Article

# On the Solution of the Schrödinger Equation with Position-Dependent Mass 

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#### Abstract

We have considered the Iwasawa and Gauss decompositions for the Lie group SL( $2, \mathrm{R}$ ). According to these decompositions, the Casimir operators of the group and the Hamiltonians with position-dependent mass were expressed. Then, the unbound state solutions of the Schrödinger equations with position-dependent mass were given.


Keywords: Lie group; Casimir operator; Schrödinger equation; decomposition

## 1. Introduction

Since Lie groups have both group and manifold structures, they are widely used in various branches of mathematics and physics. In particular, the group theory technique was used to solve the time independent Schrödinger equation for some potentials [1-8]. In recent years, interest in the Schrödinger equation with position-dependent mass has gradually increased, and various studies have been carried out [9-16]. In fact, the concept of position-dependent mass comes from the effective mass approach of many body problems in condensed matter physics. The equation has a wide range of applications in the several physical systems such as electronic properties of semiconductors, quantum dots, metal clusters and so on [17-19]. Moreover, the equation is more complicated than the equation with constant coefficient. The different methods were used in these studies, such as the point canonical transformation [15], Nikiforov-Uvarov method [16], Green's functions [20], super symmetric approach [21], group theory technique [22,23]. In group theory technique, Hamiltonian of the physical system is related to the Casimir operator of the group by $[C-q]=Q[H-E]$, where $C$ is Casimir operator, $H$ is Hamiltonian operator, $Q$ is an arbitrary function and $q, E$ are eigenvalues of the Casimir and Hamiltonian operators, respectively [24,25]. The various types of Hamiltonians with position-dependent mass are proposed in [9,13]. In this paper, we shall work with BenDaniel and Duke form [26] of the Hamiltonian, i.e.,

$$
\begin{equation*}
H=-\frac{1}{m(x)} \frac{d^{2}}{d^{2} x}+\frac{\dot{m}}{m^{2}} \frac{d}{d x}+V x, \dot{m}=\frac{d m}{d x}, m(x)=m_{0} M(x), \quad\left(2 m_{0}=\hbar=1\right), \tag{1}
\end{equation*}
$$

where $M(x)$ and $m_{0}$ denote position-dependent and constant mass, respectively [27]. Levi-Leblond showed that the Hamiltonian in the form (1) is invariant under the Galilean transformations [12].

In Reference [22], a generalized procedure has been proposed to obtain exactly solvable Hamiltonian with position-dependent mass related to the Casimir operator of the Lie group. Here, the Casimir operator of the group $\operatorname{SL}(2, R)$ is associated with the Hamiltonian of the physical system as $[C+j(j+1)]=Q[H-E]$, where $-j(j+1)$ is eigenvalue of the Casimir operator $[24,25,28]$.

Let $F$ be a Lie algebra and $J_{i}$ a set of its elements $(i=1, \ldots, r)$. If an operator commutes with all elements of the algebra, i.e., $\left[C, J_{i}\right]=0$, then it is called Casimir operator [2]. The problem of construction of the Casimir operators has been studied by many scientists [2,29-32]. The number of
independent Casimir operators is equal to the rank of the algebra. The Casimir operator is represented by the infinitesimal operators of representation of the group. The right regular representation of the group can be expressed as follows [22]

$$
\begin{equation*}
T\left(g _ { 0 } \left(f(g)=\left(h\left(g g_{0}\right) / h(g)\right)^{1 / 2} f\left(g g_{0}\right), \quad g, g_{0} \in G, h(g)>0\right.\right. \tag{2}
\end{equation*}
$$

The T representation is unitary according to the inner product

$$
\begin{equation*}
\left(f, f^{\prime}\right)=\int f^{*}(g) f^{\prime}(g) d \mu \tag{3}
\end{equation*}
$$

where $d \mu=h(g) d g$ is an invariant measure on group. In the case of $h(g)=1$, the T representation above can be written as follows $\widetilde{T}\left(g_{0}\right) \widetilde{f}(g)=\widetilde{f}\left(g g_{0}\right)$. Obviously, the $\widetilde{T}$ representation is equivalent to T representation. The W transform that provides this equivalence is given by

$$
\begin{equation*}
W: f \rightarrow \widetilde{f}=h^{1 / 2} f \tag{4}
\end{equation*}
$$

Now, we can find the infinitesimal operators of the representation and the Casimir operator.
The infinitesimal operators $J_{k}$ corresponding to one parameter subgroups $\omega_{k}$ of the regular representation of the group is given by $J_{k}=-\left.i \frac{d}{d \tau} T\left(\omega_{k}(\tau)\right)\right|_{\tau=0}, k=1,2,3$. In addition, these operators satisfy commutation relations $[22,33]$. From these infinitesimal operators the Casimir operator of the group $\operatorname{SL}(2, R)$ is given by $C=J_{1}^{2}+J_{2}^{2}-J_{3}^{2}$.

For convenience in the calculations, it is important to decompose an element of the group according to its one parameter subgroups. There exist various well-known decompositions for groups like Cartan, Iwasawa, Gauss and Bruhat and so on [1]. Some different decompositions of the group SL(2,R) have been studied in reference [34]. The most useful of these decompositions seems to be due to the factorization of the group elements in the form of $g=b a d$, where each factor is a parameter subgroup of the group. Let us suppose that only the middle element $a$ depends on position coordinate of particle [22]. Then, the Casimir operator is restricted to a one dimensional subspace of functions of the form $f(g)=h(g)^{-1 / 2} t_{r s}^{j}(g)$, where $h(g)$ a positive function and the matrix elements $t_{r s}^{j}(g)$ are solutions of the eigenvalue equation for the Casimir operator. Thus, the Casimir operator becomes a differential operator connected with only on the position coordinate. Finally, if $h(g)$ is selected appropriately, the Hamiltonian operator with position-dependent mass can be found from the $[C+j(j+1)]=Q[H-E]$.

The Schrödinger equation with position-dependent mass is given by

$$
\begin{equation*}
H \psi(x)=\left[-\frac{1}{m(x)} \frac{d^{2}}{d^{2} x}+\frac{\dot{m}}{m^{2}} \frac{d}{d x}+V(x)\right] \psi(x)=E \psi(x) \tag{5}
\end{equation*}
$$

The equation is a second order linear differential equation. In the equation, the $H$ operator is the total energy of the physical system and is also called the Hamiltonian. The solutions of the equation are given as bound state $(E<0)$ and unbound state $(E>0)$ [35,36]. Additional energy is required to change from bound state to unbound state. Here we are only interested in the unbound state solutions of the equation. For example, a bound state solution is given for a physical system expressed by Hamiltonian with position-dependent mass in reference [37].

In the next section, the information about the group SL(2,R) was given. According to the selected decompositions of the group, the Hamiltonians with position-dependent mass related to the Casimir operator and unbound state solutions were given.

## 2. The Group SL $(2, R)$

The special linear group $\operatorname{SL}(2, R)$ is group of the real $2 \times 2$ matrices with unit determinant

$$
\begin{equation*}
\mathrm{SL}(2, \mathrm{R})=\{g \in \mathrm{GL}(\mathrm{n}, \mathrm{R}): \operatorname{det}(g)=\alpha \delta-\beta \gamma=1, \quad \alpha, \beta, \gamma, \delta \in R\} . \tag{6}
\end{equation*}
$$

Its Lie algebra consists of traceless real $2 \times 2$ matrices;

$$
\begin{equation*}
\mathrm{sl}(2, \mathrm{R})=\{\mathrm{g} \in g l(n, R): \operatorname{tr}(\mathrm{g})=0\} . \tag{7}
\end{equation*}
$$

The one parameter subgroups of the group SL $(2, R)$ are parameterized as follows

$$
\begin{gather*}
\omega_{1}=\left\{\left(\begin{array}{cc}
e^{-t / 2} & 0 \\
0 & e^{t / 2}
\end{array}\right) \in \operatorname{SO}(1,1) ; t \in \mathbb{R}\right\}, \omega_{2}=\left\{\left(\begin{array}{cc}
\cos (t / 2) & -\sin (t / 2) \\
\sin (t / 2) & \cos (t / 2)
\end{array}\right) \in \operatorname{SO}(2) ; t \in[0,2 \pi]\right\},  \tag{8}\\
\omega_{3}=\left\{\left(\begin{array}{cc}
1 & t \\
0 & 1
\end{array}\right) \in E(1) ; t \in \mathbb{R}\right\}, \omega_{4}=\left\{\left(\begin{array}{cc}
1 & 0 \\
t & 1
\end{array}\right) \in E(1) ; t \in \mathbb{R}\right\} . \tag{9}
\end{gather*}
$$

Firstly, we are considering the Iwasawa decomposition. For an element $g$ of the group $\operatorname{SL}(2, R)$, the Iwasawa decomposition can be given as follows [22]

$$
g(\varphi, t, u)=\left(\begin{array}{cc}
\cos (\varphi / 2) & \sin (\varphi / 2)  \tag{10}\\
-\sin (\varphi / 2) & \cos (\varphi / 2)
\end{array}\right)\left(\begin{array}{cc}
e^{-t / 2} & 0 \\
0 & e^{t / 2}
\end{array}\right)\left(\begin{array}{ll}
1 & u \\
0 & 1
\end{array}\right)
$$

where the ranges of the parameters are $0 \leq \varphi<2 \pi,-\infty<t, u<\infty, t=t(x)$. According to this decomposition, the infinitesimal operators can be found as follows

$$
\begin{gather*}
i J_{1}=-\frac{1}{\dot{t}} \frac{\partial}{\partial x}+u \frac{\partial}{\partial u}-\frac{\dot{h}}{2 h \dot{t}}, \quad i J_{2}=\frac{u}{\dot{t}} \frac{\partial}{\partial x}-e^{-t} \frac{\partial}{\partial \varphi}+\left(1-u^{2}+e^{-2 t}\right) \frac{\partial}{\partial u}+\frac{\dot{h} u}{2 h \dot{t}},  \tag{11}\\
i J_{3}=\frac{u}{\dot{t}} \frac{\partial}{\partial x}-e^{-t} \frac{\partial}{\partial \phi}-\left(1+u^{2}-e^{-2 t}\right) \frac{\partial}{\partial u}+\frac{\dot{h} u}{2 h \dot{t}} . \tag{12}
\end{gather*}
$$

From the above infinitesimal operators, the Casimir operator of the group is found as follows

$$
\begin{equation*}
C=-\frac{1}{\dot{t}^{2}} \frac{\partial^{2}}{\partial x^{2}}-\frac{1}{\dot{t}^{2}}\left(\frac{\dot{h}}{h}-\frac{\ddot{t}}{\dot{t}}+\dot{t}\right) \frac{\partial}{\partial x}+2 e^{-t} \frac{\partial^{2}}{\partial \varphi \partial u}-e^{-2 t} \frac{\partial^{2}}{\partial u^{2}}+\frac{\dot{h}}{2 h \dot{t}^{2}}\left(\frac{\dot{h}}{2 h}-\frac{\ddot{h}}{2 \dot{h}}+\frac{\ddot{t}}{\dot{t}}-\dot{t}\right) . \tag{13}
\end{equation*}
$$

We can choose the one-dimensional subspace $\mathfrak{J}_{n \lambda}$ as subspace of $L^{2}(S L(2, R))$ consisting of functions

$$
\begin{equation*}
f(g)=h^{-1 / 2} t_{n \lambda}^{j}(g)=e^{-i n \varphi-i \lambda u} \psi(x) \tag{14}
\end{equation*}
$$

where $n$ and $\lambda$ take values integer. The eigenvalue equation can be given as $c f(g)=-j(j+1) f(g)$. Then the Casimir operator restricted to subspace $\mathfrak{J}_{n \lambda}$ becomes a differential operator in $x$ alone and is represented by $\left.C\right|_{\mathfrak{J}_{n \lambda}}$

$$
\begin{equation*}
\left.[C+j(j+1)]\right|_{\mathfrak{J}_{n \lambda}}=\frac{m}{\dot{t}^{2}}\left\{-\frac{1}{m} \frac{d^{2}}{d x^{2}}-\frac{1}{m}\left(\frac{\dot{h}}{h}-\frac{\ddot{t}}{\dot{t}}+\dot{t}\right) \frac{d}{d x}+\frac{\dot{t}^{2}}{m}\left(-2 n \lambda e^{-t}+\lambda^{2} e^{-2 t}\right)+\frac{\dot{h}}{2 m h}\left(\frac{\dot{h}}{2 h}-\frac{\ddot{h}}{\dot{h}}+\frac{\ddot{t}}{\dot{t}}-\dot{t}\right)+\frac{\dot{t}^{2}}{m} j(j+1)\right\} . \tag{15}
\end{equation*}
$$

In order to obtain the Hamiltonian (1) form, the following equation must be provided

$$
\begin{equation*}
-\frac{\dot{h}}{h}+\frac{\ddot{t}}{\dot{t}}-\dot{t}=\frac{\dot{m}}{m} \tag{16}
\end{equation*}
$$

If this equation is solved, we find $h=\dot{t} e^{-t} / m$. Hence, we can write the operator $\left.[C+j(j+1)]\right|_{\mathfrak{J}_{n \lambda}}$ as follows

$$
\begin{equation*}
\left.[C+j(j+1)]\right|_{\mathfrak{I}_{n \lambda}}=\frac{m z^{2}}{\dot{z}^{2}}\left\{-\frac{1}{m} \frac{d^{2}}{d x^{2}}+\frac{\dot{m}}{m^{2}} \frac{d}{d x}+\frac{\dot{z}^{2}}{4 m z^{2}}\left[4 \lambda^{2} z^{2}-8 n \lambda z+4\left(j+\frac{1}{2}\right)^{2}-1\right]-\frac{\dddot{z}}{2 m \dot{z}}+\frac{3}{4 m} \frac{\ddot{z}^{2}}{z^{2}}+\frac{\ddot{m}}{2 m^{2}}-\frac{3}{4} \frac{\dot{m}^{2}}{m^{3}}\right\} \tag{17}
\end{equation*}
$$

where $z=e^{-t}$. The relations between the $n, \lambda$ and $j$ parameters and energy eigenvalue can be selected as follows $[38,39]$

$$
\begin{equation*}
-8 n \lambda=a-c_{1} E, 4 \lambda^{2}=f-h_{1} E, \quad 1-4(j+1 / 2)^{2}=c_{0} E-h_{0} \tag{18}
\end{equation*}
$$

where $a, f, c_{0}, c_{1}, h_{0}, h_{1}$ are six real parameters.The parameters $n$ and $\lambda$ are irreducible representations of the $\mathrm{SO}(2)$ and $\mathrm{E}(1)$ groups. It corresponds to physical rotation and translation.

The parameter $j$ is eigenvalue. At the same time, quantum numbers correspond to the eigenvalues of the group's generators. Thus, parameters $n$ and $j$ are quantum numbers.

In this case the operator (17) is as follows

$$
\begin{equation*}
\left.[C+j(j+1)]\right|_{\mathfrak{I}_{n \lambda}}=\frac{m z^{2}}{\dot{z}^{2}}\left\{-\frac{1}{m} \frac{d^{2}}{d x^{2}}+\frac{\dot{m}}{m^{2}} \frac{d}{d x}-\frac{\dot{z}^{2}}{4 m z^{2}} R(z) E+\frac{\dot{z}^{2}}{4 m z^{2}}\left(f z^{2}+a z+h_{0}\right)-\frac{\dddot{z}}{2 m \dot{z}}+\frac{3}{4 m} \frac{\dot{z}^{2}}{\dot{z}^{2}}+\frac{\ddot{m}}{2 m^{2}}-\frac{3}{4} \frac{\dot{m}^{2}}{m^{3}}\right\} \tag{19}
\end{equation*}
$$

Thus, the Hamiltonian from the equality $[C+j(j+1)]=Q[H-E]$ is obtained as

$$
\begin{equation*}
H=-\frac{1}{m} \frac{d^{2}}{d x^{2}}+\frac{\dot{m}}{m^{2}} \frac{d}{d x}+V(x)=-\frac{1}{m} \frac{d^{2}}{d x^{2}}+\frac{\dot{m}}{m^{2}} \frac{d}{d x}+\frac{f z^{2}+a z+h_{0}+1}{R}+\frac{z^{2}}{R^{2}}\left[-\frac{5\left(c_{1}^{2}-4 h_{1} c_{0}\right)}{4 R}-h_{1}+\frac{c_{1}}{z}\right]+\frac{\ddot{m}}{4 m^{2}}-\frac{7}{16} \frac{\dot{m}^{2}}{m^{3}} \tag{20}
\end{equation*}
$$

where $Q=\frac{m z^{2}}{\dot{z}^{2}}, \frac{\dot{z}^{2} R(z)}{4 m z^{2}}=1, R(z)=h_{1} z^{2}+c_{1} z+c_{0}$.
The potential term $V(x)$ in the Hamiltonian is the potential type of Natanzon. The Natanzon potentials are most commonly given as follows ( $a, f, c_{0}, c_{1}, h_{0}, h_{1}$ ) depending on the six parameters [40].

The unitary irreducible representation matrix elements $t_{n \lambda}^{j}(g)$ of the group $\operatorname{SL}(2, R)$ is given as follows [33]

$$
\begin{equation*}
t_{n \lambda}^{j}(g)=\frac{2^{j+1} e^{-n \pi i}|\lambda|^{-j-1}}{\pi \Gamma(-j+n)} e^{-i n \varphi-i \lambda u} W_{-n, j+1 / 2}\left(2|\lambda| e^{-t}\right) \tag{21}
\end{equation*}
$$

In this case the unbound state solution from the expressed (14) can be written as

$$
\begin{equation*}
\psi(t)=\frac{2^{j+1} e^{-n \pi i}|\lambda|^{-j-1}}{\pi \Gamma(-j+n)} m^{1 / 4} e^{t / 2} W_{-n, j+1 / 2}\left(2|\lambda| e^{-t}\right) \tag{22}
\end{equation*}
$$

where $W_{-n, j+1 / 2}\left(2|\lambda| e^{-t}\right)$ is the Whittaker function.
The reflection and transmission coefficients are important in unbound state solutions. These coefficients can be found by examining the asymptotic behavior of the wave function of the physical system.

Potentials known for the specific values of the parameters from the general expression of the potential can be obtained. For example, we can select the special values of the parameters as $c_{1}=c_{0}=0$. Substituting these parameters into the equation in (18) and $\frac{\dot{z}^{2} R(z)}{4 m z^{2}}=1$, we get

$$
\begin{equation*}
z=2 s(x) / \sqrt{h_{1}}, s(x)=\int \sqrt{m(x)} d x,-8 n \lambda=a, \quad 4 \lambda^{2}=f-h_{1} E, \quad 1-4(j+1 / 2)^{2}=-h_{0} \tag{23}
\end{equation*}
$$

Hence, from the potential expression $V(x)$ we obtain Kratzer type potential [41]

$$
\begin{equation*}
V(x)=\frac{j(j+1)}{s(x)^{2}}-\frac{4 n \lambda / \sqrt{h_{1}}}{s(x)}+\frac{f}{h_{1}}+\frac{\ddot{m}}{4 m^{2}}-\frac{7 \dot{m}^{2}}{16 m^{3}} \tag{24}
\end{equation*}
$$

The expression of the potential according to the position-dependent mass on the selected can be rewritten.

Secondly, we consider the Gauss decomposition [22]. For an element $g$ of the group $\operatorname{SL}(2, R)$, the Gauss decomposition can be given as follows

$$
g(v, t, u)=\left(\begin{array}{ll}
1 & 0  \tag{25}\\
v & 1
\end{array}\right)\left(\begin{array}{cc}
e^{-t / 2} & 0 \\
0 & e^{t / 2}
\end{array}\right)\left(\begin{array}{ll}
1 & u \\
0 & 1
\end{array}\right),
$$

where the ranges of the parameters are $0<t<\infty,-\infty<v, u<\infty$. The infinitesimal operators respect to this parameterization can be found as

$$
\begin{equation*}
i J_{1}=\frac{1}{\dot{t}} \frac{\partial}{\partial x}-u \frac{\partial}{\partial u}+\frac{\dot{h}}{2 h \dot{t}}, \quad i J_{2}=\frac{2 u}{\dot{t}} \frac{\partial}{\partial x}+e^{-t} \frac{\partial}{\partial v}-u^{2} \frac{\partial}{\partial u}+\frac{\dot{h} u}{h \dot{t}}, i J_{3}=\frac{\partial}{\partial u} \tag{26}
\end{equation*}
$$

Then, the Casimir operator of the group is found as

$$
\begin{equation*}
C=-\frac{1}{\dot{t}^{2}} \frac{\partial^{2}}{\partial x^{2}}-\frac{1}{\dot{t}^{2}}\left(\frac{\dot{h}}{h}-\frac{\ddot{t}}{\dot{t}}+\dot{t}\right) \frac{\partial}{\partial x}-e^{-t} \frac{\partial^{2}}{\partial u \partial v}+\frac{\dot{h}}{2 h \dot{t}^{2}}\left(\frac{\dot{h}}{2 h}-\frac{\ddot{h}}{\dot{h}}+\frac{\ddot{t}}{\dot{t}}-\dot{t}\right) . \tag{27}
\end{equation*}
$$

Let $\mathfrak{J}_{\mu \lambda}$ be a subspace of $L^{2}(S L(2, R))$ consisting of functions $f(g)$ such that

$$
\begin{equation*}
f(g)=h^{-1 / 2} t_{\mu \lambda}^{j}(g)=e^{-i \mu v-i \lambda u} \psi(x) \tag{28}
\end{equation*}
$$

where $\mu$ and $\lambda$ take values integer. Then the Casimir operator restricted to the subspace $\mathfrak{J}_{\mu \lambda}$ becomes a differential operator in $x$ alone

$$
\begin{equation*}
\left.[C+j(j+1)]\right|_{J_{\mu \lambda}}=\frac{m}{\dot{t}^{2}}\left\{-\frac{1}{m} \frac{d^{2}}{d x^{2}}-\frac{1}{m}\left(\frac{\dot{h}}{h}-\ddot{\ddot{t}}+\dot{\dot{t}}\right) \frac{d}{d x}+\frac{\dot{t}^{2}}{m} \mu \lambda e^{-t}+\frac{\dot{h}}{2 m h}\left(\frac{\dot{h}}{2 h}-\frac{\ddot{h}}{\dot{h}}+\frac{\ddot{t}}{\dot{t}}-\dot{t}\right)+\frac{\dot{t}^{2}}{m} j(j+1)\right\} . \tag{29}
\end{equation*}
$$

We want to obtain the Hamiltonian in the form (1). For this the following equality must be achieved

$$
\begin{equation*}
-\frac{\dot{h}}{h}+\frac{\ddot{t}}{\dot{t}}-\dot{t}=\frac{\dot{m}}{m} \tag{30}
\end{equation*}
$$

From the above equation, $h$ has the form $h=\dot{t} e^{-t} / m$. In this case, the operator $\left.[C+j(j+1)]\right|_{\mathfrak{J}_{\mu \lambda}}$ is written with respect to $z=e^{-t}$ as

$$
\begin{equation*}
\left.[C+j(j+1)]\right|_{\mathfrak{S}_{\mu \lambda}}=\frac{m z^{2}}{\dot{z}^{2}}\left\{-\frac{1}{m} \frac{d^{2}}{d x^{2}}+\frac{\dot{m}}{m^{2}} \frac{d}{d x}+\frac{\dot{z}^{2}}{m z^{2}}[\mu \lambda z+j(j+1)]-\frac{\dddot{z}}{2 m \dot{z}}+\frac{3 \ddot{z}^{2}}{4 m z^{2}}+\frac{\ddot{m}}{2 m^{2}}-\frac{3 \dot{\dot{m}}^{2}}{4 m^{3}}\right\} . \tag{31}
\end{equation*}
$$

The relations between the $\lambda, \mu$ and $j$ parameters and energy eigenvalue can be selected as follows

$$
\begin{equation*}
\lambda \mu=-a E+f, \quad j(j+1)=c_{0} E+c_{1} \tag{32}
\end{equation*}
$$

Then we have

$$
\begin{equation*}
\left.[C+j(j+1)]\right|_{\mathfrak{J}_{\mu \lambda}}=\frac{m z^{2}}{\dot{z}^{2}}\left\{-\frac{1}{m} \frac{d^{2}}{d x^{2}}+\frac{\dot{m}}{m^{2}} \frac{d}{d x}+\frac{\left(f z+\mathcal{c}_{1}\right) \dot{z}^{2}}{m z^{2}}-\frac{\dddot{z}}{2 m \dot{z}}+\frac{3}{4 m} \frac{\ddot{z}^{2}}{\dot{z}^{2}}+\frac{\ddot{m}}{2 m^{2}}-\frac{3}{4} \frac{\dot{m}^{2}}{m^{3}}-\frac{\dot{z}^{2}\left(a z-c_{0}\right)}{m z^{2}} E\right\} . \tag{33}
\end{equation*}
$$

Hence, the Hamiltonian has the form

$$
\begin{equation*}
H=-\frac{1}{m} \frac{d^{2}}{d x^{2}}+\frac{\dot{m}}{m^{2}} \frac{d}{d x}+V(x)=-\frac{1}{m} \frac{d^{2}}{d x^{2}}+\frac{\dot{m}}{m^{2}} \frac{d}{d x}+\frac{f z+c_{1}-3 / 4}{R}-\frac{3 a^{2}}{16 m R^{2}}+\frac{3 \dot{m} a^{2}}{16 m^{2} R}+\frac{\ddot{m}}{4 m^{2}}-\frac{15}{16} \frac{\dot{m}^{2}}{m^{3}} \tag{34}
\end{equation*}
$$

where $Q=m z^{2} / \dot{z}^{2}, \frac{\dot{z}^{2} R(z)}{m z^{2}}=1, R(z)=a z-c_{0}$.
The unbound state solution the from the expressed (28) can be written as

$$
\begin{equation*}
\psi(t)=N_{1} m^{1 / 4} e^{t / 2} K_{2 j+1}\left(2 \sqrt{\lambda \mu} e^{-t / 2}\right), j=-1 / 2+i \rho, \quad \rho \geq 0, \quad \lambda \mu>0 \tag{35}
\end{equation*}
$$

where $N_{1}$ is the normalization constant and $K_{2 j+1}\left(2 \sqrt{\lambda \mu} e^{-t / 2}\right)$ is the Macdonald's function.
For special value of the parameters in the statement potential $V(x)$, potential functions can be written clearly. For example, we choose $c_{1}=3 / 4, \quad a=0, \quad c_{0}=-1$. Based on these parameters, from the (32) equalities and the $\frac{\dot{z}^{2} R(z)}{m z^{2}}=1$ equation, we get $z=e^{s(x)}, s(x)=\int \sqrt{m(x)} d x, \lambda \mu=f, E=$ $-(j+1 / 2)^{2}+1$.

Thus, we obtain the Toda type potential [42,43]

$$
\begin{equation*}
V(x)=\lambda \mu e^{s(x)}+\frac{\ddot{m}}{4 m^{2}}-\frac{15 \dot{m}^{2}}{16 m^{3}} \tag{36}
\end{equation*}
$$

## 3. Results

For Cartan and Gauss separations of SL(2,R) group, Casimir operators and Hamiltonians associated with these Casimir operators and were found respectively as follows,

$$
\begin{gather*}
C=-\frac{1}{\dot{t}^{2}} \frac{\partial^{2}}{\partial x^{2}}-\frac{1}{\dot{t}^{2}}\left(\frac{\dot{h}}{h}-\frac{\ddot{t}}{\dot{t}}+\dot{t}\right) \frac{\partial}{\partial x}+2 e^{-t} \frac{\partial^{2}}{\partial \varphi \partial u}-e^{-2 t} \frac{\partial^{2}}{\partial u^{2}}+\frac{\dot{h}}{2 h t^{2}}\left(\frac{\dot{h}}{2 h}-\frac{\ddot{h}}{2 \dot{h}}+\frac{\ddot{t}}{\dot{t}}-\dot{t}\right),  \tag{37}\\
C=-\frac{1}{\dot{t}^{2}} \frac{\partial^{2}}{\partial x^{2}}-\frac{1}{\dot{t}^{2}}\left(\frac{\dot{h}}{h}-\frac{\ddot{t}}{\dot{t}}+\dot{t}\right) \frac{\partial}{\partial x}-e^{-t} \frac{\partial^{2}}{\partial u \partial v}+\frac{\dot{h}}{2 h \dot{t}^{2}}\left(\frac{\dot{h}}{2 h}-\frac{\ddot{h}}{\dot{h}}+\frac{\ddot{t}}{\dot{t}}-\dot{t}\right),  \tag{38}\\
H=-\frac{1}{m} \frac{d^{2}}{d x^{2}}+\frac{\dot{m}}{m^{2}} \frac{d}{d x}+\frac{f z^{2}+a z+h_{0}+1}{R}+\frac{z^{2}}{R^{2}}\left[-\frac{5\left(c_{1}^{2}-4 h_{1} c_{0}\right)}{4 R}-h_{1}+\frac{c_{1}}{z}\right]+\frac{\ddot{m}}{4 m^{2}}-\frac{7}{16} \frac{\dot{m}^{2}}{m^{3}},  \tag{39}\\
H=-\frac{1}{m} \frac{d^{2}}{d x^{2}}+\frac{\dot{m}}{m^{2}} \frac{d}{d x}+\frac{f z+c_{1}-3 / 4}{R}-\frac{3 a^{2}}{16 m R^{2}}+\frac{3 \dot{m} a^{2}}{16 m^{2} R}+\frac{\ddot{m}}{4 m^{2}}-\frac{15}{16} \frac{\dot{m}^{2}}{m^{3}} . \tag{40}
\end{gather*}
$$

The unbound state solutions from the eigenvalue equation created by the Casimir operator were found respectively as follows

$$
\begin{gather*}
\psi(t)=\frac{2^{j+1} e^{-n \pi i}|\lambda|^{-j-1}}{\pi \Gamma(-j+n)} m^{1 / 4} e^{t / 2} W_{-n, j+1 / 2}\left(2|\lambda| e^{-t}\right)  \tag{41}\\
\psi(t)=N_{1} m^{1 / 4} e^{t / 2} K_{2 j+1}\left(2 \sqrt{\lambda \mu} e^{-t / 2}\right) \tag{42}
\end{gather*}
$$

In this study, we obtained the Casimir operators of the $\mathrm{SL}(2, R)$ group and Hamiltonians operators with position mass from the equality $[C+j(j+1)]=Q[H-E]$. As it is known, equations created by these operators (Eigenvalue problem) are frequently seen in physical problems. The solution of the Schrödinger equation is often sought for a given potential. But, in the group theory approach, an algebraic solution of the Schrödinger equation is obtained from the symmetry property of the physical system. A more general expression of the potential function is obtained. Important potential types (Pöschl-Teller, Rosen-Morse, Morse), energy eigenvalues and wave functions can be obtained for the special values of the parameters in the potential expression [15,44,45]. As an example, we gave the potential functions Toda and Kratzer with position-dependent mass, which play important role in quantum mechanics [43,46,47]. Depending on the potential discussed, the physical system may have bound and unbound state solutions. The solutions (matrix elements) are solutions of the eigenvalue equation created by the Casimir operator. Thus, wave functions of the Schrödinger equation with position-dependent mass are given by the matrix elements of the group. Specifically, if we select the mass $m(x)$ as constant, the solutions given above will be the solutions of the physical system created with Toda and Kratzer potential functions. The parameters $n, \lambda, \mu$ appearing in the solutions and Casimir operators are the parameters of the group. Also, the Hamiltonians contain arbitrary real parameters and these parameters can be specially selected. In the expression of potential and solution,
these arbitrary parameters are written in terms of the group's parameters, because the main parameters are the parameters of the group.

## 4. Discussion

Group theory is the mathematics of symmetry [24,48]. Therefore, it occupies an important place in physical applications. Especially, analytically solvable potentials are very useful in physics. For example, the Morse potential is widely used in molecular physics. The Schrödinger equation with position-dependent mass has a wide range of applications in various physical systems such as quantum liquids, compositionally graded crystals. In this paper, we have dealt with the one-dimensional Schrödinger equation with position-dependent mass. For this, we have obtained the infinitesimal operators of the regular representation of the group $\operatorname{SL}(2, R)$ and the Casimir operators. Then, we have obtained Hamiltonians with position-dependent mass which are connected to the Casmir operators. In conclusion the unbound state solutions for the Schrödinger equation are given. The solutions of the considered physical systems can be found for different expressions of the position-dependent mass $m(x)$.

In this work, the Schrödinger equation with position-dependent mass is independent of time.
The equation can be considered with time and position-dependent mass. In this case, Schrödinger equation is written as $L \psi=\left(i \frac{\partial}{\partial t}-H_{k}-V\right) \psi=0$ [49]. The fact that the equation depends on both position-dependent mass and time makes its solution even more difficult. Even so, this situation can be studied with the group theory technique.

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