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Gravity Models with Nonlinear Symmetry Realization

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Abstract: Validity of three gravity models with non-linear realization of conformal symmetry previously discussed in literature is addressed. Two models are found to be equivalent up to a change of coset coordinates. It was found that models contain ghost degrees of freedom that may be excluded by an introduction of an additional symmetry to the target space. One model found to be safe in early universe. The others found to lack spin-2 degrees of freedom and to have peculiar coupling to matter degrees of freedom.

Keywords: non-linear symmetry realization; modified gravity; conformal symmetry

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1. Introduction

Conformal symmetry occupies a special place in gravity physics due to the well-known Ogievetsky theorem [1]. The theorem states that any generator of the infinitely-dimensional coordinate transformation group is presented as series of commutators of generators from the conformal group $C(1,3)$ and the affine one $A(4)$. Therefore, the conformal group is strongly related with coordinate transformations and should have a certain influence on the structure of a gravity theory.

Particular implementations of conformal symmetry for gravity models were studied in multiple papers, so we only mention a few key results. In paper [2], it was shown that any metric gravity theory can be viewed as a theory with a combined non-linear realization of conformal and affine symmetry (see also [3–6]). Explicit non-linear realization of the conformal symmetry within AdS/CFT correspondence was constructed in [7]. It also should be mentioned that there are models with a linear realization of the conformal symmetry [8,9], but their applicability is debatable [10–13].

There are a few physical reasons to study models with non-linear realizations of the conformal symmetry. Early Universe is naturally associated with the conformal symmetry. Firstly, it is reasonable to expect the early state of the Universe to have Planck scale temperature. Because of this all conceivable particles can be considered as massless and the conformal symmetry will emerge naturally. It is also possible to make a realistic case for a scenario with a conformal fixed point reached in the early Universe [14–16]. An alternative evidence supporting this reasoning comes from the inflation theory. If an inflation is driven by a scalar field potential then the potential has an area where it is approximately flat to be consistent with the slow-roll scenario. However, in this area the potential will admit a scalar field shift symmetry $\phi \to \phi + c$ which, in turn, excludes relevant dimensional parameters and enforces the conformal symmetry on the model.

During the universe evolution the conformal symmetry will inevitable be spontaneously broken down to the Poincare group. It is known that spontaneous symmetry breaking and non-linear symmetry realization are the same thing (except for a few very special cases that are irrelevant within the context of this paper). Therefore, it is natural to study models with non-linear realizations of the conformal symmetry that they are expected to appear natural in the context of a cosmological evolution originated from the conformal phase.

This framework points on an interesting opportunity to obtain a united description of both inflationary and post-inflationary expansion within a single model with a non-linear realized conformal symmetry. In full accordance with the Goldstone theorem, if the conformal symmetry is broken the corresponding model develops new massive and massless degrees of freedom (DoF). If the conformal symmetry is broken down to the Lorentz group of dimension ten one can try to associate ten corresponding massless Goldstone modes with ten components of the metric tensor. The massive components, in turn, can be associated with the inflation field. The corresponding mass, in turn, will define the inflation scale.

This reasoning highlights two perspective research direction. The first one is to search for models with a non-linear realization of the conformal symmetry which contain at least one scalar DoF and to verify its possibility to drive inflation. The other direction is to check if a given model with a non-linear symmetry realization has massless spin-2 degrees of freedom that could be associated with gravitons.

The main goal of this paper is to examine three particular models with the non-linear conformal symmetry realization presented in [17]. It must be noted that the manifold of models with a non-linear conformal symmetry realization is extremely wide [7,18–20]. The discussed models are chosen, firstly, for their simplicity and, secondly, because their scrutiny was already begun in [21].

In paper [17] three models were presented. The first one is defined by the Lagrangian:

$$\begin{aligned}\mathcal{L}_I = {} & \frac{1}{2}\left[1+\frac{\sigma^2}{\varepsilon^2}\left\{f_2\left(\frac{\psi}{\varepsilon}\right)\right\}^2\right]\eta^{\mu\nu}\,\partial_\mu\psi\,\partial_\nu\psi+\frac{1}{2}\left[f_1\left(\frac{\psi}{\varepsilon}\right)\right]^2\eta^{\mu\nu}\,\eta_{(\alpha)(\beta)}\,\partial_\mu\sigma^{(\alpha)}\,\partial_\nu\sigma^{(\beta)} \\ & -\frac{1}{\varepsilon}f_1\left(\frac{\psi}{\varepsilon}\right)f_2\left(\frac{\psi}{\varepsilon}\right)\eta^{\mu\nu}\,\partial_\mu\psi\,\sigma^{(\alpha)}\partial_\nu\sigma^{(\beta)}\,\eta_{(\alpha)(\beta)}.\end{aligned} \tag{1}$$

Here, ψ and $\sigma^{(\alpha)}$ ($(\alpha)=0,\cdots,4$) are scalar fields associated with target space coordinates on which the conformal symmetry acts non-linearly. Field indices $\sigma^{(\alpha)}$ taken in brackets should not be confused with Lorentz indices. All fields have the canonical mass dimension and ε is a mass parameter corresponding to the conformal symmetry breaking scale. Finally, functions $f_i(x)$ are

$$\begin{aligned}f_1(x) &= \frac{e^x-1}{x}, \\ f_2(x) &= \frac{e^x-x-1}{x^2}.\end{aligned} \tag{2}$$

Therefore, the model (1) describes five scalar fields propagating in a flat space-time. Fields $\sigma^{(\alpha)}$ have the Minkowski space as the target space so their indices are contracted with the Minkowski target metric $\eta_{(\alpha)(\beta)}$.

A comment of a derivation of this model is due. Conformal group admits the following generators $L_{(\mu)(\nu)}$, $P_{(\mu)}$, $R_{(\mu)(\nu)}$, $K_{(\mu)}$, and D. The first two correspond to Lorentz transformations and coordinate shifts and constitute the Poincare algebra; the former three operators extend the Poincare algebra. Dynamical variables of model (1) are subjected to the following non-linear symmetry realization

$$\exp\left[\frac{i}{2}\theta^{(\mu)(\nu)}L_{(\mu)(\nu)} + i\,\theta D + i\theta^{(\mu)}K_{(\mu)}\right]\,\exp\left[i\phi D + i\sigma^{(\alpha)}K_{(\alpha)}\right] \quad (3)$$
$$= \exp\left[i\phi' D + i\sigma'^{(\alpha)}K_{(\alpha)}\right]\exp\left[\frac{i}{2}u^{(\mu)(\nu)}L_{(\mu)(\nu)}\right].$$

Here, ϕ is related with ψ as $\psi = \varepsilon\phi$; $\theta^{(\mu)(\nu)}$, $\theta^{(\mu)}$, and θ are transformation parameters; $u^{(\mu)(\nu)} = u^{(\mu)(\nu)}(\theta, \theta^{(\alpha)}, \theta^{(\alpha)(\beta)})$ are parameters of the associated Lorentz group transformations. In other words, this equitation defined a non-linear action of the conformal group on variables ψ and σ while $u^{(\mu)(\nu)}$ is responsible for a non-linear action of the conformal group on all objects subjected to the Lorentz group.

Let us put a special emphasis on the fact that the conformal symmetry acts on all objects subjected to Lorentz transformations, but its action is indistinguishable from the standard Lorentz group action. This allows one to include the regular matter in the model without an explicit violation of the conformal symmetry. This fact also holds for gravity. One of the cornerstones of gravitational theory is the equivalence between curved geometry and physical force. Usually this equivalence is used to justify a usage of geometric quantities, such as the Riemann tensor, for a description of gravity. However the equivalence works in both directions, so we can righteously consider gravity as a theory of a physical field $h_{\mu\nu}$ propagating about the flat spacetime. We will return to this issue in the next section and discuss it in more details.

In summary, (1) provides a model of five scalar fields subjected to a non-linear realization of the conformal symmetry. The model can be extended with the regular matter including gravity in a way consistent with the non-linear symmetry realization. The present scalar degrees of freedom serve as natural candidates for inflaton field and we will address this opportunity further. At the same time the model has no DoF that can be associated with gravitons, so we will discuss this model in the context of inflation only.

The second model has the following Lagrangian

$$\mathcal{L} = \eta^{\mu\nu}\nabla_\mu h^{(\alpha)(\beta)}\,\nabla_\nu h^{(\rho)(\sigma)}\,\eta_{(\alpha)(\rho)}\,\eta_{(\beta)(\sigma)}, \quad (4)$$

where the covariant derivatives are

$$\frac{i}{2}\nabla_\mu h^{(\alpha)(\beta)} = \frac{i}{2}\partial_\mu h^{(\alpha)(\beta)} + \sum_{n=1}^{\infty}\frac{i}{(2n+1)!}\left(\mathrm{ad}_h^{2n-1}\,h\partial_\mu h\right)^{(\alpha)(\beta)} \quad (5)$$
$$= \frac{i}{2}\left[\partial_\mu h^{(\alpha)(\beta)} - \eta_{(\nu)(\sigma)}\eta_{(\mu)(\lambda)}\left(\frac{1}{3}h^{(\alpha)(\nu)}\,h^{(\beta)(\lambda)}\partial_\mu h^{(\sigma)(\mu)} - \frac{1}{3}h^{(\alpha)(\nu)}\,h^{(\sigma)(\mu)}\partial_\mu h^{(\lambda)(\beta)}\right)\right. \quad (6)$$
$$\left. + \mathcal{O}(h^5)\right].$$

Here, all repeated bracket indices are contracted with the Minkowski metric of the target space and ad operator is defined as:

$$\left(\mathrm{ad}_h\,\partial_\mu h\right)^{(\mu)(\nu)} = [h, \partial_\mu h]^{(\mu)(\nu)} = h^{(\mu)(\rho)}\partial_\mu h^{(\sigma)(\nu)}\eta_{(\rho)(\sigma)} - \partial_\mu h^{(\mu)(\rho)}h^{(\sigma)(\nu)}\eta_{(\rho)(\sigma)}. \quad (7)$$

Degrees of freedom $h_{(\mu)(\nu)}$ are associated with the coset coordinates and their transformation under the non-linear conformal symmetry action is defined by the following formula:

$$\exp\left[\frac{i}{2}\theta^{(\mu)(\nu)}L_{(\mu)(\nu)}\right]\exp\left[\frac{i}{2}h^{(\mu)(\nu)}R_{(\mu)(\nu)}\right] \\ =\exp\left[\frac{i}{2}h'^{(\mu)(\nu)}R_{(\mu)(\nu)}\right]\exp\left[\frac{i}{2}u^{(\mu)(\nu)}L_{(\mu)(\nu)}\right]. \tag{8}$$

Here, $\theta^{(\mu)(\nu)}$ are transformation parameters, and $u^{(\mu)(\nu)}$ realize a non-linear conformal group action on the Lorentz group.

Unlike the previous case the model has no natural candidates for an inflaton field, but $h_{(\mu)(\nu)}$ appear to be similar to small metric perturbations. In this paper, we consider such an equivalence and argue that despite their similarity they cannot be directly associated with gravitons.

Finally, the last model discussed in this paper in given by the following Lagrangian:

$$\mathcal{L}=\eta^{\mu\nu}\nabla_\mu h^{(\alpha)(\beta)}\,\nabla_\nu h^{(\rho)(\sigma)}\,\eta_{(\alpha)(\rho)}\,\eta_{(\beta)(\sigma)}+\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{9}$$

The same definition of covariant derivatives is used, and ϕ is a scalar. The model is similar to the previous one as it is constructed on the following non-linear realization:

$$\exp\left[\frac{i}{2}\theta^{(\mu)(\nu)}L_{(\mu)(\nu)}\right]\exp\left[\frac{i}{2}h^{(\mu)(\nu)}R_{(\mu)(\nu)}+i\,\phi D\right] \\ =\exp\left[\frac{i}{2}h'^{(\mu)(\nu)}R_{(\mu)(\nu)}+i\,\phi' D\right]\exp\left[\frac{i}{2}u^{(\mu)(\nu)}L_{(\mu)(\nu)}\right]. \tag{10}$$

The formula shows that model (9) use a wider coset than (4). Coset of model (4) is founded on $R_{(\mu)(\nu)}$ operators while coset of (9) is founded on $R_{(\mu)(\nu)}$ and D operators. As it was shown in the previous paper [17], the scalar ϕ has the trivial covariant derivative $\nabla_\mu\phi=\partial_\mu\phi$ so it does not require any coupling neither to the regular matter nor to $h_{(\mu)(\nu)}$. Note that such a coupling can be introduced in the model manually, but this case lies beyond the scope of this paper. In order to highlight the fact that ϕ does not require any coupling we will call it as a sterile scalar.

Similarly to the previous case, the model has no obvious candidates for an inflaton field. Therefore, we will only study an opportunity to associate $h^{(\mu)(\nu)}$ with the gravitational DoF. However, as we will show further, models (4) and (9) are equivalent. Because of this we will mainly focus on model (4).

The paper is split in two parts. In the first part we will discuss cosmological behavior of model (1) as it is the only model with a suitable candidates for inflatons. In the second part, we will study DoF of models (4) and (9) in order to establish if $h_{(\mu)(\nu)}$ can be associated with gravitons. Following this logic the paper is organized as follows. In Section 2, we discuss cosmological regimes described by the model (1) and show that scalar degrees of freedom are decoupled. Therefore, they do not influence the cosmological behavior in a meaningful way. We discuss a possible relation between this phenomenon and the conformal symmetry together with implications for realistic cosmological scenarios. In Section 3, we discuss the field content of models (4) and (9). It is shown that these models actually equivalent up to a coordinate redefinition on the target space. At the same time these models contain vector ghost degrees of freedom. We argue that these degrees of freedom can be excluded via an introduction of an additional symmetry to the target space. However, this symmetry should be agreed with the non-linear realization of the conformal symmetry which may influence the used non-linear realization in a meaningful way. Section 4 contains our conclusions which extend previous results [21]. Finally, in Appendix A we briefly show that DoF of the second model is also coupled to matter degrees of freedom in a peculiar way. We provide an expression for a covariant derivative of a vector field (A3) which defines such a coupling.

2. Cosmological Behavior

To study the cosmological behavior of the model (1) one has to introduce gravitational degrees of freedom because the number of DoF in (1) is not enough to describe gravitons. This can be consistently done because of the following.

Let us examine the non-linear realization of the conformal symmetry (3). The given formula defines transformations of ϕ and $\sigma^{(\mu)}$ under the non-linear symmetry action

$$\begin{aligned} \phi &\to \phi'(\phi,\theta,\theta^{(\mu)},\theta^{(\mu)(\nu)}), \\ \sigma^{(\alpha)} &\to \sigma^{(\alpha)}(\sigma,\theta^{(\mu)},\theta^{(\mu)(\nu)}) . \end{aligned} \tag{11}$$

However, it also defines parameter of Lorentz transformations (i.e., linear action of the Lorentz group) through which the conformal symmetry acts on all objects subjected to the Lorentz transformations:

$$u^{(\alpha)(\beta)} = u^{(\alpha)(\beta)}(\theta,\theta^{(\mu)},\theta^{(\mu)(\nu)}) . \tag{12}$$

This allows one to extend model (1) with arbitrary matter.

Gravity can be introduced this way and be consistent with the conformal symmetry. Due to the equivalence principle gravity can be considered either as a force (acting in a flat background spacetime) and as a geometry of a spacetime. This equivalence allows one to treat gravity as a gauge theory of symmetric tensor $h_{\mu\nu}$ in a flat spacetime. The gauge symmetry ensures that the corresponding action has an infinite number of terms and they can be rearranged in geometrical quantities such as the scalar curvature R. Such an approach to gravity description is more suitable for weak gravitational field when spacetime perturbations are weak and one can only account for a few leading terms in the action. However, it is also consistent with non-perturbative phenomena, such as the cosmological expansion. In that case the gravitational field $h_{\mu\nu}$ should be considered excited in all points of spacetime and one would be forced to deal with the whole infinite number of terms of the action. The geometric approach to gravity simply provides a tool to summarize this infinite series (to geometric quantities) and to obtain equations with a finite number of terms.

These reasons provide us with the following opportunity to introduce gravity consistent with the non-linear symmetry realization. One starts with general relativity given in a perturbative framework. This means that we consider general relativity action as an action with an infinite number of terms that describe gauge field $h_{\mu\nu}$ about a flat spacetime. Because of the equivalence principle this action is viewed only as a particular parameterization of general relativity. The gauge field $h_{\mu\nu}$ is subjected to the standard Lorentz transformations with parameters $u^{\mu\nu}$. In order to subject the theory to the non-linear conformal group action (3) one simply replace the standard Lorentz group parameters $u^{\mu\nu}$ with those obtained from (3). Therefore, each element of the conformal group is mapped on an element of the Lorentz group with the mapping given by (3). Hence, all quantities invariant with respect to the Lorentz group (including the scalar curvature R) are made to be invariant under the non-linear action of the conformal group. Finally, let us highlight that such a relation does take place only because the non-linear group action (3) also spawns a linear Lorentz group action through which it can act on the regular matter.

Because of the discussed reasons one can use the following model to study the cosmological expansions:

$$\mathcal{S} = \int d^4x \sqrt{-g} \Bigg\{ -\frac{2}{\kappa^2} R + \frac{1}{2} \left[1 + \frac{\sigma^2}{\varepsilon^2} \left\{ f_2\left(\frac{\psi}{\varepsilon}\right) \right\}^2 \right] g^{\mu\nu} \, \partial_\mu \psi \, \partial_\nu \psi \\ + \frac{1}{2} \left[f_1\left(\frac{\psi}{\varepsilon}\right) \right]^2 g^{\mu\nu} \, \eta_{(\alpha)(\beta)} \, \partial_\mu \sigma^{(\alpha)} \, \partial_\nu \sigma^{(\beta)} - \frac{1}{\varepsilon} f_1\left(\frac{\psi}{\varepsilon}\right) f_2\left(\frac{\psi}{\varepsilon}\right) g^{\mu\nu} \, \partial_\mu \psi \, \sigma^{(\alpha)} \partial_\nu \sigma^{(\beta)} \, \eta_{(\alpha)(\beta)} \Bigg\}, \tag{13}$$

where κ is related with the Newton constant G_N as $\kappa^2 = 32\pi G_N$. Fields ψ and $\sigma^{(\alpha)}$ transform under the non-linear conformal group action with the target space remaining flat.

Let us consider the cosmological behavior of (13) with the open Friedmann space-time:

$$ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu = dt^2 - a^2(t) \Big(dx^2 + dy^2 + dz^2 \Big). \tag{14}$$

Here, $a(t)$ is the scale factor. The corresponding Einstein equations read

$$G_{\mu\nu} = \frac{\kappa^2}{4} C_{\mu\nu}{}^{\alpha\beta} \Bigg[\frac{1}{2} \left[1 + \frac{\sigma^2}{\varepsilon^2} \left\{ f_2\left(\frac{\psi}{\varepsilon}\right) \right\}^2 \right] \partial_\alpha \psi \, \partial_\beta \psi + \frac{1}{2} \left[f_1\left(\frac{\psi}{\varepsilon}\right) \right]^2 \eta_{(\mu)(\nu)} \, \partial_\alpha \sigma^{(\mu)} \, \partial_\beta \sigma^{(\nu)} \\ - \frac{1}{\varepsilon} f_1\left(\frac{\psi}{\varepsilon}\right) f_2\left(\frac{\psi}{\varepsilon}\right) \partial_\alpha \psi \, \sigma^{(\mu)} \partial_\beta \sigma^{(\nu)} \, \eta_{(\mu)(\nu)} \Bigg] \tag{15}$$

where

$$C_{\mu\nu}{}^{\alpha\beta} \stackrel{\text{def}}{=} \delta^\alpha_\mu \delta^\beta_\nu + \delta^\beta_\mu \delta^\alpha_\nu - g_{\mu\nu} g^{\alpha\beta} \,. \tag{16}$$

These equations have two non-vanishing components which can be reduced to:

$$-3\frac{\ddot{a}}{a} = \frac{1}{4}\kappa^2 \left(\frac{1}{2} \left[1 + \frac{\sigma^2}{\varepsilon^2} \left\{ f_2\left(\frac{\psi}{\varepsilon}\right) \right\}^2 \right] \dot{\psi}^2 + \left[f_1\left(\frac{\psi}{\varepsilon}\right) \right]^2 \dot{\sigma}^{(\alpha)} \dot{\sigma}_{(\alpha)} - \frac{2}{\varepsilon} f_1\left(\frac{\psi}{\varepsilon}\right) f_2\left(\frac{\psi}{\varepsilon}\right) \dot{\psi} \, \dot{\sigma}_{(\alpha)} \sigma^{(\alpha)} \right), \\ 2\dot{a}^2 + a\ddot{a} = 0 \,. \tag{17}$$

The second equation from (17) does not contain scalar fields and describes the behavior of the scalar factor by itself. It can be solved analytically with the result:

$$a(t) = c_2 (c_1 + 3\,t)^{\frac{1}{3}} \,, \tag{18}$$

where c_1 and c_2 are the integration constants defined by the boundary conditions.

Now, we analyze the result (18). First of all we note that here the universe has only a decelerated expansion hence the model has no room for an inflationary phase.

Secondly, one can assume that the matter content of the model (i.e., the scalar fields) is described by the standard equation of state (EoS) $p = w\rho$. Here, p is a pressure of scalar fields, ρ is an energy density of the matter, and w is the EoS parameter. Solutions with such EoS are well known [22,23], so one can easily restore w from the form of asymptotic of (18). At large values of time the scale factor is proportional to $t^{\frac{1}{3}}$ corresponding to EoS parameter $w = 1$. That is why, despite the fact that scalar fields in this model admit vanishing masses, their behavior do not correspond to a relativistic matter one with EoS parameter $w = 1/3$. The reason is that the discussed scalar fields have a non-trivial interaction sector which influence their EoS.

Finally, the fact that the model admits a decelerating solution deserves a special attention as it has ghost degrees of freedom. These ghost degrees of freedom appear due to the metric on scalar field target space $\eta_{(\mu)(\nu)}$:

$$g^{\mu\nu}\,\eta_{(\alpha)(\beta)}\,\partial_\mu\sigma^{(\alpha)}\,\partial_\nu\sigma^{(\beta)} = \partial^\mu\sigma^{(0)}\,\partial_\mu\sigma^{(0)} - \sum_{i=1}^{3}\partial^\mu\sigma^{(i)}\,\partial_\mu\sigma^{(i)}\,. \tag{19}$$

Note that these ghost degrees of freedom do not manifest themselves at the level of cosmological solutions.

Despite the fact the analytical solution (18) can be obtained, there are no reasons to believe that it is stable. Equations (17) take a simple form due to a cancellation of the stress-energy tensor of scalar fields. This cancellation, in turn, is possible because both metric and scalar fields depend only on the time variable. As soon as one considers metric and scalar field perturbations propagating around the background a similar cancellation becomes impossible.

The existence of ghost degrees of freedom obstructs possible implementations of model (13). Let us discuss possible opportunities to exclude ghosts. The first opportunity is to use the inverse Higgs mechanism [24]. Unfortunately, this procedure is not applicable in the considered case because the discussed non-linear symmetry realization does no satisfy the necessary criteria. Another opportunity to exclude ghosts is to introduce an additional symmetry at the scalar field target space which would make $\sigma^{(1)} = \sigma^{(2)} = \sigma^{(3)} = 0$. Consequently, only a single massless sterile scalar field (i.e., it has neither self-interaction nor potential) ψ remains and such a field can hardly be applied in realistic scenarios. The best opportunity would be to find a mechanism excluding ghost degrees of freedom from the model's physical spectrum, but allowing them to propagate only in loops. As it was shown in [25], for such a case the scalar field ψ develops non-trivial interaction at the loop level. However, such a mechanism has yet to be found.

In summary, we conclude that the model (1) can be used for realistic scenarios after ghost degrees of freedom being excluded. For the time being it is possible to find a cosmological solutions (18) which shows that the scalar fields act as matter with EoS parameter $w = 1$ and cannot drive an inflation. This makes the model safe in the early Universe. It cannot drive inflation but it also cannot influence an inflationary scenario driven by another inflation field. Therefore the model should be extended in order to tame ghost degrees of freedom.

3. Field Content

Now we switch to a discussion of the field content of second (4) and third (9) models. Firstly, they have at least ten degrees of freedom from the symmetric matrix $h^{(\alpha)(\beta)}$, therefore they may describe spin-2 massless degrees of freedom associated with gravitons. Let us explain this feature in detail. Within the model discussed in the previous section there were non degrees of freedom that can be associated with gravitons (with small metric perturbations of a background spacetime). Because of this, one has no choice but to introduce gravity alongside the regular matter degrees of freedom. In turn, degrees of freedom present in the model can only be considered as generic scalar field that, at best, can drive an inflation. The models to be addressed in this sections, on the contrary, have degrees of freedom $h^{(\alpha)(\beta)}$ that do look like small metric perturbations. In order for them to actually be spin-2 massless degrees of freedom they must describe the correct number of DoF and be subjected to certain equations. Consequently, the main aim of this section is to examine if $h^{(\alpha)(\beta)}$ can in actuality be associated with gravity. Because of this we will not introduce any additional degrees of freedom and will only focus on $h^{(\alpha)(\beta)}$. The second issue we will address is the fact that $h^{(\alpha)(\beta)}$ may also contain two scalar degrees of freedom (associated with its determinant and trace) which could serve as inflatons. Finally, the third model (9) contains a sterile scalar appearing as a consequence of the properties of operator D of the conformal group [17]. These issues are clarified further.

First and foremost, we shall address the issue related with the sterile scalar of model (9). In the original article [17] it was missed that the sterile scalar is related with the trace of $h^{(\alpha)(\beta)}$. The reason behind this is due to the relation between the conformal group generators. Namely, as degrees of freedom $h^{(\alpha)(\beta)}$ are associated with the coset coordinates along $R_{(\alpha)(\beta)}$ direction the sterile scalar ϕ is associated with the coset coordinates along D direction also. However, operators $R_{(\alpha)(\beta)}$ are dependent and coupled as [1,2,17]:

$$\eta^{(\alpha)(\beta)}\,R_{(\alpha)(\beta)} = 2D\,, \tag{20}$$

which follows from definitions of generators $R_{(\alpha)(\beta)}$ and D:

$$\begin{aligned} D &= x^{\mu} P_{\mu}, \\ R_{(\mu)(\nu)} &= x_{\mu} P_{\nu} + x_{\nu} P_{\mu}, \end{aligned} \tag{21}$$

where $P_\mu = i\partial_\mu$ is the generator of translations.

Therefore, the corresponding coset coordinates are also dependent and the trace $\eta_{(\alpha)(\beta)}h^{(\alpha)(\beta)}$ should be associated with ϕ. Hence, the trace component of $h^{(\alpha)(\beta)}$ acts as a sterile massless scalar and, therefore, cannot drive the inflation. Moreover, one can treat $h^{(\alpha)(\beta)}$ as a traceless matrix with 9 independent components reducing the number of valuable degrees of freedom in the model. On the other hand it guarantees the traceless of $h^{(\alpha)(\beta)}$ similar to GR degrees of freedom.

Finally, note that this result can be obtained independently via the direct verification. The definition $h^{(\alpha)(\beta)} = \frac{1}{4}\eta^{(\alpha)(\beta)}\,h$ with $h = \eta_{(\alpha)(\beta)}h^{(\alpha)(\beta)}$ causes the covariant derivative (5) to be completely reduced to the regular ones hence the Lagrangian (4) describes only a single massless sterile scalar.

Now, we switch to $\bar{h}^{(\alpha)(\beta)}$ as the traceless part of $h^{(\alpha)(\beta)}$ and start to study its Lagrangian. The corresponding covariant derivative (5) is:

$$\begin{aligned} &\frac{i}{2}\nabla_{\mu}\bar{h}^{(\alpha)(\beta)} \\ &= \frac{i}{2}\bigg[\partial_{\mu}\bar{h}^{(\alpha)(\beta)} - \eta_{(\nu)(\sigma)}\eta_{(\mu)(\lambda)}\left(\frac{1}{3}\bar{h}^{(\alpha)(\nu)}\,\bar{h}^{(\beta)(\lambda)}\partial_{\mu}\bar{h}^{(\sigma)(\mu)} - \frac{1}{3}\bar{h}^{(\alpha)(\nu)}\,\bar{h}^{(\sigma)(\mu)}\partial_{\mu}\bar{h}^{(\lambda)(\beta)}\right) + \mathcal{O}(h^5)\bigg]. \end{aligned} \tag{22}$$

This derivative matches the expression (5) because the trace component is contained only in $\partial_\mu h^{(\alpha)(\beta)}$. Therefore, the expression (4) determines the traceless part of $h^{(\alpha)(\beta)}$ without changing its form.

Further, as the target space metric is not positively defined the model may contain ghosts. Let us demonstrate this explicitly. The Lagrangian density $\mathcal{L}$ of (4) given up to $\mathcal{O}(h^3)$ reads:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\eta_{(\alpha)(\rho)}\eta_{(\beta)(\sigma)}\,\partial_{\mu}\bar{h}^{(\alpha)(\beta)}\,\partial^{\mu}\bar{h}^{(\rho)(\sigma)} \\ &= \frac{1}{2}\partial_{\mu}\bar{h}^{(0)(0)}\partial^{\mu}\bar{h}^{(0)(0)} + \frac{1}{2}\sum_{a,b=1}^{3}\partial_{\mu}\bar{h}^{(a)(b)}\,\partial^{\mu}\bar{h}^{(a)(b)} - \sum_{s=1}^{3}\partial_{\mu}\bar{h}^{(0)(s)}\,\partial^{\mu}\bar{h}^{(0)(s)}\,. \end{aligned} \tag{23}$$

It is clearly seen that $\bar{h}^{(0)(s)}$ with $s = 1,2,3$ have the wrong sign kinetic term and describe ghost degrees of freedom. Because of the off-diagonal elements $h^{(0)(i)}$ the model cannot be considered realistic until these degrees of freedom are excluded. These ghosts cannot be excluded via the inverse Higgs mechanism as the corresponding generators do not satisfy the required conditions [24].

One can introduce an additional symmetry in the target space. The choice of the target space coordinates as $\zeta^{(\alpha)}$ causes the following form of the target space metric:

$$g = \eta_{(\alpha)(\beta)}\, d\zeta^{(\alpha)} \otimes d\zeta^{(\beta)}\,. \tag{24}$$

If one demands a symmetry with respect to $\zeta^{(0)}$ inversion ($\zeta^{(0)} \to -\zeta^{(0)}$) then the field $h^{(\alpha)(\beta)}$ must have the vanishing component $h^{(0)(i)}$ to be consistent with the considering symmetry. This symmetry condition should be considered independently from the particular choice of the non-linear conformal symmetry realization. However, it is unclear what physical reason can justify the introduction of such a symmetry. Therefore, for the time being we consider it as an ad hoc prescription.

With the discussed prescription the traceless part of $h^{(\alpha)(\beta)}$ includes six independent components which can fit one spin-2 degree of freedom and one spin-0 degree of freedom. To obtain the vanishing trace we set

$$\overline{h}^{(0)(0)} = \sum_{a=1}^{3} \overline{h}^{(a)(a)}. \tag{25}$$

In such a presentation, the only non-vanishing components are $h^{(a)(b)}$ where $a, b = 1, 2, 3$. Within such a setup the original Lagrangian (4) up to $\mathcal{O}(h^4)$ order looks as:

$$\begin{aligned}
\mathcal{L} = \;& \partial_\mu h^{(1)(1)}\, \partial^\mu h^{(1)(1)} + \partial_\mu h^{(2)(2)}\, \partial^\mu h^{(2)(2)} + \partial_\mu h^{(3)(3)}\, \partial^\mu h^{(3)(3)} \\
& + \partial_\mu h^{(1)(2)}\, \partial^\mu h^{(1)(2)} + \partial_\mu h^{(2)(3)}\, \partial^\mu h^{(2)(3)} + \partial_\mu h^{(3)(1)}\, \partial^\mu h^{(3)(1)} \\
& \partial_\mu h^{(1)(1)}\, \partial^\mu h^{(2)(2)} + \partial_\mu h^{(2)(2)}\, \partial^\mu h^{(3)(3)} + \partial_\mu h^{(3)(3)}\, \partial^\mu h^{(1)(1)} + \mathcal{O}(h^4)\,.
\end{aligned} \tag{26}$$

Generically this Lagrangian is non-diagonal. It could be made diagonal in the following representation:

$$\begin{aligned}
\zeta_1 &= \frac{1}{\sqrt{6}}\left[h^{(1)(1)} + h^{(2)(2)} + h^{(3)(3)}\right], & \zeta_4 &= h^{(1)(2)}\,, \\
\zeta_2 &= h^{(1)(1)} - h^{(3)(3)}\,, & \zeta_5 &= h^{(2)(3)}\,, \\
\zeta_3 &= h^{(1)(1)} - h^{(2)(2)}\,, & \zeta_6 &= h^{(3)(1)}\,.
\end{aligned} \tag{27}$$

The quadratic part of the diagonal Lagrangian reads

$$\mathcal{L} = \sum_{i=1}^{6} \partial_\mu \zeta_i\, \partial^\mu \zeta_i \tag{28}$$

so the corresponding field equations are reduced to the Klein–Gordon equation

$$\Box\, \zeta_i = 0\,. \tag{29}$$

As a result the model describes massless degrees of freedom. However, one lacks a condition which can fix their chirality.

Mass and chirality of a particle are fixed by eigenvalues of the Poincare group Casimir operators [26,27]. The D'Alamber operator $\Box$ defined to the mass operator which is one of the two Poincare group Casimir operators. The second Casimir operator is the Pauli–Lubanski vector

$$W^\mu = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta}\, P_\nu\, M_{\alpha\beta} \tag{30}$$

where $M_{\alpha\beta}$ are Lorentz generators building Lorentz group action on a given degree of freedom. The degrees of freedom $h^{(\alpha)(\beta)}$ should be mixed to diagonalize the Lagrangian, therefore, they can be only scalars of the trivial Lorenz group action. Hence, the model has no room for spin-2 degrees of freedom.

This section could be summarized as follows: Both discussed models (4) and (9) really are equal because of the relations between the conformal group generators with the influence on the coset coordinates and the corresponding non-linear symmetry realization. Secondly, the model (4) contains ghost degrees of freedom that can be excluded from the model only with an ad hoc prescription of auxiliary symmetry of the model target space. Such a prescription makes the model to be not natural. Nonetheless, the model with the additional symmetry is healthy and describes massless degrees of freedom appearing to be scalar ones. All these factors make the considered model not realistic.

4. Discussion and Summary

Three models with particular non-linear conformal symmetry realizations [17] were studied. We extend the consideration started in [21] and demonstrate that the discussed models could become realistic only after significant modifications.

The first model (1) seems not to be realistic. Firstly, the original model contains five degrees of freedom propagating in a flat space-time. In [17], it was argued that an extension to a curved space-time case may realize a viable inflationary scenario. Following [21] we show that in this model a universe expands with a deceleration (EoS parameter $w = 1$) and, therefore, there is no inflation in it. Secondly, the model contains ghost degrees of freedom. Although they do not appear at the cosmological solution it is reasonable to expect that they make the solution unstable. Therefore, the model is not applicable unless ghosts are excluded from the perturbation spectrum.

Secondly, (4) and (9) were analyzed and found to be equivalent. This feature was missed both in [17,21]. The models are equivalent up to a parameterization of dynamical variables. As it is pointed in the previous section the trace component of coset coordinates $h^{(\alpha)(\beta)}$ present in model (4) is the coset coordinate ϕ present in model (9). The coset coordinate ϕ is associated with operator D which is related with the trace of operators $R_{(\alpha)(\beta)}$ which, in turn, are associated with coordinated $h_{(\alpha)(\beta)}$ (see (20)). Therefore, models (4) and (9) merely provide different parameterization of the same model.

Finally, we analyzed the field content of model (4). The trace component of $h_{(\alpha)(\beta)}$ acts as a sterile massless scalar which excludes its practical application. The traceless part $\overline{h}_{(\alpha)(\beta)}$ contains nine degrees of freedom. They are three ghosts $\overline{h}^{(0)(s)}$ with $s = 1, 2, 3$ which cannot be excluded with the inverse Higgs mechanism [24]. An additional symmetry of the target space may exclude the ghosts. Therefore, an opportunity to introduce such a symmetry should be studied.

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Abbreviations

The following abbreviations are used in this manuscript:

GR General relativity

Appendix A

Here, it is necessary to demonstrate some features of an interaction between $h^{(\mu)(\nu)}$ and matter degrees of freedom within model (4). The issue provides an additional reason to believe that the model can hardly be considered realistic.

For the sake of simplicity we consider a single massless vector field (which can be associated with the electromagnetic field). Accordingly to [17] such a vector field is described by the standard Lagrangian

$$\mathcal{L}_{\text{vec}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,. \tag{A1}$$

However, a definition of the field tensor contains covariant derivatives:

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \,. \tag{A2}$$

The covariant derivative, in turn, reads

$$\begin{aligned} \nabla_\mu A^\nu &= \partial_\mu A^\nu + \sum_{n=0}^{\infty} \frac{i}{(2n+2)!} (ad_h^{2n} h \partial_\mu h)^{(\alpha)(\beta)} (M_{(\alpha)(\beta)})^\nu_\sigma A^\sigma \\ &= \partial_\mu A^\nu + \frac{i}{2} \eta_{(\gamma)(\sigma)} h^{(\alpha)(\gamma)} \partial_\mu h^{(\beta)(\sigma)} (M_{(\alpha)(\beta)})^\nu_\sigma A^\sigma + \mathcal{O}(h^2) \,. \end{aligned} \tag{A3}$$

Here, $\left(M_{(\alpha)(\beta)}\right)_{\mu\nu} = i(\eta_{(\alpha)\mu}\eta_{(\beta)\nu} - \eta_{(\alpha)\nu}\eta_{(\beta)\mu})$ is the vector representation of Lorentz group generators.

The corresponding Lagrangian has a few peculiar features. First and foremost, at the leading order the model describes interaction between two modes $h^{(\mu)(\nu)}$ and two vectors A^μ. Within GR, in contrast, the leading order interaction between gravity and a massless vector field is a cubic interaction describing an interaction between two vectors and a graviton. Secondly, in contrast with GR the interaction can contain only odd powers of $h^{(\mu)(\nu)}$.

Implications of these features lead to significant differences between GR and (4). The most interesting one is related with $2 \to 2$ scattering. Within GR there are tree-level amplitudes describing gravitational scattering of massless vectors. Within model (4) such a scattering appears only at the one-loop level.

It is necessary to point out that to pinpoint the exact difference in such scattering processes a more detailed analysis is required. However, even at the current level the model (4) appears to be inconsistent with GR.

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