

Article Aspects of Gauss-Bonnet Scalarisation of Charged Black Holes

Carlos A. R. Herdeiro D, Alexandre M. Pombo *D and Eugen Radu

Departamento de Matemática da Universidade de Aveiro and Centre for Research and Development in Mathematics and Applications (CIDMA), Campus de Santiago, 3810-183 Aveiro, Portugal; herdeiro@ua.pt (C.A.R.H.); eugen.radu@ua.pt (E.R.)

* Correspondence: pomboalexandremira@ua.pt

Abstract: The general relativity vacuum black holes (BHs) can be scalarised in models where a scalar field non-minimally couples to the Gauss-Bonnet (GB) invariant. Such GB scalarisation comes in two flavours, depending on the GB sign that triggers the phenomenon. Hereafter these two cases are termed GB^{\pm} scalarisation. For vacuum BHs, only GB^+ scalarisation is possible in the static case, while GB^- scalarisation is *spin* induced. But for electrovacuum BHs, GB^- is also *charged* induced. We discuss the GB^- scalarisation of Reissner-Nordström and Kerr-Newman BHs, discussing zero modes and constructing fully non-linear solutions. Some comparisons with GB^+ scalarisation are given. To assess the generality of the observed features, we also briefly consider the GB^{\pm} scalarisation of stringy dilatonic BHs and coloured BHs which provide qualitative differences with respect to the electrovacuum case, namely on the distribution and existence of regions triggering GB^- scalarisation.

Keywords: black holes; Gauss-Bonnet; scalar fields; scalarisation



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1. Introduction

It is conceivable that deviations from General Relativity (GR) occur *only* for sufficiently large curvatures. One explicit realization of this idea is the phenomenon of spontaneous scalarisation. The original idea proposed the scalarisation of neutron stars in scalar-tensor models [1]. More recently, it gained a new guise in which the scalarisation of the GR vacuum BH solutions becomes possible, in the context of extended scalar-tensor models that include the Gauss-Bonnet (GB) quadratic curvature invariant R_{GB}^2 , as first pointed out in [2–4]. In the following we shall dub the latter *GB scalarisation*.



GB scalarisation circumvents well-known no-hair theorems (see [5] for a review) due to a certain class of non-minimal couplings between a real scalar field ϕ and the GB invariant. The phenomenon occurs for BHs in an appropriate mass range, defined by a dimensionful coupling in the model. Moreover, it can be triggered either if $R_{GB}^2 > 0$ —hereafter dubbed GB⁺ scalarisation [6–21]—or if $R_{GB}^2 < 0$ —hereafter dubbed GB⁻ scalarisation. For the Kerr family of GR, the latter only occurs for sufficiently fast spinning BHs [22–24], which justifies the terminology *spin-induced scalarisation* [22]. By contrast, in the case of GB⁺ scalarisation, Kerr BHs can also scalarise, but rotation actually suppresses the effects of scalarisation [17,25].

Enlarging the model to include charged BHs, however, GB⁻ scalarization ceases to rely solely on rotation. This can already be illustrated in electrovacuum GR. The Kerr-Newman solution develops a negative GB invariant for either sufficiently large dimensionless angular momentum *j* or sufficiently large dimensionless charge *q*. Thus, sufficiently near extremality, Kerr-Newman BHs develop regions with $R_{GB}^2 < 0$ —Figure 1. One may expect that the boundary of the region with $R_{GB}^2 < 0$ marks the onset of the solutions prone to GB⁻ scalarisation, as for the Kerr case [26]. We shall confirm this expectation below, explicitly constructing some of the GB⁻ scalarised Kerr-Newman solutions and comparing them with the corresponding GB⁺ scalarised solutions.



Figure 1. GB invariant (in units of mass) of a Kerr-Newman BH with mass *M*, angular momentum *J* and charge *Q*, evaluated at the horizon's poles ($r = r_h$ and $\theta = 0$), as a function of the dimensionless parameters $j = J/M^2$ and q = Q/M. Near extremality, $R_{GB}^2 < 0$.

The discussion of the previous paragraph shows that, within electrovacuum BHs, GB⁻ scalarisation can be spin-induced or charge-induced (or both). Let us remark, however, that such GB charge-induced scalarisation is different from the scalarisation of charged BHs introduced in [27], where the non-minimal coupling occurs between the scalar field and the Maxwell field, with the GB term (or any curvature corrections) being absent.

Additionally, the aforementioned observations on the GB⁻ scalarisation of Kerr-Newman BHs show this process occurs even for spherically symmetric, non-spinning Reissner-Nordström (RN) BHs [28]. Moreover, the negative GB invariant always occurs in the immediate vicinity of the horizon. One may ask whether these features are generic. Is any charged BH model prone to GB⁻ scalarisation sufficiently close to the maximal allowed charge? And is the $R_{GB}^2 < 0$ region supporting the scalarisation always occuring in the immediate vicinity of the horizon? Interestingly, neither of these features is generic, as we shall illustrate by considering two alternative models of charged (spherical) BHs.

This paper is organized as follows. In Section 2 we detail the Einstein-Maxwell-scalar-GB model and its equations of motion. A general discussion on the tachyonic instability associated with the scalarisation is given, followed by the relevant physical quantities for describing scalarised BHs. In Section 3 we consider the GB[±] scalarisation of the RN

BH, first discussing the sign of the GB invariant for the electrovacuum RN BH and then constructing both the linear scalar clouds and non-linear scalarised BHs. In Section 4 we consider the GB^{\pm} scalarisation of the Kerr-Newman BH. After a brief discussion on the sign of the GB invariant, we discuss the construction of the solutions and provide a sample of numerical results, focusing on the GB⁻ scalarisation case. In Section 5 we briefly consider Einstein-Maxwell-dilaton and Einstein-non-Abelian BHs, which yield two valuable lessons concerning GB^{\pm} scalarisation of charged BHs. We conclude with a brief discussion and final remarks in Section 6.

2. The Einstein-Maxwell-Scalar-GB Model

We wish to consider the Einstein-Maxwell-scalar-GB (EMsGB) model, described by the following action

$$\mathcal{S}_{\text{EMsGB}} = \int d^4x \sqrt{-g} \left[\frac{1}{4}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \epsilon\frac{\lambda^2}{4}f(\phi)R_{\text{GB}}^2 \right],\tag{1}$$

where R is the Ricci scalar with respect to the spacetime metric $g_{\mu\nu}$, R_{GB}^2 is the GB invariant

$$R_{\rm GB}^2 \equiv R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\alpha\beta}R^{\alpha\beta} + 4R^2 , \qquad (2)$$

 $R_{\alpha\beta\mu\nu}$ is the Riemann tensor, $R_{\alpha\beta}$ is the Ricci tensor, $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ is the Maxwell field strength tensor where $A = A_{\mu}dx^{\mu}$ is the U(1) gauge potential, $f(\phi)$ is a coupling function of the real scalar field ϕ , λ is a constant of the theory with dimension of length and $\epsilon = \pm 1$ is chosen for GB^{ϵ} scalarisation.

Varying the action (1) with respect to the metric tensor $g_{\mu\nu}$, we obtain the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 2 T_{\mu\nu}^{(\text{eff})} .$$
(3)

The effective energy-momentum tensor $T_{\mu\nu}^{(eff)}$ has three distinct components:

$$T_{\mu\nu}^{(\text{eff})} = T_{\mu\nu}^{(\text{s})} + T_{\mu\nu}^{(\text{M})} + T_{\mu\nu}^{(\text{GB})} , \qquad (4)$$

consisting of the (pure) scalar and Maxwell parts, respectively,

$$T_{\mu\nu}^{(s)} = \partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}g_{\mu\nu}\partial_{\alpha}\phi\partial^{\alpha}\phi , \quad T_{\mu\nu}^{(M)} = F_{\mu\alpha}F_{\nu}^{\ \alpha} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} , \tag{5}$$

and a third contribution due to the scalar-GB term in (1)

$$T_{\mu\nu}^{(\text{GB})} = -2\epsilon\lambda^2 P_{\mu\gamma\nu\alpha}\nabla^{\alpha}\nabla^{\gamma}f(\phi) , \qquad (6)$$

where

$$P_{\alpha\beta\mu\nu} = -\frac{1}{4} \varepsilon_{\alpha\beta\rho\sigma} R^{\rho\sigma\gamma\delta} \varepsilon_{\mu\nu\gamma\delta} = R_{\alpha\beta\mu\nu} + g_{\alpha\nu} R_{\beta\mu} - g_{\alpha\mu} R_{\beta\nu} + g_{\beta\mu} R_{\alpha\nu} - g_{\beta\nu} R_{\alpha\mu} + \frac{1}{2} (g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}) R_{\alpha\mu\nu}$$

and $\varepsilon_{\alpha\beta\rho\sigma}$ is the Levi-Civita tensor. The scalar field equation is

$$\nabla^2 \phi + \epsilon \frac{\lambda^2}{4} \frac{df(\phi)}{d\phi} R_{\rm GB}^2 = 0 , \qquad (7)$$

while the (source-free) Maxwell equations have the usual form

$$\nabla_{\mu}F^{\mu\nu} = 0.$$
(8)

2.1. GB^{ϵ} Scalarisation of Electrovacuum Solutions

Spontaneous scalarisation manifests itself at the linear level as a tachyonic instability. Let us assume that $\phi = 0$ solves (7), which will hold for a class of coupling functions. Then, the field equations reduce to those of electrovacuum GR, and the corresponding solutions provide solutions of the full model (1) as well. In particular, the Kerr-Newman geometry will be a solution of this model. Thus, for concreteness we shall refer to the scalarisation of the Kerr-Newman solution in the following; but a similar discussion would hold for any GR electrovacuum solution.

Next, we consider scalar perturbations of the Kerr-Newman solution within the full model (1). Assuming a small- ϕ expansion for the coupling function

$$f(\phi) = f|_{\phi=0} + \frac{1}{2} \frac{d^2 f}{d\phi^2} \Big|_{\phi=0} \phi^2 + \mathcal{O}(\phi^3) , \qquad (9)$$

the linearised scalar field Equation (7) around the Kerr-Newman solution becomes

$$(\Box - \mu_{\text{eff}}^2)\phi = 0, \quad \text{where} \quad \mu_{\text{eff}}^2 = -\epsilon \frac{\lambda^2}{4} \frac{d^2 f}{d\phi^2} \Big|_{\phi=0} R_{\text{GB}}^2, \quad (10)$$

where \Box and R_{GB}^2 are computed for the scalar-free Kerr-Newman solution.

If μ_{eff}^2 is not strictly positive, the scalar field possesses a (spacetime dependent) tachyonic mass. Wherever this tachyonic mass is supported, such region potentially supports a spacetime instability, which is precisely the GB scalarisation. To simplify the discussion, we assume without any loss of generality that $d^2f/d\phi^2\Big|_{\phi=0}$ is strictly positive. Then the condition $\mu_{\text{eff}}^2 < 0$ is equivalent to

$$\epsilon R_{\rm GB}^2 > 0 \,. \tag{11}$$

When this condition is obeyed in some region(s) outside the BH horizon, GB^{ϵ} scalarisation is triggered.

If the Kerr-Newman BH reduces to a Schwarzschild BH of mass *M*, then

$$\mu_{\rm eff}^2 = -\epsilon \frac{\lambda^2}{4} \frac{d^2 f}{d\phi^2} \Big|_{\phi=0} \frac{48M^2}{r^6} , \qquad (12)$$

and only GB^+ scalarisation is possible.

2.2. Physical Quantities of Interest for Scalarised BHs

When the above instability is present there is also a different class of solutions for the model (1), besides the electrovacuum ones. These are the scalarised solutions. We are interested in the case of stationary BHs. Generically, these solutions posses three global charges: the mass M, the electric charge Q and the angular momentum J. For the solutions in this work, there is also a "scalar charge" Q_s , which is not, however, associated with a conservation law. There are also a number of relevant horizon quantities: the Hawking temperature T_H , the horizon area A_H , the entropy S, and the horizon angular velocity Ω_H . The BH entropy is the sum of two terms,

$$S = S_{\rm E} + S_{\rm sGB} , \text{ with } S_{\rm E} = \frac{1}{4} A_H , \qquad S_{\rm sGB} = \epsilon \frac{1}{2} \lambda^2 \int_H d^2 x \sqrt{h} f(\phi) \mathbf{R}^{(2)} , \qquad (13)$$

where $\mathbb{R}^{(2)}$ is the Ricci scalar of the induced horizon metric *h*. The solutions satisfy the Smarr law

$$M = 2\Omega_H J + 2T_H S + \Phi Q + M_s , \qquad (14)$$

where Φ is the electrostatic potential and M_s is a contribution of the scalar field

$$M_s = \frac{1}{2} \int d^3g \sqrt{-g} (\partial_a \phi)^2 \,. \tag{15}$$

Also, the solutions satisfy the first law of BH thermodynamics

$$dM = T_H dS + \Omega_H dJ + \Phi dQ , \qquad (16)$$

in which there is no contribution from the scalar field.

In this work we shall focus on the quadratic coupling function,¹

$$f(\phi) = \frac{\phi^2}{2} , \qquad (17)$$

which is the simplest choice of $f(\phi)$ that guarantees that $\phi = 0$ satisfies the scalar Equation (7).

It is useful to observe that the equations of the model are invariant under the transformation

$$r \to \alpha r$$
, $\lambda \to \alpha \lambda$, (18)

with *r* the radial coordinate and $\alpha > 0$ an arbitrary positive constant. Only quantities invariant under (18) (e.g., Q/M or Q/λ) have a physical meaning. Following standard terminology, we define the following *reduced* quantities:

$$q \equiv \frac{Q}{M} , \qquad a_H \equiv \frac{A_H}{16\pi M^2} , \qquad t_H \equiv 8\pi T_H M , \qquad j \equiv \frac{J}{M^2} , \qquad (19)$$

which will be considered in what follows.

3. GB^e Scalarisation of Reissner-Nordström BHs

Let us start by considering the spinless limit of the Kerr-Newman family, the RN BH. The corresponding metric and gauge field can be written as (see e.g., [29])

$$ds^{2} = -N(r)\sigma(r)^{2}dt^{2} + \frac{dr^{2}}{N(r)} + r^{2}(d\theta^{2} + \sin^{2}\theta d\varphi^{2}), \qquad V = V(r)dt,$$
(20)

where

$$\sigma(r) = 1$$
, $N(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$, $V(r) = \frac{Q}{r}$. (21)

This BH possess an event horizon at

$$r = r_h = M + \sqrt{M^2 - Q^2} \,. \tag{22}$$

Thus, $0 \le q \le 1$ and q = 1 for the extremal RN BH. The GB invariant of the RN metric reads

$$R_{\rm GB}^2 = \frac{8}{r^8} \left[6M^2r^2 - 12Q^2Mr + 5Q^4 \right].$$
⁽²³⁾

For $Q \neq 0$ this always becomes negative for some region with r > 0. This region, however, is cloaked by the horizon unless the largest root of the quadratic equation in (the square brackets in) (23) exceeds r_h . This condition is

$$qQ\left(1+\frac{1}{\sqrt{6}}\right) > r_h \,. \tag{24}$$

Using (22) one can easily show that is possible for

$$q > q_c \simeq 0.957058$$
 (25)

Thus, for $q_c < q \leq 1$, a RN BH can undergo GB⁻ scalarisation.

3.1. The Linear Scalar Clouds

At the onset of the tachyonic instability, the linearised scalar field Equation (10) on the RN background allows solutions known as *scalar clouds*. These occur for a discrete set of RN solutions, each solution corresponding to a particular harmonic scalar field mode. To see this, we perform a harmonic decomposition of the scalar field,

$$\phi = U(r)Y_{\ell m}(\theta,\varphi) , \qquad (26)$$

where $Y_{\ell m}$ are the real spherical harmonics and ℓ , m are the associated quantum numbers with the usual ranges, $\ell = 0, 1, ...$ and $-\ell \leq m \leq \ell$. For a RN BH background (20) and (21), the linearised scalar Equation (10) becomes a radial equation

$$\left[r^2 N(r) U'(r)\right]' = \ell(\ell+1) U(r) - \epsilon \frac{\lambda^2}{r^2} \left(\frac{12M^2}{r^2} + \frac{10Q^4}{r^4} - \frac{24MQ^2}{r^3}\right),$$
(27)

where a prime denotes a derivative *w.r.t*. the radial coordinate *r*.

This equation has the following asymptotic solutions: near the horizon

$$U(r) = u_0 + \frac{r_h}{Q^2 - r_h^2} \left[-\ell(\ell+1) + \epsilon \frac{\lambda^2}{r_h^2} \left(3 - \frac{6Q^2}{r_h^2} + \frac{Q^4}{r_h^4} \right) \right] u_0(r - r_h) + \dots , \qquad (28)$$

where u_0 is an arbitrary nonzero constant which, in numerics, we set to 1; and near spatial infinity

$$U(r) = \frac{Q_s}{r^{\ell+1}} + \dots , \qquad (29)$$

where Q_s is the scalar charge for $\ell = 0$.

Solving (27) with the above asymptotic behaviours can be viewed as an eigenvalue problem. For a given ℓ , requiring the scalar field to smoothly interpolate between the asymptotics (28) and (29), crossing *n* times the *r*-axis, selects a discrete set of RN parameters. Thus, each scalar cloud is characterized by three 'quantum numbers' (n, ℓ, m) , with the radial function being degenerate in terms of *m* as a consequence of the spherical symmetry of the RN background. In this work we shall report results on nodeless spherically symmetric fundamental solutions only², i.e., with $\ell = m = n = 0$.

Having chosen the clouds quantum numbers, taking λ as fixed scale set in the action, and fixing the reduced charged q, the radial equation has a solution for a specific dimensionless ratio λ/M . For instance, for $\epsilon = +1$, $\ell = m = n = 0$ and q = 0 the selected value is $\lambda/M \sim 1.704$, corresponding to the initial point of the blue dotted curve in Figure 2 (left panel). This is the zero mode of the GB⁺ instability of Schwarzschild. It selects a mass scale. Smaller masses (larger λ/M) describe BHs unstable against scalarisation; larger masses (smaller λ/M) correspond to stable BHs.

The variation of λ/M with increasing *q* can be interpreted as follows. GB scalarisation of Schwarzschild BHs may be attributed to a repulsive gravitational effect of the GB term, which only becomes dominant for sufficiently small BHs (in terms of λ). Adding electric charge introduces two competing effects. On the one hand the electric charge provides a repulsive gravitational effect for RN BHs. This facilitates scalarisation, making it available for larger BHs (larger *M*, smaller λ/M). On the other hand, the repulsive gravitational effect of the GB invariant, which is at the source of the scalarisation phenomenon, becomes supressed (and eventually the GB term even changes sign in some region), when increasing

q. This supresses scalarisation, making it available only for smaller BHs (smaller *M*, larger λ/M). The trend observed in the blue dotted curve in the main panel of Figure 2 (left panel) suggest that for small *q* the RN charge repulsion dominates and for large *q* the GB charge suppression becomes dominant. The inset, on the other hand, gives the behaviour for $\epsilon = -1$. In this case, as *q* increases, in the allowed (large *q*) interval, the GB behaviour dominates, and due to the opposite sign it provides an ever more significant repulsive contribution, thus faciliating GB⁻ scalarisation, which therefore is available for larger masses. We also remark that, for GB⁻ scalarisation, the ratio λ/M appears to diverge as $q \to q_c$, while it stays finite as $q \to 1$.



Figure 2. (Left panel) Dimensionless ratio λ/M of the set of RN solutions supporting the $\ell = m = n = 0$ scalar cloud vs. the reduced electric charge *q* for $\epsilon = \pm 1$. (**Right panel**) Typical radial profiles of the spherical, nodeless scalar clouds on a RN BH background.

The profiles of typical scalar clouds are shown in Figure 2 (right panel). For $\epsilon = +1$ and moderate q we recover the picture found in the Schwarzschild case, Q = 0: a monotonically decreasing profile starting with some finite value at the horizon (see the inset). This is also true for the $\epsilon = -1$ case (red curve in main panel). But for $q > q_c$ and $\epsilon = +1$, a new qualitative behaviour emerges: the maximal value of the scalar cloud can be attained outside the horizon³—see the blue curve in the main panel.

3.2. The Non-Linear Spherically Symmetric Scalarised BHs

The linear scalar clouds just discussed can be continued to the non-linear regime. Their backreaction originates scalarised BHs. We shall now discuss their construction for the case of the spherical, nodeless scalar clouds.

The ansatz to obtain the scalarised BH solutions is (20), together with a radial scalar field

$$\phi \equiv \phi(r) \,. \tag{30}$$

As in the usual electrovacuum case, the Maxwell Equation (8) yields a total derivative, leading to a first integral

$$V'(r) = -\frac{Q\sigma(r)}{r^2} \,. \tag{31}$$

This introduces the electric charge measured at infinity, *Q*. The scalar field satisfies the equation

$$\phi'' + \left(\frac{2}{r} + \frac{N'}{N} + \frac{\sigma'}{\sigma}\right)\phi' - \frac{\epsilon\lambda^2}{r^2 N\sigma} \Big[(3-5N)N'\sigma' + \sigma((1-N)N'' - N'^2) + 2(1-N)N\sigma'' \Big]\phi = 0,$$
(32)

while the equations for the metric functions N, σ are too involved and shall not be displayed here.

We are interested in BH solutions with an event horizon located at $r = r_h > 0$. The equations of the model are subject to the following boundary conditions.

$$N|_{r_{h}} = 0, \quad \sigma|_{r_{h}} = \sigma_{h}, \quad \phi|_{r_{h}} = \phi_{h}, \quad V|_{r_{h}} = 0; \quad N|_{\infty} = 1, \quad \sigma|_{\infty} = 1, \quad \phi|_{\infty} = 0, \quad V|_{\infty} = \Phi,$$
(33)

where σ_h , ϕ_h are constants fixed by numerics and Φ is the electrostatic potential at infinity. The horizon data fix the Hawking temperature, the horizon area and the entropy of the solutions,

$$T_H = \frac{\sigma_h N'(r_h)}{4\pi} , \qquad A_H = 4\pi r_h^2 , \qquad S = \pi r_h^2 + \epsilon \lambda^2 f(\phi_h) . \tag{34}$$

A local solution compatible with these asymptotics can be constructed both close to the horizon (as a power series in $(r - r_h)$) and at infinity (as a power series in 1/r). For example, the first terms in the far field expression of the solutions read

$$N = 1 - \frac{2M}{r} + \frac{Q^2 + Q_s^2}{r^2} + \dots, \quad \phi(r) = \frac{Q_s}{r} + \dots, \quad V(r) = \Phi - \frac{Q}{r} + \dots, \quad \sigma(r) = 1 - \frac{Q_s^2}{2r^2} + \dots.$$
(35)

With the details just laid out we have numerically constructed the nonlinear continuation of the scalar clouds solving the full equations of the EMsGB model, for both signs of ϵ .

Technically, the construction of the scalarised BHs is a one parameter shooting problem in terms of the value of the scalar field at the horizon ϕ_h . The input parameters are r_h , Qand λ . Fixing the length scale λ , this leads to a two dimensional parameter space for the problem. The numerical results for several values of the ratio Q/λ are shown in Figure 3.



Figure 3. Charge-temperature (main panels panel) and charge-horizon area (insets) diagrams, in units set by mass, for several illustrative families of GB^{ϵ} scalarised RN solutions, for both $\epsilon = +1$ (**left panel**) and $\epsilon = -1$ (**right panel**). The branches of scalarised solutions bifurcate from the electrovacuum RN BHs (blue line) and terminate in critical configurations (red circles).

Figure 3 shows that for a given ratio Q/λ and both values of ϵ , one finds a continuum of solutions which bifurcate from the corresponding RN BH supporting a scalar cloud with these parameters. This line has a finite extent, ending in a critical configuration where the numerical process fails to converge. A general explanation for this behaviour can be traced back to the fact that the radicand of a square root in the horizon expansion of the scalar field vanishes as the critical set is approached. This is a generic feature of GB-scalar models. An exception here are the $\epsilon = +1$ solutions emerging from RN BHs with $q > q_c$, in which case the critical configurations seem to possess a curvature singularity for some radius outside the event horizon (see [28] for a discussion).

From Figure 3 one may highlight two qualitatively different features when comparing $\epsilon = \pm 1$. First, scalarisation reduces (increases) a_H for $\epsilon = +1$ ($\epsilon = -1$). Secondly, "overcharged" solutions with q > 1 exist for $\epsilon = -1$ only.

4. GB^{*e*} Scalarisation of Kerr-Newman BHs

Let us now address the GB^{ϵ} scalarisation of the spinning, charged electrovacuum Kerr-Newman BH. This is a solution of the model (1), with the coupling (17), together with a vanishing scalar field, $\phi = 0$. This BH is described by its ADM mass M, total angular momentum per unit mass a = J/M and electric charge Q. In Boyer-Lindquist coordinates it reads (see e.g., [29])

$$ds^{2} = -\frac{\Delta}{\Sigma} \left(dt - a\sin^{2}\theta d\varphi \right)^{2} + \Sigma \left(\frac{dr^{2}}{\Delta} + d\theta^{2} \right) + \frac{\sin^{2}\theta}{\Sigma} \left[adt - \left(\Sigma + a^{2}\sin^{2}\theta \right) d\varphi \right]^{2}, \quad (36)$$

and

$$A_{\mu}dx^{\mu} = -\frac{Qr}{\Sigma} \left(dt - a\sin^2\theta d\varphi \right) , \qquad (37)$$

where

$$\Delta \equiv r^2 - 2Mr + a^2 + Q^2 , \qquad \Sigma \equiv r^2 + a^2 \cos^2 \theta .$$
(38)

The event horizon of this solution is located at

$$r_h = M \left(1 + \sqrt{1 - j^2 - q^2} \right).$$
 (39)

This implies the Kerr-Newman bound, $j^2 + q^2 \le 1$; an extremal BH saturates this bound. The Kerr-Newman metric has a GB invariant

$$R_{\rm GB}^2 = \frac{48M^2}{\Sigma^3} \left(1 - \frac{2a^2}{\Sigma^3} \left(3r^2 - a^2\cos^2\theta \right)^2\cos^2\theta \right) + \frac{8Q^2}{\Sigma^6} \left\{ r^4 (5Q^2 - 12Mr) + a^2\cos^2\theta \left[2r^2 (-19Q^2 + 60Mr) + 5a^2(Q^2 - 12Mr)\cos^2\theta \right] \right\}.$$
(40)

We have studied⁴ the sign of this quantity as a function of the parameters (j, q)—Figure 1, observing that the qualitative picture found in the Kerr (q = 0) or RN (j = 0) cases is still valid for a Kerr-Newman BH. While R_{GB}^2 is positive for large values of the radial coordinate, its sign close to the event horizon depends on the value of (j, q). That is, for fixed j (or q), the GB invariant R_{GB}^2 always becomes negative in a region outside the horizon, for large enough values of q (or j). In the presence of rotation, this region is located around the poles of the horizon, $\theta = 0$, π —Figure 4. In Figure 1 we show the region in the (j, q)-domain where the GB invariant takes a negative sign at the poles of the horizon. Kerr-Newman BHs with $R_{GB}^2 < 0$ have the potential to be scalarised for both signs of ϵ .



Figure 4. The GB invariant at the horizon (in units of mass) as a function of the θ -coordinate for several Kerr-Newman BHs.

4.1. Construction of the Scalarised Kerr-Newman BHs

To construct the GB^{ε} scalarised Kerr-Newman BHs, we shall use the ansatz in [23,25], supplemented with a nonzero gauge field. The metric ansatz reads:⁵

$$ds^{2} = -e^{2F_{0}}Ndt^{2} + e^{2F_{1}}\left(\frac{dr^{2}}{N} + r^{2}d\theta^{2}\right) + e^{2F_{2}}r^{2}\sin^{2}\theta(d\varphi - Wdt)^{2}, \quad \text{with} \quad N = 1 - \frac{r_{h}}{r}, \quad (41)$$

where the metric functions F_i , W, as well as the scalar field ansatz $\phi = \phi(r, \theta)$, depend on r, θ only. The ansatz for the U(1) potential is

$$A = A_{\varphi} \sin \theta (d\varphi - Wdt) + Vdt , \qquad (42)$$

with two potentials, a magnetic one A_{φ} and an electric one *V*, both depending on *r*, θ .

Setting $A_{\varphi} = V = 0$ results in the spontaneously scalarised Kerr BHs in [23,25], albeit with a different coupling function. The limit $W = A_{\varphi} = 0$ results in the scalarised RN BHs [27,31,32] discussed above, albeit for a different radial coordinate.

The general problem is solved subject to the following boundary conditions. Asymptotic flatness requires

$$\lim_{r \to \infty} F_i = \lim_{r \to \infty} W = 0, \quad \text{and} \quad \lim_{r \to \infty} \phi = \lim_{r \to \infty} A_{\varphi} = 0, \quad \lim_{r \to \infty} V = \Phi.$$
(43)

Axial symmetry and regularity impose the following boundary conditions on the symmetry axis, i.e., at $\theta = 0, \pi$:

$$\partial_{\theta}F_{i} = \partial_{\theta}W = \partial_{\theta}\phi = \partial_{\theta}V = A_{\varphi} = 0.$$
(44)

Moreover, the absence of conical singularities implies also that $F_1 = F_2$ on the symmetry axis.

The event horizon is located at a constant value of $r = r_h > 0$. Only non-extremal solutions can be studied within the metric ansatz (41). We proceed by introducing a new radial coordinate $x = \sqrt{r^2 - r_H^2}$, which simplifies the boundary conditions at the horizon and also the numerical treatment of the problem. This results in the following boundary conditions at the horizon

$$\partial_x F_i\big|_{r=r_h} = \partial_x \phi\big|_{r=r_h} = 0, \qquad W\big|_{r=r_h} = \Omega_H, \qquad \partial_x A_\varphi\big|_{r=r_h} = V\big|_{r=r_h} = 0, \qquad (45)$$

where the constant $\Omega_H > 0$ is the horizon angular velocity. Then, an approximate expansion of the solution compatible with these boundary conditions can easily be constructed.

Specializing some of the aforementioned physical quantities of interest for the ansatz in use, we obtain that the Hawking temperature and the event horizon area are determined by the following horizon data,

$$T_H = \frac{1}{4\pi r_h} e^{F_0^{(0)}(\theta) - F_1^{(0)}(\theta)} , \qquad A_H = 2\pi r_h^2 \int_0^\pi d\theta \sin\theta \ e^{F_1^{(0)}(\theta) + F_2^{(0)}(\theta)} , \qquad (46)$$

with the near horizon expansion $F_i = F_i^{(0)}(\theta) + x^2 F_i^{(2)}(\theta) + \dots$, and i = 0, 1, 2. The ADM mass *M*, the angular momentum *J*, the scalar 'charge' Q_s , together with the

The ADM mass *M*, the angular momentum *J*, the scalar 'charge' Q_s , together with the the magnetic dipole momentum q_m , the electrostatic potential Φ and the electric charge *Q* are read off from the far field asymptotic of the metric and matter functions

$$g_{tt} = -1 + \frac{2M}{r} + \dots, \ g_{\varphi t} = -\frac{2J}{r}\sin^2\theta + \dots, \ \phi = -\frac{Q_s}{r} + \dots, \ A_{\varphi} = \frac{q_m \sin\theta}{r} + \dots, \ V = \Phi - \frac{Q}{r} + \dots$$

We remark that both the metric functions and the scalar field are invariant *w.r.t.* the transformation $\theta \rightarrow \pi - \theta$.

4.2. Numerical Results

With the setup just described, we have employed a numerical approach similar to the one in [23,25]. The typical numerical errors for the solutions so obtained and reported below are of the order of 10^{-3} .

The scalarised Kerr-Newman solutions possess four independent charges: the three global charges shared with their electrovacuum counterparts, (M, J, Q), plus the scalar charge Q_s . In our approach, the input parameters are: the event horizon radius r_h , the horizon angular velocity Ω_H , the asymptotic value of the electrostatic potential Φ (or the electric charge Q), together with the coupling constant λ (which specifies the theory). Therefore, after fixing the scale λ , we are left with a three dimensional parameter space. A full scanning of such large parameter space of scalarised Kerr-Newman BHs is therefore a time consuming task. Here we focus on illustrative sets of solutions which already capture the generic behaviour. Moreover, although we have verified that spinning scalarised solutions exist for both signs of ϵ , we shall focus on the results for the more novel case of spin/charge scalarisation, $\epsilon = -1$. In practice we have scanned the parameter space by varying both r_h and Q, for several different values of Ω_H . Alternatively, we have varied both r_h and Ω_H , for fixed values of Q.

Our numerical results suggest that the solutions share most of the properties of the scalarised Kerr BHs discussed in [23,24]. In Figure 5 we display the reduced quantities (q, j) (left panel) and $(j, M/\lambda)$ (right panel) for solutions with $\Omega_H \lambda = 0.469$ and illustrative values of the ratio Q/λ . These scalarised solutions emerge from a Kerr-Newman BH supporting a zero mode solution of the scalar equation—a scalar cloud—corresponding to the blue squares in Figure 5. The profiles of such clouds (with the maximal value of the scalar field normalised to one) are shown in Figure 6, for two different Kerr-Newman BHs. One can see that, while for small values of the reduced angular momentum *j*, the profile of the scalar field looks rather similar to that found in spherical case (see Figure 2 (right)), the large-*j* ones have a strong θ -dependence, the minimal value of ϕ being always found in the equatorial plane. The nonlinear continuation of the clouds (with constant Q/λ) results in a sequence of solutions terminating at a critical configuration, as in the RN case reported before, corresponding to the red circles in Figure 5. The main trend observed in Figure 5 is that fixing Q and Ω_H in units of λ , the scalarised BHs have more mass (and thus smaller *q* and λ/M) and larger *j*.



Figure 5. Branches of GB⁻ scalarised Kerr-Newman BHs in a *q* vs. *j* plot (**left panel**) and *j* vs. λ/M plot (**right panel**). The branches are for a specific choice of $\Omega_H \lambda = 0.469$ and different choices of Q/λ . The blue squares correspond to Kerr-Newman BHs with a vanishing scalar field, while the red circles correspond to critical configurations.



Figure 6. The profile of typical scalar clouds are shown for two Kerr-Newman backgrounds with different values of the reduced charge and angular momentum (q, j).

Although our scanning of the full parameter space was limited, extrapolating the existing numerical data, we antecipate that the (three dimensional) domain of existence of $\epsilon = -1$ spinning, charged scalarised BHs is bounded by four sets of solutions: (i) the *existence surface*, which corresponds to the set of Kerr-Newman solutions supporting scalar clouds; (ii) the set of *critical solutions*, which form again a two dimensional surface; (iii) the *static configurations*, J = 0, which corresponds to the $\epsilon = -1$ scalarised RN solutions discussed in the previous Section; and (iv) the *neutral configurations*, Q = 0, which were studied with the choice of the coupling function (17) in [24]. As for the Q = 0 case, the *existence surface* is universal for any expression of the coupling function allowing for scalarisation. Concerning the set (ii) (*critical solutions*) the numerical process fails to converge as it is approached, as in the static limit. The explanation for this behaviour can again be traced back to the fact that the radicand of a square root in the horizon expansion of the solutions vanishes as the critical set is approached. Note also that the sets (ii)–(iv) are not universal; they depend on the choice of the coupling function $f(\phi)$.

5. Lessons from Alternative Charged BHs

It is interesting to test the generality of some of the results above concerning the interplay between the introduction of charge and the scalarisation phenomenon. For this purpose, in this Section we shall be considering the simpler case of static, spherically symmetric, charged BHs in some alternative models, rather than electrovacuum.

As a first observation, we remark that for a finite mass, asymptotically flat solution, the GB invariant R_{GB}^2 is strictly positive for large enough r. As with the RN BH, a sufficient condition for the occurrence of GB induced scalarization for *both* signs of ϵ is that $R_{GB}^2 < 0$ at the horizon. In general, however, the sign of the GB invariant at the horizon depends on the matter content. Indeed, for a generic spherically symmetric BH spacetime and using the metric ansatz (20), a straightforward computation leads to the simple relation

$$R_{\rm GB}^2\big|_{r=r_h} = \frac{12}{r_h^2} + 16\rho_{(H)}^2 - \frac{16}{r_h^2} \Big[2\rho_{(H)} + p_{\theta(H)}\Big], \tag{47}$$

where r_h is the horizon radius, $\rho_{(H)} = -T_t^t(r_h)$ and $p_{\theta(H)} = T_{\theta}^{\theta}(r_h)$. One can easily see that, for a generic matter content, the above quantity has no definite sign.

One may then ask the following two questions: (1) is $R_{GB}^2|_{r=r_h} < 0$ close to the maximal charge for any charged BH model? (2) is $R_{GB}^2|_{r=r_h} < 0$ a *necessary* condition for GB⁻ scalarisation for any charged BH model?

We will now show, by concrete illustrations, that both these questions have a *negative* answer.

5.1. Einstein-Maxwell-Dilaton BHs

To answer question (1) above, we have investigated the sign of the GB invariant (together with its behaviour in the bulk) for the stringy generalisation of the RN BH—the Gibbons-Maeda-Garfinkle-Horowitz-Strominger (GMGHS) family of BHs [33,34]. In our context, these solutions are found for an action of the form

$$\mathcal{S}_{\text{EMdsGB}} = \int d^4x \sqrt{-g} \left[\frac{1}{4} R - \frac{1}{2} \partial_\mu \psi \partial^\mu \psi - \frac{1}{4} e^{-2\alpha\psi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \epsilon \frac{\lambda^2}{4} f(\phi) R_{\text{GB}}^2 \right], \tag{48}$$

describing an Einstein-Maxwell-dilaton-scalar-GB model, which includes an extra scalar field (the dilaton ψ) with a non-minimally coupling with the Maxwell term, where $\alpha \ge 0$ is a constant of the theory. The GMGHS solution is found for $\phi = 0$, with $f(\phi)$ satisfying the condition (9). It is easy to prove that the behaviour found in the RN case ($\alpha = 0, \psi = 0$) is recovered for small enough values of α . In that case, for large enough values of the electric charge R_{GB}^2 becomes negative in a region between horizon and some maximal value of the radial coordinate. However, a direct computation shows that, for $\alpha > 0.9036$ (for the conventions used in [34]), R_{GB}^2 is strictly positive at the horizon and also in the bulk, irrespective of the value of the electric charge. Thus, as with Schwarzschild vacuum BHs, GB⁻ scalarisation of the GMGHS with large enough values of the dilaton coupling constant becomes impossible. This suggests that it would be interesting to check the status of GB⁻ scalarisation of the rotating counterpart of the GMGHS BH, the well known Kerr-Sen BH [35].

5.2. Einstein-Yang-Mills BHs

To answer question (2) above, we have investigated the sign of the GB invariant for the case of Einstein–Yang-Mills (EYM) BHs with SU(2) non-Abelian hair (nA) [36–38]. In our context, these solutions are found for an action of the form

$$\mathcal{S}_{\text{EYMsGB}} = \int d^4x \sqrt{-g} \left[\frac{1}{4}R - \frac{1}{4}F^{(a)}_{\mu\nu}F^{\mu\nu(a)} - \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi + \epsilon \frac{\lambda^2}{4}f(\phi)R^2_{\text{GB}} \right],\tag{49}$$

with $F_{\mu\nu}^{(a)}$ the nA field strength and a = 1, 2, 3. These so called *coloured* BHs are asymptotically flat and possess a *single* global "charge"—the ADM mass, despite the presence of a local magnetic field (see [39,40] for reviews). Another striking difference with respect to their (magnetic) RN Abelian counterparts is the existence of a smooth solitonic limit [41], obtained as the horizon size shrinks to zero. At the same time, there is no upper bound on their horizon size. However, the large EYM BHs are essentially Schwarzschild solutions; the contribution of the YM fields to the total ADM mass becomes negligible, albeit these fields are still nontrivial.

Contact with question (2) above comes from observing that the GB invariant is always positive at the horizon for these solutions. However, R_{GB}^2 may take negative values in a shell which does not touch the horizon, i.e., for some range of the radial coordinate $r_h < r_1 < r < r_2$ —see Figure 7 (left panel). This feature occurs for small enough BHs: using the metric form (20) and the conventions in [39], we confirmed such shell is present for $0 < r_h \leq 0.71$, or, equivalently, $0 < a_H \leq 0.158$.

Given this qualitative difference with the RN case, one could ask whether GB^{ϵ} scalarisation is still possible for both signs of ϵ . The answer is positive, and we have constructed the corresponding GB scalar clouds, i.e., solved Equation (10) for a large set of EYM BH backgrounds and both values of ϵ —see Figure 7 (right panel). As expected, $\epsilon = +1$ scalar clouds exist for all non-Abelian BHs. The value of the ratio $\lambda/M \simeq 1.704$ corresponding to a Schwarzschild BH is approach asymptotically, as $a_H \rightarrow 1$ (i.e., large EYM BHs). Also, scalar clouds with $\epsilon = -1$ exist for *all* BHs with $R_{GB}^2 < 0$ in a shell outside the horizon. Nonlinear continuations of these scalar clouds should exist, but we did not attempt to construct them.



This example makes clear that GB^{ϵ} scalarisation of BHs is not necessarily supported and triggered in the immediate vicinity of the event horizon.

Figure 7. (Left panel) GB invariant as a function of the radial coordinate for an EYM BH with horizon radius $r_h = 0.6$. One notices a shell with $R_{GB}^2 < 0$, located outside the horizon. (**Right panel**) λ/M of the set of EYM BHs supporting scalar clouds as a function of the reduced horizon area for $\epsilon = \pm 1$.

6. Further Remarks

There is a well known analogy between the spinning vacuum Kerr BHs and the electrically charged static RN solutions.⁶ They possess many similar properties at the level of a thermodynamical description; in particular, both RN and Kerr BHs have an extremal limit, with a finite horizon size. In the context of this work, it is interesting to note that the GB invariant of a RN BH changes sign if *q* is large enough as the Kerr one changes sign for sufficiently large *j*. Thus, it is natural to conjecture that the qualitative picture found concerning the GB^{ϵ} scalarisation of Kerr BHs [22–25] should be essentially recovered when replacing rotation by electric charge, with the existence of both $\epsilon = \pm 1$ scalarised solutions.

In this work we have confirmed that this conjecture is true, and construct the corresponding scalarised RN BHs. That is, we provide evidence for the following scenario: given an expression for the coupling function $f(\phi)$, two classes of charged RN-scalarised solutions may exist for the same global charges. The first one has $\epsilon = 1$, and can be viewed as a generalization of the Q = 0 solutions in [2–4]. The second has $\epsilon = -1$, and in this case the condition $\mu_{\text{eff}}^2 < 0$ is supported by a large enough charge to mass ratio $q > q_c = 0.957058$, which implies $R_{\text{GB}}^2 < 0$ for some region outside the horizon. We have also presented a preliminary investigation of the spinning generalisations of the above BHs, i.e., the scalarised Kerr-Newman BHs.

In the last part of this work, we have addressed the generality of these results. First, we established that the GB invariant of the stringy dilatonic generalisation of the RN BH becomes strictly positive for large enough values of the dilaton coupling constant. Also, we pointed out the possibility that the $\epsilon = -1$ BH scalarisation may also appear in situations where R_{GB}^2 is negative in a spherical *shell* outside and disconnected from the horizon. This is the case of the coloured BHs in EYM theory.

In this paper, to simplify the picture, we have assumed the absence of a self-interaction term for the scalar field in the action (1). GB⁺ scalarisation of spherical BHs including such self-interactions is discussed e.g., in [42,43]. Our results could be generalised to include such self-interactions.

The results in this paper emphasise that GB⁻ scalarisation is not only spin induced, but can also be induced by electromagnetic fields. Whereas electric charge is typically neglected for astrophysical BHs, the astrophysical environment surrounding astrophysical BHs has non-trivial electromagnetic fields, by virtue of the ionized matter in accretion disks, sourcing, for instance, magnetic fields. Thus, it would be interesting to consider in detail if and under which conditions such environments, if sufficiently "heavy", trigger GB⁻ scalarisation.

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Notes

- ¹ For spherical symmetry, we have also explored the exponential coupling studied in [3,8], and observed that the behaviour is qualitatively similar.
- ² Similar solutions are likely to exist for any other values of the quantum numbers, some preliminary results being found for the $\ell = 1$, m = n = 0 case. An investigation of Q = 0, $\ell = 1$ static solutions has been reported in [17].
- ³ This feature can be explained by studying Equation (27) [28].
- ⁴ An alternative expression for (40), in terms of $P_1 \equiv (1 + \sqrt{1 j^2 q^2})r/r_h$ and $P_2 \equiv j \cos \theta$, is

$$R_{\rm GB}^2 = \frac{48}{M^4} \frac{1}{(P_1^2 + P_2^2)^3} \left[1 - \frac{2}{(P_1^2 + P_2^2)^3} \left(P_2^2 (3P_1^2 - P_2^2)^2 + q^2 (P_1(P_1^4 - 10P_1^2P_2^2 + 5P_2^4) - \frac{q^2}{12} \left(5P_1^4 - 38P_1^2P_2^2 + 5P_2^4 \right) \right) \right],$$

a form which has been employed in our study.

- ⁵ The Kerr-Newman BH can also be written in this coordinate system. The corresponding expressions in the Kerr limit can be found in [30].
- ⁶ Since the electric-magnetic duality is still valid for the (Abelian) models in this work, the solutions possess a dual magnetic description.

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