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Deformed General Relativity and Quantum Black Holes Interior

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Abstract: Effective models of black holes interior have led to several proposals for regular black holes. In the so-called polymer models, based on effective deformations of the phase space of spherically symmetric general relativity in vacuum, one considers a deformed Hamiltonian constraint while keeping a non-deformed vectorial constraint, leading under some conditions to a notion of deformed covariance. In this article, we revisit and study further the question of covariance in these deformed gravity models. In particular, we propose a Lagrangian formulation for these deformed gravity models where polymer-like deformations are introduced at the level of the full theory prior to the symmetry reduction and prior to the Legendre transformation. This enables us to test whether the concept of deformed covariance found in spherically symmetric vacuum gravity can be extended to the full theory, and we show that, in the large class of models we are considering, the deformed covariance cannot be realized beyond spherical symmetry in the sense that the only deformed theory which leads to a closed constraints algebra is general relativity. Hence, we focus on the spherically symmetric sector, where there exist non-trivial deformed but closed constraints algebras. We investigate the possibility to deform the vectorial constraint as well and we prove that non-trivial deformations of the vectorial constraint with the condition that the constraints algebra remains closed do not exist. Then, we compute the most general deformed Hamiltonian constraint which admits a closed constraints algebra and thus leads to a well-defined effective theory associated with a notion of deformed covariance. Finally, we study static solutions of these effective theories and, remarkably, we solve explicitly and in full generality the corresponding modified Einstein equations, even for the effective theories which do not satisfy the closeness condition. In particular, we give the expressions of the components of the effective metric (for spherically symmetric black holes interior) in terms of the functions that govern the deformations of the theory.

Keywords: black holes; loop quantum gravity; modified gravity

1. Introduction

Black holes are iconic predictions of general relativity. Their classical description is very well established, and strong evidence for their existence is now regularly reported through gravitational wave astronomy. However, their quantum description is still an open problem. The presence of a classical singularity in their core prevents having a complete description of their interior geometry, and more generally of their formation. It is widely believed that, once a consistent UV completion of general relativity is available, classical singularities will be replaced by a well defined "quantum geometry", allowing for a consistent unitary evolution of the degrees of freedom at the quantum level. Since a complete and consistent quantum theory of gravity is still missing, important efforts towards the understanding of the quantum description of black holes have been focused on developing instead effective approaches. The construction of effective regular black holes geometries have triggered an important activity over the last decades, initiated by the work of Bardeen (see [1] for a review). However, such ad hoc models suffer from several inconsistencies, and in particular, fail to provide a framework to discuss completely the end point of the evaporation[1], as they predict generically an infinite time of evaporation [3].

Alternative approaches to quantum black holes rely on canonical quantizations of the phase space of spherically symmetric gravity in vacuum (see [4–6] for early works on this topic). The polymer quantization, inspired by loop quantum gravity, has led to several proposals for the effective description of the Schwarzschild interior. Early works in this direction, motivated by previous results on the polymer quantization of cosmological backgrounds, were presented in [7–15], suggesting new scenarios for the end of the collapse [16,17]. Since then, the polymer quantization of this model has been revisited by several authors [18–22], with several improvements along the years. Later on, effective approaches based on a "quantum" deformation of the classical phase space were used to provide effective and regular metrics for the Schwarzschild interior. Such effective models appear as a shortcut to investigate possible quantum corrections to the interior geometry without going through the whole quantization procedure[2]. Following this strategy, one found effective geometries where the singular vacuum Schwarzschild interior has been replaced by a black-to-white hole bounce, first in [25,26], and more recently in [27,28]. These new solutions have been discussed in [29–31] and an alternative effective polymer model of black-to-white hole transition, providing several improvements, has been introduced in [32–34] (see also [35–44] for additional investigations on polymer black hole geometries and [45–48] for yet another class of effective models of black-to-white hole transition). While these classical polymer constructions provide a straightforward and interesting platform to investigate quantum corrected effective metrics, the strategy employed in these Hamiltonian constructions turns out to suffer from several shortcomings related to the covariance of the quantum corrections.

To explain this problem, we start by giving the basic building bocks of these polymer effective models. First, let us recall that the Hamiltonian formulation of vacuum spherically symmetric gravity, where there are no local degrees of freedom, leads to two first class constraints, the Hamiltonian and (radial) vectorial constraints, which generate time and radial diffeomorphisms. Then, the common strategy employed in these effective polymer models is to introduce some regularization at the phase space level through some modifications of the Hamiltonian constraint known as point-wise holonomy corrections, while keeping the vectorial constraint unchanged. In general, this regularization breaks the covariance of the system, as the algebra of the first class constraints is no longer closed. Then, the solutions to the modified equations of motion one can extract from this modified Hamiltonian system do not have a classical space-time interpretation, as covariance is lost. In models such as those in [25–28] as well as in [32], where only the homogeneous interior region is considered, the issue of covariance is not visible. However, if one tries to extend these models to take into account inhomogeneities, the issue of the covariance breaking becomes important.

Interestingly, it was realized in a series of articles [49–54] that, under suitable conditions, one can construct regularizations where the deformed algebra of constraints remain closed [3]. The consequences of this deformed covariance was then investigated in depth in the context of spherically symmetric

[1] See however [2] for a new construction with potentially interesting features.

[2] The common underlying assumption of these effective polymer models is that the modified Hamiltonian constraint actually corresponds to the expectation value of the Hamiltonian operator in some suitable coherent states at the quantum level. However, it is fair to say that the concrete realization of this assumption for polymer black holes is usually left aside, and one only deals with a modified classical system. Recently, efforts towards computing this expectation value of the Hamiltonian operator for spherically symmetric quantum geometries were presented by [23], based on the framework of quantum reduced loop quantum gravity. Consequences for the black hole interior geometry were discussed by [24].

[3] Even more recently, it was realized that, if one starts from the spherically symmetric vacuum gravity phase space in terms of the self-dual variables, for which the Barbero–Immirzi parameter $\gamma = \pm i$, one obtains polymer-like modifications while keeping the algebra of constraints closed and undeformed [55]. This construction was generalized to the Gowdy model and

backgrounds [60–62] and then extended to more general symmetry reduced models in [63–68]. Phenomenological implications were explored both for inhomogeneous cosmological backgrounds as well as for black holes in [69–71]. More recently, exact solutions for black holes interior in a very large class of gravitational systems with deformed covariance were found in [72,73], and relations between these deformed theories and modified gravity (in the framework of scalar–tensor theories) have been investigated in [74–80].

In this paper, we further investigate the issue of covariance in effective black holes models. As mentioned above, there exist modifications of spherically symmetric general relativity, inspired from loop quantum gravity, which lead to a deformed but closed constraints algebra. This property makes the effective theories particularly interesting because they are invariant under a deformed covariance, thus there is no effective quantum anomalies, and then the classical solutions have a clear geometrical meaning. Now, we can ask the question whether one can extend such a construction beyond spherical symmetry. We explore this question in Section 2. We consider a large class of modified theories of gravity whose deformation is not only inspired from (holonomy corrections induced by) loop quantum gravity but is also a natural generalization of the deformation introduced in the context of spherical symmetry. In particular, these theories are invariant under three-dimensional space-like diffeomorphisms. We prove that any theory of this class which has a closed constraints algebra reduces, after some canonical transformations, to general relativity with a cosmological constant. Hence, at the level of the full theory, we show that there is no non-trivial deformations of general relativity in a large class of modified gravity theories with the condition that they produce a closed constraints algebra. On the one hand, this result clearly contrasts with what happens when space-times are reduced to spherical symmetry[4]. On the other hand, this finding parallels the well known Hojman–Kuchar–Teteilboim theorem, which allows reconstructing uniquely, from the Dirac's hypersurface algebra, the Einstein–Hilbert action [81].

In Section 3, we revisit and study new aspects of loop (holonomy) deformations of spherically symmetric general relativity. We start with a quick Hamiltonian analysis of general relativity reduced to spherical symmetry using Ashtekar-like variables, and we obtain the two expected constraints, the Hamiltonian constraint and the radial diffeomorphisms constraint. Then, we ask the question whether one can deform these constraints to keep their algebra closed. First, we show that there is no non-trivial deformations (up to canonical transformations) of the sub-algebra generated by the radial diffeomorphisms constraint. In other words, space-like diffeomorphisms cannot be consistently deformed in an effective theory of general relativity and they remain, in that sense, classical. This result is consistent with loop quantum gravity where kinematical quantum states are invariant under classical space-like diffeomorphisms. However, it is possible to deform the Hamiltonian constraint in a way that the constraints algebra is deformed but closed. We compute and recover the most general deformation which keeps the algebra closed.

In Section 4, we compute the equations of motion for the effective metric components and we look for their solutions corresponding to "static" black holes interiors, which means that the metric components are time dependent only. Notice that, in that case, the closeness of the constraints algebra is not an issue and one can safely relax this condition. Hence, we consider in this section very general deformations of the Hamiltonian constraint with the only restriction that it still transforms as a scalar under radial diffeomorphisms. Remarkably, we completely solve the deformed Einstein equations in that very general case and we give the expressions of the components of the effective metric in terms of the functions that govern the deformations. For concreteness, we illustrate this general result with simple examples.

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perturbative inhomogeneous cosmological background. Moreover, the polymer-like modifications can be implemented within the $\bar{\mu}$-scheme [56–58]. These results have been generalized recently in [59].

[4] Or cylindrical symmetry.

We conclude the article with a brief summary and a discussion of the results. We also present some interesting perspectives.

2. Deformed General Relativity

In this section, we recall some aspects and also give new results on modified gravity, which are useful in the following sections when we study modifications of gravity induced by loop quantum gravity. In particular, we start by introducing a large class of theories of modified gravity which share some features with effective theories of loop quantum gravity when it is reduced to spherical symmetry. However, contrary to effective loop quantum gravity, these modified theories of gravity are defined in full generality and not only in the case of symmetry reduced backgrounds. Then, we discuss these theories and show in particular the impossibility to have a closed deformed algebra of constraints in 3+1 dimensions in this class of theories. This result raises the question of the possibility to extend the construction of effective loop quantum gravity theories with a closed deformed algebra of constraints beyond spherical symmetry.

2.1. From (Effective) Loop Quantum Gravity to Modified Gravity

As any theory of quantum gravity, loop quantum gravity is expected to modify Einstein equations at the effective level. Such modifications have been computed and studied for homogeneous and spherically symmetric space-times to understand the effects of quantum gravity in early cosmology and black holes physics. In general, one distinguishes between two types of deformations, known as holonomy and inverse-triad corrections. Here, we are concerned only with (point-)holonomy corrections, which are local modifications in the sense that they do not produce non-local equations of motion.

These (point-)holonomy corrections are assumed to modify only the Hamiltonian constraint and do not affect the vectorial constraints. Indeed, in the construction of the kinematical Hilbert space of loop quantum gravity, the invariance under diffeomorphisms is imposed "classically" from the action of diffeomorphisms on the graphs of spin-network states. Hence, in general, one does not use holonomies to quantize the vectorial constraints. For this reason, it is natural to require that any effective theory of loop quantum gravity remains invariant under (undeformed) spatial diffeomorphisms. Furthermore, we show in the case of spherical symmetry that it is impossible to construct a non-trivial deformation of the vectorial constraint, which leaves the deformed diff-algebra closed (see Section 3.2).

To construct such effective theories, we assume that the space-time $\mathcal{M}$ if of the form $\Sigma \times \mathbb{R}$ where Σ is a space-like slice, and we consider the ADM parameterization of the metric,

$$ds^2 \quad = \quad -N^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,, \tag{1}$$

where γ_{ij} is the spatial metric on Σ, N is the lapse function, and N^i the shift vector. We also need to introduce the second fundamental form (extrinsic curvature tensor) whose components are given by

$$K_{ij} = \frac{1}{2N}\left(\dot{\gamma}_{ij} - D_i N_j - D_j N_i\right)\,, \tag{2}$$

where D_i denotes the spatial covariant derivative associated with the spatial metric h_{ij}. In this parameterization, the Einstein–Hilbert action takes the form

$$S_{EH}[g] = \int d^4x\,\sqrt{-g}\,\mathcal{R} \;=\; \int d^4x\, N\sqrt{\gamma}\left(K_{ij}K^{ij} - K^2 + R\right)\,, \tag{3}$$

where $\mathcal{R}$ and R are, respectively, the four-dimensional and the three-dimensional Ricci scalars and g and γ are the determinants of the metrics g_{ij} and γ_{ij}, respectively. In the following, we omit the boundary contribution to the action even though a complete analysis would require including them.

To mimic loop quantum gravity type modifications, we now introduce a class of modified theories of gravity defined by an action of the form

$$S[\gamma_{ij}, N, N^i] \equiv \int d^4x \, N\sqrt{\gamma} \left[F(K^i_j) + R \right] , \tag{4}$$

where F is an arbitrary three-dimensional scalar constructed from the components of the extrinsic curvature K_i^j. Only the part of the action which depends on K_i^j is modified. This choice is motivated by the fact that, as we recall below (for spherically symmetric spacetimes), (point-)holonomy corrections affect mainly the components of the extrinsic curvature, whereas the components of the three-dimensional Ricci tensor are left unchanged[5]. Interestingly, these modifications contrast with deformations à la Horava gravity [82,83] where the spatial "Ricci" part of the action is strongly modified, whereas the "K-part" is (almost) unchanged compared to general relativity (up to a detuning relative coefficient between $K_{ij}K^{ij}$ and K^2).

Hence, we consider the actions in Equation (4) as models for effective loop quantum gravity theories in any background. As K^i_j can be viewed as a three-dimensional matrix, the most general function $F(K^i_j)$ which transforms as a scalar under space-like diffeomorphisms can be written as

$$F(K^i_j) \;=\; F\left(\mathrm{tr}(K), \mathrm{tr}(K^2), \mathrm{tr}(K^3)\right) , \tag{5}$$

where $\mathrm{tr}(M) \equiv M_{ij}\gamma^{ij} = M^i_i$ for any matrix M. This is a direct consequence of the Cayley–Hamilton theorem. For simplicity, we are using the same notation for the two (different) functions F in the l.h.d. and r.h.s. of Equation (5).

Before analyzing further this class of theories, let us make a couple of remarks. First, modified theories of gravity which are invariant under spatial diffeomorphisms have only been studied intensively these last years on many aspects. Their Lagrangians involve not only the extrinsic curvature tensor, but also the three-dimensional Ricci tensor, the lapse and the shift, and their covariant derivatives. Horava gravity is an example of such theories with remarkable ultra-violet properties [82,83]. Scalar–tensor theories in the unitary gauge (where the scalar is a function of time only) are other interesting examples which have been applied to late time cosmology [84–86].

The second remark concerns the relation between the theories in Equation (4) and effective loop quantum gravity. Once again, let us emphasize that the choice of such theories to model the effects of loop quantum gravity at an effective level is motivated by results in symmetry reduced situations where modifications affect the components of the momenta of the metric (see, for instance, [7,9,11,18,25,28,62,68]). More rigorous constructions of effective theories (based on the full canonical or covariant quantum theory) with no symmetry reduction have been proposed recently and they lead to much more complete and more involved descriptions [23,24]. For the purposes of this paper where we study the possibility to extend the symmetry reduced effective theories to general (non-symmetry reduced) effective theories, the action in Equation (4) is the most general starting point provided that we require that the modifications are local and affect only the extrinsic curvature.

2.2. Canonical Analysis and Deformed Hamiltonian Constraint

The aim of this subsection is to make a canonical analysis of Equation (4) to see how the Hamiltonian constraint is modified compared to general relativity. For simplicity, we first assume that the function F in Equation (5) depends on $\mathrm{tr}(K)$ and $\mathrm{tr}(K^2)$ only. We discuss later the general case where F depends also on $\mathrm{tr}(K^3)$.

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[5] Notice that the standard boundary term has been ignored here, but it would be interesting to investigate its role in such deformed theory. A careful investigation of this boundary term is nevertheless needed to properly understand how the quasi-local observables are affected in such deformed gravity theory.

We start by introducing the pairs of canonical variables,

$$\{\gamma_{ij}(x), \pi^{kl}(y)\} = \frac{1}{2}(\delta_i^k\delta_j^l + \delta_j^k\delta_i^l)\,\delta^3(x-y)\,, \quad \{N(x), \pi_N(y)\} = \delta^3(x-y)\,. \tag{6}$$

As the theory is invariant under spatial diffeomorphisms, we do not need to introduce momenta associated to the shift vector N^i. In the deformed theories, the lapse is still not dynamical and we get the primary constraint $\pi_N = 0$.

An immediate calculation shows that the momenta π^{ij} are given by

$$\pi^{ij} \;\equiv\; \sqrt{\gamma}p^{ij} \;\equiv N\sqrt{\gamma}\frac{\partial F}{\partial \dot{\gamma}_{ij}} \;=\; \sqrt{\gamma}\left(\frac{1}{2}F_{(1)}\gamma^{ij} + F_{(2)}K^{ij}\right)\,, \tag{7}$$

where $F_{(a)}$ is the partial derivative of F with respect to its first $(a=1)$ or second $(a=2)$ variable. From the expression of the momenta in Equation (7), it is immediate to see that

$$\mathrm{tr}(p) = \frac{3}{2}F_{(1)} + \mathrm{tr}(K)F_{(2)}\,, \quad \mathrm{tr}(p^2) = \frac{3}{4}F_{(1)}^2 + \mathrm{tr}(K^2)F_{(2)}^2 + \mathrm{tr}(K)F_{(1)}F_{(2)}\,, \tag{8}$$

which implies, as expected, that $\mathrm{tr}(p)$ and $\mathrm{tr}(p^2)$ are scalars that can be expressed in terms of $\mathrm{tr}(K)$ and $\mathrm{tr}(K^2)$. Locally (if F satisfies regularity conditions which we assume to be true), one can reverse these relations as follows

$$\mathrm{tr}(K) = A\left(\mathrm{tr}(p), \mathrm{tr}(p^2)\right)\,, \quad \mathrm{tr}(K^2) = B\left(\mathrm{tr}(p), \mathrm{tr}(p^2)\right)\,, \tag{9}$$

where the explicit form of the functions A and B is not needed here. Then, one makes use of these relations to invert the equations between the velocities and the momenta, and to solve K^{ij} in terms of p^{ij},

$$K^{ij} = \frac{1}{F_{(2)}}\left(p^{ij} - \frac{F_{(1)}}{2}\gamma^{ij}\right)\,. \tag{10}$$

In these equations, $F_{(1)}$ and $F_{(2)}$ are viewed as functions of $\mathrm{tr}(p)$ and $\mathrm{tr}(p^2)$ using Equation (9). Finally, the Hamiltonian can be obtained immediately, and after a short calculation, we show that

$$\begin{aligned} H &= \int d^3x\sqrt{\gamma}\left(p^{ij}\dot{\gamma}_{ij} - N\left[F\left(\mathrm{tr}(K), \mathrm{tr}(K^2)\right) + R\right]\right) && (11)\\ &= \int d^3x\sqrt{\gamma}\left(N\mathcal{H} + N^i\mathcal{H}_i\right)\,, && (12) \end{aligned}$$

where $\mathcal{H}_i \equiv -2D^j p_{ij}$ is the usual vectorial constraint, whereas $\mathcal{H}$ is a deformed Hamiltonian constraint given by

$$\mathcal{H} \equiv G(\mathrm{tr}(p), \mathrm{tr}(p^2)) - R\,, \qquad \text{where} \quad G \equiv \frac{2}{F_{(2)}}\mathrm{tr}(p^2) - \frac{F_{(1)}}{F_{(2)}}\mathrm{tr}(p) - F\,. \tag{13}$$

Here again, the function F and its derivative are viewed as functions of $\mathrm{tr}(p)$ and $\mathrm{tr}(p^2)$ using Equation (9). As there are no restrictions on the function F (up to some regularity conditions which allow the inversion in Equation (9)), the function G is also arbitrary, and the action in Equation (4) leads to a generic loop-like deformation of the Hamiltonian constraint where only the momenta component of the Hamiltonian is affected. Hence, as mentioned above, it provides a very general model to test effective loop quantum gravity theories where only the Hamiltonian constraint is modified. The theory remains invariant under spatial diffeomorphisms, and the vectorial constraint is unchanged.

Notice that, if we had started with a function F which would have depended also on the variable $\mathrm{tr}(K^3)$, we would have obtained an expression for the Hamiltonian constraint similar to Equation (13) with a function G depending also on $\mathrm{tr}(p^3)$. In the rest of the paper, we restrict our study to the case in Equation (13).

2.3. Deformed Hamiltonian Constraint vs. Closed Algebra of Constraints

Once the Hamiltonian has been computed, one continues the Hamiltonian analysis to see whether the deformed Hamiltonian constraint is first or second class. As $\mathcal{H}$ is a scalar with respect to spatial diffeomorphisms, it commutes obviously with the vectorial constraint and studying the stability of the Hamiltonian constraint under time evolution reduces to computing the Poisson algebra

$$C[u,v] \equiv \{\mathcal{H}[u], \mathcal{H}[v]\}\,, \quad \text{with} \qquad \mathcal{H}[u] \equiv \int d^3x \sqrt{\gamma}\, u\, (G-R)\,, \tag{14}$$

for any functions u, v on Σ. If $C[u,v]$ is weakly vanishing, $\mathcal{H}$ is first class and the canonical analysis stops here with the conclusion that the theory propagates two degrees of freedom as in general relativity. On the other hand, when $C[u,v]$ is not weakly vanishing, the theory admits a new (secondary) constraint and one has to analyze further the properties of this constraint to know the number of degrees of freedom.

The calculation of $C[u,v]$ can be done as follows. From the very definition of the Poisson bracket, we have

$$C[u,v] = \int d^3z \left[\frac{\delta}{\delta\gamma^{ij}(z)} \int d^3x \sqrt{\gamma(x)}\, u(x) R(x)\right] \left[\int d^3y \sqrt{\gamma(y)}\, v(y) \frac{\delta G(y)}{\delta \pi_{ij}(z)}\right] - (u \leftrightarrow v)\,, \tag{15}$$

where $(u \leftrightarrow v)$ means that we add the same expression where the roles of the functions u and v are inverted, so that $C[u,v]$ is skew-symmetric. Then, using the definition of the three-dimensional Ricci scalar, we obtain

$$C[u,v] = \int d^3x \sqrt{\gamma} \left[D^i D^j u - \gamma^{ij} \Delta u\right] v \left[G_{(1)} \gamma_{ij} + 2 G_{(2)} p_{ij}\right] - (u \leftrightarrow v)\,, \tag{16}$$

where $\Delta = D^i D_i$ is the Laplacian, and $G_{(a)}$ is the derivative of G with respect to its first $(a=1)$ or second $(a=2)$ variable. Later, we also use the notation $G_{(ab)}$ for the second partial derivatives of G. The initial three integrals in Equation (15) reduce to only the one integral in Equation (16) after some integrations by part. The rest of the calculation is straightforward and one obtains finally

$$C[u,v] = 2 \int d^3x \sqrt{\gamma} (u D^j v - v D^j u) \left[G_{(2)} D^i p_{ij} + \mathcal{C}_j\right]\,, \tag{17}$$

where

$$\mathcal{C}_j \equiv p_{ij}\, D^i G_{(2)} - D_j (G_{(1)} + \mathrm{tr}(p) G_{(2)})\,. \tag{18}$$

Thus, $C[u,v]$ reduces to a sum of two terms in the last parenthesis in Equation (17). The first one is nothing but the vectorial constraint, which is obviously weakly vanishing. In general, the second term in Equation (18) is not vanishing and leads to new constraints in the theory. Before discussing these new constraints, let us ask the question whether one can get a closed algebra of constraints with a deformed Hamiltonian constraint in a non-symmetry reduced case. This happens only when $\mathcal{C}_j$ vanishes (at least weakly). An immediate calculation shows that $\mathcal{C}_j$ can be written as a sum of four terms,

$$\begin{aligned} \mathcal{C}_j \;=\; & -[G_{(11)} + G_{(2)} + \mathrm{tr}(p) G_{(12)}] D_j \mathrm{tr}(p) + G_{(12)} p_{ij} D^i [\mathrm{tr}(p)] \\ & -[G_{(12)} + G_{(22)} \mathrm{tr}(p)] D_j \mathrm{tr}(p^2) + G_{(22)} p_{ij} D^i [\mathrm{tr}(p^2)]\,, \end{aligned} \tag{19}$$

which have independent tensorial structures. Hence, $\mathcal{C}_j$ vanishes only if each term vanishes independently, which leads to the following four conditions of the function G,

$$G_{(11)} + G_{(22)} + \text{tr}(p)G_{(12)} = 0\,, \quad G_{(12)} + \text{tr}(p)G_{(22)} = 0\,, \quad G_{(12)} = G_{(22)} = 0\,. \tag{20}$$

The general solution, which can be computed immediately, reads (up to a global constant factor)

$$G = \text{tr}(p^2) - \frac{1}{2}\text{tr}(p)^2 + \alpha\text{tr}(p) + \lambda\,, \tag{21}$$

where α and λ are constant. Without loss of generality, one can always fix α to zero from a canonical transformation of the form $p_{ij} \mapsto p_{ij} + \beta\gamma_{ij}$, whereas γ_{ij} is unchanged. In that case, we recover the Hamiltonian constraint of general relativity supplemented with a cosmological constant.

As a consequence, none of the theories in the class in Equation (4), which is different from general relativity with a cosmological constant, admits a closed algebra of constraints generated by the usual vectorial constraints and a deformed Hamiltonian constraint. Furthermore, the stability under time evolution of the deformed Hamiltonian constraint leads to a differential equation for the lapse function,

$$\mathcal{S} \equiv ND^i\mathcal{C}_i + 2\mathcal{C}_iD^iN \ \approx\ 0\,, \tag{22}$$

which has to be understood as a new constraint in the theory. We use the standard notation $\approx$ for the weak equality (equality on the constraints surface). Not only $\mathcal{S}$ does not commute with π_N, but also their Poisson bracket is non-local in the sense that it involves derivatives of delta distributions. Such a situation is pathological and makes the theory ill-defined with an undefined number of degrees of freedom.

The conclusion of this analysis is that one cannot extend the condition of having a closed algebra of constraints associated to an effective loop quantum gravity theory for an arbitrary (non-symmetry reduced) background which satisfies the following properties: first, it is invariant under spatial diffeomorphisms; second, the deformation does not involve the three-dimensional Ricci tensor but only the extrinsic curvature tensor; and, finally, the effective theory is a local theory of the metric. One should relax one of these hypothesis to construct an effective theory of loop quantum gravity. As the fundamental variables in loop quantum gravity are holonomies of a connection, one may expect to get instead a non-local or a non-metric deformation.

Nonetheless, requiring a closed algebra of constraints for an effective theory of loop quantum gravity (satisfying the previous properties) for spherically symmetric space-times becomes possible for some reasons we explain below. Their constructions lead to very interesting scenarios, as we see in the next two sections.

3. Deformations of Spherically Symmetric Gravitation

From now on, we restrict our study to (non-static) spherically symmetric space-times. Even if the deformed gravity theory in this symmetry reduced sector cannot be embedded into a fully covariant theory, it is still of interest to consider such specific models for at least two reasons. First, it provides a rare enough example of a gravitational system with deformed covariance, and leads to interesting peculiar effects, such as transition between Lorentzian and Euclidean regimes in the deep interior for which we have explicit solutions. Second, this deformed covariance is not tied to spherical symmetry, but is also realized in more general systems, such as cylindrical symmetry reduced gravity (or Gowdy systems). Hence, while deformed covariance can only be realized in symmetry reduced systems, it provides an interesting testbed to understand the spacetime description beyond standard GR.

Focusing therefore on the spherically symmetric sector, one can choose a coordinate system such that the spatial line element $d\ell^2$ takes the simple form

$$d\ell^2 \equiv \gamma_{ij}dx^idx^j = \gamma_{rr}dr^2 + \gamma_{\theta\theta}(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{23}$$

where γ_{rr} and $\gamma_{\theta\theta}$ are functions of (t, r) only. We also assume that the lapse function N and the (radial component of the) shift vector N^r are functions of (t, r) only.

We start this section with a quick review of the Hamiltonian analysis of general relativity reduced to spherical symmetry. This gives us the opportunity to introduce some notations. Afterwards, we study the possibility to deform the theory à la loop quantum gravity. We adopt a Hamiltonian point of view and ask the question of the possibility to deform the constraints keeping a closed Poisson algebra. First, we prove the impossibility to deform the vectorial constraint without introducing anomalies (in the algebra of constraints). However, as is well known, we show on the other hand that it is possible to deform the Hamiltonian constraint keeping a closed algebra, and we classify these deformations. This results contrasts with the results obtained in the previous section where we show the impossibility to deform (with some hypothesis) the Hamiltonian constraint for a generic background with no particular symmetries. We discuss why reducing to spherical symmetry makes the deformation possible.

3.1. Reduction to Spherical Symmetry

We start with a review of the Hamiltonian analysis of general relativity for non-static spherically symmetric geometries. The ADM parameterization of the metric is given by Equation (1) where N^r is the only non-trivial component of the shift vector and the spatial metric is Equation (23). The only non-vanishing components of the extrinsic curvature in Equation (2) are

$$K^r_r = \frac{1}{2N}\left(\frac{\dot\gamma_{rr}}{\gamma_{rr}} - N_r\frac{\partial_r\gamma_{rr}}{\gamma_{rr}^2} - 2\frac{\partial_r N_r}{\gamma_{rr}}\right), \tag{24}$$

$$K^\theta_\theta = K^\varphi_\varphi = \frac{1}{2N}\left(\frac{\dot\gamma_{\theta\theta}}{\gamma_{\theta\theta}} + N_r\frac{\partial_r\gamma_{\theta\theta}}{\gamma_{rr}\gamma_{\theta\theta}}\right). \tag{25}$$

Therefore, the Einstein–Hilbert action in Equation (3) reduces to (up to an irrelevant global constant that we neglect but can be computed from the integration over the angles θ and φ)

$$S_{EH} = \int dr\,dt\,L_{EH} = \int dr\,dt\,N\sqrt{\gamma_{rr}}\gamma_{\theta\theta}\left(-2(K^\theta_\theta)^2 - 4K^\theta_\theta K^r_r + R\right). \tag{26}$$

We recall that R is the three-dimensional Ricci scalar for spherically symmetric background whose expression is given below.

To go further, it is convenient to introduce the "electric fields" E^r and E^φ associated to the metric in Equation (23), which are defined by the relations,

$$\gamma_{rr} \equiv \frac{(E^\varphi)^2}{E^r}, \qquad \gamma_{\theta\theta} \equiv E^r, \tag{27}$$

with the condition that $E^\varphi > 0$. Hence, Equation (26) is now considered as an action for the dynamical variables E^r and E^φ. To perform its canonical analysis, one introduces the two pairs of conjugate momenta,

$$\{E^r(x), \pi_r(y)\} = \delta(x-y), \qquad \{E^\varphi(x), \pi_\varphi(y)\} = \delta(x-y), \tag{28}$$

whereas N and N^r are considered as Lagrange multipliers. Notice that the notations x and y refer to the radial coordinate. The momenta are easily computed and read

$$\pi_\varphi = -4\sqrt{E^r}K^\varphi_\varphi, \qquad \pi_r = -2\frac{E^\varphi}{2\sqrt{E^r}}K^r_r. \tag{29}$$

Then, we compute the Hamiltonian of the theory,

$$H \equiv \int dr\ \left(\pi_r \dot{E}^r + \pi_\varphi \dot{E}^\varphi - L_{EH}\right) = \int dr\ \left(N\mathcal{H} + N^r \mathcal{H}_r\right) \tag{30}$$

where the Hamiltonian and vectorial constraints are, respectively, given by,

$$\mathcal{H} \equiv -\frac{E^\varphi}{2\sqrt{E^r}}\pi_\varphi^2 - 2\sqrt{E^r}\pi_\varphi \pi_r - 4E^\varphi \sqrt{E^r} R\,, \quad \mathcal{H}_r \equiv E^\varphi \pi_\varphi' - \pi_r (E^r)'\,, \tag{31}$$

up to a rescaling of the lapse function, with the following expression of the Ricci scalar,

$$4R = \frac{1}{2E^r} - \frac{((E^\varphi)')^2}{8(E^\varphi)^2 E^r} + \frac{(E^r)'(E^\varphi)'}{2(E^\varphi)^3} - \frac{(E^r)''}{2(E^\varphi)^2}\,. \tag{32}$$

We use the notation f' for the derivative of any function f with respect to r.

It is well-known that $\mathcal{H}$ and $\mathcal{H}_r$ define first class constraints, generate the invariance under diffeomorphisms for non-static spherically symmetric backgrounds, and satisfy the closed Poisson algebra,

$$\begin{aligned} \{\mathcal{H}_r[u], \mathcal{H}_r[v]\} &= \mathcal{H}_r[u'v - uv']\,, && (33)\\ \{\mathcal{H}_r[u], \mathcal{H}[v]\} &= -\mathcal{H}[uv']\,, && (34)\\ \{\mathcal{H}[u], \mathcal{H}[v]\} &= \mathcal{H}_r[\gamma^{rr}(uv' - vu')]\,. && (35) \end{aligned}$$

where $\mathcal{H}_r[u]$ and $\mathcal{H}[u]$ are the smeared constraints, u being a regular function of r. As a consequence, the Dirac analysis of the constraints stops here, and the theory propagates no degrees of freedom.

One expects loop quantum gravity to modify, at the effective level, the Einstein equations. In the Hamiltonian framework, this means that one expects modifications of the constraints of the theory. Even though we show in the previous section that it is not possible to obtain a non-trivial effective deformation of the full theory with local modifications of the Einstein–Hilbert action (of the form in Equation (4)) which keep the constraints algebra closed, we can ask the same question when we consider spherically symmetric space-times. It is well-known that this problem has a solution in that case. Now, we derive the conditions for deformed vectorial and Hamiltonian constraints to still have a closed Poisson algebra.

3.2. Deformation of the Vectorial Constraint

From the construction of kinematical states in loop quantum gravity, we know that one has to keep the invariance under spatial diffeomorphisms, or at least to keep a closed algebra for an eventual deformation of the vectorial constraint (to avoid having anomalies). Indeed, spatial diffeomorphisms are still symmetries of the kinematical Hilbert space of loop quantum theory, and there is no reason to violate such a symmetry at the effective level. However, one can wonder if the (spatial) diffeomorphism algebra could be eventually deformed (compared to the classical case) but still closed. We explore this problem and show that, under some general hypothesis, this is not possible.

3.2.1. First Necessary Condition

For that purpose, we start with the simple remark that π_φ and E^r transform as scalars under the action of the vectorial constraint $\mathcal{H}_r$, whereas E^φ and π_r are densities. Hence, for convenience (in this subsection only), we introduce the notations,

$$q_1 \equiv \pi_\varphi\,, \quad p_1 \equiv -E^\varphi\,, \quad q_2 \equiv E^r\,, \quad p_2 \equiv \pi_r\,, \tag{36}$$

so that

$$\{q_i(x), p_j(y)\} = \delta_{ij}\delta(x-y)\,, \qquad \mathcal{H}_r = -(p_1 q_1' + p_2 q_2')\,. \tag{37}$$

From the point of view of the vectorial constraint, the two degrees of freedom (q_1, p_1) and (q_2, p_2) are decoupled (there are no cross terms involved), and we look for modifications which conserve this decoupling.

Hence, we ask the question whether there exists a function $\mathcal{D}(q, q', p, p')$ such that the deformed vectorial constraint,

$$\mathcal{H}_{r,\text{def}} \equiv \mathcal{D}(q_1, q_1', p_1, p_1') + \mathcal{D}(q_2, q_2', p_2, p_2') \;\approx\; 0\,, \tag{38}$$

satisfies a closed Poisson algebra. We can treat the two components separately and compute the Poisson brackets between the smeared function

$$\mathcal{D}[u] \equiv \int dr\, u(r)\, \mathcal{D}(q, q', p, p')\,, \tag{39}$$

where we omit, for simplicity, the labels (1 or 2) for the position and momentum variables. Variations of $\mathcal{D}[u]$ induced by variations δq and δp of p and q, respectively, are easily computed and read

$$\delta\mathcal{D}[u] \;=\; \int dr\, \left[\delta q\, (u\mathcal{D}_q - (u\mathcal{D}_{q'})') + \delta p\, (u\mathcal{D}_p - (u\mathcal{D}_{p'})')\right]\,, \tag{40}$$

where $\mathcal{D}_z$ denotes the derivative of $\mathcal{D}$ with respect to $z \in \{q, q', p, p'\}$. Hence, after an immediate calculation, we show that

$$\begin{aligned} \{\mathcal{D}[u], \mathcal{D}[v]\} \;=\; & \int dr\, (uv' - u'v)\, \Big[\mathcal{D}_q\mathcal{D}_{p'} - \mathcal{D}_{q'}\mathcal{D}_p + \mathcal{D}_{q'}(\mathcal{D}_{p'q}q' + \mathcal{D}_{p'p}p' + \mathcal{D}_{p'q'}q'' + \mathcal{D}_{p'p'}p'') \\ & - \mathcal{D}_{p'}(\mathcal{D}_{q'q}q' + \mathcal{D}_{q'p}p' + \mathcal{D}_{q'q'}q'' + \mathcal{D}_{q'p'}p'')\Big]\,. \end{aligned} \tag{41}$$

For the algebra to be closed, the terms proportional to p'' and q'' in Equation (41) must vanish, which implies the two necessary conditions

$$\mathcal{D}_{q'}\mathcal{D}_{p'q'} - \mathcal{D}_{p'}\mathcal{D}_{q'q'} = 0 = \mathcal{D}_{q'}\mathcal{D}_{p'p'} - \mathcal{D}_{p'}\mathcal{D}_{q'p'}\,. \tag{42}$$

These two equations are obviously similar and then are solved in the same way. We concentrate on the first one, which is solved as follows:

$$\frac{\mathcal{D}_{q'q'}}{\mathcal{D}_{q'}} - \frac{\mathcal{D}_{q'p'}}{\mathcal{D}_{p'}} = 0 \Longleftrightarrow \frac{\partial}{\partial q'}\left(\ln \mathcal{D}_{q'} - \ln \mathcal{D}_{p'}\right) = 0 \Longleftrightarrow \mathcal{D}_{p'} = A(q, p, p')\mathcal{D}_{q'}\,, \tag{43}$$

where the function A is arbitrary. Similarly, the second condition leads to

$$\mathcal{D}_{p'} = B(q, p, q')\mathcal{D}_{q'}\,, \tag{44}$$

where the function B is also arbitrary. As a consequence, $A(q, p, p') = B(q, p, q') = C(q, p)$ and the general solution of the two previous conditions is

$$\mathcal{D}(q, q', p, p') = D\left(q' + C(q, p)p'\right)\,, \tag{45}$$

where D is an arbitrary function of one variable. Hence, any deformation of the diffeomorphism algebra is necessarily of the form in Equation (45). Before going further and solving the remaining conditions, we make a canonical transformation that drastically simplifies the analysis.

3.2.2. Canonical Transformation: Simplification of the Problem

Indeed, we introduce a new pair of canonically conjugate variables (Q, P) related to (q, p) via the generating function $\Phi(p, Q)$ by the relations

$$q = \frac{\partial \Phi}{\partial p} \equiv \alpha(p, Q)\,, \qquad P = \frac{\partial \Phi}{\partial Q} \equiv \beta(p, Q)\,. \tag{46}$$

Hence, the combination $q' + C(q, p)p'$ that appears in Equation (45) transforms under this canonical transformation according to

$$q' + C(q, p)p' = \alpha_Q Q' + \left[\alpha_p + C(\alpha(p, Q), p)\right] p'\,, \tag{47}$$

and then we can always choose a generating function Φ such that its partial derivative α satisfies the condition

$$\alpha_p(p, Q) + C\left(\alpha(p, Q), p\right) = 0\,, \tag{48}$$

so that $q' + C(q, p)p' = \alpha_Q Q'$ does not depend on P', and the deformed constraint takes the simple form

$$\mathcal{D}(q, q', p, p') = D(\alpha_Q(Q, P)Q')\,, \tag{49}$$

where we use the shorthand $\alpha_Q(Q, P)$ for $\alpha_Q(p(Q, P), Q)$. When the function C is regular enough, the first-order partial differential equation (Equation (48)) always admits a solution at least locally.

As a consequence, up to canonical transformations, the deformed constraint can be reduced, without loss of generality, to the simple form

$$\mathcal{D}(q, q', p, p') \;=\; D(X)\,, \qquad X \equiv J(q, p)q'\,, \tag{50}$$

where J is an arbitrary function.

3.2.3. No-Go: No Closed Algebra for Deformed Diffeomorphisms Constraints

At this stage, we return to the expression of the Poisson bracket in Equation (41) between the deformed constraints and we immediately see that it simplifies considerably when $\mathcal{D}$ is of the form in Equation (50),

$$\{\mathcal{D}[u], \mathcal{D}[v]\} \;=\; \int dx\, (u'v - uv')\mathcal{D}_{q'}\mathcal{D}_p\,. \tag{51}$$

Hence, the full constraint in Equation (38) has a closed Poisson algebra if and only if

$$\mathcal{D}_{q'}\mathcal{D}_p = \lambda \mathcal{D}\,, \tag{52}$$

where λ is necessarily a constant (independent of q_1, p_1, q_2, p_2, and their derivatives). Substituting Equation (50) into this equation leads to

$$\frac{XD_X^2}{D} = \frac{\lambda}{J_p}\,, \tag{53}$$

which can be solved easily for the function D that is given by

$$D = \lambda \frac{X}{J_p}\,. \tag{54}$$

As D is a function of X only by definition, J_p is necessarily a constant. Finally, we arrive at the conclusion that the only constraints in Equation (38) which admit a closed Poisson algebra are such that

$$\mathcal{D}(q,q',p,p') = \lambda(p+\mu(q))q' \,, \tag{55}$$

where we recall that λ is a constant, and the arbitrary function $\mu(q)$ can be set to zero without loss of generality by a simple canonical transformation. Hence, there is no deformation of the diffeomorphisms Poisson algebra which preserves the decoupling in Equation (38).

3.3. Deformation of the Scalar Constraint

Now, we review the possibility to deform the Hamiltonian constraint keeping a closed (but deformed) constraints Poisson algebra. Hence, we look for deformed Hamiltonian constraints $\mathcal{H}_{\rm def}$, which are functions on the phase space, such that Equation (34) remains unchanged and Equation (35) is deformed but closed.

3.3.1. General Point-Holonomy Deformation of the Scalar Constraint

First, the fact that the Poisson bracket in Equation (34) is unchanged implies that the deformed Hamiltonian constraint $\mathcal{H}_{\rm def}$ is a scalar density of weight of +1. Furthermore, as we consider point-holonomy corrections only, the Ricci part of the classical Hamiltonian constraint in Equation (31) is unchanged and then $\mathcal{H}_{\rm def}$ can be written in the form

$$\mathcal{H}_{\rm def} \;=\; \sqrt{\gamma}\,\left[S(\pi_\varphi,\frac{\pi_r}{E^\varphi},E^r) - 4R\right] \,, \tag{56}$$

where S is an arbitrary function of the three independent scalars,

$$X_1 \equiv \pi_\varphi \,, \quad X_2 \equiv \sqrt{E^r}\,\frac{\pi_r}{\sqrt{\gamma}} = \frac{\pi_r}{E^\varphi} \,, \quad X_3 \equiv E^r \,, \tag{57}$$

which generate the algebra of scalar functions in the phase space. Notice that the factor 4 that comes with the Ricci scalar R has been introduced to make the comparisons with the classical constraint in Equation (31) easier.

Now, we compute the Poisson bracket between two deformed Hamiltonian constraints in Equation (56). For simplicity, we introduce the notation

$$G(\pi_\varphi,\pi_r,E^\varphi,E^r) \equiv \sqrt{\gamma}\,S(\pi_\varphi,\frac{\pi_r}{E^\varphi},E^r) \,, \tag{58}$$

and we show that, for any pair of functions (u,v), the Poisson bracket between the smeared deformed constraints $\mathcal{H}_{\rm def}[u]$ and $\mathcal{H}_{\rm def}[v]$ is given by

$$\{\mathcal{H}_{\rm def}[u],\mathcal{H}_{\rm def}[v]\} = \int dr\,ds\;[u(r)v(s)-u(s)v(r)]\left\{G(r),-\sqrt{h(s)}R(s)\right\} \,, \tag{59}$$

which, after a long but straightforward calculation, reduces to

$$\{\mathcal{H}_{\rm def}[u],\mathcal{H}_{\rm def}[v]\} = \int dr\,(uv'-u'v)\,\frac{\sqrt{E^r}}{2E^\varphi}\left[\frac{(E^r)'}{E^\varphi}\frac{\partial G}{\partial \pi_\varphi} - \left(\frac{\partial G}{\partial \pi_r}\right)'\right] . \tag{60}$$

If one substitutes the expression of G in terms of the function S in Equation (58), this Poisson bracket becomes

$$\begin{aligned}\{\mathcal{H}_{\rm def}[u],\mathcal{H}_{\rm def}[v]\} \;&=\; \int dr\,(uv'-u'v)\frac{\sqrt{E^r}}{2E^\varphi}\left[(E^r)'\sqrt{E^r}S_{(1)}-\frac{(E^r)'}{2\sqrt{E^r}}S_{(2)}\right.\\ &\left.\quad -\sqrt{E^r}\left(S_{(12)}\pi'_\varphi+S_{(22)}\left(\frac{\pi_r}{E^\varphi}\right)'+S_{(23)}(E^r)'\right)\right], \qquad (61)\end{aligned}$$

where $S_{(a)}\equiv\partial S/\partial X_a$ and $S_{(ab)}\equiv\partial^2 S/(\partial X_a\partial X_b)$ denote the partial derivatives of the function S with respect to the variables X_a in Equation (57).

3.3.2. Closeness of the Deformed Algebra

The constraints algebra is closed if and only if

$$\{\mathcal{H}_{\rm def}[u],\mathcal{H}_{\rm def}[v]\} \;=\; \mathcal{H}_r[\gamma^{rr}_{\rm def}(uv'-u'v)]\,, \qquad (62)$$

where $\gamma^{rr}_{\rm def}$ could be an arbitrary function on the phase space which represents the deformation of the (inverse) metric component γ^{rr}. As a consequence, from Equation (61), we immediately see that a necessary condition for the algebra to be closed is that $S_{(22)}=0$, which implies that S is an affine function of X_2, and then, as $\sqrt{\gamma}=E^\varphi\sqrt{E^r}$,

$$S(\pi_\varphi,\frac{\pi_r}{E^\varphi},E^r)=A(\pi_\varphi,E^r)+B(\pi_\varphi,E^r)\frac{\pi_r}{E^\varphi}\,, \qquad (63)$$

where A and B are arbitrary functions. In that case, it is easy to see that Equation (61) reduces to

$$\begin{aligned}\{\mathcal{H}_{\rm def}[u],\mathcal{H}_{\rm def}[v]\} \;&=\; \mathcal{H}_r[\gamma^{rr}_{\rm def}(uv'-u'v)]\\ &\quad+\int dr\,(uv'-u'v)\frac{E^r(E^r)'}{2E^\varphi}\left[\frac{\partial A}{\partial\pi_\varphi}-\frac{\partial B}{\partial E^r}-\frac{B}{2E^r}\right], \qquad (64)\end{aligned}$$

with the deformed (inverse) metric coefficient given by

$$\gamma^{rr}_{\rm def}\equiv-\frac{E^r}{2(E^\varphi)^2}\frac{\partial B}{\partial\pi_\varphi}\,. \qquad (65)$$

As a consequence, the algebra is closed if and only if the following condition is satisfied

$$\frac{\partial A}{\partial\pi_\varphi}-\frac{\partial B}{\partial E^r}-\frac{B}{2E^r} \;=\; 0\,, \qquad (66)$$

otherwise there is an anomaly and the Hamiltonian constraint is no longer first class, which means that it does not generate any symmetries. This conclusion is consistent with what has already been found in the literature [62,66,68,87].

Before going further and discussing this condition, notice that in general relativity the Hamiltonian constraint is of the form in Equation (56) with Equation (63) where the functions A and B are explicitly given by

$$A \;=\; -\frac{\pi_\varphi^2}{2E^r}\,,\qquad B \;=\; -2\pi_\varphi\,, \qquad (67)$$

which obviously satisfy the closeness condition in Equation (66). Furthermore, the deformed inverse metric in Equation (65) reduces, in that case, to the classical inverse metric γ^{rr} in Equation (27).

3.3.3. Discussion

We find deformations of spherically symmetric reduced general relativity that lead to a closed constraints algebra where the vectorial constraint is unchanged but the Hamiltonian constraint is deformed by (point-)holonomy-like corrections. These theories admit two different symmetries, which are the classical invariance under diffeomorphisms and an invariance under a deformed time reparameterisation. Hence, they have no local degrees of freedom, exactly as general relativity when it is reduced to spherical symmetry. The deformed symmetry has been discussed recently in [73].

The possibility to have a closed (deformed) constraints algebra relies on the fact that we consider spherically symmetric backgrounds only. The reason is simply that there is only one spatial diffeomorphisms constraint in the symmetry reduced theory, whereas there are obviously three of them in the full theory. As shown in Section 2.3, requiring the invariance under three-dimensional spatial diffeomorphisms leads to no-go results and to the impossibility of having a non-trivial Hamiltonian (local and point-holonomy-like) deformation of general relativity.

To illustrate better the specificity of spherically symmetric models compared to fully (three-dimensional) covariant theories, let us study the spherically symmetric reduced version of Equation (4) when F is of the form in Equation (5) with no $\text{tr}(K^3)$ dependency. For that purpose, we use the same notations as in Section 3.1 and we compute the momenta canonically conjugated to E^r and E^φ, which are given by the relations

$$\frac{E^r\pi_r}{\sqrt{\gamma}} = \frac{1}{2}F_{(1)} + (2K^\varphi_\varphi - K^r_r)F_{(2)}\,, \qquad \frac{E^\varphi\pi_\varphi}{\sqrt{\gamma}} = F_{(1)} + 2K^r_r F_{(2)}\,, \tag{68}$$

where $F_{(1)}$ and $F_{(2)}$ are the derivatives of F with respect to $\text{tr}(K)$ and $\text{tr}(K^2)$, respectively. From these expressions, we immediately show the two following relations

$$\frac{E^r\pi_r}{\sqrt{\gamma}} + \frac{E^\varphi\pi_\varphi}{\sqrt{\gamma}} = \frac{3}{2}F_{(1)} + \text{tr}(K)F_{(2)}\,, \tag{69}$$

$$\left(\frac{E^\varphi\pi_\varphi}{\sqrt{\gamma}}\right)^2 + 2\left(\frac{E^r\pi_r}{\sqrt{\gamma}} + \frac{1}{2}\frac{E^\varphi\pi_\varphi}{\sqrt{\gamma}}\right)^2 = 2F_{(1)}^2 + 4\text{tr}(K)F_{(1)}F_{(2)} + 4\text{tr}(K^2)F_{(2)}^2\,, \tag{70}$$

which imply that $\text{tr}(K)$ and $\text{tr}(K^2)$ can be (at least implicitly) expressed in terms of the combinations $E^r\pi_r/\sqrt{\gamma}$ and $E^\varphi\pi_\varphi/\sqrt{\gamma}$. As a consequence, one shows after a long but straightforward calculation that the Hamiltonian is of the usual form in Equation (30) with a classical vectorial constraint and the deformed Hamiltonian constraint in Equation (56) with

$$S(\pi_\varphi, \frac{\pi_r}{\sqrt{\gamma}}, E^r) = \tilde{S}\left(\frac{E^r\pi_r}{\sqrt{\gamma}}, \frac{E^\varphi\pi_\varphi}{\sqrt{\gamma}}\right) \equiv F_{(1)}\text{tr}(K) + 2F_{(2)}\text{tr}(K^2)\,, \tag{71}$$

where, in the last equation, $\text{tr}(K)$ and $\text{tr}(K^2)$ are expressed in terms of

$$\frac{E^r\pi_r}{\sqrt{\gamma}} = E^r\frac{\pi_r}{\sqrt{\gamma}}\,, \qquad \frac{E^\varphi\pi_\varphi}{\sqrt{\gamma}} = \frac{\pi_\varphi}{\sqrt{E^r}}\,. \tag{72}$$

As a consequence, these theories have a deformed Hamiltonian constraint of the type in Equation (56). Hence, they fall in the class of theories with a closed deformed constraints algebra if the conditions in Equations (63) and (66) are fulfilled, even though its fully covariant constraints algebra has an anomaly. Thus, this analysis illustrates clearly that a fully covariant theory with a non-closed constraints algebra can lead, in the spherical symmetric sector, to a reduced theory with a closed constraints algebra. This remark raises the possibility to extend the closeness property beyond spherical symmetry in a fully covariant effective (local and metric) action for loop quantum gravity.

3.3.4. The Case of a Non-Closed Constraints Algebra

When the conditions in Equations (63) and (66) for having a closed constraints algebra are not fulfilled, the conservation under time evolution of the deformation Hamiltonian constraint leads to a new constraint given by,

$$\mathcal{C} \equiv (E^r)'\sqrt{E^r}S_{(1)} - \frac{(E^r)'}{2\sqrt{E^r}}S_{(2)} - \sqrt{E^r}\left(S_{(12)}\pi_\varphi' + S_{(22)}\left(\frac{\pi_r}{E^\varphi}\right)' + S_{(23)}(E^r)'\right) \approx 0\,, \tag{73}$$

which reduces to the constraint

$$\frac{\partial A}{\partial \pi_\varphi} - \frac{\partial B}{\partial E^r} - \frac{B}{2E^r} \approx 0\,, \tag{74}$$

when only the condition in Equation (63) is satisfied and $(E^r)' \neq 0$. Generically, the new constraint $\mathcal{C}$ has a non-vanishing Poisson bracket with the deformed Hamiltonian constraint,

$$\{\mathcal{H}_{\mathrm{def}}[u], \mathcal{C}\} \neq 0\,, \tag{75}$$

for any non-vanishing function u. As a consequence, in that situation, the deformed Hamiltonian constraint and the secondary constraint in Equation (73) form a pair of second-class constraints. There is no longer invariance under deformed time reparameterization and the lapse function is a Lagrange multiplier which is fixed by the requirement that the secondary constraint has to be stable under time evolution. Hence, the equation satisfied by the lapse is simply given by

$$\dot{\mathcal{C}} = \{\mathcal{C}, \mathcal{H}_{\mathrm{def}}[N]\} = 0\,, \tag{76}$$

which is, in general, a partial differential equation. Clearly, $N = 0$ is a solution, but it is physically unacceptable. A necessary condition for the theory to be physically relevant is the existence of a non-vanishing solution for the lapse function.

As in the previous case, these theories do not have local degrees of freedom because one is replacing the first class constraint $\mathcal{H}_{\mathrm{def}}$ by a pair of second class constraints $(\mathcal{H}_{\mathrm{def}}, \mathcal{C})$. Let us emphasize again that, in this case, the lapse function is not free and can be, in principle, expressed in terms of the phase space variables. Contrary to the fully covariant case where the non-closeness of the constraints algebra leads to inconsistencies, spherically symmetric models with a non-closed constraints algebra is well-defined, even though it no longer exhibits any deformed reparameterization invariance. Furthermore, it is not clear whether these theories have, in general, physically relevant solutions.

4. Effective Black Holes Interior Solutions

The purpose of this section is to study (interior) black hole solutions of the theory defined by a deformed Hamiltonian constraint of the form in Equation (56) while the vectorial constraint is unchanged. First, we compute the equations of motion. Then, we show how to resolve them in full generality. Finally, we give some concrete examples, and we drawn comparisons with solutions already found in the literature.

4.1. Equations of Motion

Here, we look for static spherically symmetric black hole solutions where the metric components in Equation (23), the lapse, and the shift vector depend on the "radial" coordinate only. Furthermore, as the deformations of general relativity are induced by quantum gravity, we are interested in the geometry inside the black hole where quantum gravity effects are supposed to become important (at least close enough to the classical singularity). In the black hole interior (behind the horizon), the radial coordinate plays the role of time and we denote it t instead of r. In other words, the role of these variables changes when one crosses the horizon, which corresponds to considering time-dependent degrees of freedom only.

Thus, the equations of motion and the constraints reduce to ordinary differential equations involving time derivatives only. Hence, the vectorial constraint is trivially satisfied and one can formally forget about the closedness property of the constraints algebra. Furthermore, the secondary constraint $\mathcal{C}$ itself, introduced in Equation (73), is also trivially satisfied when radial derivatives vanish, and then the lapse function is free.

Now, let us compute the equations of motion. For that, we recall that the time evolution of any function $\mathcal{O}$ on the phase space is defined from the Poisson bracket by

$$\dot{\mathcal{O}} = \{\mathcal{O}, \mathcal{H}_{\rm def}[N] + \mathcal{H}_r[N^r]\}\,. \tag{77}$$

Hence, the equations of motion for the phase space variables in Equation (28) are easily obtained, and they are partial differential equations that we cannot solve in full generality. As mentioned when introducing this section, we restrict our study to time dependent solutions only. Therefore, one can drop all radial dependency, and the Hamiltonian constraint together with the effective Einstein equations dramatically simplify. Indeed, the Hamiltonian constraint becomes

$$\mathcal{H}_{\rm def} \;=\; \sqrt{E^r}E^\varphi S\left(\pi_\varphi, \frac{\pi_r}{E^\varphi}, E^r\right) - \frac{E^\varphi}{2\sqrt{E^r}}\,, \tag{78}$$

and the Einstein equations are

$$\begin{aligned}
\dot{E}^r &= N\sqrt{E^r}S_{(2)}\,, && (79)\\
\dot{E}^\varphi &= N\sqrt{E^r}E^\varphi S_{(1)}\,, && (80)\\
\dot{\pi}_r &\approx -N\sqrt{E^r}E^\varphi\left(S_{(3)} + \frac{1}{2(E^r)^2}\right)\,, && (81)\\
\dot{\pi}_\varphi &\approx \frac{N\sqrt{E^r}\pi_r}{E^\varphi}S_{(2)}\,, && (82)
\end{aligned}$$

where the weak equality $\approx$ means an equality up to a term proportional to the constraint $\mathcal{H}_{\rm def}$. As the dynamical system reduces to a classical mechanical system (and not a classical field theory because the radial dependency has disappeared), the time evolution of the deformed Hamiltonian constraint is trivially satisfied, which can be verified explicitly. As a consequence, one can forget about the secondary constraint in Equation (73) and still have a consistent dynamical system. Indeed, one immediately sees that Equation (73) is trivially satisfied when radial derivatives are vanishing.

4.2. General Solution of the Equations of Motion

Let us show how to solve these equations in full generality. What makes the system "solvable" is that it admits a "triangular" structure in the sense that one can first decouple the variable E^r from the other variables, then solve the equation for E^r, and then successively decouple the equations for π_φ, E^φ, and π_r that can be solved in principle (at least numerically). This method works for any function S provided that $S_{(2)}$ does not vanish and generalizes the results of [72].

4.2.1. Resolution of the System

As announced above, we start by solving E^r. For that, we fix the lapse function by the condition

$$NS_{(2)} = 2\,, \tag{83}$$

which is equivalent to change the time variable t into τ such that $2d\tau = NS_{(2)}dt$. In that case, the equation for E^r (Equation (79)) simplifies and can solved explicitly according to

$$E^r(t) = (t+a)^2\,, \tag{84}$$

where a is an integration constant that we fix to $a = 0$ in order to recover the Schwarzschild solution at the classical limit.

Then, we concentrate on the equation for π_φ (Equation (82)), which becomes,

$$\dot{\pi}_\varphi = 2t\frac{\pi_r}{E^\varphi}\,, \tag{85}$$

where, to simplify notations, we replaced the weak equality $\approx$ by a standard equality. To go further and solve this equation, we use the Hamiltonian constraint in Equation (78), which enables us to write the relation

$$2t^2\, S(\pi_\varphi, \frac{\pi_r}{E^\varphi}, t^2) = 1\,. \tag{86}$$

As $S_{(2)}$ is supposed not to vanish, we can formally (and locally) invert this equation and solve π_r/E^φ in terms of π_φ and t (by virtue of the implicit function theorem) according to,

$$\frac{\pi_r}{E^\varphi} = P(\pi_\varphi, t^2)\,, \tag{87}$$

where P is the (implicit) function defined by the relation

$$2t^2\, S(\pi_\varphi, P(\pi_\varphi, t^2), t^2) = 1\,. \tag{88}$$

Interestingly, when the necessary condition in Equation (63) to have a closed constraints algebra is satisfied, the function P can be explicitly computed and it is given by the expression

$$P(\pi_\varphi, t^2) \;=\; \frac{1-2t^2 A(\pi_\varphi, t^2)}{2t^2 B(\pi_\varphi, t^2)}\,. \tag{89}$$

In any case, the variable π_φ in Equation (85) decouples and satisfies the equation

$$\dot{\pi}_\varphi = 2tP(\pi_\varphi, t^2)\,, \tag{90}$$

which can be solved, at least numerically. Below, we propose examples where it can be solved analytically.

We continue with the equation for E^φ in Equation (80), which can now be solved according to

$$E^\varphi \;=\; \exp\left(2\int^t du\, u\, \frac{S_{(1)}(u)}{S_{(2)}(u)}\right)\,, \tag{91}$$

where $S_{(a)}(t) = S_{(a)}(\pi_\varphi(t), P(\pi_\varphi(t), t^2), t^2)$ and $\pi_\varphi(t)$ is given by the solution of Equation (90). Of course, E^φ is defined up to a constant which can be fixed by physical conditions. Finally, the remaining variable π_r is immediately obtained from Equation (87).

4.2.2. Summary of the Results: Metric in the Black Hole Interior

To conclude this section, we summarize the results. The resolution relies on the choice of the lapse function in Equation (83) (which is equivalent to a change of coordinate) and on the (implicit) inversion of the Hamiltonian constraint, which state the existence of a function P such that

$$\frac{\pi_r}{E^\varphi} = P(\pi_\varphi, E^r)\,, \qquad \text{where} \quad 2E^r S(\pi_\varphi, P(\pi_\varphi, E^r), E^r) \;=\; 1\,. \tag{92}$$

In general, P is implicitly defined only and can be computed locally (at the vicinity of any points in the phase space when $S_{(2)} \neq 0$). In some cases, the function P can be found explicitly. Then, the general solution is

$$E^r = t^2\,, \quad \dot{\pi}_\varphi = 2tP(\pi_\varphi, t^2)\,, \quad E^\varphi = \exp\left(2\int^t du\, u\, \frac{S_{(1)}(u)}{S_{(2)}(u)}\right)\,, \quad \pi_r = E^\varphi P(\pi_\varphi, t^2)\,, \tag{93}$$

which depends on two integration constants, the first one coming from the integration of π_φ, and the second one coming in the integral defining E^φ. Notice that the constant a, which appears in the integration of E^r in Equation (84), has already been fixed to $a = 0$. Finally, the metric inside the spherical black hole is given by

$$ds^2 = -\frac{1}{F(t)}dt^2 + G(t)\,dr^2 + t^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\,. \tag{94}$$

where the two functions F and G are defined by

$$F(t) \equiv \frac{S_{(2)}^2}{4}\,, \qquad G(t) \equiv \frac{(E^\varphi)^2}{t^2}\,. \tag{95}$$

Such a metric corresponds to a black hole interior if it admits at least one event horizon, which imposes conditions on the functions F and G.

4.3. Conditions for Describing a Trapped Interior Region

At this stage, we have obtained an exact expression of the most general homogeneous solution of the very general effective theory introduced before. The components of the effective metric are expressed in terms of the functions that govern the deformation of the theory. Nonetheless, a priori, this solution could describe a cosmological background or a homogeneous black hole interior. Here, we are interested in the case of black holes interiors.

In order for this solution to describe a well defined black hole interior, and therefore a trapped region, one has to impose some conditions.

- The geometry is bounded by an outer horizon located at t_{h}.
- This outer horizon is null, and therefore it can be interpreted as a black hole horizon.
- The geometry can be consistently extended through this horizon, in the outer-communication region corresponding to $t > t_{\mathrm{h}}$.

To impose these three conditions, it is useful to introduce the Kodama vector defined for any spherically symmetric geometry as follows. Following [88], a general spherically symmetric spacetime can always be written as,

$$ds^2 = \sigma_{ab}dx^a dx^b + R^2(t,r)d\Omega^2, \tag{96}$$

where σ_{ab} is the two-dimensional metric on the base space with standard coordinates (t,r), the scalar function $R(t,r)$ is the physical radius, and $d\Omega^2$ is the metric on the normalized two-sphere. With these notations, the Kodama vector is defined by

$$k^\alpha := \epsilon_\perp^{\alpha\beta}\nabla_\alpha R\,, \tag{97}$$

where $\epsilon_\perp^{\alpha\beta} = \epsilon^{\alpha\beta}/\sqrt{|\sigma|}$ is the densitized two-dimensional Levi–Civita tensor (see [88,89] for details as well as [90–92]). In time-dependent and inhomogeneous geometries, where there is no time-like Killing vector, this vector turns out to be especially useful to locate the horizons. For asymptotically flat spacetimes, it coincides with the time-like Killing vector as spatial infinity. The crucial property of the Kodama vector is that it encodes the causal structure of the spacetime. In particular, assuming that there is a single horizon at t_h, surrounding a trapped region for $t < t_\mathrm{h}$, the Kodama vector turns out to be time-like, null, and space-like for $t > t_\mathrm{h}$, $t = t_\mathrm{h}$, and $0 < t < t_\mathrm{h}$, respectively. As such, the Kodama vector becomes space-like in a trapped region.

Hence, we compute the Kodama vector in the homogeneous interior metric we found, and we obtain that

$$k^\alpha\partial_\alpha = \frac{1}{\sqrt{|g_{tt}|g_{rr}}}\frac{\partial}{\partial r} = -\sqrt{\frac{F(t)}{G(t)}}\frac{\partial}{\partial r} \qquad \Rightarrow \qquad k^\alpha k_\alpha = F(t)\,. \tag{98}$$

Imposing the first two conditions above implies that there exists t_h such that

$$F(t_\mathrm{h}) = 0\,, \tag{99}$$

which signals the presence of an outer horizon. Now, imposing that there is a coordinate system in which one can extend the spacetime beyond the horizon is equivalent to demanding that the determinant of the metric remains regular on the horizon and keeps the same sign both inside and outside the horizon, which ensures that the metric remains Lorentzian in the whole spacetime. As the determinant is trivially given by

$$\det(g) = -\frac{G}{F}\,t^4\sin^2\theta\,, \tag{100}$$

at the horizon where $F(t_\mathrm{h}) = 0$, one has to impose the condition

$$0 < \lim_{t\to t_\mathrm{h}} C(t) < +\infty\,, \qquad C(t) \equiv \frac{G(t)}{F(t)}\,. \tag{101}$$

As a consequence, one obtains in addition that $g_{rr}(t_\mathrm{h}) = G(t_\mathrm{h}) = 0$, which ensures that the outer horizon is indeed a black hole horizon. Moreover, one has to impose that the deformation are such that the curvature invariants remain regular at the horizon.

Once we have identified the generic conditions for the solution in Equation (94) to describe a black hole interior, bounded by at least an outer horizon, it would be interesting to derive the conditions for this geometry to be regular. However, the general expression of the metric in Equation (94) prevents us from providing sharp and useful conditions on this issue. We leave therefore this interesting direction for future works.

4.4. Examples

To illustrate the previous results, we consider theories where S is of the form in Equation (63), i.e., it is an affine function of π_r/E^φ. In that case, the function P entering in Equation (93) exists globally and can be computed explicitly. Furthermore, we assume that

$$A(\pi_\varphi, E^r) = -\frac{f_1(\pi_\varphi)}{2E^r}\,, \qquad B(\pi_\varphi, E^r) = -2f_2(\pi_\varphi)\,, \tag{102}$$

where f_1 and f_2 are functions of π_φ only. If one imposes the closeness of the constraints algebra, then A and B satisfy the condition in Equation (74), which translates into $2f_2 = df_1/d\pi_\varphi$, and one recovers obviously the well-known anomaly-free condition [62,72,87]. To be general, we assume for the moment that f_1 and f_2 are independent.

Let us compute the building blocks to find the explicit form of the line element in Equation (94). A direct calculation shows that

$$P = -\frac{1}{4E^r}\frac{1+f_1(\pi_\varphi)}{f_2(\pi_\varphi)}\,, \quad S_{(1)} = \frac{1}{2E^r}\left(-f_1' + \frac{1+f_1(\pi_\varphi)}{f_2(\pi_\varphi)}f_2'\right)\,, \quad S_{(2)} = -2f_2(\pi_\varphi)\,, \tag{103}$$

where f_a' $(a=1,2)$ is the derivative of f_a with respect to π_φ. Using $E^r = t^2$ and, after a short calculation, one shows that the equation for $\pi_\varphi(t)$ can be reformulated as

$$\frac{2f_2(\pi_\varphi)}{1+f_1(\pi_\varphi)}\dot{\pi}_\varphi \;=\; -\frac{1}{t}\,. \tag{104}$$

On integrates easily this equation and shows that π_φ is obtained by inverting the relation

$$1+f(\pi_\varphi) = \frac{r_s}{t}\,, \qquad \text{with} \quad f(\pi_\varphi) \equiv \exp\left(\int^{\pi_\varphi} \frac{2f_2(x)}{f_1(x)+1}dx\right) - 1\,, \tag{105}$$

where r_s is a constant of integration. Notice that the constant that comes from the integral in the r.h.s. of Equation (105) can be reabsorbed into r_s.

The expression of E^φ follows immediately and, after direct calculations, one obtains

$$E^\varphi(t) = \frac{2b\,f_2(\pi_\varphi)}{1+f_1(\pi_\varphi)} = \frac{b}{1+f(\pi_\varphi)}f'(\pi_\varphi) = \frac{bt}{r_s}f'(\pi_\varphi)\,, \tag{106}$$

where $\pi_\varphi(t)$ is given by Equation (105) and b is a constant of integration. Without loss of generality, we can fix $2b = r_s$ (as in [72]) by a redefinition of the constant in the integral defining f as in Equation (105). As a conclusion, the line element is given by Equation (94) with

$$F(t) = \left[(f_2\circ f^{-1})(\frac{r_s}{t}-1)\right]^2\,, \qquad G(t) = \frac{1}{4}\left[(f'\circ f^{-1})(\frac{r_s}{t}-1)\right]^2\,. \tag{107}$$

As $f' = 2f_2/(f_1+1)$, we can simplify further the expression of $G(t)$, which can be written in the form $G(t) = C(t)F(t)$ as in Equation (101) with

$$C(t) = \frac{r_s^2}{t^2}\left[1+(f_1\circ f^{-1})(\frac{r_s}{t}-1)\right]^{-2}\,. \tag{108}$$

Interestingly, as f_1 and f_2 are generically independent, F and G are themselves independent and one obtains the most general "static" and spherically symmetry solution for the metric. Such a geometry describes effectively a black hole interior if it admits an event horizon that imposes the conditions we describe in the previous subsection. In particular, if we assume that the horizon is located at $t = r_s$ (as for the Schwarzchild solution), these conditions simplify and read

$$(f_2\circ f^{-1})(0) = 0\,, \qquad (f_1\circ f^{-1})(0) \neq -1\,. \tag{109}$$

In the case where the condition $2f_2 = f_1'$ to have a closed constraints algebra is satisfied, $f+1 = \lambda(f_1+1)$ where λ is an integration constant. A quick calculation shows that Equation (108) reduces to $C(t) = \lambda$ and one recovers the result of [72]:

$$F(t) = G(t) = \frac{1}{4}\left[(f_1'\circ f_1^{-1})(\frac{r_s}{t}-1)\right]^2\,, \tag{110}$$

where λ has been fixed to $\lambda = 1$.

5. Discussion

In this article, we revisit and study further the issue of deformed covariance in the so-called polymer models of quantum spherically symmetric general relativity in vacuum. We obtain several new results.

First, in Section 2, we introduce a large class of modified gravity theories which mimic, at the level of the full theory (hence, beyond spherical symmetry), the loop quantum gravity (holonomy-like) deformations that one uses in standard symmetry reduced polymer models. In these theories, the invariance under space-time diffeomorphisms is broken, whereas three-dimensional diffeomorphisms remain symmetries, as it is expected from loop quantum gravity, and only the extrinsic curvature part of the Lagrangian is deformed compared to general relativity. Obviously, these theories have a deformed Hamiltonian constraint. Hence, we ask the question whether it is possible that the constraints' algebra remains closed, even though it is deformed, and we show that this is impossible. In the full class of these theories of modified gravity, only general relativity leads to a closed constraints algebra, and such a no-go result is very similar to the uniqueness Hojmann–Kuchar–Teitelboim theorem [81]. As a consequence, it seems that the notion of deformed covariance found in spherically symmetric model and discussed in detail in [62] could be a peculiar consequence of the spherical symmetry. Nevertheless, we must emphasize that the effective models we introduce are based on the implicit assumption that quantum gravity effects can be described in terms of a local action whose dynamical variable is still a metric. Hence, it is in principle possible to evade the no-go result considering instead non-local actions or non-metric and more exotic variables.

Second, we focus on spherically symmetric models. We consider the Hamiltonian theory whose phase space consists of two pairs of conjugate variables (schematically the two free functions in the most general spherically symmetric metric together with their conjugate momenta) which satisfy two constraints, the Hamiltonian and the vectorial constraint. We ask the question to which extent one can deform these constraints, compared to general relativity, with the condition that their algebra remains closed. We show that there is no non-trivial deformation of the vectorial constraint, which provides a very strong argument for keeping the vectorial constraint undeformed, as is usually done in the context of polymer models. Then, we compute the most general deformation of the Hamiltonian constraint and we recover known results in the literature. Our construction provides a large class of effective theories with a deformed covariance that encompasses most of the polymer models introduced in the literature. Notice that the model introduced recently in [27,32] does not fall in this class mainly because the corresponding Hamiltonian constraint is, a priori, not a scalar with respect to the vectorial constraint.

Finally, we compute the equations of motion and look for spherically symmetric static solutions. Remarkably, we find an explicit, exact and very general solution. In particular, we give the expressions of the components of the effective metric in terms of the functions that govern the deformations of the theory. This exact solution, given by Equations (94) and (95), can describe both a cosmological or a black hole interior solution. Assuming that the solution corresponds to the interior of a black hole, we gave the general conditions in Equations (99) and (101) in order for the effective geometry to come with a well-defined trapped region.

This work opens many directions. First, one can extend our analysis to include different models as the ones introduced in [27,32]. One can try to find how to characterize more generally such models and to compute their general static solutions, as done in the present work. Moreover, we now have a framework to address the question of the stability of the solutions we found with respect to linear (and spherical) perturbations. To our knowledge, stability properties of the perturbed polymer black holes have been investigated only in [13,41]. Let us finally stress that the present modified gravity model could also be used to derive polymer-like deformations of the spherically symmetric exterior geometry, and could be extended to describe deformations of the axisymmetric vacuum gravity phase space.

This would allow us to discuss the effective dynamics of polymer deformations of the Kerr black hole (see [93] for current efforts in this direction). We leave these open directions for future works.

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