)]
53. Blas, D.; Lim, E. Phenomenology of theories of gravity without Lorentz invariance: The preferred frame case. *Int. J. Mod. Phys. D* **2015**, *23*, 1443009. [[CrossRef](#)]
54. Misonoh, Y.; Maeda, K.-I. Black holes and Thunderbolt singularities with Lifshitz scaling terms. *Phys. Rev. D* **2015**, *92*, 084049. [[CrossRef](#)]
55. Tian, M.; Wang, X.-W.; da Silva, M.F.; Wang, A. Gravitational collapse and formation of universal horizons. *arXiv* **2015**, arXiv:1501.04134.
56. Ding, C.; Wang, A.; Wang, X.-W.; Zhu, T. Hawking radiation of charged Einstein-aether black holes at both Killing and universal horizons. *Nucl. Phys. B* **2016**, *913*, 694. [[CrossRef](#)]
57. Liberati, S.; Pacilio, C. Smarr formula for Lovelock black holes: A Lagrangian approach. *Phys. Rev. D* **2016**, *93*, 084044. [[CrossRef](#)]
58. Lin, K.; Satheeshkumar, V.H.; Wang, A. Static and rotating universal horizons and black holes in gravitational theories with broken Lorentz invariance. *Phys. Rev. D* **2016**, *93*, 124025. [[CrossRef](#)]
59. Maciel, A. Quasilocal approach to general universal horizons. *Phys. Rev. D* **2016**, *93*, 104013. [[CrossRef](#)]
60. Lin, K.; Mukohyama, S.; Wang, A.; Zhu, T. No static black hole hairs in gravitational theories with broken Lorentz invariance. *Phys. Rev. D* **2017**, *95*, 124053. [[CrossRef](#)]
61. Gasperini, M. Singularity prevention and broken Lorentz symmetry. *Class. Quantum Grav.* **1987**, *4*, 485. [[CrossRef](#)]

62. Kostelecký, V.A.; Samuel, S. Spontaneous breaking of Lorentz symmetry in string theory. *Phys. Rev. D* **1989**, *39*, 683. [[CrossRef](#)] [[PubMed](#)]
63. Bailey, Q.G.; Kostelecký, V.A. Signals for Lorentz violation in post-Newtonian gravity. *Phys. Rev. D* **2006**, *74*, 045001. [[CrossRef](#)]
64. Baumgarte, T.W.; Shapiro, S.L. *Numerical Relativity: Solving Einstein's Equations on the Computer*; Cambridge University Press: Cambridge, UK, 2010.
65. Hawking, S.W.; Ellis, G.F.R. *The Large Scale Structure of Spacetime*; Cambridge University Press: Cambridge, UK, 1973.
66. Seifert, M.D. Stability of spherically symmetric solutions in modified theories of gravity. *Phys. Rev. D* **2007**, *76*, 064002. [[CrossRef](#)]
67. D'Inverno, R. *Introducing Einstein's Relativity*; Clarendon Press: Oxford, UK, 2003.
68. Zhu, T.; Wang, A. Observational tests of the self-dual spacetime in loop quantum gravity. *Phys. Rev. D* **2020**, *102*, 124042. [[CrossRef](#)]
69. Visser, M. *Lorentzian Wormholes: From Einstein to Hawking*; American Institute of Physics: Melville, NY, USA, 1996.