

Article

# Reflection Identities of Harmonic Sums of Weight Four

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**Abstract:** In attempt to find a proper space of function expressing the eigenvalue of the color-singlet BFKL equation in  $N = 4$  SYM, we consider an analytic continuation of harmonic sums from positive even integer values of the argument to the complex plane. The resulting meromorphic functions have pole singularities at negative integers. We derive the reflection identities for harmonic sums at weight four decomposing a product of two harmonic sums with mixed pole structure into a linear combination of terms each having a pole at either negative or non-negative values of the argument. The pole decomposition demonstrates how the product of two simpler harmonic sums can build more complicated harmonic sums at higher weight. We list a minimal irreducible set of bilinear reflection identities at weight four, which represents the main result of the paper. We also discuss how other trilinear and quadlinear reflection identities can be constructed from our result with the use of well known quasi-shuffle relations for harmonic sums.

**Keywords:** BFKL equation; analytic continuation; Carlson's theorem; functional identities; harmonic sums

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## 1. Introduction

The Balitsky–Fadin–Kuraev–Lipatov (BFKL) equation was formulated about four decades ago [1–5] in attempt to describe the leading Regge trajectory (Pomeron) in the framework of the perturbative gauge theory, in particular the Quantum Chromodynamics (QCD). The BFKL approach is based on identifying the leading contributions in the perturbative expansion, namely the terms that are accompanied by the large logarithm of center-of-mass energy. This way one separates the dynamics of the longitudinal and transverse degrees of freedom, where the longitudinal momentum contributes to the large parameter (logarithm of center-of-mass energy) and plays a role of the “time” parameter<sup>1</sup> of the evolution, while the transverse momenta defines the time-independent Hamiltonian. The BFKL equation is derived by considering all possible Feynman diagrams in the perturbative QCD selecting those that give the leading order (LO) in power of logarithm of the center-of-mass energy. The resulting LO BFKL equation can be viewed also as a propagation of a bound state of two Reggeized gluons in the  $t$ -channel. The BFKL equation initiated a lot of activity in the field of analytic perturbative calculations as well phenomenological studies and comparison to the experimental data. For more details on the BFKL equation, its derivation, self-consistency, bootstrap, the issue of unitarity violation, etc., the reader is referred to a profound review book by Ioffe, Fadin and Lipatov [6]. For the purpose of the present discussion, we only want to emphasize the following important aspects of this approach. The BFKL equation is formulated through the perturbative expansion that give a solid

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<sup>1</sup> More precisely, the logarithm of the center-of-mass energy is similar to the imaginary time in the Schrödinger-like equation.

background and good control of the calculations, but it also makes it extremely difficult to go to higher orders of the perturbative expansion. In fact, at the moment, we only know the leading (LO) and next-to-leading (NLO) order of the BFKL equation and the full solution in both QCD and its supersymmetric extensions. The exact analytic solution was found due to an interesting feature of the BFKL equation; it can be reformulated in terms of the Heisenberg spin chain and thus integrable [7]. The color singlet BFKL equation is of particular interest because it is related to the pomeron in the Regge theory and not much is known about it beyond next-to-leading order due to involved calculations in the old-fashioned diagrammatic expansion. Some good news may come from maximally super-symmetric Yang–Mills gauge theory, called  $N = 4$  SYM, where the calculation may be done by exploiting spin chain analogy of the BFKL dynamics.

In this paper, we continue discussion of our previous study [8] aiming at generalizing recent results by Gromov, Levkovich-Maslyuk and Sizov [9], Caron Huot and Herraren [10], and Alfimov, Gromov and Sizov [11] on the color singlet next-to-next-to-leading (NLO) BFKL eigenvalue for a case of arbitrary values of conformal spin and anomalous dimension. Our approach is to consider all possible nonlinear combinations of the functions, which are obtained from the analytic continuation of the harmonic sums to the complex plane. The resulting meromorphic functions enjoy the reflection identities, which we derive in this paper.

In this section, we introduce all necessary definitions and concepts and then in Section 2 we show some examples of use of the reflection identities derived below. This puts the present discussion in the broader context of well known state-of-art results.

The reflection identities for the analytically continued harmonic sums represent a product of two harmonic sums of argument  $z$  and  $-1 - z$  is expressed through a linear combination of other harmonic sums of the same arguments, i.e.,

$$S_{a_1, a_2, \dots}(z) S_{b_1, b_2, \dots}(-1 - z) = S_{c_1, c_2, \dots}(z) + \dots + S_{d_1, d_2, \dots}(-1 - z) + \dots \quad (1)$$

We consider an analytic continuation of harmonic sums from positive even integer values of the argument to the complex plane except for isolated pole singularities at negative integers. This analytic continuation is done using the generalization of Carlson's theorem in terms of the Mellin transform of the Harmonic Polylogarithms (HPL). We derive the reflection identities for the resulting meromorphic functions (analytically continued harmonic sums) at weight four decomposing a product of two harmonic sums with mixed pole structure into a linear combination of terms each having a pole at either negative or non-negative values of the argument.

The reflection identities at weight two are not new and were known a long time ago in the context of functions related to the Euler Gamma function. To the best of our knowledge, they appear the earliest in Chapter 20 of the book by Nielsen [12]. At weight three, they were recently calculated by the author [8]. This paper deals with weight four.

After the analytic continuation to the complex plane, the harmonic sums have pole singularities at negative integers. The reflection identities present a pole separation for a product of two sums with mixed pole structure. We call those functional relations the reflection identities because the argument of the harmonic sums is reflected with respect to the point  $z + \frac{(-1 - z)}{2} = -\frac{1}{2}$ . The reflection identities up to weight three were published in our previous study [8] and here we present them at weight four. The harmonic sums are defined through a nested summation with their argument being the upper limit in the outermost sum [13–17]

$$S_{a_1, a_2, \dots, a_k}(n) = \sum_{n \geq i_1 \geq i_2 \geq \dots \geq i_k \geq 1} \frac{\text{sign}(a_1)^{i_1}}{i_1^{|a_1|}} \dots \frac{\text{sign}(a_k)^{i_k}}{i_k^{|a_k|}}, \quad n \in \mathbb{N}^* \quad (2)$$

In this paper, we consider the harmonic sums with only real integer values of  $a_i$ , which build the alphabet of the possible negative and positive indices. In Equation (2),  $k$  is the depth and  $w = \sum_{i=1}^k |a_i|$  is the weight of the harmonic sum  $S_{a_1, a_2, \dots, a_k}(n)$ .

The indices of harmonic sums  $a_1, a_2, \dots, a_k$  can be either positive or negative integers and label uniquely  $S_{a_1, a_2, \dots, a_k}(n)$  for any given weight. However, there is no unique way of building the functional basis for a given weight because the harmonic sums are subject to so-called quasi-shuffle relations, where a linear combination of  $S_{a_1, a_2, \dots, a_k}(n)$  with the same argument but all possible permutations of indices can be expressed through a nonlinear combinations of harmonic sums at lower weight. There is also some freedom in choosing the irreducible minimal set of  $S_{a_1, a_2, \dots, a_k}(n)$  that builds those nonlinear combinations. The quasi-shuffle relations make a connection between the linear and nonlinear combinations of the harmonic sums of the same argument. For example, the quasi-shuffle relation at depth two reads

$$S_{a,b}(z) + S_{b,a}(z) = S_a(z)S_b(z) + S_{\text{sign}(a)\text{sign}(b)(|a|+|b|)}(z) \quad (3)$$

The quasi-shuffle relations of the harmonic sums is closely connected to the quasi-shuffle algebra of the harmonic polylogarithms [17].

There is another type of identity called the duplication identities where a combination of harmonic sums of argument  $n$  can be expressed through a harmonic sum of the argument  $2n$ . The duplication identities introduce another freedom in choosing the functional basis. In this paper, we consider the analytic continuation of the harmonic sums to from positive integer values of the argument to the complex plane. The resulting meromorphic functions is defined on the complex plane except for isolated poles at negative integer values of the argument. The analytic continuation is done in terms of the Mellin transform of corresponding Harmonic Polylogarithms and was recently used by Gromov, Levkovich-Maslyuk and Sizov [9,11] and Caron Huot and Herraren [10] for expressing the eigenvalue Balitsky–Fadin–Kuraev–Lipatov (BFKL) equation using the principle of Maximal Transcedentality [18] in super Yang–Mills  $\mathcal{N} = 4$  field theory. We plan to use their results together with analysis done by one of the authors and collaborators [19,20] to understand the general structure of the BFKL equation in QCD and beyond. The Mellin transform allows making the analytic continuation to the complex plane. This method is well known and implemented in the dedicated Mathematica package by Gromov, Levkovich-Maslyuk and Sizov [9,11], which we use for our calculations. Here, we present a simple example that illustrates the general idea. We follow the lines of papers of Kotikov, Lipatov, Onishchenko and Velizhanin [21] as well as Kotikov and Velizhanin [22] introducing the analytic continuation of harmonic sums from positive integer values of the argument to the complex plane. We consider the harmonic sum

$$S_{-1}(z) = \sum_{k=1}^z \frac{(-1)^k}{k}, \quad z \in \mathbb{N}^* \quad (4)$$

The corresponding Mellin transform reads

$$\int_0^1 \frac{1}{1+x} x^z = (-1)^z (S_{-1}(z) + \ln 2) \quad (5)$$

One can see that  $S_{-1}(z)$  on its own is not an analytic function because of the term  $(-1)^z$  and we impose that we start from even integer values of the argument  $z$ . In this case, we define its analytic continuation from even positive integers to all positive integers through [21,22]

$$\bar{S}_{-1}^+(z) = (-1)^z S_{-1}(z) + ((-1)^z - 1) \ln 2 \quad (6)$$

and thus we can write

$$\bar{S}_{-1}^+(z) = \int_0^1 \frac{1}{1+x} x^z - \ln 2 \quad (7)$$

This way we defined  $\bar{S}_{-1}^+(z)$  using the Mellin transform of ratio function  $\frac{1}{1+x}$ . In more complicated cases of other harmonic sums, one includes also Harmonic Polylogarithms on top of the ratio functions, but the general procedure is very similar and largely covered in a number of publications [16,23–26]. It is worth mentioning that there is another analytic continuation for the harmonic sum, from odd positive integer values of the argument, which is different for harmonic sums with at least one negative index and denoted by  $\bar{S}_{a_1,a_2,\dots}^-(z)$ . Both analytic continuations are equally valid. Our goal is to find a closed expression of the BFKL eigenvalue for all possible values of anomalous dimension and conformal spin, so that we follow the notation of Gromov, Levkovich-Maslyuk and Sizov [9], and use  $\bar{S}^+(z)$  throughout the text. For simplicity of presentation, in this paper, we write everywhere  $S_{a_1,a_2,\dots}(z)$  instead of  $\bar{S}_{a_1,a_2,\dots}^+(z)$ .

As mentioned above, there is no unique way of defining a minimal irreducible set of harmonic sums due to the functional relations between them. For example, one can use the quasi-shuffle relations and then the minimal irreducible basis would include quadratic terms  $S_{a_1}(z)S_{a_2}(z)$  in place of either  $S_{a_1,a_2}(z)$  or  $S_{a_2,a_1}(z)$ . It is convenient to use quasi-shuffle relations to remove from the minimal basis the harmonic sums with the first index being equal 1, because those are divergent as  $z \rightarrow \infty$ . Then, the remaining harmonic sums give transcendental constants at  $z \rightarrow \infty$ . Most constants are reducible and one is free to choose an irreducible set of transcendental constants at any given weight. We use the set implemented in the HarmonicSums package [24,25]

$$C_1 = \{\log(2)\} \quad (8)$$

$$C_2 = \left\{ \pi^2, \log^2(2) \right\} \quad (9)$$

$$C_3 = \left\{ \pi^2 \log(2), \log^3(2), \zeta_3 \right\} \quad (10)$$

$$C_4 = \left\{ \pi^4, \pi^2 \log^2(2), \log^4(2), \text{Li}_4\left(\frac{1}{2}\right), \zeta_3 \log(2) \right\} \quad (11)$$

where  $C_w$  stands for a minimal set of irreducible constants at given weight  $w$ . There is only one of those at  $w = 1$ , two at  $w = 2$ , three at  $w = 3$  and five irreducible constants at weight  $w = 4$ . We choose to use a linear minimal set of the harmonic sums to represent our results. In this set, we do not apply quasi-shuffle relations and thus all the terms of the basis are linear in  $S_{a_1,a_2,\dots}(z)$ . This choice is dictated mostly by a convenience and was also used by Caron Huot and Herraren [10], on which we would like to rely in our future calculations. The minimal linear set of harmonic sums we use is as follows

$$B_1 = \{S_{-1}, S_1\} \quad (12)$$

$$B_2 = \{S_{-2}, S_2, S_{-1,1}, S_{1,-1}, S_{1,1}, S_{-1,-1}\} \quad (13)$$

$$\begin{aligned} B_3 = & \{S_{-3}, S_3, S_{-2,-1}, S_{-2,1}, S_{2,-1}, S_{2,1}, S_{-1,1,-1}, S_{-1,1,1}, S_{1,-2}, S_{1,2}, S_{1,-1,-1}, S_{1,-1,1}, \\ & S_{1,1,-1}, S_{1,1,1}, S_{-1,-2}, S_{-1,2}, S_{-1,-1,-1}, S_{-1,-1,1}\} \end{aligned} \quad (14)$$

$$\begin{aligned}
B_4 = & \{S_{-4}, S_4, S_{-3,-1}, S_{-3,1}, S_{2,-2}, S_{3,-1}, S_{3,1}, S_{-2,-1,-1}, S_{-2,-1,1}, S_{-2,1,-1}, S_{-2,1,1}, \\
& S_{2,-1,-1}, S_{2,-1,1}, S_{2,1,-1}, S_{2,1,1}, S_{-1,1,-1,-1}, S_{-1,1,-1,1}, S_{-1,1,1,1}, S_{1,-3}, S_{1,3}, S_{1,-2,-1}, \\
& S_{1,-2,1}, S_{1,-1,-2}, S_{1,-1,2}, S_{1,1,-2}, S_{1,1,2}, S_{1,2,-1}, S_{1,2,1}, S_{1,-1,-1,-1}, S_{1,-1,-1,1}, S_{1,-1,1,-1}, \\
& S_{1,-1,1,1}, S_{1,1,-1,-1}, S_{1,1,-1,1}, S_{1,1,1,-1}, S_{1,1,1,1}, S_{-1,-3}, S_{-1,3}, S_{-1,-2,-1}, S_{-1,-2,1}, \\
& S_{-1,-1,-2}, S_{-1,-1,2}, S_{-1,1,-2}, S_{-1,1,2}, S_{-1,2,-1}, S_{-1,2,1}, S_{-1,-1,-1,-1}, S_{-1,-1,-1,1}, \\
& S_{-1,-1,1,-1}, S_{-1,-1,1,1}, S_{-1,1,1,-1}, S_{-2,-2}, S_{-2,2}, S_{2,2}\} \tag{15}
\end{aligned}$$

A comprehensive discussion on harmonic sums, irreducible constants, functional identities and possible choice of the minimal set of functions at given weight was presented by J. Ablinger [24]. In this paper, we focus only at the reflection identities for harmonic sums at weight  $w = 4$  analytically continued from even positive points to complex plane. In the next section, we discuss them in more details along the method we use in our calculations. In the next section, we discuss the physical motivation for the reflection identities focusing on the well known color singlet NLO BFKL eigenvalue in  $\mathcal{N} = 4$  SYM. We demonstrate how its full functional form can be restored just from two specific values of the conformal spin while keeping the dependence on the anomalous dimension.

## 2. Motivation: Restoring NLO Eigenvalue

In this section, we illustrate how the reflection identities derived in this paper can be used for restoring the full functional dependence on the conformal spin of BFKL eigenvalue. The BFKL eigenvalue is a function of two variables: the anomalous dimension  $\nu$ , which takes real continuous values, and the conformal spin  $n$ , which is defined at discrete integers. In some specific applications, one can consider analytic continuation to the complex plane, but those cases are beyond the scope of the present paper.

As mentioned above, it is a very non-trivial task to calculate higher order corrections to the BFKL eigenvalue using a well-established perturbation theory. Instead, one can consider analogous systems based on the modern integrability approaches and have exact analytic results in some regions of the  $(\nu, n)$  plane. This was done by N. Gromov, F. Levkovich-Maslyuk and G. Sizov [9] using Quantum Spectral Curves and the resulting singlet NNLO BFKL eigenvalue in  $\mathcal{N} = 4$  SYM was calculated for a specific value of the conformal spin  $n = 0$  while keeping the full dependence on the anomalous dimension  $\nu$ .

Caron Huot and Herraren [10] extended this analysis to the other, but still specific values of the conformal spin  $n = 1, 2, \dots$  providing infinitely many lines at the  $(\nu, n)$  plane. Despite plenty of data, we are still lacking the full functional form of the singlet NNLO BFKL eigenvalue in  $\mathcal{N} = 4$  SYM mainly because we do not know the space of functions it is built of. A natural candidate would be the harmonic sums of one variable considered in the present paper, but the problem is that we need to express the function of two variables in terms of the function of one variable and there are many ways to do that.

In both cases in References [9,10], the authors considered the pseudo-holomorphic separable form of the singlet NNLO BFKL eigenvalue for any specific value of the conformal spin. However, this pseudo-holomorphic separable form differs from value to value of the conformal spin and it seems to be not suitable for writing a closed expression valid for all values of the conformal spin.

It follows from the form of NLO BFKL eigenvalue that one needs to relax the condition of pseudo-holomorphic separability and consider bilinear forms of mixed functions of holomorphic and antiholomorphic variables. This approach clearly works at the NLO level, where we know the exact closed expression for arbitrary values of the conformal spin while preserving the full dependence on the anomalous dimension. In the following, we illustrate this idea for the case of the singlet NLO BFKL eigenvalue in  $\mathcal{N} = 4$  SYM using the reflection identities for harmonic sums at weight  $w = 3$  derived by the author in Reference [8]. In the rest of this section, we use the expressions and the notation of Kotikov and Lipatov [27] for the case of  $\mathcal{N} = 4$  SYM.

For simplicity of presentation, let us define a complex variable

$$z = -\frac{1}{2} + i\nu + \frac{|n|}{2}, \quad \bar{z} = -\frac{1}{2} - i\nu + \frac{|n|}{2}. \quad (16)$$

Then, the leading-order (LO) and the next-to-leading (NLO) BFKL eigenvalue in the color singlet channel reads (see Section 3 of Reference [27])

$$\omega = 4\hat{a} \left[ \chi(z, \bar{z}) + \delta(z, \bar{z}) \hat{a} \right].$$

The LO function is given by

$$\chi(z, \bar{z}) = 2\psi(1) - \psi(1+z) - \psi(1+\bar{z}) = -S_1(z) - S_1(\bar{z}), \quad (17)$$

where the digamma function defined by

$$\psi(z) = \frac{d \ln \Gamma(z)}{dz}, \quad (18)$$

for  $\Gamma(z)$  is a Euler gamma function. In Equation (17), we use the fact the digamma function is the analytic continuation of the harmonic sum  $S_1(z)$ .

The function that determines the NLO part of the BFKL eigenvalue reads

$$\delta(z, \bar{z}) = \phi(1+z) + \phi(1+\bar{z}) - 2\chi(z, \bar{z}) \left( \beta'(1+z) + \beta'(1+\bar{z}) + \zeta_2 \right), \quad (19)$$

where

$$\phi(z) = 3\zeta(3) + \Psi''(z) - 2\Phi_2(z) + 2\beta'(z) (\Psi(1) - \Psi(z)). \quad (20)$$

The most complicated term appearing in Equation (20) can be written as follows

$$\Phi_2(z) = \sum_{k=0}^{\infty} \frac{(\beta'(k+1) + (-1)^k \Psi'(k+1))}{k+z} - \sum_{k=0}^{\infty} \frac{(-1)^k (\Psi(k+1) - \Psi(1))}{(k+z)^2} \quad (21)$$

for the derivative of the Dirichlet beta function

$$\beta'(z) = \frac{1}{4} \left[ \Psi' \left( \frac{z+1}{2} \right) - \Psi' \left( \frac{z}{2} \right) \right] = - \sum_{r=0}^{\infty} \frac{(-1)^r}{(z+r)^2} = -S_{-2}(z-1) - \frac{\zeta_2}{2}. \quad (22)$$

In Equation (22), we use the fact the derivative of the Dirichlet beta function is related to the analytic continuation of the harmonic sum  $S_{-2}(z-1)$  from even positive values of the argument to the complex plane. In a similar way, we write the expression in Equation (20) in terms of the analytically continued harmonic sums as follows

$$\phi(z) = 4S_{1,-2}(z-1) - 2S_{-3}(z-1) + \frac{1}{3}\pi^2 S_1(z-1) + 2S_3(z-1). \quad (23)$$

Then, the NLO function in Equation (19) reads

$$\delta(z, \bar{z}) = \phi(1+z) + \phi(1+\bar{z}) - 2 \left( S_1(z) + S_1(\bar{z}) \right) \left( S_{-2}(z) + S_{-2}(\bar{z}) \right) \quad (24)$$

At this point, we have the exact expression for the LO and NLO BFKL eigenvalue in the color singlet state in  $\mathcal{N} = 4$  SYM expressed in terms of the analytically continued harmonic sums. It is no different from the one given in the paper of Kotikov and Lipatov [27].

Now, we consider a special case of zero conformal spin for expression in Equation (24). For  $n = 0$ , the holomorphic variables  $z$  and  $\bar{z}$  are not independent anymore and related by  $\bar{z} = -1 - z$  suggesting that it would be more natural to write  $\delta(z, \bar{z})$  in Equation (24) as a sum of two parts

$$\delta(z, \bar{z}) = F_2(z) + F_2(-1 - z), \quad (25)$$

where

$$\frac{F_2(z)}{4} = -\frac{3}{2}\zeta_3 + \pi^2 \ln 2 + \frac{\pi^2}{3}S_1(z) + 2S_3(z) + \pi^2 S_{-1}(z) - 4S_{-2,1}(z), \quad (26)$$

using the reflection identity derived by the author in Reference [8]

$$\begin{aligned} S_1(z)S_{-2}(-1 - z) &= -\frac{1}{2}\pi^2 \log(2) + \frac{3\zeta(3)}{4} - \frac{1}{4}\pi^2 S_{-1}(-1 - z) - \frac{1}{4}\pi^2 S_{-1}(z) \\ &\quad + \frac{1}{12}\pi^2 S_1(-1 - z) - \frac{1}{12}\pi^2 S_1(z) + S_{-2,1}(z) + S_{1,-2}(-1 - z) \end{aligned} \quad (27)$$

for the mixed term  $S_1(z)S_{-2}(\bar{z})$  appearing in the last term of Equation (24).

In fact, this example reproduces the starting point of References [9,10], which laid a path to the corresponding expression for the NNLO case expressed in terms of the harmonic sums at weight  $w = 5$ . The only difference is that the authors of References [9,10] did not use the reflection identities rather a direct pole decomposition. We use the same pole decomposition for deriving the reflection identities effectively; there is no difference between the two approaches at this point.

The need for the reflection identities becomes obvious when we try to go back from the expression for  $n = 0$  in Equation (25) in attempt to restore the full functional dependence on the conformal spin in Equation (24). In this case, we need to consider all possible combinations of  $S_{\{a\}}(z)S_{\{b\}}(\bar{z})$  appearing as a direct product of harmonic sums at weight  $w = 1$  and weight  $w = 2$  and then use the corresponding reflection identities  $S_{\{a\}}(z)S_{\{b\}}(-1 - z)$  in the following way. Let us assume we do not know a closed expression valid for an arbitrary value of the conformal spin, i.e., we do not know the form of the expression in Equation (23). We do know that this expression must be built of the harmonic sums of the uniform weight  $w = 3$  because this corresponds to the highest transcendentality principle formulated for  $\mathcal{N} = 4$  SYM by Kotikov and Lipatov [18]. Then, we can consider the following terms that should build our ansatz

$$\beta_i \left( S_{\{a_i\}}(z)S_{\{b_i\}}(\bar{z}) - S_{\{a_i\}}(z)S_{\{b_i\}}(-1 - z) \right), \quad (28)$$

where  $\beta_i$  is the unknown parameter to be fixed and the index  $i$  runs over all possible terms in the direct product of harmonic sums at weight  $w = 1$  and weight  $w = 2$ . Obviously, at  $n = 0$ , Equation (28) vanishes and we cannot fix the coefficients  $\beta_i$  using only information from Equation (25), but if we add to it all possible harmonic sums of either  $z$  or  $\bar{z}$  as well as constants of uniform weight  $w = 3$  we can fix the coefficients of those, merely based on the pole expansion of Equation (25). Thus, we are left with only a limited set of unknown parameters  $\beta_i$  and to fix those we need to consider only one additional value of the conformal spin, say  $n = 1$ . Provided we have this expression, we can easily calculate  $\beta_i$  using pole expansion of the harmonic sums and comparing them for the two expressions. Fortunately, the analytic expressions in terms of harmonic sums for both  $n = 0$  and  $n = 1$  as well as many others are available in References [9,10] and we only need to consider all possible reflection identities up to weight  $w = 5$  following the steps we have outlined at the example of the NLO BFKL eigenvalue. In this paper, we derive the reflection identities at weight  $w = 4$ , which adds to our previous studies

for  $w = 2$  and  $w = 3$  [8]. The missing part is the reflection identities at weight  $w = 5$  still need to be calculated; there are 216 of them compared to 57 at weight  $w = 4$ .

In the next section, we discuss methods and results of our calculations focusing on some technical aspects of building the proper space of functions, performing the pole expansion of the harmonic sums and building coefficient equations for the reflection identities.

### 3. Methods and Results

The reflection identities at weight  $w = 4$  are obtained by taking a product of harmonic sums of argument  $z$  at weight  $w = 1$  and harmonic sums of argument  $-1 - z$  at weight  $w = 3$ , i.e.,  $B_1 \otimes \bar{B}_3$ , and also by taking a product of harmonic sums of argument  $z$  and  $-1 - z$  at weight  $w = 2$ , i.e.,  $B_2 \otimes \bar{B}_2$ .

The number of basis harmonic sums in  $B_1$ ,  $B_2$  and  $B_3$  is given by

$$\text{Length}(B_1) = 2, \quad \text{Length}(B_2) = 6, \quad \text{Length}(B_3) = 18 \quad (29)$$

so that the number of elements in the products  $B_1 \otimes \bar{B}_3$  and  $B_2 \otimes \bar{B}_2$  reads

$$B_1 \otimes \bar{B}_3 = 2 \times 18 = 36 \quad (30)$$

and

$$B_2 \otimes \bar{B}_2 = \frac{6 \times (6 - 1)}{2} + 6 = 21, \quad (31)$$

resulting in the total number of irreducible reflections identities at weight  $w = 4$  being equal to  $21 + 36 = 57$ .

To calculate the reflection identities at weight  $w = 4$ , we use the basis harmonic sums at  $w = 4$  listed in Equation (15) together with basis harmonic sums at lower weight listed in the work of J. Ablinger in Equations (12)–(14) multiplied by irreducible constants at corresponding weight listed in Equations (8)–(10). This should be supplemented by irreducible constants at weight  $w = 4$  listed in Equation (11). The number of basis sums in  $B_4$  equals

$$\text{Length}(B_4) = 54 \quad (32)$$

so that the total number of terms in the expansion ansatz at  $w = 4$

$$B_4 + B_3 \otimes C_1 + B_2 \otimes C_2 + B_1 \otimes C_3 + C_4 \quad (33)$$

is given by

$$\text{Length}(\text{ANZ}_4) = 54 + 18 \times 1 + 6 \times 2 + 2 \times 3 + 5 = 95 \quad (34)$$

The full expansion ansatz at  $w = 4$  is given by

$$\begin{aligned} \text{ANZ}_4 = & \left\{ \pi^4, \pi^2 \log^2(2), \log^4(2), \text{Li}_4\left(\frac{1}{2}\right), S_{-4}, S_{-3} \log(2), \pi^2 S_{-2}, S_{-2} \log^2(2), \right. \\ & \pi^2 S_{-1} \log(2), S_{-1} \log^3(2), \pi^2 S_1 \log(2), S_1 \log^3(2), \pi^2 S_2, S_2 \log^2(2), \\ & S_3 \log(2), S_4, S_{-3,-1}, S_{-3,1}, S_{-2,-2}, \log(2)S_{-2,-1}, \log(2)S_{-2,1}, S_{-2,2}, \\ & S_{-1,-3}, \log(2)S_{-1,-2}, \pi^2 S_{-1,-1}, \log^2(2)S_{-1,-1}, \pi^2 S_{-1,1}, \log^2(2)S_{-1,1}, \\ & \log(2)S_{-1,2}, S_{-1,3}, S_{1,-3}, \log(2)S_{1,-2}, \pi^2 S_{1,-1}, \log^2(2)S_{1,-1}, \pi^2 S_{1,1}, \\ & \log^2(2)S_{1,1}, \log(2)S_{1,2}, S_{1,3}, S_{2,-2}, \log(2)S_{2,-1}, \log(2)S_{2,1}, S_{2,2}, S_{3,-1}, \\ & S_{3,1}, S_{-2,-1,-1}, S_{-2,-1,1}, S_{-2,1,-1}, S_{-2,1,1}, S_{-1,-2,-1}, S_{-1,-2,1}, S_{-1,-1,-2}, \\ & \log(2)S_{-1,-1,-1}, \log(2)S_{-1,-1,1}, S_{-1,-1,2}, S_{-1,1,-2}, \log(2)S_{-1,1,-1}, \\ & \log(2)S_{-1,1,1}, S_{-1,1,2}, S_{-1,2,-1}, S_{-1,2,1}, S_{1,-2,-1}, S_{1,-2,1}, S_{1,-1,-2}, \\ & \log(2)S_{1,-1,-1}, \log(2)S_{1,-1,1}, S_{1,-1,2}, S_{1,1,-2}, \log(2)S_{1,1,-1}, \log(2)S_{1,1,1}, \\ & S_{1,1,2}, S_{1,2,-1}, S_{1,2,1}, S_{2,-1,-1}, S_{2,-1,1}, S_{2,1,-1}, S_{2,1,1}, S_{-1,-1,-1}, S_{-1,-1,-1,1}, \\ & S_{-1,-1,1,-1}, S_{-1,-1,1,1}, S_{-1,1,-1,-1}, S_{-1,1,-1,1}, S_{-1,1,1,-1}, S_{-1,1,1,1}, S_{1,-1,-1,-1}, \\ & S_{1,-1,-1,1}, S_{1,-1,1,-1}, S_{1,-1,1,1}, S_{1,1,-1,-1}, S_{1,1,-1,1}, S_{1,1,1,-1}, S_{1,1,1,1}, \zeta_3 \log(2), \\ & \zeta_3 S_{-1}, \zeta_3 S_1 \} \end{aligned} \quad (35)$$

It is worth emphasizing that we do not reduce  $\text{ANZ}_4$  using quasi-shuffle relations because we want to stick to the *linear basis* used by Caron Huot and Herraren [10] for calculations of the color singlet NNLO BFKL eigenvalue. It should also be mentioned that the total number of the basis elements in either linear  $\text{ANZ}_4$  or nonlinear basis obtained with the use of quasi-shuffle relations is the same. The choice of the basis does not effects the final result and is solely a matter of convenience.

The expansion of the product of two functions of argument  $z$  and argument  $-1 - z$  we search in terms of two sets of  $\text{ANZ}_4$ , one of argument  $z$  and the other of argument  $-1 - z$ . The total number of elements in this expression equals  $95 \times 2 - 5 = 185$ , where we remove five redundant constants at  $w = 4$  because they are the same for both arguments. We fix the 185 free coefficients using pole expansion of the product  $s_{a_1, a_2, \dots}(z)s_{b_1, b_2, \dots}(-1 - z)$  around negative integers of  $z = -5, \dots, -1$  and expanding to the second order of the expansion parameter. It turns out that, to fix all 185 free coefficients, we need only expansion up to first order and we use the second order of the expansion to double check our results. We check our results listed in the Appendix A by a direct numerical calculation at the accuracy  $10^{-10}$ .

Below, we give two examples of reflection identities,  $S_1(z)S_{2,1}(-1 - z)$  for harmonic sums with positive indices and  $S_{-1}(z)S_{-2,-1}(-1 - z)$  for harmonic sums with negative indices. All 57 irreducible reflection identities at weight four are listed in the Appendix A. The two examples are

$$\begin{aligned} S_1(z)S_{2,1}(-1 - z) = & \frac{6\zeta_2^2}{5} - S_2(z)\zeta_2 + S_2(-1 - z)\zeta_2 + 2S_1(z)\zeta_3 - 2\zeta_3 S_1(-1 - z) + S_{3,1}(z) \\ & - S_{3,1}(-1 - z) - S_{2,1,1}(z) + S_{1,2,1}(-1 - z) + S_{2,1,1}(-1 - z) \end{aligned} \quad (36)$$

and

$$\begin{aligned}
S_{-1}(z)S_{-2,-1}(-1-z) = & -\frac{\ln^4(2)}{6} - S_{-2}(z)\ln^2(2) + S_2(z)\ln^2(2) + \zeta_2\ln^2(2) \\
& - S_{-2}(-1-z)\ln^2(2) + S_2(-1-z)\ln^2(2) - S_{-3}(z)\ln(2) \\
& + S_3(z)\ln(2) - 6\zeta_3\ln(2) - S_{-3}(-1-z)\ln(2) + S_3(-1-z)\ln(2) \\
& - 2S_{-2,-1}(z)\ln(2) - 2S_{-2,-1}(-1-z)\ln(2) + \frac{33\zeta_2^2}{20} - 4\text{Li}_4\left(\frac{1}{2}\right) \\
& + \frac{1}{2}S_{-2}(z)\zeta_2 + \frac{1}{4}S_{-1}(z)\zeta_3 + \frac{1}{2}\zeta_2S_{-2}(-1-z) + \frac{1}{4}\zeta_3S_{-1}(-1-z) \\
& + S_{3,-1}(z) + S_{3,-1}(-1-z) - S_{-2,-1,-1}(z) - S_{-2,-1,-1}(-1-z) \\
& - S_{-1,-2,-1}(-1-z)
\end{aligned} \tag{37}$$

One can see that the reflection identities for harmonic sums with negative indices are more complicated than those with only positive indices and this happens mostly due to appearance of constant  $\ln(2)$ , which originates from sign alternating summation in  $S_{-1}(z)$  absent for positive indices.

In the present paper, we consider only bilinear reflection identities expressing a product of two harmonic sums of argument  $z$  and  $-1-z$  in terms of a linear combination of other sums of the same arguments. One can consider also trilinear and quadlinear identities, but whose reducible and form a linear combination of the bilinear identities presented in this paper. For example, we can consider a trilinear term  $S_1(z)S_1(-1-z)S_2(-1-z)$  and write it as

$$S_1(z)S_1(-1-z)S_2(-1-z) = S_1(z)S_{1,2}(-1-z) + S_1(z)S_{2,1}(-1-z) - S_1(z)S_3(-1-z) \tag{38}$$

where we used a quasi-shuffle identity from Equation (3)

$$S_{1,2}(z) + S_{2,1}(z) - S_3(z) = S_1(z)S_2(z) \tag{39}$$

The expression for  $S_1(z)S_{1,2}(-1-z)$  is given in Equation (A28) and for  $S_1(z)S_3(-1-z)$  in Equation (A20). Plugging those together with Equation (36) into Equation (38), we get

$$\begin{aligned}
S_1(z)S_1(-1-z)S_2(-1-z) = & -\zeta_2S_{1,1}(-1-z) - S_{1,3} - S_{2,2}(-1-z) - S_{3,1}(-1-z) \\
& + 2S_{1,1,2}(-1-z) + S_{1,2,1}(-1-z) + S_{2,1,1}(-1-z) \\
& + 2\zeta_2S_2(-1-z) + \zeta_3S_1(-1-z) + \zeta_2S_{1,1}(z) + S_{3,1}(z) \\
& - S_{1,2,1}(z) - S_{2,1,1}(z) + \frac{4\zeta_2^2}{5} - \zeta_2S_2(z) + 2\zeta_3S_1(z)
\end{aligned} \tag{40}$$

In a similar way, one can build any trilinear or quadlinear reflection identity using quasi-shuffle relations for harmonic sums and the bilinear reflection identities listed in the Appendix A. All possible quasi-shuffle relations required for the present discussion are available in the HarmonicSums package by J. Ablinger. The quasi-shuffle relations before and after analytic continuation of the harmonic sums to the complex plane are the same.

#### 4. Discussion and Conclusions

We consider meromorphic functions obtained by analytic continuation of the harmonic sums from positive integers to the complex plane except for isolated pole singularities at negative integer values of the argument. We call those functions the analytically continued harmonic sums or simply the harmonic sums for clarity of the presentation. We discuss the reflection identities for harmonic sums of weight four. There are 57 irreducible bilinear identities listed in the Appendix A. All other bilinear reflection identities are easily obtained by a trivial change of argument  $z \leftrightarrow -1-z$ . The trilinear and quadlinear identities for a product of three and four harmonic sums are obtained from the identities

listed in the Appendix A using the quasi-shuffle relations for harmonic sums. In our analysis, we use the linear basis for harmonic sums and limit ourselves to harmonic sums analytically continued from even integer values of the argument to the complex plane. The analytic continuation from odd integers is beyond the scope of the present study.

For deriving the reflection identities presented in this paper, we used Harmonic Sums package by Ablinger [24], HPL package by Maitre [28] and dedicated Mathematica package for pomeron NNLO eigenvalue by Gromov, Levkovich-Maslyuk and Sizov [9].

We expanded around positive and negative integer points the product of two harmonic sums  $S_{a_1, a_2, \dots}(z)S_{b_1, b_2, \dots}(-1 - z)$  and the functional basis built of pure Harmonic Sums with constants of relevant weight. Then, we compared the coefficients of the irreducible constants of a given weight and solved the resulting set of coefficient equations. We used higher order expansion to cross check our results. The bilinear reflection identities presented here are derived from the pole expansion based on the Mellin transform and then checked them against the quasi-shuffle identities and numerical calculations of the corresponding harmonic sums on the complex plane.

We attach a Mathematica notebook with our results.

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## Abbreviations

The following abbreviations are used in this manuscript:

BFKL equation	Balitsky–Fadin–Kuraev–Lipatov equation
LO	leading order
NLO	next-to-leading order
NLO	next-to-leading order
NNLO	next-to-next-to-leading order

## Appendix A

Here, we list the irreducible reflection identities at weight  $w = 4$ . In all our expressions, we use the linear minimal set of harmonic sums given in Equations (12)–(15).

We use a compact notation where  $s_{a_1, a_2, \dots}$  stands for  $S_{a_1, a_2, \dots}(z)$ , whereas  $\bar{s}_{a_1, a_2, \dots}$  stands for  $S_{a_1, a_2, \dots}(-1 - z)$ .

The constants are also written in a compact and readable way  $\ln_2 = \ln 2 \simeq 0.693147$  and  $\text{LiHalf}_4 = \text{Li}_4\left(\frac{1}{2}\right) \simeq 0.517479$ . For example, in this notation, Equation (36) is written as Equation (A24).

All other bilinear reflection identities are obtained by a trivial change of argument  $z \leftrightarrow -1 - z$ .

### Appendix A.1. Reflection Identities Originating from $B_1 \otimes \bar{B}_3$

$$\begin{aligned} s_{-1}\bar{s}_{-3} &= \frac{7\zeta_2^2}{5} - \frac{s_2\zeta_2}{2} - \frac{1}{2}\bar{s}_2\zeta_2 - \ln_2 s_{-3} + \ln_2 s_3 - \frac{3\ln_2\zeta_3}{2} - \frac{3}{4}s_{-1}\zeta_3 \\ &\quad - \ln_2\bar{s}_{-3} - \frac{3}{4}\zeta_3\bar{s}_{-1} - \ln_2\bar{s}_3 - s_{-3,-1} - \bar{s}_{-1,-3} \end{aligned} \tag{A1}$$

$$\begin{aligned} s_{-1}\bar{s}_3 &= -\frac{3\zeta_2^2}{5} + \frac{1}{2}s_{-2}\zeta_2 - \frac{1}{2}\bar{s}_{-2}\zeta_2 - \ln_2 s_{-3} + \ln_2 s_3 - \frac{3\ln_2\zeta_3}{2} \\ &\quad - \frac{3}{4}s_{-1}\zeta_3 - \ln_2 \bar{s}_{-3} - \frac{3}{4}\zeta_3\bar{s}_{-1} - \ln_2 \bar{s}_3 + s_{3,-1} - \bar{s}_{-1,3} \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} s_{-1}\bar{s}_{-2,-1} &= -\frac{\ln_2^4}{6} - s_{-2}\ln_2^2 + s_2\ln_2^2 + \zeta_2\ln_2^2 - \bar{s}_{-2}\ln_2^2 + \bar{s}_2\ln_2^2 - s_{-3}\ln_2 + s_3\ln_2 \\ &\quad - 6\zeta_3\ln_2 - \bar{s}_{-3}\ln_2 + \bar{s}_3\ln_2 - 2s_{-2,-1}\ln_2 - 2\bar{s}_{-2,-1}\ln_2 + \frac{33\zeta_2^2}{20} - 4\text{LiHalf}_4 \\ &\quad + \frac{1}{2}s_{-2}\zeta_2 + \frac{1}{4}s_{-1}\zeta_3 + \frac{1}{2}\zeta_2\bar{s}_{-2} + \frac{1}{4}\zeta_3\bar{s}_{-1} + s_{3,-1} + \bar{s}_{3,-1} - s_{-2,-1,-1} \\ &\quad - \bar{s}_{-2,-1,-1} - \bar{s}_{-1,-2,-1} \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} s_{-1}\bar{s}_{-2,1} &= -\frac{\ln_2^4}{12} - \frac{1}{2}s_{-2}\ln_2^2 + \frac{1}{2}s_2\ln_2^2 + \frac{1}{2}\zeta_2\ln_2^2 - \frac{1}{2}\bar{s}_{-2}\ln_2^2 + \frac{1}{2}\bar{s}_2\ln_2^2 - s_{-3}\ln_2 + s_3\ln_2 \\ &\quad - 3\zeta_3\ln_2 - s_{-2,-1}\ln_2 + s_{-2,1}\ln_2 - \bar{s}_{-2,-1}\ln_2 - \bar{s}_{-2,1}\ln_2 + \frac{61\zeta_2^2}{40} - 2\text{LiHalf}_4 \\ &\quad - \frac{s_2\zeta_2}{2} - \frac{5}{8}s_{-1}\zeta_3 - \frac{1}{2}\zeta_2\bar{s}_{-2} - \frac{5}{8}\zeta_3\bar{s}_{-1} - \frac{1}{2}\zeta_2\bar{s}_2 - s_{-3,-1} + \bar{s}_{3,1} + s_{-2,1,-1} \\ &\quad - \bar{s}_{-2,-1,1} - \bar{s}_{-1,-2,1} \end{aligned} \quad (\text{A4})$$

$$\begin{aligned} s_{-1}\bar{s}_{2,-1} &= -\frac{\ln_2^4}{6} - s_{-2}\ln_2^2 + s_2\ln_2^2 + \zeta_2\ln_2^2 + \bar{s}_{-2}\ln_2^2 - \bar{s}_2\ln_2^2 - s_{-3}\ln_2 \\ &\quad + s_3\ln_2 + \bar{s}_{-3}\ln_2 - \bar{s}_3\ln_2 + 2s_{2,-1}\ln_2 - 2\bar{s}_{2,-1}\ln_2 + \frac{5\zeta_2^2}{4} \\ &\quad - 4\text{LiHalf}_4 - \frac{s_2\zeta_2}{2} + \frac{1}{4}s_{-1}\zeta_3 + \frac{1}{4}\zeta_3\bar{s}_{-1} + \frac{1}{2}\zeta_2\bar{s}_2 - s_{-3,-1} \\ &\quad + \bar{s}_{-3,-1} + s_{2,-1,-1} - \bar{s}_{-1,2,-1} - \bar{s}_{2,-1,-1} \end{aligned} \quad (\text{A5})$$

$$\begin{aligned} s_{-1}\bar{s}_{2,1} &= \frac{\ln_2^4}{6} - \frac{1}{2}s_{-2}\ln_2^2 + \frac{1}{2}s_2\ln_2^2 - \zeta_2\ln_2^2 + \frac{1}{2}\bar{s}_{-2}\ln_2^2 - \frac{1}{2}\bar{s}_2\ln_2^2 \\ &\quad - s_{-3}\ln_2 + s_3\ln_2 + \frac{9\zeta_3\ln_2}{4} + s_{2,-1}\ln_2 - s_{2,1}\ln_2 - \bar{s}_{2,-1}\ln_2 - \bar{s}_{2,1}\ln_2 - \frac{7\zeta_2^2}{4} \\ &\quad + 4\text{LiHalf}_4 + \frac{1}{2}s_{-2}\zeta_2 - \frac{5}{8}s_{-1}\zeta_3 - \frac{1}{2}\zeta_2\bar{s}_{-2} - \frac{5}{8}\zeta_3\bar{s}_{-1} - \frac{1}{2}\zeta_2\bar{s}_2 \\ &\quad + s_{3,-1} + \bar{s}_{-3,1} - s_{2,1,-1} - \bar{s}_{-1,2,1} - \bar{s}_{2,-1,1} \end{aligned} \quad (\text{A6})$$

$$\begin{aligned} s_{-1}\bar{s}_{-1,1,-1} &= -\frac{7\ln_2^4}{24} - s_{-2}\ln_2^2 + s_2\ln_2^2 + \frac{9}{4}\zeta_2\ln_2^2 - \frac{1}{2}\bar{s}_{-2}\ln_2^2 + \frac{1}{2}\bar{s}_2\ln_2^2 \\ &\quad - \frac{1}{2}s_{-1,-1}\ln_2^2 + \frac{3}{2}s_{-1,1}\ln_2^2 + \frac{1}{2}\bar{s}_{-1,-1}\ln_2^2 - \frac{1}{2}\bar{s}_{-1,1}\ln_2^2 - s_{-3}\ln_2 + s_3\ln_2 - 3\zeta_3\ln_2 \\ &\quad + \frac{3}{2}\zeta_2\bar{s}_{-1}\ln_2 - 2s_{-2,-1}\ln_2 - s_{-1,-2}\ln_2 + s_{-1,2}\ln_2 + \bar{s}_{-1,-2}\ln_2 \\ &\quad - \bar{s}_{-1,2}\ln_2 + 2s_{-1,1,-1}\ln_2 - 2\bar{s}_{-1,1,-1}\ln_2 + \frac{13\zeta_2^2}{40} - \text{LiHalf}_4 + \frac{1}{2}s_{-2}\zeta_2 \\ &\quad + \frac{1}{4}s_{-1}\zeta_3 - \frac{1}{8}\zeta_3\bar{s}_{-1} - \frac{1}{2}\zeta_2s_{-1,1} + s_{3,-1} + \frac{1}{2}\zeta_2\bar{s}_{-1,1} - s_{-2,-1,-1} \\ &\quad - s_{-1,-2,-1} + \bar{s}_{-1,-2,-1} + \bar{s}_{2,1,-1} + s_{-1,1,-1,-1} - 2\bar{s}_{-1,-1,1,-1} - \bar{s}_{-1,1,-1,-1} \end{aligned} \quad (\text{A7})$$

$$\begin{aligned}
s_{-1}\bar{s}_{-1,1,1} &= -\frac{\ln_2^4}{6} - \frac{1}{3}s_{-1}\ln_2^3 - \frac{1}{2}s_{-2}\ln_2^2 + \frac{1}{2}s_2\ln_2^2 + \frac{3}{4}\zeta_2\ln_2^2 - \frac{1}{2}s_{-1,-1}\ln_2^2 \\
&\quad + \frac{1}{2}s_{-1,1}\ln_2^2 + \frac{1}{2}\bar{s}_{-1,-1}\ln_2^2 - \frac{1}{2}\bar{s}_{-1,1}\ln_2^2 - s_{-3}\ln_2 + s_3\ln_2 + s_{-1}\zeta_2\ln_2 \\
&\quad - \frac{7\zeta_3\ln_2}{8} - s_{-2,-1}\ln_2 + s_{-2,1}\ln_2 - s_{-1,-2}\ln_2 + s_{-1,2}\ln_2 + s_{-1,1,-1}\ln_2 \\
&\quad - s_{-1,1,1}\ln_2 - \bar{s}_{-1,1,-1}\ln_2 - \bar{s}_{-1,1,1}\ln_2 - \frac{27\zeta_2^2}{40} + 4\text{LiHalf}_4 - \frac{s_2\zeta_2}{2} \\
&\quad - \frac{9}{8}s_{-1}\zeta_3 - \zeta_3\bar{s}_{-1} - s_{-3,-1} + \frac{1}{2}\zeta_2s_{-1,-1} - \frac{1}{2}\zeta_2\bar{s}_{-1,-1} - \frac{1}{2}\zeta_2\bar{s}_{-1,1} \\
&\quad + s_{-2,1,-1} + s_{-1,2,-1} + \bar{s}_{-1,-2,1} + \bar{s}_{2,1,1} - s_{-1,1,1,-1} \\
&\quad - 2\bar{s}_{-1,-1,1,1} - \bar{s}_{-1,1,-1,1}
\end{aligned} \tag{A8}$$

$$\begin{aligned}
s_{-1}\bar{s}_{1,-2} &= -\frac{\ln_2^4}{6} + \frac{3}{2}\zeta_2\ln_2^2 - s_{-3}\ln_2 + s_3\ln_2 + \frac{1}{2}s_{-1}\zeta_2\ln_2 - \frac{1}{2}s_1\zeta_2\ln_2 - \frac{15\zeta_3\ln_2}{4} \\
&\quad + \frac{1}{2}\zeta_2\bar{s}_{-1}\ln_2 - \frac{1}{2}\zeta_2\bar{s}_1\ln_2 + s_{1,-2}\ln_2 - s_{1,2}\ln_2 - \bar{s}_{1,-2}\ln_2 - \bar{s}_{1,2}\ln_2 + 2\zeta_2^2 \\
&\quad - 4\text{LiHalf}_4 - \frac{s_2\zeta_2}{2} - \frac{1}{8}s_{-1}\zeta_3 + \frac{5s_1\zeta_3}{8} + \frac{1}{2}\zeta_2\bar{s}_{-2} - \frac{1}{8}\zeta_3\bar{s}_{-1} + \frac{13}{8}\zeta_3\bar{s}_1 \\
&\quad - \frac{1}{2}\zeta_2\bar{s}_2 - s_{-3,-1} - \frac{1}{2}\zeta_2s_{1,-1} + \bar{s}_{-2,-2} - \frac{1}{2}\zeta_2\bar{s}_{1,-1} + s_{1,-2,-1} \\
&\quad - \bar{s}_{-1,1,-2} - \bar{s}_{1,-1,-2}
\end{aligned} \tag{A9}$$

$$\begin{aligned}
s_{-1}\bar{s}_{1,2} &= -\frac{\ln_2^4}{12} + \zeta_2\ln_2^2 - s_{-3}\ln_2 + s_3\ln_2 + \frac{1}{2}s_{-1}\zeta_2\ln_2 - \frac{1}{2}s_1\zeta_2\ln_2 \\
&\quad - \frac{15\zeta_3\ln_2}{4} + \frac{1}{2}\zeta_2\bar{s}_{-1}\ln_2 - \frac{1}{2}\zeta_2\bar{s}_1\ln_2 + s_{1,-2}\ln_2 - s_{1,2}\ln_2 \\
&\quad - \bar{s}_{1,-2}\ln_2 - \bar{s}_{1,2}\ln_2 + \frac{7\zeta_2^2}{8} - 2\text{LiHalf}_4 + \frac{1}{2}s_{-2}\zeta_2 - s_{-1}\zeta_3 \\
&\quad + \frac{s_1\zeta_3}{4} - \zeta_3\bar{s}_{-1} - \zeta_3\bar{s}_1 - \frac{1}{2}\zeta_2s_{1,-1} + s_{3,-1} + \bar{s}_{-2,2} \\
&\quad - \frac{1}{2}\zeta_2\bar{s}_{1,-1} - s_{1,2,-1} - \bar{s}_{-1,1,2} - \bar{s}_{1,-1,2}
\end{aligned} \tag{A10}$$

$$\begin{aligned}
s_{-1}\bar{s}_{-1,-1,-1} &= \frac{\ln_2^4}{8} - \frac{1}{6}s_{-1}\ln_2^3 + \frac{7}{6}s_1\ln_2^3 - \frac{1}{6}\bar{s}_{-1}\ln_2^3 + \frac{1}{6}\bar{s}_1\ln_2^3 - s_{-2}\ln_2^2 \\
&\quad + s_2\ln_2^2 - \frac{1}{2}\zeta_2\ln_2^2 - \bar{s}_{-2}\ln_2^2 + \bar{s}_2\ln_2^2 + 2s_{1,-1}\ln_2^2 - s_{-3}\ln_2 \\
&\quad + s_3\ln_2 - s_1\zeta_2\ln_2 - 3\zeta_3\ln_2 - 2s_{-2,-1}\ln_2 + s_{1,-2}\ln_2 - s_{1,2}\ln_2 \\
&\quad - \bar{s}_{1,-2}\ln_2 + \bar{s}_{1,2}\ln_2 + 2s_{1,-1,-1}\ln_2 - 2\bar{s}_{1,-1,-1}\ln_2 + \frac{6\zeta_2^2}{5} \\
&\quad - 3\text{LiHalf}_4 + \frac{1}{2}s_{-2}\zeta_2 - \frac{1}{4}s_{-1}\zeta_3 + \frac{s_1\zeta_3}{2} - \frac{1}{4}\zeta_3\bar{s}_{-1} - \frac{3}{4}\zeta_3\bar{s}_1 \\
&\quad - \frac{1}{2}\zeta_2s_{1,-1} + s_{3,-1} + \frac{1}{2}\zeta_2\bar{s}_{1,-1} - s_{-2,-1,-1} - s_{1,2,-1} \\
&\quad + \bar{s}_{-2,-1,-1} + \bar{s}_{1,2,-1} + s_{1,-1,-1,-1} - \bar{s}_{-1,1,-1,-1} - 2\bar{s}_{1,-1,-1,-1}
\end{aligned} \tag{A11}$$

$$\begin{aligned}
s_{-1}\bar{s}_{1,-1,1} = & -\frac{\ln_2^4}{12} - \frac{1}{6}s_{-1}\ln_2^3 + \frac{5}{6}s_1\ln_2^3 - \frac{1}{6}\bar{s}_{-1}\ln_2^3 + \frac{1}{6}\bar{s}_1\ln_2^3 \\
& - \frac{1}{2}s_{-2}\ln_2^2 + \frac{1}{2}s_2\ln_2^2 - \frac{1}{2}\zeta_2\ln_2^2 - \frac{1}{2}\bar{s}_{-2}\ln_2^2 + \frac{1}{2}\bar{s}_2\ln_2^2 \\
& + s_{1,-1}\ln_2^2 - s_{-3}\ln_2 + s_3\ln_2 - s_1\zeta_2\ln_2 - \frac{19\zeta_3\ln_2}{8} \\
& - \frac{3}{2}\zeta_2\bar{s}_1\ln_2 - s_{-2,-1}\ln_2 + s_{-2,1}\ln_2 + s_{1,-2}\ln_2 - s_{1,2}\ln_2 \\
& + s_{1,-1,-1}\ln_2 - s_{1,-1,1}\ln_2 - \bar{s}_{1,-1,-1}\ln_2 - \bar{s}_{1,-1,1}\ln_2 + \frac{49\zeta_2^2}{20} \\
& - 6\text{LiHalf}_4 - \frac{s_2\zeta_2}{2} + \frac{1}{8}s_{-1}\zeta_3 + \frac{3s_1\zeta_3}{4} + \frac{1}{2}\zeta_2\bar{s}_{-2} + \frac{1}{8}\zeta_3\bar{s}_{-1} \\
& + \frac{3}{2}\zeta_3\bar{s}_1 - \frac{1}{2}\zeta_2\bar{s}_2 - s_{-3,-1} - \frac{1}{2}\zeta_2s_{1,-1} - \zeta_2\bar{s}_{1,-1} + s_{-2,1,-1} \\
& + s_{1,-2,-1} + \bar{s}_{-2,-1,1} + \bar{s}_{1,2,1} - s_{1,-1,1,-1} \\
& - \bar{s}_{-1,1,-1,1} - 2\bar{s}_{1,-1,-1,1}
\end{aligned} \tag{A12}$$

$$\begin{aligned}
s_{-1}\bar{s}_{1,1,-1} = & -\frac{\ln_2^4}{3} - \frac{1}{6}s_{-1}\ln_2^3 - \frac{1}{6}s_1\ln_2^3 - \frac{1}{6}\bar{s}_{-1}\ln_2^3 + \frac{1}{6}\bar{s}_1\ln_2^3 - s_{-2}\ln_2^2 \\
& + s_2\ln_2^2 + \frac{11}{4}\zeta_2\ln_2^2 + \frac{1}{2}\bar{s}_{-2}\ln_2^2 - \frac{1}{2}\bar{s}_2\ln_2^2 + \frac{1}{2}s_{1,-1}\ln_2^2 \\
& - \frac{3}{2}s_{1,1}\ln_2^2 + \frac{1}{2}\bar{s}_{1,-1}\ln_2^2 - \frac{1}{2}\bar{s}_{1,1}\ln_2^2 - s_{-3}\ln_2 + s_3\ln_2 + \frac{1}{2}s_{-1}\zeta_2\ln_2 \\
& + \frac{1}{2}s_1\zeta_2\ln_2 - 4\zeta_3\ln_2 + \frac{1}{2}\zeta_2\bar{s}_{-1}\ln_2 + \zeta_2\bar{s}_1\ln_2 + s_{1,-2}\ln_2 - s_{1,2}\ln_2 \\
& + 2s_{2,-1}\ln_2 + \bar{s}_{1,-2}\ln_2 - \bar{s}_{1,2}\ln_2 - 2s_{1,1,-1}\ln_2 - 2\bar{s}_{1,1,-1}\ln_2 + \frac{8\zeta_2^2}{5} \\
& - 4\text{LiHalf}_4 - \frac{s_2\zeta_2}{2} - \frac{s_1\zeta_3}{4} - \frac{1}{8}\zeta_3\bar{s}_1 - s_{-3,-1} + \frac{1}{2}\zeta_2s_{1,1} + \frac{1}{2}\zeta_2\bar{s}_{1,1} \\
& + s_{1,-2,-1} + s_{2,-1,-1} + \bar{s}_{-2,1,-1} + \bar{s}_{1,-2,-1} - s_{1,1,-1,-1} \\
& - \bar{s}_{-1,1,1,-1} - \bar{s}_{1,-1,1,-1} - \bar{s}_{1,1,-1,-1}
\end{aligned} \tag{A13}$$

$$\begin{aligned}
s_{-1}\bar{s}_{1,1,1} = & -\frac{\ln_2^4}{24} - \frac{1}{6}s_{-1}\ln_2^3 + \frac{1}{6}s_1\ln_2^3 - \frac{1}{6}\bar{s}_{-1}\ln_2^3 + \frac{1}{6}\bar{s}_1\ln_2^3 - \frac{1}{2}s_{-2}\ln_2^2 \\
& + \frac{1}{2}s_2\ln_2^2 + \frac{1}{4}\zeta_2\ln_2^2 + \frac{1}{2}s_{1,-1}\ln_2^2 - \frac{1}{2}s_{1,1}\ln_2^2 + \frac{1}{2}\bar{s}_{1,-1}\ln_2^2 - \frac{1}{2}\bar{s}_{1,1}\ln_2^2 \\
& - s_{-3}\ln_2 + s_3\ln_2 + \frac{1}{2}s_{-1}\zeta_2\ln_2 - \frac{1}{2}s_1\zeta_2\ln_2 - \frac{7\zeta_3\ln_2}{8} + \frac{1}{2}\zeta_2\bar{s}_{-1}\ln_2 \\
& - \frac{1}{2}\zeta_2\bar{s}_1\ln_2 + s_{1,-2}\ln_2 - s_{1,2}\ln_2 + s_{2,-1}\ln_2 - s_{2,1}\ln_2 - s_{1,1,-1}\ln_2 \\
& + s_{1,1,1}\ln_2 - \bar{s}_{1,1,-1}\ln_2 - \bar{s}_{1,1,1}\ln_2 - \frac{11\zeta_2^2}{40} + \text{LiHalf}_4 + \frac{1}{2}s_{-2}\zeta_2 \\
& - \frac{7}{8}s_{-1}\zeta_3 + \frac{s_1\zeta_3}{4} - \frac{7}{8}\zeta_3\bar{s}_{-1} - \frac{1}{8}\zeta_3\bar{s}_1 - \frac{1}{2}\zeta_2s_{1,-1} + s_{3,-1} - \frac{1}{2}\zeta_2\bar{s}_{1,-1} \\
& - \frac{1}{2}\zeta_2\bar{s}_{1,1} - s_{1,2,-1} - s_{2,1,-1} + \bar{s}_{-2,1,1} + \bar{s}_{1,-2,1} + s_{1,1,1,-1} \\
& - \bar{s}_{-1,1,1,1} - \bar{s}_{1,-1,1,1} - \bar{s}_{1,1,-1,1}
\end{aligned} \tag{A14}$$

$$\begin{aligned}
s_{-1}\bar{s}_{-1,-2} &= \zeta_2 \ln_2^2 - s_{-3} \ln_2 + s_3 \ln_2 + s_{-1}\zeta_2 \ln_2 + \frac{3\zeta_3 \ln_2}{2} - s_{-1,-2} \ln_2 \\
&\quad + s_{-1,2} \ln_2 - \bar{s}_{-1,-2} \ln_2 - \bar{s}_{-1,2} \ln_2 - \frac{11\zeta_2^2}{10} + \frac{1}{2}s_{-2}\zeta_2 - \frac{3}{4}s_{-1}\zeta_3 \\
&\quad - \frac{1}{2}\zeta_2\bar{s}_{-2} + \frac{3}{2}\zeta_3\bar{s}_{-1} + \frac{1}{2}\zeta_2\bar{s}_2 + \frac{1}{2}\zeta_2s_{-1,-1} + s_{3,-1} \\
&\quad - \frac{1}{2}\zeta_2\bar{s}_{-1,-1} + \bar{s}_{2,-2} - s_{-1,-2,-1} - 2\bar{s}_{-1,-1,-2}
\end{aligned} \tag{A15}$$

$$\begin{aligned}
s_{-1}\bar{s}_{-1,2} &= \frac{\ln_2^4}{3} - \zeta_2 \ln_2^2 - s_{-3} \ln_2 + s_3 \ln_2 + s_{-1}\zeta_2 \ln_2 + \frac{3\zeta_3 \ln_2}{2} \\
&\quad - s_{-1,-2} \ln_2 + s_{-1,2} \ln_2 - \bar{s}_{-1,-2} \ln_2 - \bar{s}_{-1,2} \ln_2 - \frac{8\zeta_2^2}{5} \\
&\quad + 8\text{LiHalf}_4 - \frac{s_2\zeta_2}{2} - \frac{5}{4}s_{-1}\zeta_3 - 2\zeta_3\bar{s}_{-1} - s_{-3,-1} \\
&\quad + \frac{1}{2}\zeta_2s_{-1,-1} - \frac{1}{2}\zeta_2\bar{s}_{-1,-1} + \bar{s}_{2,2} + s_{-1,2,-1} - 2\bar{s}_{-1,-1,2}
\end{aligned} \tag{A16}$$

$$\begin{aligned}
s_{-1}\bar{s}_{-1,-1,-1} &= -\frac{\ln_2^4}{2} - \frac{4}{3}s_{-1} \ln_2^3 - s_{-2} \ln_2^2 + s_2 \ln_2^2 + \bar{s}_{-2} \ln_2^2 - \bar{s}_2 \ln_2^2 \\
&\quad - 2s_{-1,-1} \ln_2^2 - s_{-3} \ln_2 + s_3 \ln_2 + s_{-1}\zeta_2 \ln_2 + 2\zeta_3 \ln_2 \\
&\quad - s_{-1,-2} \ln_2 + s_{-1,2} \ln_2 + 2s_{2,-1} \ln_2 - \bar{s}_{-1,-2} \ln_2 \\
&\quad + \bar{s}_{-1,2} \ln_2 - 2s_{-1,-1,-1} \ln_2 - 2\bar{s}_{-1,-1,-1} \ln_2 - \frac{21\zeta_2^2}{20} \\
&\quad + 4\text{LiHalf}_4 - \frac{s_2\zeta_2}{2} - \frac{3}{4}s_{-1}\zeta_3 - \zeta_3\bar{s}_{-1} - s_{-3,-1} \\
&\quad + \frac{1}{2}\zeta_2s_{-1,-1} + \frac{1}{2}\zeta_2\bar{s}_{-1,-1} + s_{-1,2,-1} + s_{2,-1,-1} \\
&\quad + \bar{s}_{-1,2,-1} + \bar{s}_{2,-1,-1} - s_{-1,-1,-1,-1} - 3\bar{s}_{-1,-1,-1,-1}
\end{aligned} \tag{A17}$$

$$\begin{aligned}
s_{-1}\bar{s}_{-1,-1,1} &= -\frac{13\ln_2^4}{24} - s_{-1} \ln_2^3 - \frac{1}{2}s_{-2} \ln_2^2 + \frac{1}{2}s_2 \ln_2^2 + \frac{1}{2}\bar{s}_{-2} \ln_2^2 - \frac{1}{2}\bar{s}_2 \ln_2^2 \\
&\quad - s_{-1,-1} \ln_2^2 - s_{-3} \ln_2 + s_3 \ln_2 + s_{-1}\zeta_2 \ln_2 + \frac{17\zeta_3 \ln_2}{8} - \frac{3}{2}\zeta_2\bar{s}_{-1} \ln_2 \\
&\quad - s_{-1,-2} \ln_2 + s_{-1,2} \ln_2 + s_{2,-1} \ln_2 - s_{2,1} \ln_2 - s_{-1,-1,-1} \ln_2 \\
&\quad + s_{-1,-1,1} \ln_2 - \bar{s}_{-1,-1,-1} \ln_2 - \bar{s}_{-1,-1,1} \ln_2 - \frac{47\zeta_2^2}{40} + \text{LiHalf}_4 \\
&\quad + \frac{1}{2}s_{-2}\zeta_2 - \frac{5}{8}s_{-1}\zeta_3 - \frac{1}{2}\zeta_2\bar{s}_{-2} + \frac{13}{8}\zeta_3\bar{s}_{-1} + \frac{1}{2}\zeta_2\bar{s}_2 + \frac{1}{2}\zeta_2s_{-1,-1} \\
&\quad + s_{3,-1} - \zeta_2\bar{s}_{-1,-1} - s_{-1,-2,-1} - s_{2,1,-1} + \bar{s}_{-1,2,1} \\
&\quad + \bar{s}_{2,-1,1} + s_{-1,-1,1,-1} - 3\bar{s}_{-1,-1,-1,-1}
\end{aligned} \tag{A18}$$

$$\begin{aligned}
s_1\bar{s}_{-3} &= \frac{\ln_2^4}{6} - \zeta_2 \ln_2^2 - \frac{37\zeta_2^2}{20} + 4\text{LiHalf}_4 + s_{-2}\zeta_2 - \frac{7}{4}s_{-1}\zeta_3 - \frac{3s_1\zeta_3}{4} - \zeta_2\bar{s}_{-2} \\
&\quad - \frac{7}{4}\zeta_3\bar{s}_{-1} + \frac{3}{4}\zeta_3\bar{s}_1 - s_{-3,1} + \bar{s}_{1,-3}
\end{aligned} \tag{A19}$$

$$s_1\bar{s}_3 = \frac{8\zeta_2^2}{5} - s_2\zeta_2 - \bar{s}_2\zeta_2 + s_1\zeta_3 - \zeta_3\bar{s}_1 + s_{3,1} + \bar{s}_{1,3} \tag{A20}$$

$$\begin{aligned}
s_1 \bar{s}_{-2,-1} = & -\frac{\ln_2^4}{4} - \frac{1}{2}s_{-2}\ln_2^2 + \frac{1}{2}s_2\ln_2^2 + \frac{9}{2}\zeta_2\ln_2^2 - \frac{1}{2}\bar{s}_{-2}\ln_2^2 + \frac{1}{2}\bar{s}_2\ln_2^2 + \frac{3}{2}s_{-1}\zeta_2\ln_2 \\
& + \frac{3}{2}s_1\zeta_2\ln_2 - 3\zeta_3\ln_2 - \bar{s}_{-3}\ln_2 + \frac{3}{2}\zeta_2\bar{s}_{-1}\ln_2 - \frac{3}{2}\zeta_2\bar{s}_1\ln_2 + \bar{s}_3\ln_2 - s_{-2,-1}\ln_2 \\
& - s_{-2,1}\ln_2 - \bar{s}_{-2,-1}\ln_2 + \bar{s}_{-2,1}\ln_2 + \frac{15\zeta_2^2}{8} - 6\text{LiHalf}_4 - \frac{1}{2}s_{-2}\zeta_2 - \frac{s_2\zeta_2}{2} \\
& - \frac{5s_1\zeta_3}{8} + \frac{5}{8}\zeta_3\bar{s}_1 - \frac{1}{2}\zeta_2\bar{s}_2 + s_{3,1} - \bar{s}_{-3,-1} - s_{-2,-1,1} + \bar{s}_{-2,1,-1} + \bar{s}_{1,-2,-1} \quad (\text{A21})
\end{aligned}$$

$$\begin{aligned}
s_1 \bar{s}_{-2,1} = & \frac{\ln_2^4}{4} - \frac{3}{2}\zeta_2\ln_2^2 - \frac{11\zeta_2^2}{8} + 6\text{LiHalf}_4 + s_{-2}\zeta_2 - \frac{21}{8}s_{-1}\zeta_3 - \frac{5s_1\zeta_3}{8} \\
& + \zeta_2\bar{s}_{-2} - \frac{21}{8}\zeta_3\bar{s}_{-1} + \frac{5}{8}\zeta_3\bar{s}_1 - s_{-3,1} - \bar{s}_{-3,1} + s_{-2,1,1} + \bar{s}_{-2,1,1} + \bar{s}_{1,-2,1} \quad (\text{A22})
\end{aligned}$$

$$\begin{aligned}
s_1 \bar{s}_{2,-1} = & -\frac{\ln_2^4}{12} - \frac{1}{2}s_{-2}\ln_2^2 + \frac{1}{2}s_2\ln_2^2 - \frac{5}{2}\zeta_2\ln_2^2 + \frac{1}{2}\bar{s}_{-2}\ln_2^2 - \frac{1}{2}\bar{s}_2\ln_2^2 - \frac{3}{2}s_{-1}\zeta_2\ln_2 \\
& - \frac{3}{2}s_1\zeta_2\ln_2 - \frac{9\zeta_3\ln_2}{4} + \bar{s}_{-3}\ln_2 - \frac{3}{2}\zeta_2\bar{s}_{-1}\ln_2 + \frac{3}{2}\zeta_2\bar{s}_1\ln_2 - \bar{s}_3\ln_2 + s_{2,-1}\ln_2 \\
& + s_{2,1}\ln_2 - \bar{s}_{2,-1}\ln_2 + \bar{s}_{2,1}\ln_2 + \frac{21\zeta_2^2}{40} - 2\text{LiHalf}_4 + \frac{1}{2}s_{-2}\zeta_2 + \frac{s_2\zeta_2}{2} + \frac{7}{8}s_{-1}\zeta_3 \\
& + \frac{s_1\zeta_3}{4} - \frac{1}{2}\zeta_2\bar{s}_{-2} + \frac{7}{8}\zeta_3\bar{s}_{-1} - \frac{1}{4}\zeta_3\bar{s}_1 - s_{-3,1} - \bar{s}_{3,-1} + s_{2,-1,1} \\
& + \bar{s}_{1,2,-1} + \bar{s}_{2,1,-1} \quad (\text{A23})
\end{aligned}$$

$$s_1 \bar{s}_{2,1} = \frac{6\zeta_2^2}{5} - s_2\zeta_2 + \bar{s}_2\zeta_2 + 2s_1\zeta_3 - 2\zeta_3\bar{s}_1 + s_{3,1} - \bar{s}_{3,1} - s_{2,1,1} + \bar{s}_{1,2,1} + \bar{s}_{2,1,1} \quad (\text{A24})$$

$$\begin{aligned}
s_1 \bar{s}_{-1,1,-1} = & -\frac{\ln_2^4}{4} + \frac{1}{6}s_{-1}\ln_2^3 - \frac{1}{6}s_1\ln_2^3 - \frac{1}{6}\bar{s}_{-1}\ln_2^3 + \frac{1}{6}\bar{s}_1\ln_2^3 - \frac{1}{2}s_{-2}\ln_2^2 + \frac{1}{2}s_2\ln_2^2 \\
& + \frac{3}{2}\zeta_2\ln_2^2 - \frac{1}{2}\bar{s}_{-2}\ln_2^2 + \frac{1}{2}\bar{s}_2\ln_2^2 + s_{-1,1}\ln_2^2 + \frac{5}{2}s_{-1}\zeta_2\ln_2 + \frac{1}{2}s_1\zeta_2\ln_2 \\
& - \frac{17\zeta_3\ln_2}{8} - \zeta_2\bar{s}_{-1}\ln_2 - \frac{1}{2}\zeta_2\bar{s}_1\ln_2 - s_{-2,-1}\ln_2 - s_{-2,1}\ln_2 + \bar{s}_{-1,-2}\ln_2 \\
& - \bar{s}_{-1,2}\ln_2 + s_{-1,1,-1}\ln_2 + s_{-1,1,1}\ln_2 - \bar{s}_{-1,1,-1}\ln_2 + \bar{s}_{-1,1,1}\ln_2 + \frac{19\zeta_2^2}{20} \\
& - 2\text{LiHalf}_4 - \frac{1}{2}s_{-2}\zeta_2 - \frac{s_2\zeta_2}{2} - \frac{3}{4}s_{-1}\zeta_3 + \frac{s_1\zeta_3}{8} + \frac{1}{4}\zeta_3\bar{s}_{-1} - \frac{1}{8}\zeta_3\bar{s}_1 \\
& + \frac{1}{2}\zeta_2s_{-1,-1} + \frac{1}{2}\zeta_2s_{-1,1} + s_{3,1} - \frac{1}{2}\zeta_2\bar{s}_{-1,-1} - s_{-2,-1,1} - s_{-1,-2,1} \\
& - \bar{s}_{-2,1,-1} - \bar{s}_{-1,2,-1} + s_{-1,1,-1,1} + 2\bar{s}_{-1,1,1,-1} + \bar{s}_{1,-1,1,-1} \quad (\text{A25})
\end{aligned}$$

$$\begin{aligned}
s_1 \bar{s}_{-1,1,1} = & -\frac{\ln_2^4}{8} - \frac{1}{6}s_{-1}\ln_2^3 - \frac{1}{6}s_1\ln_2^3 - \frac{1}{6}\bar{s}_{-1}\ln_2^3 + \frac{1}{6}\bar{s}_1\ln_2^3 + \frac{1}{2}s_{-1}\zeta_2\ln_2 \\
& + \frac{1}{2}s_1\zeta_2\ln_2 + \frac{1}{2}\zeta_2\bar{s}_{-1}\ln_2 - \frac{1}{2}\zeta_2\bar{s}_1\ln_2 - \frac{13\zeta_2^2}{20} + 3\text{LiHalf}_4 + s_{-2}\zeta_2 \\
& - \frac{23}{8}s_{-1}\zeta_3 - \frac{7s_1\zeta_3}{8} + \frac{1}{8}\zeta_3\bar{s}_{-1} + \frac{7}{8}\zeta_3\bar{s}_1 - s_{-3,1} - \zeta_2s_{-1,1} + \zeta_2\bar{s}_{-1,1} + s_{-2,1,1} \\
& + s_{-1,2,1} - \bar{s}_{-2,1,1} - \bar{s}_{-1,2,1} - s_{-1,1,1,1} + 2\bar{s}_{-1,1,1,1} + \bar{s}_{1,-1,1,1} \quad (\text{A26})
\end{aligned}$$

$$\begin{aligned}
s_1 \bar{s}_{1,-2} = & \frac{\ln_2^4}{6} + \frac{1}{2} \zeta_2 \ln_2^2 + \frac{3}{2} s_{-1} \zeta_2 \ln_2 - \frac{3}{2} s_1 \zeta_2 \ln_2 + \frac{3}{2} \zeta_2 \bar{s}_{-1} \ln_2 - \frac{3}{2} \zeta_2 \bar{s}_1 \ln_2 \\
& - \frac{21 \zeta_2^2}{10} + 4 \text{LiHalf}_4 + s_{-2} \zeta_2 - \frac{7}{4} s_{-1} \zeta_3 + \frac{s_1 \zeta_3}{2} + \frac{1}{2} \zeta_2 \bar{s}_{-2} - \frac{7}{4} \zeta_3 \bar{s}_{-1} \\
& + \frac{1}{4} \zeta_3 \bar{s}_1 - \frac{1}{2} \zeta_2 \bar{s}_2 - s_{-3,1} - \frac{3}{2} \zeta_2 s_{1,-1} - \frac{1}{2} \zeta_2 s_{1,1} - \frac{3}{2} \zeta_2 \bar{s}_{1,-1} + \frac{1}{2} \zeta_2 \bar{s}_{1,1} \\
& - \bar{s}_{2,-2} + s_{1,-2,1} + 2 \bar{s}_{1,1,-2}
\end{aligned} \tag{A27}$$

$$s_1 \bar{s}_{1,2} = \frac{6 \zeta_2^2}{5} - s_2 \zeta_2 + s_{1,1} \zeta_2 - \bar{s}_{1,1} \zeta_2 + s_1 \zeta_3 + 2 \zeta_3 \bar{s}_1 + s_{3,1} - \bar{s}_{2,2} - s_{1,2,1} + 2 \bar{s}_{1,1,2} \tag{A28}$$

$$\begin{aligned}
s_1 \bar{s}_{1,-1,-1} = & \frac{\ln_2^4}{3} + s_1 \ln_2^3 - \frac{1}{2} s_{-2} \ln_2^2 + \frac{1}{2} s_2 \ln_2^2 + \frac{11}{4} \zeta_2 \ln_2^2 - \bar{s}_{-2} \ln_2^2 + \bar{s}_2 \ln_2^2 \\
& + \frac{3}{2} s_{1,-1} \ln_2^2 + \frac{1}{2} s_{1,1} \ln_2^2 + \frac{1}{2} \bar{s}_{1,-1} \ln_2^2 - \frac{1}{2} \bar{s}_{1,1} \ln_2^2 + \frac{1}{2} s_{-1} \zeta_2 \ln_2 \\
& + \frac{3}{2} s_1 \zeta_2 \ln_2 - \frac{5 \zeta_3 \ln_2}{8} + \frac{1}{2} \zeta_2 \bar{s}_{-1} \ln_2 - \frac{1}{2} \zeta_2 \bar{s}_1 \ln_2 - s_{-2,-1} \ln_2 \\
& - s_{-2,1} \ln_2 - \bar{s}_{1,-2} \ln_2 + \bar{s}_{1,2} \ln_2 + s_{1,-1,-1} \ln_2 + s_{1,-1,1} \ln_2 - \bar{s}_{1,-1,-1} \ln_2 \\
& + \bar{s}_{1,-1,1} \ln_2 + \frac{19 \zeta_2^2}{40} - \frac{1}{2} s_{-2} \zeta_2 - \frac{s_2 \zeta_2}{2} - \frac{5 s_1 \zeta_3}{8} + \zeta_3 \bar{s}_1 + \frac{1}{2} \zeta_2 s_{1,-1} \\
& + \frac{1}{2} \zeta_2 s_{1,1} + s_{3,1} - \frac{1}{2} \zeta_2 \bar{s}_{1,1} - s_{-2,-1,1} - s_{1,2,1} - \bar{s}_{1,-2,-1} - \bar{s}_{2,-1,-1} \\
& + s_{1,-1,-1,1} + \bar{s}_{1,-1,1,-1} + 2 \bar{s}_{1,1,-1,-1}
\end{aligned} \tag{A29}$$

$$\begin{aligned}
s_1 \bar{s}_{1,-1,1} = & \frac{11 \ln_2^4}{24} + \frac{2}{3} s_1 \ln_2^3 + \frac{3}{4} \zeta_2 \ln_2^2 - \frac{1}{2} \bar{s}_{-2} \ln_2^2 + \frac{1}{2} \bar{s}_2 \ln_2^2 + \frac{1}{2} s_{1,-1} \ln_2^2 \\
& + \frac{1}{2} s_{1,1} \ln_2^2 + \frac{1}{2} \bar{s}_{1,-1} \ln_2^2 - \frac{1}{2} \bar{s}_{1,1} \ln_2^2 + \frac{1}{2} s_{-1} \zeta_2 \ln_2 - \frac{3}{2} s_1 \zeta_2 \ln_2 \\
& + \frac{1}{2} \zeta_2 \bar{s}_{-1} \ln_2 + \zeta_2 \bar{s}_1 \ln_2 - \frac{87 \zeta_2^2}{40} + 5 \text{LiHalf}_4 + s_{-2} \zeta_2 - \frac{13}{8} s_{-1} \zeta_3 \\
& - \frac{s_1 \zeta_3}{8} + \frac{1}{2} \zeta_2 \bar{s}_{-2} - \frac{13}{8} \zeta_3 \bar{s}_{-1} - \frac{1}{2} \zeta_2 \bar{s}_2 - s_{-3,1} - \frac{3}{2} \zeta_2 s_{1,-1} \\
& - \frac{1}{2} \zeta_2 s_{1,1} + \frac{1}{2} \zeta_2 \bar{s}_{1,-1} + \frac{1}{2} \zeta_2 \bar{s}_{1,1} + s_{-2,1,1} + s_{1,-2,1} - \bar{s}_{1,-2,1} \\
& - \bar{s}_{2,-1,1} - s_{1,-1,1,1} + \bar{s}_{1,-1,1,1} + 2 \bar{s}_{1,1,-1,1}
\end{aligned} \tag{A30}$$

$$\begin{aligned}
s_1 \bar{s}_{1,1,-1} = & -\frac{\ln_2^4}{8} - \frac{1}{3} s_1 \ln_2^3 - \frac{1}{2} s_{-2} \ln_2^2 + \frac{1}{2} s_2 \ln_2^2 - \frac{3}{2} \zeta_2 \ln_2^2 + \frac{1}{2} \bar{s}_{-2} \ln_2^2 \\
& - \frac{1}{2} \bar{s}_2 \ln_2^2 - s_{1,1} \ln_2^2 - 2 s_1 \zeta_2 \ln_2 - \frac{25 \zeta_3 \ln_2}{8} - \frac{3}{2} \zeta_2 \bar{s}_1 \ln_2 \\
& + s_{2,-1} \ln_2 + s_{2,1} \ln_2 + \bar{s}_{1,-2} \ln_2 - \bar{s}_{1,2} \ln_2 - s_{1,1,-1} \ln_2 - s_{1,1,1} \ln_2 \\
& - \bar{s}_{1,1,-1} \ln_2 + \bar{s}_{1,1,1} \ln_2 + \frac{13 \zeta_2^2}{40} - \text{LiHalf}_4 + \frac{1}{2} s_{-2} \zeta_2 + \frac{s_2 \zeta_2}{2} \\
& - \frac{1}{8} s_{-1} \zeta_3 + \frac{3 s_1 \zeta_3}{4} - \frac{1}{8} \zeta_3 \bar{s}_{-1} + \frac{1}{4} \zeta_3 \bar{s}_1 - s_{-3,1} - \frac{1}{2} \zeta_2 s_{1,-1} \\
& - \frac{1}{2} \zeta_2 s_{1,1} - \frac{1}{2} \zeta_2 \bar{s}_{1,-1} + s_{1,-2,1} + s_{2,-1,1} - \bar{s}_{1,2,-1} \\
& - \bar{s}_{2,1,-1} - s_{1,1,-1,1} + 3 \bar{s}_{1,1,1,-1}
\end{aligned} \tag{A31}$$

$$\begin{aligned} s_1 \bar{s}_{1,1,1} &= \frac{8\zeta_2^2}{5} - s_2 \zeta_2 + s_{1,1} \zeta_2 + \bar{s}_{1,1} \zeta_2 + 2s_1 \zeta_3 + \zeta_3 \bar{s}_1 + s_{3,1} - s_{1,2,1} \\ &\quad - s_{2,1,1} - \bar{s}_{1,2,1} - \bar{s}_{2,1,1} + s_{1,1,1,1} + 3\bar{s}_{1,1,1,1} \end{aligned} \quad (\text{A32})$$

$$\begin{aligned} s_1 \bar{s}_{-1,-2} &= \frac{\ln_2^4}{6} - \frac{1}{2} \zeta_2 \ln_2^2 + 2s_{-1} \zeta_2 \ln_2 - s_1 \zeta_2 \ln_2 - \zeta_2 \bar{s}_{-1} \ln_2 + \zeta_2 \bar{s}_1 \ln_2 \\ &\quad + 4\text{LiHalf}_4 - s_2 \zeta_2 - \frac{5}{8} s_{-1} \zeta_3 + \frac{13s_1 \zeta_3}{8} - \frac{1}{2} \zeta_2 \bar{s}_{-2} + \frac{1}{8} \zeta_3 \bar{s}_{-1} \\ &\quad - \frac{13}{8} \zeta_3 \bar{s}_1 + \frac{1}{2} \zeta_2 \bar{s}_2 + \frac{3}{2} \zeta_2 s_{-1,-1} + \frac{1}{2} \zeta_2 s_{-1,1} + s_{3,1} - \bar{s}_{-2,-2} \\ &\quad - \frac{3}{2} \zeta_2 \bar{s}_{-1,-1} + \frac{1}{2} \zeta_2 \bar{s}_{-1,1} - s_{-1,-2,1} + \bar{s}_{-1,1,-2} + \bar{s}_{1,-1,-2} \end{aligned} \quad (\text{A33})$$

$$\begin{aligned} s_1 \bar{s}_{-1,2} &= \frac{\ln_2^4}{12} + \frac{3}{2} \zeta_2 \ln_2^2 + \frac{1}{2} s_{-1} \zeta_2 \ln_2 + \frac{1}{2} s_1 \zeta_2 \ln_2 + \frac{1}{2} \zeta_2 \bar{s}_{-1} \ln_2 - \frac{1}{2} \zeta_2 \bar{s}_1 \ln_2 \\ &\quad - \frac{9\zeta_2^2}{8} + 2\text{LiHalf}_4 + s_{-2} \zeta_2 - 2s_{-1} \zeta_3 - s_1 \zeta_3 + \zeta_3 \bar{s}_{-1} + \zeta_3 \bar{s}_1 - s_{-3,1} \\ &\quad - \zeta_2 s_{-1,1} - \bar{s}_{-2,2} - \zeta_2 \bar{s}_{-1,1} + s_{-1,2,1} + \bar{s}_{-1,1,2} + \bar{s}_{1,-1,2} \end{aligned} \quad (\text{A34})$$

$$\begin{aligned} s_1 \bar{s}_{-1,-1,-1} &= -\frac{5 \ln_2^4}{8} - \frac{7}{6} s_{-1} \ln_2^3 - \frac{1}{6} s_1 \ln_2^3 - \frac{1}{6} \bar{s}_{-1} \ln_2^3 + \frac{1}{6} \bar{s}_1 \ln_2^3 - \frac{1}{2} s_{-2} \ln_2^2 \\ &\quad + \frac{1}{2} s_2 \ln_2^2 - \frac{5}{4} \zeta_2 \ln_2^2 + \bar{s}_{-2} \ln_2^2 - \bar{s}_2 \ln_2^2 - \frac{3}{2} s_{-1,-1} \ln_2^2 - \frac{1}{2} s_{-1,1} \ln_2^2 \\ &\quad + \frac{1}{2} \bar{s}_{-1,-1} \ln_2^2 - \frac{1}{2} \bar{s}_{-1,1} \ln_2^2 - \frac{3}{2} s_{-1} \zeta_2 \ln_2 - \frac{1}{2} s_1 \zeta_2 \ln_2 - \frac{\zeta_3 \ln_2}{8} \\ &\quad - \frac{1}{2} \zeta_2 \bar{s}_{-1} \ln_2 + \frac{1}{2} \zeta_2 \bar{s}_1 \ln_2 + s_{2,-1} \ln_2 + s_{2,1} \ln_2 - \bar{s}_{-1,-2} \ln_2 + \bar{s}_{-1,2} \ln_2 \\ &\quad - s_{-1,-1,-1} \ln_2 - s_{-1,-1,1} \ln_2 - \bar{s}_{-1,-1,-1} \ln_2 + \bar{s}_{-1,-1,1} \ln_2 + \frac{3\zeta_2^2}{40} \\ &\quad - \text{LiHalf}_4 + \frac{1}{2} s_{-2} \zeta_2 + \frac{s_2 \zeta_2}{2} + \frac{3}{8} s_{-1} \zeta_3 - \frac{s_1 \zeta_3}{4} + \frac{3}{4} \zeta_3 \bar{s}_{-1} + \frac{1}{4} \zeta_3 \bar{s}_1 \\ &\quad - s_{-3,1} - \frac{1}{2} \zeta_2 s_{-1,-1} - \frac{1}{2} \zeta_2 s_{-1,1} - \frac{1}{2} \zeta_2 \bar{s}_{-1,1} + s_{-1,2,1} + s_{2,-1,1} \\ &\quad - \bar{s}_{-2,-1,-1} - \bar{s}_{-1,-2,-1} - s_{-1,-1,-1,1} + \bar{s}_{-1,-1,1,-1} \\ &\quad + \bar{s}_{-1,1,-1,-1} + \bar{s}_{1,-1,-1,1} \end{aligned} \quad (\text{A35})$$

$$\begin{aligned} s_1 \bar{s}_{-1,-1,1} &= -\frac{\ln_2^4}{2} - \frac{5}{6} s_{-1} \ln_2^3 - \frac{1}{6} s_1 \ln_2^3 - \frac{1}{6} \bar{s}_{-1} \ln_2^3 + \frac{1}{6} \bar{s}_1 \ln_2^3 + \frac{3}{4} \zeta_2 \ln_2^2 \\ &\quad + \frac{1}{2} \bar{s}_{-2} \ln_2^2 - \frac{1}{2} \bar{s}_2 \ln_2^2 - \frac{1}{2} s_{-1,-1} \ln_2^2 - \frac{1}{2} s_{-1,1} \ln_2^2 + \frac{1}{2} \bar{s}_{-1,-1} \ln_2^2 \\ &\quad - \frac{1}{2} \bar{s}_{-1,1} \ln_2^2 + \frac{3}{2} s_{-1} \zeta_2 \ln_2 - \frac{1}{2} s_1 \zeta_2 \ln_2 + \zeta_2 \bar{s}_{-1} \ln_2 + \frac{1}{2} \zeta_2 \bar{s}_1 \ln_2 \\ &\quad + \frac{3\zeta_2^2}{5} - s_2 \zeta_2 + \frac{1}{4} s_{-1} \zeta_3 + \frac{7s_1 \zeta_3}{4} - \frac{1}{2} \zeta_2 \bar{s}_{-2} + \frac{1}{8} \zeta_3 \bar{s}_{-1} - \frac{7}{4} \zeta_3 \bar{s}_1 \\ &\quad + \frac{1}{2} \zeta_2 \bar{s}_2 + \frac{3}{2} \zeta_2 s_{-1,-1} + \frac{1}{2} \zeta_2 s_{-1,1} + s_{3,1} + \frac{1}{2} \zeta_2 \bar{s}_{-1,-1} + \frac{1}{2} \zeta_2 \bar{s}_{-1,1} \\ &\quad - s_{-1,-2,1} - s_{2,1,1} - \bar{s}_{-2,-1,1} - \bar{s}_{-1,-2,1} + s_{-1,-1,1,1} + \bar{s}_{-1,-1,1,1} \\ &\quad + \bar{s}_{-1,1,-1,1} + \bar{s}_{1,-1,-1,1} \end{aligned} \quad (\text{A36})$$

### Appendix A.2. Reflection Identities Originating from $B_2 \otimes \bar{B}_2$

$$s_{-2}\bar{s}_{-2} = \frac{8\zeta_2^2}{5} - s_2\zeta_2 - \bar{s}_2\zeta_2 + s_{-2,-2} + \bar{s}_{-2,-2} \quad (\text{A37})$$

$$\begin{aligned} s_{-2}\bar{s}_2 &= \frac{\ln_2^4}{3} - 2\zeta_2 \ln_2^2 - \frac{29\zeta_2^2}{10} + 8\text{LiHalf}_4 + \frac{1}{2}s_{-2}\zeta_2 - \frac{s_2\zeta_2}{2} - \frac{7}{2}s_{-1}\zeta_3 - \frac{1}{2}\zeta_2\bar{s}_{-2} \\ &\quad - \frac{7}{2}\zeta_3\bar{s}_{-1} - \frac{1}{2}\zeta_2\bar{s}_2 - s_{2,-2} + \bar{s}_{-2,2} \end{aligned} \quad (\text{A38})$$

$$\begin{aligned} s_{-2}\bar{s}_{-1,1} &= \frac{\ln_2^4}{12} + \frac{1}{2}s_{-2}\ln_2^2 + \frac{1}{2}s_2\ln_2^2 - \frac{1}{2}\bar{s}_{-2}\ln_2^2 + \frac{1}{2}\bar{s}_2\ln_2^2 + \frac{3}{2}s_{-1}\zeta_2\ln_2 - \frac{3}{2}\zeta_2\bar{s}_{-1}\ln_2 \\ &\quad - \frac{11\zeta_2^2}{40} + 2\text{LiHalf}_4 - s_2\zeta_2 - \frac{1}{8}s_{-1}\zeta_3 + \frac{1}{2}\zeta_2\bar{s}_{-2} + \frac{5}{8}\zeta_3\bar{s}_{-1} + \frac{1}{2}\zeta_2\bar{s}_2 \\ &\quad + s_{-2,-2} + \frac{3}{2}\zeta_2s_{-1,-1} - \frac{1}{2}\zeta_2s_{-1,1} - \frac{3}{2}\zeta_2\bar{s}_{-1,-1} - \frac{1}{2}\zeta_2\bar{s}_{-1,1} - \bar{s}_{3,1} \\ &\quad - s_{-1,1,-2} + \bar{s}_{-2,-1,1} + \bar{s}_{-1,-2,1} \end{aligned} \quad (\text{A39})$$

$$\begin{aligned} s_{-2}\bar{s}_{1,-1} &= -\frac{\ln_2^4}{4} - \frac{1}{2}s_{-2}\ln_2^2 - \frac{1}{2}s_2\ln_2^2 + 4\zeta_2\ln_2^2 + \frac{1}{2}\bar{s}_{-2}\ln_2^2 - \frac{1}{2}\bar{s}_2\ln_2^2 \\ &\quad + \frac{3}{2}s_{-1}\zeta_2\ln_2 - s_1\zeta_2\ln_2 - \frac{15\zeta_3\ln_2}{4} - \bar{s}_{-3}\ln_2 + \frac{3}{2}\zeta_2\bar{s}_{-1}\ln_2 \\ &\quad + \bar{s}_3\ln_2 - s_{1,-2}\ln_2 - s_{1,2}\ln_2 + \bar{s}_{1,-2}\ln_2 - \bar{s}_{1,2}\ln_2 + \frac{17\zeta_2^2}{8} \\ &\quad - 6\text{LiHalf}_4 + \frac{1}{2}s_{-2}\zeta_2 - \frac{s_2\zeta_2}{2} + \frac{13s_1\zeta_3}{8} + \frac{5}{8}\zeta_3\bar{s}_1 - \frac{1}{2}\zeta_2\bar{s}_2 \\ &\quad + s_{-2,-2} - \frac{1}{2}\zeta_2s_{1,-1} - \bar{s}_{-3,-1} - \frac{1}{2}\zeta_2\bar{s}_{1,-1} - s_{1,-1,-2} \\ &\quad + \bar{s}_{-2,1,-1} + \bar{s}_{1,-2,-1} \end{aligned} \quad (\text{A40})$$

$$\begin{aligned} s_{-2}\bar{s}_{1,1} &= \frac{\ln_2^4}{4} + \frac{3}{2}s_{-1}\zeta_2\ln_2 - \frac{3}{2}s_1\zeta_2\ln_2 + \frac{3}{2}\zeta_2\bar{s}_{-1}\ln_2 - \frac{3}{2}\zeta_2\bar{s}_1\ln_2 - \frac{13\zeta_2^2}{8} \\ &\quad + 6\text{LiHalf}_4 + \frac{1}{2}s_{-2}\zeta_2 - \frac{s_2\zeta_2}{2} - \frac{21}{8}s_{-1}\zeta_3 + \frac{s_1\zeta_3}{8} + \zeta_2\bar{s}_{-2} \\ &\quad - \frac{21}{8}\zeta_3\bar{s}_{-1} + \frac{5}{8}\zeta_3\bar{s}_1 - \frac{3}{2}\zeta_2s_{1,-1} + \frac{1}{2}\zeta_2s_{1,1} - s_{2,-2} - \bar{s}_{-3,1} \\ &\quad - \frac{3}{2}\zeta_2\bar{s}_{1,-1} - \frac{1}{2}\zeta_2\bar{s}_{1,1} + s_{1,1,-2} + \bar{s}_{-2,1,1} + \bar{s}_{1,-2,1} \end{aligned} \quad (\text{A41})$$

$$\begin{aligned} s_{-2}\bar{s}_{-1,-1} &= \frac{\ln_2^4}{6} + s_{-2}\ln_2^2 + s_2\ln_2^2 - \bar{s}_{-2}\ln_2^2 + \bar{s}_2\ln_2^2 + s_{-1}\zeta_2\ln_2 - \frac{3\zeta_3\ln_2}{2} \\ &\quad + \bar{s}_{-3}\ln_2 - \bar{s}_3\ln_2 + s_{-1,-2}\ln_2 + s_{-1,2}\ln_2 + \bar{s}_{-1,-2}\ln_2 - \bar{s}_{-1,2}\ln_2 \\ &\quad - \frac{23\zeta_2^2}{20} + 4\text{LiHalf}_4 + \frac{1}{2}s_{-2}\zeta_2 - \frac{s_2\zeta_2}{2} - \frac{5}{2}s_{-1}\zeta_3 - \frac{1}{2}\zeta_2\bar{s}_{-2} \\ &\quad - \frac{1}{4}\zeta_3\bar{s}_{-1} + \frac{1}{2}\zeta_2s_{-1,-1} - s_{2,-2} - \frac{1}{2}\zeta_2\bar{s}_{-1,-1} - \bar{s}_{3,-1} + s_{-1,-1,-2} \\ &\quad + \bar{s}_{-2,-1,-1} + \bar{s}_{-1,-2,-1} \end{aligned} \quad (\text{A42})$$

$$s_2\bar{s}_2 = \frac{12\zeta_2^2}{5} - s_{2,2} - \bar{s}_{2,2} \quad (\text{A43})$$

$$\begin{aligned}
s_2 \bar{s}_{-1,1} = & \frac{\ln_2^4}{6} + \frac{1}{2}s_{-2}\ln_2^2 + \frac{1}{2}s_2\ln_2^2 + \zeta_2\ln_2^2 - \frac{1}{2}\bar{s}_{-2}\ln_2^2 + \frac{1}{2}\bar{s}_2\ln_2^2 + \frac{3}{2}s_{-1}\zeta_2\ln_2 \\
& + \frac{3}{2}\zeta_2\bar{s}_{-1}\ln_2 - \frac{5\zeta_2^2}{4} + 4\text{LiHalf}_4 - \frac{1}{2}s_{-2}\zeta_2 - \frac{s_2\zeta_2}{2} - \frac{29}{8}s_{-1}\zeta_3 - \frac{1}{2}\zeta_2\bar{s}_{-2} \\
& - \frac{5}{8}\zeta_3\bar{s}_{-1} - \frac{1}{2}\zeta_2\bar{s}_2 + s_{-2,2} + \zeta_2s_{-1,1} + \bar{s}_{-3,1} + \zeta_2\bar{s}_{-1,1} - s_{-1,1,2} \\
& - \bar{s}_{-1,2,1} - \bar{s}_{2,-1,1}
\end{aligned} \tag{A44}$$

$$\begin{aligned}
s_2 \bar{s}_{1,-1} = & \frac{\ln_2^4}{12} - \frac{1}{2}s_{-2}\ln_2^2 - \frac{1}{2}s_2\ln_2^2 - \zeta_2\ln_2^2 + \frac{1}{2}\bar{s}_{-2}\ln_2^2 - \frac{1}{2}\bar{s}_2\ln_2^2 + \frac{1}{2}s_1\zeta_2\ln_2 - \frac{15\zeta_3\ln_2}{4} \\
& - \bar{s}_{-3}\ln_2 - \frac{3}{2}\zeta_2\bar{s}_1\ln_2 + \bar{s}_3\ln_2 - s_{1,-2}\ln_2 - s_{1,2}\ln_2 + \bar{s}_{1,-2}\ln_2 - \bar{s}_{1,2}\ln_2 - \frac{11\zeta_2^2}{40} \\
& + 2\text{LiHalf}_4 - \frac{7}{8}s_{-1}\zeta_3 - s_1\zeta_3 + \frac{1}{2}\zeta_2\bar{s}_{-2} - \frac{7}{8}\zeta_3\bar{s}_{-1} + \frac{1}{4}\zeta_3\bar{s}_1 + s_{-2,2} - \frac{1}{2}\zeta_2s_{1,-1} \\
& - \frac{1}{2}\zeta_2\bar{s}_{1,-1} + \bar{s}_{3,-1} - s_{1,-1,2} - \bar{s}_{1,2,-1} - \bar{s}_{2,1,-1}
\end{aligned} \tag{A45}$$

$$\begin{aligned}
s_2 \bar{s}_{1,1} = & \frac{4\zeta_2^2}{5} - \bar{s}_2\zeta_2 - s_{1,1}\zeta_2 + \bar{s}_{1,1}\zeta_2 + s_1\zeta_3 + 2\zeta_3\bar{s}_1 - s_{2,2} + \bar{s}_{3,1} + s_{1,1,2} \\
& - \bar{s}_{1,2,1} - \bar{s}_{2,1,1}
\end{aligned} \tag{A46}$$

$$\begin{aligned}
s_2 \bar{s}_{-1,-1} = & -\frac{\ln_2^4}{6} + s_{-2}\ln_2^2 + s_2\ln_2^2 + 2\zeta_2\ln_2^2 - \bar{s}_{-2}\ln_2^2 + \bar{s}_2\ln_2^2 + s_{-1}\zeta_2\ln_2 - \frac{3\zeta_3\ln_2}{2} \\
& + \bar{s}_{-3}\ln_2 - \bar{s}_3\ln_2 + s_{-1,-2}\ln_2 + s_{-1,2}\ln_2 + \bar{s}_{-1,-2}\ln_2 - \bar{s}_{-1,2}\ln_2 + \frac{7\zeta_2^2}{4} \\
& - 4\text{LiHalf}_4 + s_{-1}\zeta_3 + \frac{1}{4}\zeta_3\bar{s}_{-1} + \frac{1}{2}\zeta_2\bar{s}_2 + \frac{1}{2}\zeta_2s_{-1,-1} - s_{2,2} + \bar{s}_{-3,-1} \\
& - \frac{1}{2}\zeta_2\bar{s}_{-1,-1} + s_{-1,-1,2} - \bar{s}_{-1,2,-1} - \bar{s}_{2,-1,-1}
\end{aligned} \tag{A47}$$

$$\begin{aligned}
s_{-1,1} \bar{s}_{-1,1} = & -\frac{\ln_2^4}{4} - \frac{1}{2}s_{-2}\ln_2^2 + \frac{1}{2}s_2\ln_2^2 - \frac{1}{2}\zeta_2\ln_2^2 - \frac{1}{2}\bar{s}_{-2}\ln_2^2 + \frac{1}{2}\bar{s}_2\ln_2^2 + s_{-1,1}\ln_2^2 \\
& + \bar{s}_{-1,1}\ln_2^2 - \frac{3}{2}s_{-1}\zeta_2\ln_2 - \frac{3}{2}\zeta_2\bar{s}_{-1}\ln_2 - \frac{19\zeta_2^2}{20} + 6\text{LiHalf}_4 + \frac{1}{2}s_{-2}\zeta_2 + \frac{s_2\zeta_2}{2} \\
& - \frac{3}{8}s_{-1}\zeta_3 + \frac{1}{2}\zeta_2\bar{s}_{-2} - \frac{3}{8}\zeta_3\bar{s}_{-1} + \frac{1}{2}\zeta_2\bar{s}_2 - 2\zeta_2s_{-1,-1} - \zeta_2s_{-1,1} - s_{3,1} \\
& - 2\zeta_2\bar{s}_{-1,-1} - \zeta_2\bar{s}_{-1,1} - \bar{s}_{3,1} + s_{-2,-1,1} + 2s_{-1,-2,1} + s_{2,1,1} + \bar{s}_{-2,-1,1} \\
& + 2\bar{s}_{-1,-2,1} + \bar{s}_{2,1,1} - 2s_{-1,-1,1,1} - s_{-1,1,-1,1} - 2\bar{s}_{-1,-1,1,1} - \bar{s}_{-1,1,-1,1}
\end{aligned} \tag{A48}$$

$$\begin{aligned}
s_{-1,1}\bar{s}_{1,-1} = & -\frac{\ln_2^4}{3} - \frac{2}{3}s_{-1}\ln_2^3 + \frac{2}{3}s_1\ln_2^3 - \frac{1}{3}\bar{s}_{-1}\ln_2^3 + \frac{1}{3}\bar{s}_1\ln_2^3 - \frac{1}{2}\zeta_2\ln_2^2 - s_{-1,1}\ln_2^2 \\
& + \bar{s}_{1,-1}\ln_2^2 - \frac{3}{2}s_{-1}\zeta_2\ln_2 - 2s_1\zeta_2\ln_2 - \frac{\zeta_3\ln_2}{4} - \bar{s}_{-3}\ln_2 + 2\zeta_2\bar{s}_{-1}\ln_2 \\
& - \frac{1}{2}\zeta_2\bar{s}_1\ln_2 + \bar{s}_3\ln_2 + s_{-2,-1}\ln_2 + s_{-2,1}\ln_2 - \bar{s}_{-2,-1}\ln_2 + \bar{s}_{-2,1}\ln_2 \\
& - \bar{s}_{-1,-2}\ln_2 + \bar{s}_{-1,2}\ln_2 + \bar{s}_{1,-2}\ln_2 - \bar{s}_{1,2}\ln_2 - s_{-1,1,-1}\ln_2 - s_{-1,1,1}\ln_2 \\
& - s_{1,-1,-1}\ln_2 - s_{1,-1,1}\ln_2 + \bar{s}_{-1,1,-1}\ln_2 - \bar{s}_{-1,1,1}\ln_2 + \bar{s}_{1,-1,-1}\ln_2 - \bar{s}_{1,-1,1}\ln_2 \\
& + \frac{47\zeta_2^2}{40} - 4\text{LiHalf}_4 + s_{-2}\zeta_2 + \frac{3}{4}s_{-1}\zeta_3 + \frac{3s_1\zeta_3}{2} - \frac{1}{4}\zeta_3\bar{s}_{-1} + \frac{3}{4}\zeta_3\bar{s}_1 \\
& - \frac{1}{2}\zeta_2\bar{s}_2 - \frac{1}{2}\zeta_2s_{-1,-1} - \frac{1}{2}\zeta_2s_{-1,1} - \zeta_2s_{1,-1} - s_{3,1} - \bar{s}_{-3,-1} + \frac{1}{2}\zeta_2\bar{s}_{-1,-1} \\
& - \frac{1}{2}\zeta_2\bar{s}_{1,-1} + 2s_{-2,-1,1} + s_{-1,-2,1} + s_{1,2,1} + 2\bar{s}_{-2,1,-1} + \bar{s}_{-1,2,-1} \\
& + \bar{s}_{1,-2,-1} - s_{-1,1,-1,1} - 2s_{1,-1,-1,1} - 2\bar{s}_{-1,1,1,-1} - \bar{s}_{1,-1,1,-1}
\end{aligned} \tag{A49}$$

$$\begin{aligned}
s_{-1,1}\bar{s}_{1,1} = & -\frac{\ln_2^4}{8} - \frac{1}{3}s_{-1}\ln_2^3 + \frac{1}{3}s_1\ln_2^3 - \frac{1}{3}\bar{s}_{-1}\ln_2^3 + \frac{1}{3}\bar{s}_1\ln_2^3 - \frac{1}{2}s_{-2}\ln_2^2 + \frac{1}{2}s_2\ln_2^2 \\
& + \frac{3}{4}\zeta_2\ln_2^2 + \frac{1}{2}s_{1,-1}\ln_2^2 - \frac{1}{2}s_{1,1}\ln_2^2 + \frac{1}{2}\bar{s}_{1,-1}\ln_2^2 + \frac{1}{2}\bar{s}_{1,1}\ln_2^2 + s_{-1}\zeta_2\ln_2 \\
& + \frac{1}{2}s_1\zeta_2\ln_2 + \zeta_2\bar{s}_{-1}\ln_2 - \zeta_2\bar{s}_1\ln_2 - \frac{39\zeta_2^2}{40} + 3\text{LiHalf}_4 - \frac{1}{2}s_{-2}\zeta_2 - \frac{s_2\zeta_2}{2} \\
& + \frac{1}{4}s_{-1}\zeta_3 + \frac{s_1\zeta_3}{8} + \zeta_2\bar{s}_{-2} - \frac{11}{4}\zeta_3\bar{s}_{-1} - \frac{1}{4}\zeta_3\bar{s}_1 + s_{-3,1} + \zeta_2s_{-1,1} + \frac{1}{2}\zeta_2s_{1,-1} \\
& + \frac{1}{2}\zeta_2s_{1,1} - \bar{s}_{-3,1} - \zeta_2\bar{s}_{-1,1} - \frac{3}{2}\zeta_2\bar{s}_{1,-1} - \frac{1}{2}\zeta_2\bar{s}_{1,1} - s_{-2,1,1} - s_{-1,2,1} \\
& - s_{1,-2,1} - s_{2,-1,1} + 2\bar{s}_{-2,1,1} + \bar{s}_{-1,2,1} + \bar{s}_{1,-2,1} + s_{-1,1,1,1} + s_{1,-1,1,1} \\
& + s_{1,1,-1,1} - 2\bar{s}_{-1,1,1,1} - \bar{s}_{1,-1,1,1}
\end{aligned} \tag{A50}$$

$$\begin{aligned}
s_{-1,1}\bar{s}_{-1,-1} = & -\frac{11\ln_2^4}{24} + \frac{11}{4}\zeta_2\ln_2^2 - \frac{1}{2}\bar{s}_{-2}\ln_2^2 + \frac{1}{2}\bar{s}_2\ln_2^2 + \frac{3}{2}s_{-1,-1}\ln_2^2 + \frac{1}{2}s_{-1,1}\ln_2^2 \\
& + \frac{1}{2}\bar{s}_{-1,-1}\ln_2^2 + \frac{1}{2}\bar{s}_{-1,1}\ln_2^2 + \frac{7}{2}s_{-1}\zeta_2\ln_2 - 2\zeta_3\ln_2 + \bar{s}_{-3}\ln_2 - \bar{s}_3\ln_2 \\
& - s_{2,-1}\ln_2 - s_{2,1}\ln_2 + 2\bar{s}_{-1,-2}\ln_2 - 2\bar{s}_{-1,2}\ln_2 - \bar{s}_{2,-1}\ln_2 + \bar{s}_{2,1}\ln_2 \\
& + 2s_{-1,-1,-1}\ln_2 + 2s_{-1,-1,1}\ln_2 + 2\bar{s}_{-1,-1,-1}\ln_2 - 2\bar{s}_{-1,-1,1}\ln_2 - \frac{33\zeta_2^2}{40} \\
& + 3\text{LiHalf}_4 - s_2\zeta_2 - \frac{9}{4}s_{-1}\zeta_3 - \frac{1}{2}\zeta_2\bar{s}_{-2} - \frac{3}{8}\zeta_3\bar{s}_{-1} + s_{-3,1} + \frac{3}{2}\zeta_2s_{-1,-1} \\
& + \frac{1}{2}\zeta_2s_{-1,1} - \frac{1}{2}\zeta_2\bar{s}_{-1,-1} + \frac{1}{2}\zeta_2\bar{s}_{-1,1} - \bar{s}_{3,-1} - 2s_{-1,2,1} - 2s_{2,-1,1} \\
& + \bar{s}_{-2,-1,-1} + 2\bar{s}_{-1,-2,-1} + \bar{s}_{2,1,-1} + 3s_{-1,-1,-1,1} \\
& - 2\bar{s}_{-1,-1,1,-1} - \bar{s}_{-1,1,-1,-1}
\end{aligned} \tag{A51}$$

$$\begin{aligned}
s_{1,-1}\bar{s}_{1,-1} &= -\frac{\ln_2^4}{12} - \frac{1}{2}s_{-2}\ln_2^2 + \frac{1}{2}s_2\ln_2^2 + \frac{15}{2}\zeta_2\ln_2^2 - \frac{1}{2}\bar{s}_{-2}\ln_2^2 + \frac{1}{2}\bar{s}_2\ln_2^2 \\
&\quad + s_{1,-1}\ln_2^2 - 2s_{1,1}\ln_2^2 + \bar{s}_{1,-1}\ln_2^2 - 2\bar{s}_{1,1}\ln_2^2 - s_{-3}\ln_2 + s_3\ln_2 \\
&\quad + s_{-1}\zeta_2\ln_2 + \frac{3}{2}s_1\zeta_2\ln_2 - 8\zeta_3\ln_2 - \bar{s}_{-3}\ln_2 + \zeta_2\bar{s}_{-1}\ln_2 + \frac{3}{2}\zeta_2\bar{s}_1\ln_2 \\
&\quad + \bar{s}_3\ln_2 + 2s_{1,-2}\ln_2 - 2s_{1,2}\ln_2 + 2s_{2,-1}\ln_2 + 2\bar{s}_{1,-2}\ln_2 - 2\bar{s}_{1,2}\ln_2 \\
&\quad + 2\bar{s}_{2,-1}\ln_2 - 4s_{1,1,-1}\ln_2 - 4\bar{s}_{1,1,-1}\ln_2 + \frac{12\zeta_2^2}{5} - 6\text{LiHalf}_4 - \frac{s_2\zeta_2}{2} \\
&\quad - \frac{3s_1\zeta_3}{8} - \frac{3}{8}\zeta_3\bar{s}_1 - \frac{1}{2}\zeta_2\bar{s}_2 - s_{-3,-1} + \zeta_2s_{1,1} - \bar{s}_{-3,-1} + \zeta_2\bar{s}_{1,1} \\
&\quad + s_{-2,1,-1} + 2s_{1,-2,-1} + s_{2,-1,-1} + \bar{s}_{-2,1,-1} + 2\bar{s}_{1,-2,-1} + \bar{s}_{2,-1,-1} \\
&\quad - s_{1,-1,1,-1} - 2s_{1,1,-1,-1} - \bar{s}_{1,-1,1,-1} - 2\bar{s}_{1,1,-1,-1}
\end{aligned} \tag{A52}$$

$$\begin{aligned}
s_{1,-1}\bar{s}_{1,1} &= \frac{\ln_2^4}{8} - \frac{1}{4}\zeta_2\ln_2^2 - \frac{1}{2}\bar{s}_{-2}\ln_2^2 + \frac{1}{2}\bar{s}_2\ln_2^2 + \frac{1}{2}s_{1,-1}\ln_2^2 - \frac{1}{2}s_{1,1}\ln_2^2 + \frac{1}{2}\bar{s}_{1,-1}\ln_2^2 \\
&\quad - \frac{3}{2}\bar{s}_{1,1}\ln_2^2 - s_{-3}\ln_2 + s_3\ln_2 + \frac{1}{2}s_{-1}\zeta_2\ln_2 - 2s_1\zeta_2\ln_2 - 4\zeta_3\ln_2 + \frac{1}{2}\zeta_2\bar{s}_{-1}\ln_2 \\
&\quad - \frac{5}{2}\zeta_2\bar{s}_1\ln_2 + 2s_{1,-2}\ln_2 - 2s_{1,2}\ln_2 + s_{2,-1}\ln_2 - s_{2,1}\ln_2 + \bar{s}_{2,-1}\ln_2 + \bar{s}_{2,1}\ln_2 \\
&\quad - 2s_{1,1,-1}\ln_2 + 2s_{1,1,1}\ln_2 - 2\bar{s}_{1,1,-1}\ln_2 - 2\bar{s}_{1,1,1}\ln_2 + \frac{\zeta_2^2}{20} + \text{LiHalf}_4 + \frac{1}{2}s_{-2}\zeta_2 \\
&\quad - s_{-1}\zeta_3 + \frac{s_1\zeta_3}{2} + \frac{1}{2}\zeta_2\bar{s}_{-2} - \zeta_3\bar{s}_{-1} + \frac{5}{8}\zeta_3\bar{s}_1 + \frac{1}{2}\zeta_2\bar{s}_2 - \zeta_2s_{1,-1} + s_{3,-1} - \bar{s}_{-3,1} \\
&\quad - \zeta_2\bar{s}_{1,-1} - \zeta_2\bar{s}_{1,1} - 2s_{1,2,-1} - 2s_{2,1,-1} + \bar{s}_{-2,1,1} + 2\bar{s}_{1,-2,1} + \bar{s}_{2,-1,1} \\
&\quad + 3s_{1,1,1,-1} - \bar{s}_{1,-1,1,1} - 2\bar{s}_{1,1,-1,1}
\end{aligned} \tag{A53}$$

$$\begin{aligned}
s_{1,-1}\bar{s}_{-1,-1} &= -\frac{5\ln_2^4}{24} - \frac{2}{3}s_{-1}\ln_2^3 + \frac{2}{3}s_1\ln_2^3 - \frac{2}{3}\bar{s}_{-1}\ln_2^3 + \frac{2}{3}\bar{s}_1\ln_2^3 - \frac{1}{2}s_{-2}\ln_2^2 + \frac{1}{2}s_2\ln_2^2 \\
&\quad - \frac{9}{4}\zeta_2\ln_2^2 - \frac{1}{2}s_{-1,-1}\ln_2^2 + \frac{1}{2}s_{-1,1}\ln_2^2 + 2s_{1,-1}\ln_2^2 + \frac{1}{2}\bar{s}_{-1,-1}\ln_2^2 - \frac{3}{2}\bar{s}_{-1,1}\ln_2^2 \\
&\quad - s_{-3}\ln_2 + s_3\ln_2 - s_{-1}\zeta_2\ln_2 - \frac{3}{2}s_1\zeta_2\ln_2 + \bar{s}_{-3}\ln_2 + \frac{1}{2}\zeta_2\bar{s}_{-1}\ln_2 \\
&\quad + \frac{1}{2}\zeta_2\bar{s}_1\ln_2 - \bar{s}_3\ln_2 - 2s_{-2,-1}\ln_2 - s_{-1,-2}\ln_2 + s_{-1,2}\ln_2 + s_{1,-2}\ln_2 \\
&\quad - s_{1,2}\ln_2 + 2\bar{s}_{-2,-1}\ln_2 + \bar{s}_{-1,-2}\ln_2 - \bar{s}_{-1,2}\ln_2 - \bar{s}_{1,-2}\ln_2 + \bar{s}_{1,2}\ln_2 \\
&\quad + 2s_{-1,1,-1}\ln_2 + 2s_{1,-1,-1}\ln_2 - 2\bar{s}_{-1,1,-1}\ln_2 - 2\bar{s}_{1,-1,-1}\ln_2 - \frac{\zeta_2^2}{5} \\
&\quad + \text{LiHalf}_4 + \frac{1}{2}s_{-2}\zeta_2 - \frac{1}{8}s_{-1}\zeta_3 + \frac{s_1\zeta_3}{2} - \frac{1}{2}\zeta_2\bar{s}_{-2} - \frac{1}{2}\zeta_3\bar{s}_{-1} - \frac{3}{4}\zeta_3\bar{s}_1 \\
&\quad - \frac{1}{2}\zeta_2s_{-1,1} - \frac{1}{2}\zeta_2s_{1,-1} + s_{3,-1} + \frac{1}{2}\zeta_2\bar{s}_{-1,1} + \frac{1}{2}\zeta_2\bar{s}_{1,-1} - \bar{s}_{3,-1} \\
&\quad - s_{-2,-1,-1} - s_{-1,-2,-1} - s_{1,2,-1} - s_{2,1,-1} + 2\bar{s}_{-2,-1,-1} + \bar{s}_{-1,-2,-1} \\
&\quad + \bar{s}_{1,2,-1} + s_{-1,-1,1,-1} + s_{-1,1,-1,-1} + s_{1,-1,-1,-1} \\
&\quad - \bar{s}_{-1,1,-1,-1} - 2\bar{s}_{1,-1,-1,-1}
\end{aligned} \tag{A54}$$

$$\begin{aligned}
s_{1,1}\bar{s}_{1,1} &= \frac{12\zeta_2^2}{5} - s_2\zeta_2 - \bar{s}_2\zeta_2 + 2s_{1,1}\zeta_2 + 2\bar{s}_{1,1}\zeta_2 + 3s_1\zeta_3 + 3\zeta_3\bar{s}_1 + s_{3,1} \\
&\quad + \bar{s}_{3,1} - 2s_{1,2,1} - 2s_{2,1,1} - 2\bar{s}_{1,2,1} - 2\bar{s}_{2,1,1} + 3s_{1,1,1,1} + 3\bar{s}_{1,1,1,1}
\end{aligned} \tag{A55}$$

$$\begin{aligned}
s_{1,1}\bar{s}_{-1,-1} = & -\frac{\ln_2^4}{3} - \frac{2}{3}s_{-1}\ln_2^3 + \frac{2}{3}s_1\ln_2^3 - \frac{1}{3}\bar{s}_{-1}\ln_2^3 + \frac{1}{3}\bar{s}_1\ln_2^3 - \frac{1}{2}s_{-2}\ln_2^2 + \frac{1}{2}s_2\ln_2^2 \\
& + \zeta_2\ln_2^2 - \frac{1}{2}\bar{s}_{-2}\ln_2^2 + \frac{1}{2}\bar{s}_2\ln_2^2 - \frac{1}{2}s_{-1,-1}\ln_2^2 + \frac{1}{2}s_{-1,1}\ln_2^2 + \frac{3}{2}s_{1,-1}\ln_2^2 \\
& + \frac{1}{2}s_{1,1}\ln_2^2 + \frac{1}{2}\bar{s}_{-1,-1}\ln_2^2 - \frac{1}{2}\bar{s}_{-1,1}\ln_2^2 + \frac{1}{2}\bar{s}_{1,-1}\ln_2^2 - \frac{1}{2}\bar{s}_{1,1}\ln_2^2 + s_{-1}\zeta_2\ln_2 \\
& + s_1\zeta_2\ln_2 + \frac{\zeta_3\ln_2}{4} + \bar{s}_{-3}\ln_2 - \bar{s}_3\ln_2 - s_{-2,-1}\ln_2 - s_{-2,1}\ln_2 + \bar{s}_{-2,-1}\ln_2 \\
& - \bar{s}_{-2,1}\ln_2 + \bar{s}_{-1,-2}\ln_2 - \bar{s}_{-1,2}\ln_2 - \bar{s}_{1,-2}\ln_2 + \bar{s}_{1,2}\ln_2 + s_{-1,1,-1}\ln_2 \\
& + s_{-1,1,1}\ln_2 + s_{1,-1,-1}\ln_2 + s_{1,-1,1}\ln_2 - \bar{s}_{-1,1,-1}\ln_2 + \bar{s}_{-1,1,1}\ln_2 - \bar{s}_{1,-1,-1}\ln_2 \\
& + \bar{s}_{1,-1,1}\ln_2 + \frac{3\zeta_2^2}{20} - \frac{1}{2}s_{-2}\zeta_2 - \frac{s_2\zeta_2}{2} + \frac{1}{8}s_{-1}\zeta_3 + \frac{s_1\zeta_3}{4} + \frac{1}{4}\zeta_3\bar{s}_{-1} + \frac{1}{8}\zeta_3\bar{s}_1 \\
& + \frac{1}{2}\zeta_2\bar{s}_2 + \frac{1}{2}\zeta_2s_{-1,-1} + \frac{1}{2}\zeta_2s_{-1,1} + \frac{1}{2}\zeta_2s_{1,-1} + \frac{1}{2}\zeta_2s_{1,1} + s_{3,1} + \bar{s}_{-3,-1} \\
& - \frac{1}{2}\zeta_2\bar{s}_{-1,-1} - \frac{1}{2}\zeta_2\bar{s}_{1,1} - s_{-2,-1,1} - s_{-1,-2,1} - s_{1,2,1} - s_{2,1,1} - \bar{s}_{-2,1,-1} \\
& - \bar{s}_{-1,2,-1} - \bar{s}_{1,-2,-1} - \bar{s}_{2,-1,-1} + s_{-1,-1,1,1} + s_{-1,1,-1,1} + s_{1,-1,-1,1} \\
& + \bar{s}_{-1,1,1,-1} + \bar{s}_{1,-1,1,-1} + \bar{s}_{1,1,-1,-1} \tag{A56}
\end{aligned}$$

$$\begin{aligned}
s_{-1,-1}\bar{s}_{-1,-1} = & -\ln_2^4 + 2\zeta_2\ln_2^2 + 2s_{-1,-1}\ln_2^2 + 2\bar{s}_{-1,-1}\ln_2^2 + s_{-3}\ln_2 - s_3\ln_2 \\
& - s_{-1}\zeta_2\ln_2 - 4\zeta_3\ln_2 + \bar{s}_{-3}\ln_2 - \zeta_2\bar{s}_{-1}\ln_2 - \bar{s}_3\ln_2 + 2s_{-1,-2}\ln_2 \\
& - 2s_{-1,2}\ln_2 - 2s_{2,-1}\ln_2 + 2\bar{s}_{-1,-2}\ln_2 - 2\bar{s}_{-1,2}\ln_2 - 2\bar{s}_{2,-1}\ln_2 \\
& + 4s_{-1,-1,-1}\ln_2 + 4\bar{s}_{-1,-1,-1}\ln_2 + \frac{14\zeta_2^2}{5} - 8\text{LiHalf}_4 + \frac{s_2\zeta_2}{2} \\
& + \frac{5}{4}s_{-1}\zeta_3 + \frac{5}{4}\zeta_3\bar{s}_{-1} + \frac{1}{2}\zeta_2\bar{s}_2 + s_{-3,-1} - \zeta_2s_{-1,-1} + \bar{s}_{-3,-1} \\
& - \zeta_2\bar{s}_{-1,-1} - 2s_{-1,2,-1} - 2s_{2,-1,-1} - 2\bar{s}_{-1,2,-1} - 2\bar{s}_{2,-1,-1} \\
& + 3s_{-1,-1,-1,-1} + 3\bar{s}_{-1,-1,-1,-1} \tag{A57}
\end{aligned}$$

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