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Review

Higher Spins without (Anti-)de Sitter

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Abstract: Can the holographic principle be extended beyond the well-known AdS/CFT correspondence? During the last couple of years, there has been a substantial amount of research trying to find answers for this question. In this work, we provide a review of recent developments of three-dimensional theories of gravity with higher spin symmetries. We focus in particular on a proposed holographic duality involving asymptotically flat spacetimes and higher spin extended $\mathfrak{bms}_3$ symmetries. In addition, we also discuss developments concerning relativistic and nonrelativistic higher spin algebras. As a special case, Carroll gravity will be discussed in detail.

Keywords: higher spin; non-anti-de sitter; flat space; holography; nonrelativistic holography

1. Introduction

Higher spin theories on Anti-de Sitter (AdS) backgrounds provide many useful insights into various aspects of the holographic principle. Many of these works were inspired by the seminal work of Klebanov and Polyakov [1–3] who conjectured a holographic correspondence between the $O(N)$ vector model in three dimensions and Fradkin–Vasiliev higher spin gravity on AdS_4 [4–6] (see [7–9] for reviews and [10–16] for some key developments). There are many features of higher spin theories that make them interesting to study. In the context of holography, one of these features is that it is a weak/weak correspondence [17,18]. In contrast, the usual AdS/CFT correspondence [19–21] is a weak/strong correspondence that makes it useful for applications, but harder to check in detail since calculations are often feasible only on one side of the correspondence.

In particular, three-dimensional higher spin theories are useful in this context, since (in contrast to the higher-dimensional examples) one can truncate the otherwise infinite tower of higher spin excitations [22]. Furthermore, the equations that describe the propagation of a massless field of spin s in three dimensions imply that there are no local degrees of freedom when $s \geq 2$. Thus, one can also formulate three-dimensional higher spin theories as Chern–Simons theories [23] with specific boundary conditions [24–28]. This is a considerable simplification in comparison with the more complicated higher-dimensional case. Developments in three-dimensional higher spin theories in AdS include[1], e.g., the discovery of minimal model holography [44–46], higher spin black holes [31,47–49] and higher spin holographic entanglement entropy [50,51].

Since higher spin holography in AdS backgrounds has lead to many interesting insights, a natural question to ask is how to generalize this duality such that it involves other spacetimes or quantum field theories. Additionally, indeed, there are many applications where one has spacetimes that do not asymptote to AdS or do so in a weaker way compared to the Brown–Henneaux boundary conditions [52]. Some examples include Lobachevsky spacetimes [53–55], null warped AdS and their generalizations Schrödinger [56–60], Lifshitz spacetimes [61–63], flat space [64–67] and de Sitter

[1] Further examples can be found in [29–43].

holography [68–70]. Some of these spacetimes play an important role as gravity duals for nonrelativistic CFTs, which are a common occurrence in, e.g., condensed matter physics and thus may be able to provide new insight into these strongly interacting systems. Schrödinger spacetimes for example can be used as a holographic dual to describe cold atoms [56,57].

Even though non-AdS higher spin holography is a rather new field of research there has been quite a lot of research in this direction during the last couple of years. Our aim with this review is to give an overview of the results and ideas that have been accumulated over the years. A special focus of this review will be on higher spin theories in three-dimensional flat space as well as the construction of new higher spin theories using kinematical algebras as bulk isometries.

This review is organized as follows. In Section 2, we present an overview of non-AdS holography that makes use of non-AdS boundary conditions of certain higher spin gravity theories. In Section 3, we focus on a specific example of non-AdS higher spin holography, namely flat space higher spin gravity. Section 4 can be read independently of Sections 2 and 3 and explains a different approach of studying non-AdS higher spin theories not via boundary conditions, but rather by using different choices of gauge algebras realizing certain higher spin theories in the bulk.

2. Non-AdS through Boundary Conditions

A lot of the progress in non-AdS higher spin holography has been achieved by imposing suitable boundary conditions that in turn allow one to compare physical boundary observables with their bulk counterpart. In three dimensions, this can be done rather nicely using a first order formulation of gravity [71,72]. In order to set the stage for non-AdS higher spin holography, we give now a brief review[2] of this formulation for the case of Einstein gravity, as well as AdS higher spin gravity.

2.1. The (Higher Spin) Chern–Simons Formulation of Gravity

In many situations, it is advantageous to not describe gravity in terms of a metric formulation, but rather in terms of local orthonormal Lorentz frames. That is, one exchanges the metric $g_{\mu\nu}$ with a vielbein e and a spin connection ω. In three dimensions, the dreibein e and dualized spin connection ω can have the same index structure in their Lorentz indices. Thus, one can combine these two quantities into a single gauge field:

$$\mathcal{A} \equiv e^a \mathtt{P}_a + \omega^a \mathtt{J}_a, \tag{1}$$

where the generators $\mathtt{P}_a$ and $\mathtt{J}_a$ generate the following Lie algebra[3]:

$$[\mathtt{P}_a, \mathtt{P}_b] = \mp\frac{1}{\ell^2}\epsilon_{abc}\mathtt{J}^c, \quad [\mathtt{J}_a, \mathtt{J}_b] = \epsilon_{abc}\mathtt{J}^c, \quad [\mathtt{J}_a, \mathtt{P}_b] = \epsilon_{abc}P^c. \tag{2}$$

- For $-\frac{1}{\ell^2}$, i.e., de Sitter spacetimes, this gauge algebra is $\mathfrak{so}(3,1)$.
- For $\ell \to \infty$, i.e., flat spacetimes, this gauge algebra is $\mathfrak{isl}(2,\mathbb{R})$.
- For $+\frac{1}{\ell^2}$, i.e., anti-de Sitter spacetimes, this gauge algebra is $\mathfrak{so}(2,2) \sim \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$.

It has been shown [71,72] that the Chern–Simons action:

$$S_{\text{CS}}[\mathcal{A}] = \frac{k}{4\pi}\int_{\mathcal{M}} \left\langle \mathcal{A} \wedge \mathrm{d}\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right\rangle, \tag{3}$$

defined on a three-dimensional manifold $\mathcal{M} = \Sigma \times \mathbb{R}$, with the invariant nondegenerate symmetric bilinear form:

$$\langle \mathtt{J}_a, \mathtt{P}_b \rangle = \eta_{ab}, \quad \langle \mathtt{J}_a, \mathtt{J}_b \rangle = \langle \mathtt{P}_a, \mathtt{P}_b \rangle = 0, \tag{4}$$

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[2] Parts of this review are based on [73–75]. There is also a slight overlap with [76].
[3] We raise and lower indices with $\eta = \text{diag}(-,+,+)$ and $\epsilon_{012} = 1$.

is equivalent (up to boundary terms) to the Einstein–Hilbert–Palatini action[4] with vanishing cosmological constant, provided one identifies the Chern–Simons level k with Newton's constant G in three dimensions as:

$$k = \frac{1}{4G}. \tag{5}$$

The just mentioned bilinear form is also called an invariant metric. Its properties are important for each component of the Chern–Simons gauge field to have a kinematical term (non-degeneracy) and for the action to be invariant under gauge transformations (invariance). This is the general setup for Einstein gravity in three dimensions using the Chern–Simons formalism.

AdS spacetimes in particular have some very nice features in this formalism that allow for a very efficient treatment of many physical questions. Maybe the most convenient feature from a Chern–Simons perspective is that the isometry algebra of AdS $\mathfrak{so}(2,2)$ is a direct sum of two copies of $\mathfrak{sl}(2,\mathbb{R})$. This also means that one can split the gauge field $\mathcal{A}$ into two parts A and $\bar{A}$. On the level of the generators, this split can be made explicit by introducing the generators:

$$\mathtt{T}_a = \frac{1}{2}\left(\mathtt{J}_a + \ell \mathtt{P}_a\right), \qquad \bar{\mathtt{T}}_a = \frac{1}{2}\left(\mathtt{J}_a - \ell \mathtt{P}_a\right). \tag{6}$$

These new generators satisfy:

$$[\mathtt{T}_a, \bar{\mathtt{T}}_b] = 0, \qquad [\mathtt{T}_a, \mathtt{T}_b] = \epsilon_{abc}\mathtt{T}^c, \qquad [\bar{\mathtt{T}}_a, \bar{\mathtt{T}}_b] = \epsilon_{abc}\bar{\mathtt{T}}^c. \tag{7}$$

Both $\mathtt{T}_a$ and $\bar{\mathtt{T}}_a$ satisfy an $\mathfrak{sl}(2,\mathbb{R})$ algebra. From (4), one can immediately see that the invariant bilinear forms are given by:

$$\langle \mathtt{T}_a, \mathtt{T}_b \rangle = \frac{\ell}{2}\eta_{ab}, \qquad \langle \bar{\mathtt{T}}_a, \bar{\mathtt{T}}_b \rangle = -\frac{\ell}{2}\eta_{ab}. \tag{8}$$

The gauge field $\mathcal{A}$ in terms of this split can now be written as:

$$\mathcal{A} = A^a \mathtt{T}_a + \bar{A}^a \bar{\mathtt{T}}_a. \tag{9}$$

Thus, after implementing this explicit split of $\mathfrak{so}(2,2)$ into $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, the Chern–Simons action (3) also splits into two contributions:

$$S_{\mathrm{EH}}^{\mathrm{AdS}}[A, \bar{A}] = S_{\mathrm{CS}}[A] + S_{\mathrm{CS}}[\bar{A}], \tag{10}$$

where the invariant bilinear forms appearing in the Chern–Simons action are given by (8). Since both $\mathtt{T}_a$ and $\bar{\mathtt{T}}_a$ satisfy an $\mathfrak{sl}(2,\mathbb{R})$ algebra, it is usually practical to not distinguish between the two generators, i.e., setting $\mathtt{T}_a = \bar{\mathtt{T}}_a$. This in turn also means that the invariant bilinear form in both sectors will be the same. From (8), however, one knows that the invariant bilinear form in both sectors should have the opposite sign. This is not a real problem since this relative minus sign can be easily introduced by hand by not taking the sum, but rather the difference of the two Chern–Simons actions:

$$S_{\mathrm{EH}}^{\mathrm{AdS}} = S_{\mathrm{CS}}[A] - S_{\mathrm{CS}}[\bar{A}] = \frac{1}{16\pi G}\left[\int_{\mathcal{M}} \mathrm{d}^3x\sqrt{|g|}\left(\mathcal{R} + \frac{2}{\ell^2}\right) - \int_{\partial\mathcal{M}} \omega^a \wedge e_a\right]. \tag{11}$$

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[4] For a nice and explicit calculation, see Appendix A in [51].

As the factor of ℓ in (8) only yields an overall factor of ℓ to the action (11), one can also absorb this factor simply in the Chern–Simons level as:

$$k = \frac{\ell}{4G}. \tag{12}$$

The form of the Chern–Simons connection (11) is usually the one discussed in the literature on AdS holography in three dimensions. The big advantage of this split into an unbarred and a barred part in the case of AdS holography is that usually one only has to explicitly treat one of the two sectors, as the other sector works in complete analogy, up to possible overall minus signs.

Aside from this technical simplification, there is another reason why the Chern–Simons formulation is very often used in AdS and non-AdS holography alike. While a generalization to higher-dimensional gravity is easier in the metric formulation, higher spin extensions are more straightforward in this setup. Since a Chern–Simons gauge theory with gauge algebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ corresponds to spin-2 gravity with AdS isometries, it is natural to promote the gauge algebra to $\mathfrak{sl}(N,\mathbb{R}) \oplus \mathfrak{sl}(N,\mathbb{R})$[5] in order to describe gravity theories with additional higher spin symmetries. Indeed, in [23], it was shown that for $N \geq 3$, such a Chern–Simons theory describes the nonlinear interactions of gravity coupled to a finite tower of massless integer spin-$s \leq N$ fields.

From a holographic perspective, one point of interest is the asymptotic symmetries of these higher spin gravity theories for given sets of boundary conditions. The first set of consistent boundary conditions that lead to interesting higher spin extensions of the Virasoro algebra has been worked out in [24,25].

Aside from extending the gauge algebra, one also has to take care of the normalization of the Chern–Simons level k. This has to be done in such a way that the spin-2 part of the resulting higher spin theory coincides with Einstein gravity. In order to give the Chern–Simons description an interpretation in terms of a metric, one needs to re-extract the geometric information hidden in the gauge field $\mathcal{A}$. For AdS, as well as flat space higher spin theories (in the principal embedding), this can be done via:

$$g_{\mu\nu} = \#\langle e^z_\mu, e^z_\nu \rangle, \tag{13}$$

where # is some normalization constant and e^z_μ is the so-called zuvielbein that can be seen as a higher spin extension of the dreibein e_μ encountered previously. The expression zuvielbein is a German expression meaning "too many legs" to emphasize that the object e^z_μ now contains more geometric information than the usual dreibein found in spin-2 gravity. In the well-known AdS case, the previous equation can be equivalently written as [40,48]:

$$g_{\mu\nu} = \# \left\langle A_\mu - \bar{A}_\mu, A_\nu - \bar{A}_\nu \right\rangle. \tag{14}$$

2.2. Boundary Terms and Higher Spins

After this brief reminder about the Chern–Simons formulation of (AdS) higher spin theories in three dimensions, we now want to set the stage for the transition to non-AdS spacetimes. All of the interesting physics aside from global properties in three-dimensional gravity are governed by degrees of freedom at the boundary. Thus, it is of utmost importance to make sure that one can impose consistently fall off conditions on the gauge field[6] at the asymptotic boundary. Consistent in this context means that one still has a well-defined variational principle after imposing said boundary

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[5] To be more precise: the spectrum of the higher spin gravity theory depends on the specific embedding of $\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{sl}(N,\mathbb{R})$. A very popular choice in the literature on AdS higher spin holography is the principal embedding of $\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{sl}(N,\mathbb{R})$. This is due to the fact that all generators in that particular embedding have a conformal weight greater or equal to two and thus can be interpreted as describing fields with spin $s \geq 2$.

[6] Or the metric in a second order formulation.

conditions. This is crucial since a consistent variational principle is the core principle underlying the definition of equations of motion of a physical system described by some action, as well as the one needed in the path integral. Thus, the necessity of having such a well-defined variational principle in turn also influences the possible set of boundary conditions that can be consistently imposed.

In order to see this, take a closer look at the variation of the Chern–Simons action (3):

$$\delta S_{\mathrm{CS}}[\mathcal{A}] = \frac{k}{2\pi}\int_{\mathcal{M}} \langle \delta\mathcal{A} \wedge F \rangle + \frac{k}{4\pi}\int_{\partial\mathcal{M}} \langle \delta\mathcal{A} \wedge \mathcal{A} \rangle \,. \tag{15}$$

This expression only vanishes on-shell, i.e., when $F = 0$, if the second term on the right-hand side vanishes as well. Assuming that the boundary $\partial\mathcal{M}$ is parametrized by a timelike coordinate t and an angular coordinate φ, this amounts to:

$$\frac{k}{4\pi}\int_{\partial\mathcal{M}} \left\langle \delta\mathcal{A}_t\mathcal{A}_\varphi - \delta\mathcal{A}_\varphi\mathcal{A}_t \right\rangle . \tag{16}$$

This term vanishes, for instance, if either $\mathcal{A}_\varphi$ or $\mathcal{A}_t$ are equal to zero at the boundary. This is quite a stringent condition on possible boundary conditions. Thus, it would be nice to have a way of enlarging the possible set of consistent boundary conditions. This can be most easily done by adding a boundary term $B[\mathcal{A}]$ to the Chern–Simons action (3). One could consider for example the following boundary term:

$$B[\mathcal{A}] = \frac{k}{4\pi}\int_{\partial\mathcal{M}} \left\langle \mathcal{A}_\varphi\mathcal{A}_t \right\rangle . \tag{17}$$

Including this boundary term, the total variation of the resulting action is on-shell:

$$\delta S_{\mathrm{CS}}[\mathcal{A}]^{\mathrm{Tot}} = \frac{k}{2\pi}\int_{\partial\mathcal{M}} \left\langle \delta\mathcal{A}_t\mathcal{A}_\varphi \right\rangle . \tag{18}$$

Vanishing of the total variation then can be achieved for example via:

$$\mathcal{A}_\varphi\Big|_{\partial\mathcal{M}} = 0 \quad \text{or} \quad \delta\mathcal{A}_t\Big|_{\partial\mathcal{M}} = 0. \tag{19}$$

Choosing $\delta\mathcal{A}_t\Big|_{\partial\mathcal{M}} = 0$, one is thus able to enlarge the possible set of boundary conditions by making sure that the variation of a part of the Chern–Simons connection vanishes.

2.3. Examples of Non-AdS Spacetimes Realized with Higher Spin Symmetries

Adding a suitable boundary term to the Chern–Simons connection in order to allow for a bigger set of possible boundary conditions is one of the necessary prerequisites for doing non-AdS holography. The second one is due to an observation first made explicit in [77]. That is, higher spin isometries, i.e., isometries based on $\mathfrak{sl}(N,\mathbb{R})$, can be used to realize certain non-AdS spacetimes asymptotically.

Take for example a direct product of maximally symmetric spacetimes such as $\mathrm{AdS}_2 \times \mathbb{R}$ or $\mathbb{H}_2 \times \mathbb{R}$, where $\mathbb{H}_2$ is the two-dimensional Lobachevsky plane. Then, assume that the gauge algebra of the Chern–Simons connection is given by a direct sum of an embedding of $\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{sl}(N,\mathbb{R})$ that contains at least one singlet $\mathtt{S}$ with $\mathrm{tr}(\mathtt{S}^2) \neq 0$ and whose $\mathfrak{sl}(2,\mathbb{R})$ generators are labeled as $\mathtt{L}_n$. Furthermore, assume that the manifold $\mathcal{M}$ where the Chern–Simons theory is defined has the topology of a cylinder with radial coordinate ρ and boundary coordinates x^1 and x^2. Then, using (14) and the connection:

$$A = \mathtt{L}_0\,\mathrm{d}\rho + a_1 e^{\rho}\mathtt{L}_+\,\mathrm{d}x^1, \qquad \bar{A} = -\mathtt{L}_0\,\mathrm{d}\rho + e^{\rho}\mathtt{L}_-\,\mathrm{d}x^1 + \mathtt{S}\,\mathrm{d}x^2, \tag{20}$$

where a_1 is some non-zero constant, one obtains the following non-vanishing metric components:

$$g_{\rho\rho} = 2\mathrm{tr}(\mathtt{L}_0^2), \qquad g_{11} = -a_1\mathrm{tr}(\mathtt{L}_+\mathtt{L}_-)e^{2\rho}, \qquad g_{22} = \frac{1}{2}\mathrm{tr}(\mathtt{S}^2). \tag{21}$$

Depending on the sign of a_1, this metric is locally and asymptotically either $\mathrm{AdS}_2 \times \mathbb{R}$ or $\mathbb{H}_2 \times \mathbb{R}$.

This was a first indication that one can model Lobachevsky spacetimes using higher spin gauge-invariant Chern–Simons theories. Following up on this, a natural question to ask is whether or not one can introduce boundary conditions in this setup that lead to interesting boundary dynamics. In [53,55], it was shown that this is, indeed, possible using a very general algorithm[7]. This algorithm can roughly be summarized by the following steps:

Identify Bulk Theory and Variational Principle:

The first step in this algorithm consists of identifying the bulk theory[8] one wants to describe. After that, one has to propose a suitable generalized variational principle, i.e., add appropriate boundary terms that are consistent with the theory under consideration.

Impose Suitable Boundary Conditions:

After having chosen the bulk theory, the next step in this algorithm is choosing appropriate boundary conditions for the Chern–Simons connection $\mathcal{A}$. This is the most crucial step in the whole analysis as the boundary conditions essentially determine the physical content of the putative dual field theory at the boundary. Since one is dealing with a Chern–Simons gauge theory, one also has some gauge freedom left that can be used to simplify computations. Choosing a gauge:

$$\mathcal{A}_\mu = b^{-1}\left(\mathfrak{a}_\mu + a_\mu^{(0)} + a_\mu^{(1)}\right) b + b^{-1}\,\mathrm{d}b, \quad b = b(\rho), \tag{22}$$

one can then identify the following three contributions to the Chern–Simons connections:

- $\mathfrak{a}_\mu$ denotes the (fixed) background that was chosen in the previous step.
- $a_\mu^{(0)}$ corresponds to state-dependent leading contributions in addition to the background that contains all the physical information about the field degrees of freedom at the boundary.
- $a_\mu^{(1)}$ are subleading contributions.

Choosing suitable boundary conditions in this context thus means choosing $a_\mu^{(0)}$ and $a_\mu^{(1)}$ in such a way that there exist gauge transformations that preserve these boundary conditions, i.e.,

$$\delta_\varepsilon \mathcal{A}_\mu = \mathcal{O}\left(b^{-1} a_\mu^{(0)} b\right) + \mathcal{O}\left(b^{-1} a_\mu^{(1)} b\right), \tag{23}$$

for some gauge parameter ε, which can also be written as:

$$\varepsilon = b^{-1}\left(\epsilon^{(0)} + \epsilon^{(1)}\right) b. \tag{24}$$

The transformations $\epsilon^{(0)}$ usually generate the asymptotic symmetry algebra, while $\epsilon^{(1)}$ are trivial gauge transformations.

Perform Canonical Analysis and Check the Consistency of Boundary Conditions:

Once the boundary conditions and the gauge transformations that preserve these boundary conditions have been fixed, one has to determine the canonical boundary charges. This is a standard

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[7] See also, e.g., [52,78,79].

[8] This usually boils down to choosing an appropriate embedding of $\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{sl}(N,\mathbb{R})$ and then fixing the Chern–Simons connections A and $\bar{A}$ in such a way that they correctly reproduce the desired gravitational background.

procedure that is described in great detail for example in [78,79] and is based on the results of [80]. This procedure eventually leads to the variation of the canonical boundary charge:

$$\delta\mathcal{Q}[\varepsilon] = \frac{k}{2\pi}\int_{\partial\Sigma}\left\langle \epsilon^{(0)}\delta a_{\varphi}^{(0)}\right\rangle \mathrm{d}\varphi, \tag{25}$$

where φ parametrizes the cycle of the boundary cylinder. Of course, one also has to check whether or not the boundary conditions chosen at the beginning of the algorithm are actually physically admissible. For the three-dimensional higher spin gravity gravity examples that are treated in this review, that means that the variation of the canonical boundary charge is finite, conserved in time and integrable in field space. However, we want to stress that there are also other examples such as, e.g., [81], where one can also have physically interesting boundary conditions where the canonical boundary charges do not necessarily meet all of the previously stated conditions.

Determine Semiclassical Asymptotic Symmetry Algebra:

This step consists of working out the Dirac brackets between the canonical generators $\mathcal{G}$ that directly yield the semiclassical asymptotic symmetry algebra. There is a well-known trick that can be used to simplify calculations at this point. Assume that one has two charges with Dirac bracket $\{\mathcal{G}[\varepsilon_1],\mathcal{G}[\varepsilon_2]\}$. Then, one can exploit the fact that these brackets generate a gauge transformation as $\{\mathcal{G}[\varepsilon_1],\mathcal{G}[\varepsilon_2]\} = \delta_{\epsilon_2}\mathcal{G}$ and read of the Dirac brackets by evaluating $\delta_{\epsilon_2}\mathcal{G}$. This relation for the canonical gauge generators is on-shell equivalent to a corresponding relation only involving the canonical boundary charges:

$$\{\mathcal{Q}[\varepsilon_1],\mathcal{Q}[\varepsilon_2]\} = \delta_{\epsilon_2}\mathcal{Q}, \tag{26}$$

which in most cases is straightforward to calculate. This directly leads to the semiclassical asymptotic symmetry algebra including all possible semiclassical central extensions.

Determine the Quantum Asymptotic Symmetry Algebra:

This part of the algorithm first appeared in [24]. One insight of this paper was that the asymptotic symmetry algebra derived in the previous steps is only valid for large values of the central charges. For non-linear algebras, such as $\mathcal{W}$-algebras that are frequently encountered in higher spin holography, that means in particular that one has to think about how normal ordering affects the algebra when passing from a semi-classical to a quantum description of the asymptotic symmetries. One particularly simple way of doing this is to take the semi-classical symmetry algebra, normal order non-linear terms and add all possible deformations to the commutation relations. Requiring that the resulting algebra satisfies the Jacobi identities (see, e.g., [82]) is usually enough to fix all the structure constants yielding the quantum asymptotic symmetry algebras.

Identify the Dual Field Theory:

With the results from all the previous steps, one can then proceed in trying to identify or put possible restrictions on a quantum field theory that explicitly realizes these quantum asymptotic symmetries. Once this dual field theory is identified, one can perform further nontrivial checks of the holographic conjecture.

2.3.1. Lobachevsky Spacetimes

As an explicit example of this algorithm, let us consider the Lobachevsky case worked out in [53,55]. In this work, the non-principal embedding of $\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{sl}(3,\mathbb{R})$ was used to describe fluctuations around the background:

$$\mathrm{d}s^2 = \mathrm{d}t^2 + \mathrm{d}\rho^2 + \sinh^2\rho\,\mathrm{d}\varphi^2. \tag{27}$$

In a Chern–Simons formulation, this means that one can consider a connection of the form:

$$A_t = 0, \qquad \bar{A}_t = \sqrt{3}\,\mathtt{S}, \qquad A_\rho = \mathtt{L}_0, \qquad \bar{A}_\rho = -\mathtt{L}_0, \tag{28a}$$

$$A_\varphi = -\frac{1}{4}\,\mathtt{L}_1\, e^{\rho} + \frac{2\pi}{k_{\text{cs}}}\left(\mathcal{J}(\varphi)\mathtt{S} + \mathcal{G}^{\pm}(\varphi)\psi^{\pm}_{-\frac{1}{2}}\, e^{-\rho/2} + \mathcal{L}(\varphi)\mathtt{L}_{-1}\, e^{-\rho}\right), \tag{28b}$$

$$\bar{A}_\varphi = -\mathtt{L}_{-1}\, e^{\rho} + \frac{2\pi}{k_{\text{cs}}}\,\bar{\mathcal{J}}(\varphi)\mathtt{S}, \tag{28c}$$

where the non-principal embedding of $\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{sl}(3,\mathbb{R})$ is characterized by three $\mathfrak{sl}(2)$ generators $\mathtt{L}_n$ $(n = -1, 0, 1)$, two sets of generators $\psi^{\pm}_n$ $(n = -\frac{1}{2}, \frac{1}{2})$ and one singlet $\mathtt{S}$.

Performing the algorithm described previously, one finds that the (quantum) asymptotic symmetry algebra is given by a direct sum of the Polyakov–Bershadsky algebra [83,84] and a $\hat{u}(1)$ current algebra.

Defining $\hat{k} = -k - 3/2$ and denoting normal ordering with respect to a highest-weight representation by $: :$, the asymptotic symmetry algebra is given by:

$$[\mathtt{J}_n, \mathtt{J}_m] = \kappa\, n\, \delta_{n+m,0} = [\bar{\mathtt{J}}_n, \bar{\mathtt{J}}_m], \tag{29a}$$

$$[\mathtt{J}_n, \mathtt{L}_m] = n \mathtt{J}_{n+m}, \tag{29b}$$

$$[\mathtt{J}_n, \mathtt{G}^{\pm}_m] = \pm \mathtt{G}^{\pm}_{m+n}, \tag{29c}$$

$$[\mathtt{L}_n, \mathtt{L}_m] = (n-m)\mathtt{L}_{m+n} + \frac{c}{12}\, n(n^2-1)\,\delta_{n+m,0}, \tag{29d}$$

$$[\mathtt{L}_n, \mathtt{G}^{\pm}_m] = \left(\frac{n}{2} - m\right)\mathtt{G}^{\pm}_{n+m}, \tag{29e}$$

$$\begin{aligned}[\mathtt{G}^{+}_n, \mathtt{G}^{-}_m] = {} & \frac{\lambda}{2}\left(n^2 - \frac{1}{4}\right)\delta_{n+m,0} \\ & - (\hat{k}+3)\mathtt{L}_{m+n} + \frac{3}{2}(\hat{k}+1)(n-m)\mathtt{J}_{m+n} + 3\sum_{p\in\mathbb{Z}} :\mathtt{J}_{m+n-p}\mathtt{J}_p:,\end{aligned} \tag{29f}$$

with the $\hat{\mathfrak{u}}(1)$ level:

$$\kappa = \frac{2\hat{k}+3}{3}, \tag{30}$$

the Virasoro central charge:

$$c = 25 - \frac{24}{\hat{k}+3} - 6(\hat{k}+3), \tag{31}$$

and the central term in the $G^{\pm}$ commutator:

$$\lambda = (\hat{k}+1)(2\hat{k}+3)\,. \tag{32}$$

Looking at unitary representations of this algebra, one finds that there is only one value where there are negative norm states that are absent, and that is for $\hat{k} = -1$ and thus also $c = 1$. Hence, a natural guess for a dual quantum field theory is a free boson.

Applying the same logic to other non-principally embedded $\mathfrak{sl}(N,\mathbb{R})$ Chern–Simons theories, it became quickly clear that the requirement of having no negative norm states is a very simple tool in restricting possible values of the Chern–Simons level. Furthermore, one could also see that with increasing N, also the allowed values for the central charges started to grow. In [54,85], it was shown that a Chern–Simons theory with next-to-principally embedded $\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{sl}(N,\mathbb{R})$ allows for boundary conditions that yield a $\mathcal{W}^{(2)}_N$ Feigin–Semikhatov [86] algebra as an asymptotic symmetry algebra. Looking at negative norm states for these algebras, one finds again restrictions on the allowed values of the central charge c that depend on N in such a way that the central charge can take arbitrarily large (but finite[9]) values. This is quite an interesting result since this provides an example of a unitary theory of gravity whose boundary dynamics are covered by a dual quantum field theory that allows both for a semiclassical (large values of the central charge), as well as an ultra quantum (central charge of $\mathcal{O}(1)$) regime. Thus, this family of $\mathcal{W}_N^{(2)}$ models provides a novel class of models that may be good candidates for toy models of quantum gravity in three dimensions.

2.3.2. Lifshitz Spacetimes

Even though the Lobachevsky case was the first example where higher spin symmetries proved useful for describing asymptotics beyond AdS, it is by far not the only case considered in the literature so far. Another example that gained quite a bit of attention is the case of asymptotic Lifshitz spacetimes [61]:

$$\mathrm{d}s^2 = \ell^2 \left(\frac{\mathrm{d}r^2 + \mathrm{d}x^2}{r^2} - \frac{\mathrm{d}t^2}{r^{2z}} \right), \tag{33}$$

where $z \in \mathbb{R}$ is a scaling exponent.

The authors of [62] used the Chern–Simons higher spin formulation successfully to describe non-rotating black holes in three-dimensional Lifshitz spacetimes with $z = 2$. In addition, this allowed them also to study the thermodynamic properties of these black holes in detail.

Another very interesting aspect of describing Lifshitz spacetimes using higher spin symmetries has been explored in [63]. The starting point of the analysis was again an $\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(3,\mathbb{R})$ higher spin Chern–Simons theory with boundary conditions such that the corresponding metric asymptotes to the Lifshitz spacetime (33). Looking at the resulting form of the asymptotic symmetry algebra, the authors found two copies of a $\mathcal{W}_3$ algebra with a central charge $c = \frac{3\ell}{2G}$. This is quite an interesting result, since this is exactly what one would get starting with a spin-3 extension of AdS_3 [25,27]. It was then later argued in [60] that this may be due to the non-invertibility of the zuvielbein in the higher spin Lifshitz case, and thus, the metric interpretation of Lifshitz spacetimes in higher spin theories might be questioned.

These are not the only interesting features that have been explored in the context of Lifshitz holography using higher spin symmetries. Furthermore, very interesting relations to integrable systems have been discovered in [88–90].

2.3.3. Null Warped, Schrödinger Spacetimes

Null warped AdS:

$$\mathrm{d}s^2 = \ell^2 \left(\frac{\mathrm{d}r^2}{4r^2} + 2r\,\mathrm{d}t\,\mathrm{d}\varphi + f(r,z)\,\mathrm{d}\varphi^2 \right), \tag{34}$$

with $f(r,z) = r^z + \beta r + \alpha^2$ and where z is a real parameter and α and β constants of motion, is another case of spacetimes that have been linked to higher spin theories. In [59], the authors proposed boundary conditions that asymptote to null warped AdS and found a single copy of the $\mathcal{W}_3^{(2)}$ Polyakov–Bershadsky algebra (29) as asymptotic symmetries.

Last, but not least, we also want to mention that Schrödinger spacetimes:

$$\mathrm{d}s^2 = -r^{2z}\,\mathrm{d}t^2 - 2r^2\,\mathrm{d}t\,\mathrm{d}x^- + \frac{\mathrm{d}r^2}{r^2} + r^2\,\mathrm{d}x^i, \tag{35}$$

can be treated in a higher spin context, both in three dimensions, as well as in higher dimensions [91,92].

[9] In [87], it has been shown that any embedding of $\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{sl}(N,\mathbb{R})$ that contains a singlet contains negative norm states for $c \to \infty$.

Even though this review is focused on higher spins without anti-de Sitter, we also want to point out some work on higher spins in de Sitter [68–70], as well an example of chiral higher spin theories in AdS[10] [94].

3. Flat Space Higher Spin Theories as Specific Examples

Besides the examples of non-AdS higher spin theories that have already been mentioned in the previous section, there is another quite prominent example of a holographic correspondence involving higher spins, that is flat space. Before we go into more details regarding higher spins in flat space, we want to give a brief overview of important developments regarding flat space holography in general.

The first indications that there might be a holographic correspondence in asymptotically flat spacetimes were worked out in [95–97]. In the last decade, there has been a lot of progress in that direction especially in three spacetime dimensions. In 2006, Barnich and Compère [98] presented a consistent set of boundary conditions for asymptotically flat spacetimes at null infinity[11] that extended previous considerations of [102]. Using these boundary conditions, Barnich and Compère were able to show that the corresponding asymptotic symmetry algebra is given by the three-dimensional Bondi–Metzner–Sachs algebra ($\mathfrak{bms}_3$) [103,104]. Since the discovery of the Barnich–Compère boundary conditions, many other boundary conditions in asymptotically flat spacetimes have been found leading to either extensions of the $\mathfrak{bms}_3$ algebra as asymptotic symmetry algebra such as [105–110] or to other algebras such as a warped conformal algebra [111], Heisenberg algebras [112] or an $\mathfrak{isl}(2)_k$ algebra [113].

In particular, the Barnich–Compère boundary conditions, and the associated $\mathfrak{bms}_3$ asymptotic symmetries were used quite extensively for various non-trivial checks of a putative holographic correspondence [114–132].

The previously mentioned developments were mainly focused on either pure Einstein gravity or supersymmetric extensions thereof. Now, what about (massless) higher spin theories in flat space? In four or higher dimensional flat space, there are in fact quite a number of no-go theorems that forbid non-trivial higher spin interactions such as the Coleman–Mandula theorem [133], its generalization by Pelc and Horwitz [134], the Aragone–Deser no-go result [135], the Weinberg–Witten theorem [136], and others. For a very nice overview of all these various no-go theorems, please refer to [137]. This seems like bad news for non-trivial interacting (massless) higher spin theories. However, every no-go theorem is only as good as its premises, and as such, there are various ways of circumventing these theorems such as, e.g., having a non-zero cosmological constant [6]. Interestingly enough, the no-go theorems mentioned previously do not apply in three dimensions, and thus, it seems possible to have non-trivial interacting (massless) higher spin theories in three dimensions[12].

Indeed, in [64,65], the first consistent boundary conditions for a higher spin extension of the Poincaré algebra were found.

3.1. Flat Space Spin-3 Gravity

Higher spin theories in three-dimensional flat space can be described in a very similar fashion as in the AdS$_3$ case, that is by a suitable Chern–Simons formulation. In the AdS$_3$ case, the basic gauge symmetries of the Chern–Simons gauge field are given by a direct sum of two copies of $\mathfrak{sl}(N,\mathbb{R})$ (or more general $\mathfrak{hs}[\lambda]$). In the flat space case, the corresponding connections take values in $\mathfrak{isl}(N,\mathbb{R})$.

---

[10] The boundary conditions in this work can be seen as the spin-3 extension of the boundary conditions found in [93].

[11] These are boundary conditions for either future or past null infinity. Thus, to be more precise, one obtains one copy of $\mathfrak{bms}_3$ on future and another copy on past null infinity. For successful efforts of connecting these two algebras, see [99–101].

[12] Even though (A)dS backgrounds favor interactions of massless higher spin fields, higher spin interactions are not completely ruled out even in higher-dimensional flat space. For recent developments regarding higher spins in four or higher-dimensional flat space, see, e.g., [138–141].

The structure of $\mathfrak{isl}(N,\mathbb{R})$ is that of a semidirect sum of $\mathfrak{sl}(N,\mathbb{R})$ with an abelian ideal that is isomorphic to $\mathfrak{sl}(N,\mathbb{R})$ as a vector space. One nice thing about this structure is that it can be straightforwardly obtained by suitable İnönü–Wigner contractions [142], and thus, one has a direct way of obtaining these algebras from the well-known AdS$_3$ higher spin gauge symmetries[13]. These kinds of contractions have been used quite successfully to obtain new higher spin algebras in flat space (both isometries and asymptotic symmetries) [66,73,147], as well as flat space analogues of important formulas like the (spin-3) Cardy formula [125,148].

Thus, the starting point for a spin-3 theory in flat space is a Chern–Simons action with gauge algebra[14] $\mathfrak{isl}(3,\mathbb{R})$ equipped with an appropriate bilinear form. Then, one can choose boundary conditions as [64,65]:

$$\mathcal{A} = b^{-1}\,\mathrm{d}b + b^{-1}a(u,\varphi)b, \qquad b = e^{\frac{r}{2}\mathtt{M}_{-1}}, \tag{36}$$

with:

$$a(u,\varphi) = a_\varphi(u,\varphi)\,\mathrm{d}\varphi + a_u(u,\varphi)\,\mathrm{d}u, \tag{37}$$

where:

$$a_\varphi(u,\varphi) = \mathtt{L}_1 - \frac{\mathcal{M}}{4}\mathtt{L}_{-1} - \frac{\mathcal{N}}{2}\mathtt{M}_{-1} + \frac{\mathcal{V}}{2}\mathtt{U}_{-2} + \mathcal{Z}\mathtt{V}_{-2}, \tag{38a}$$

$$a_u(u,\varphi) = \mathtt{M}_1 - \frac{\mathcal{M}}{4}\mathtt{M}_{-1} + \frac{\mathcal{V}}{2}\mathtt{V}_{-2}. \tag{38b}$$

The operators $\mathtt{L}_n, \mathtt{M}_n$ with $n = \pm 1, 0$ and $\mathtt{U}_m, \mathtt{V}_m$ with $m = \pm 2, \pm 1, 0$ span the $\mathfrak{isl}(3,\mathbb{R})$ algebra[15] with invariant bilinear form:

$$\langle \mathtt{L}_n \mathtt{M}_m \rangle = -2\eta_{nm}, \qquad \langle \mathtt{U}_n \mathtt{V}_m \rangle = \frac{2}{3}\mathcal{K}_{nm}, \tag{39}$$

where $\eta_{nm} = \mathrm{antidiag}(1, -\frac{1}{2}, 1)$ and $\mathcal{K}_{nm} = \mathrm{antidiag}(12, -3, 2, -3, 12)$. The zuvielbein can be extracted from the gauge connection by using that $\mathcal{A} = e_n^{(2)}\mathtt{M}_n + e_n^{(3)}\mathtt{V}_n + \omega_n^{(2)}\mathtt{L}_n + \omega_n^{(3)}\mathtt{U}_n$. Using these ingredients, one can determine the metric[16]

$$g_{\mu\nu} = \eta_{ab} e_\mu^{(2)a} e_\nu^{(2)b} + \mathcal{K}_{ab} e_\mu^{(3)a} e_\nu^{(3)b}. \tag{40}$$

Thus, consequently, these boundary conditions describe the following metric and spin-3 field:

$$\mathrm{d}s^2 = \mathcal{M}\,\mathrm{d}u^2 - 2\,\mathrm{d}r\,\mathrm{d}u + 2\mathcal{N}\,\mathrm{d}u\,\mathrm{d}\varphi + r^2\,\mathrm{d}\varphi^2, \qquad \Phi_{\mu\nu\lambda}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,\mathrm{d}x^\lambda = 2\mathcal{V}\,\mathrm{d}u^3 + 4\mathcal{Z}\,\mathrm{d}u^2\,\mathrm{d}\varphi. \tag{41}$$

From a geometric point of view, the metric is nothing else than flat space in Eddington–Finkelstein coordinates. Working out the asymptotic symmetries, one finds that these boundary conditions lead to the following non-linear, centrally-extended asymptotic symmetry algebra:

---

13 Please refer to [117,119,126,132,143–146] for early, as well as recent work in flat space holography in three dimensions that rely on contractions.
14 To be more precise, it is the principal embedding of $\mathfrak{isl}(2,\mathbb{R}) \hookrightarrow \mathfrak{isl}(3,\mathbb{R})$.
15 The commutation relations are identical to the ones in (42) after restricting the mode numbers as already mentioned and in addition dropping all non-linear terms.
16 The spin-3 field can be determined in analogy by using the cubic Casimir of the $\mathfrak{sl}(3,\mathbb{R})$ subalgebra.

$$
\begin{aligned}
[\mathtt{L}_n, \mathtt{L}_m] &= (n-m)\mathtt{L}_{n-m} + \frac{c_L}{12} n(n^2-1)\delta_{n+m,0}, && \text{(42a)}\\
[\mathtt{L}_n, \mathtt{M}_m] &= (n-m)\mathtt{M}_{n-m} + \frac{c_M}{12} n(n^2-1)\delta_{n+m,0}, && \text{(42b)}\\
[\mathtt{L}_n, \mathtt{U}_m] &= (2n-m)\mathtt{U}_{n+m}, && \text{(42c)}\\
[\mathtt{L}_n, \mathtt{V}_m] &= (2n-m)\mathtt{V}_{n+m}, && \text{(42d)}\\
[\mathtt{M}_n, \mathtt{U}_m] &= (2n-m)\mathtt{V}_{n+m}, && \text{(42e)}\\
[\mathtt{U}_n, \mathtt{U}_m] &= (n-m)(2n^2+2m^2-nm-8)\mathtt{L}_{n+m} + \frac{192}{c_M}(n-m)\Lambda_{n+m} \\
&\quad - \frac{96c_L}{c_M^2}(n-m)\Theta_{n+m} + \frac{c_L}{12} n(n^2-1)(n^2-4)\delta_{n+m,0}, && \text{(42f)}\\
[\mathtt{U}_n, \mathtt{V}_m] &= (n-m)(2n^2+2m^2-nm-8)\mathtt{M}_{n+m} + \frac{96}{c_M}(n-m)\Theta_{n+m} \\
&\quad + \frac{c_M}{12} n(n^2-4)(n^2-1)\delta_{n+m,0}, && \text{(42g)}
\end{aligned}
$$

with:

$$
\Lambda_n = \sum_{p\in\mathbb{Z}} \mathtt{L}_p \mathtt{M}_{n-p}, \qquad \Theta_n = \sum_{p\in\mathbb{Z}} \mathtt{M}_p \mathtt{M}_{n-p}, \tag{43}
$$

and:

$$
c_L = 0, \qquad c_M = \frac{3}{G}. \tag{44}
$$

This algebra is usually denoted by $\mathcal{FW}_3$ to denote its role similar to the $\mathcal{W}_3$ algebra in AdS_3 higher spin holography. Furthermore, this algebra can be obtained as a specific İnönü–Wigner contraction that can be interpreted as a limit of vanishing cosmological constant of the the AdS_3 spin-3 asymptotic $\mathcal{W}_3$ symmetries.

Assuming that one starts with two copies of a quantum[17] $\mathcal{W}_3$ algebra [149] whose generators are labeled as $\mathcal{L}_n$, $\bar{\mathcal{L}}_n$ and $\mathcal{W}_n$, $\bar{\mathcal{W}}_n$, then one can define the following linear combinations:

$$
\mathtt{L}_n := \mathcal{L}_n - \bar{\mathcal{L}}_{-n}, \qquad \mathtt{M}_n := \frac{1}{\ell}\left(\mathcal{L}_n + \bar{\mathcal{L}}_{-n}\right), \tag{45a}
$$

$$
\mathtt{U}_n := \mathcal{W}_n - \bar{\mathcal{W}}_{-n}, \qquad \mathtt{V}_n := \frac{1}{\ell}\left(\mathcal{W}_n + \bar{\mathcal{W}}_{-n}\right), \tag{45b}
$$

and in the limit $\ell \to \infty$ obtain exactly (42). It should also be noted that besides the contraction (45), one can also perform a so-called nonrelativistic contraction using the following alternative linear combination:

$$
\mathtt{L}_n := \mathcal{L}_n + \bar{\mathcal{L}}_n, \qquad \mathtt{M}_n := -\epsilon\left(\mathcal{L}_n - \bar{\mathcal{L}}_n\right), \tag{46a}
$$

$$
\mathtt{U}_n := \mathcal{W}_n + \bar{\mathcal{W}}_n, \qquad \mathtt{V}_n := -\epsilon\left(\mathcal{W}_n - \bar{\mathcal{W}}_n\right), \tag{46b}
$$

that in the limit $\epsilon \to 0$ yields another kind of non-linear, centrally-extended algebra [147] that can be seen as natural (quantum) higher spin extension of the Galilean conformal algebra $\mathfrak{gca}_2$. In the spin-2 case, these two limits yield two isomorphic algebras, namely the $\mathfrak{bms}_3$ and $\mathfrak{gca}_2$ algebra, respectively. However, as soon as one adds higher spins, these two limits do not yield isomorphic algebras anymore. The reason for this is basically that each limit favors different representations. The ultrarelativistic limit that leads to the $\mathfrak{bms}_3$ algebra favors so-called (unitary) induced representations, whereas the

[17] That means that all non-linear terms are normal ordered with respect to some highest-weight representation and the central terms have $\mathcal{O}(1)$ corrections that are necessary to satisfy the Jacobi identities when the non-linear terms are normal ordered.

nonrelativistic contraction favors (generically non-unitary [66]) highest-weight representations [147]. Since normal ordering requires a notion of vacuum, what is meant by normal ordering differs as soon as there are non-linear terms present in the algebra and as such also influences the structure constants.

3.2. Flat Space Cosmologies with Spin-3 Hair

Cosmological solutions in flat space [150,151] are well known and thoroughly studied objects. As such, another very interesting thing to study in the context of higher spin theories is comprised of cosmological solutions in flat space that also carry higher spin hair. This has been done successfully first in [67] and subsequently also in [152]. The basic idea of describing such cosmological solutions is by taking the φ-part of the connection (37) and extending the u-part by arbitrary, but fixed chemical potentials in such a way that the equations of motion $F = 0$ are satisfied. By imposing suitable holonomy conditions[18], one can then determine, the inverse temperature, angular potential and higher spin chemical potentials and subsequently also the thermal entropy of cosmological solutions with additional spin-3 hair. If one denotes the spin-2 charges by $\mathcal{N}$, $\mathcal{M}$, the spin-3 charges by $\mathcal{Z}$, $\mathcal{V}$ and introducing the dimensionless ratios:

$$\frac{\mathcal{R}-1}{4\mathcal{R}^{3/2}} = \frac{|\mathcal{V}|}{\mathcal{M}^{3/2}}, \qquad \mathcal{R} > 3, \qquad \text{and} \qquad \mathcal{P} = \frac{\mathcal{Z}}{\sqrt{\mathcal{M}\mathcal{N}}}, \tag{47}$$

one obtains the following formula for the thermal entropy of cosmological solutions with spin-3 hair:

$$S_{\text{Th}} = \frac{\pi}{2G}\frac{|\mathcal{N}|}{\sqrt{\mathcal{M}}}\frac{2\mathcal{R}-3-12\mathcal{P}\sqrt{\mathcal{R}}}{(\mathcal{R}-3)\sqrt{4-3/\mathcal{R}}}. \tag{48}$$

This result can also be understood in terms of a limiting procedure of the AdS spin-3 results for the thermal entropy of BTZsolutions with spin-3 hair [148]. In complete analogy to the limiting procedure of the BTZ black hole entropy, one has to consider the following expression that can be seen as a inner horizon entropy formula of spin-3 charged BTZ black holes:

$$S_{\text{inner}} = 2\pi\left|\sqrt{\frac{c\,\mathcal{L}}{6}}\sqrt{1-\frac{3}{4C}} - \sqrt{\frac{\bar{c}\,\bar{\mathcal{L}}}{6}}\sqrt{1-\frac{3}{4\bar{C}}}\right|, \tag{49}$$

where $c = \bar{c} = \frac{3\ell}{2G}$ and the dimensionless ratios C and $\bar{C}$ are given by:

$$\sqrt{\frac{c}{6\mathcal{L}^3}}\frac{\mathcal{W}}{4} = \frac{C-1}{C^{\frac{3}{2}}}, \qquad \sqrt{\frac{\bar{c}}{6\bar{\mathcal{L}}^3}}\frac{\bar{\mathcal{W}}}{4} = \frac{\bar{C}-1}{\bar{C}^{\frac{3}{2}}}. \tag{50}$$

In order to take the limit, one also has to introduce suitable relation of the AdS spin-2 and spin-3 charges $\mathcal{L}$, $\bar{\mathcal{L}}$, $\mathcal{W}$, $\bar{\mathcal{W}}$ with their flat space counterparts $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{Z}$, $\mathcal{V}$. If one chooses the relations as:

$$\mathcal{M} = 12\left(\frac{\mathcal{L}}{c}+\frac{\bar{\mathcal{L}}}{\bar{c}}\right), \qquad \mathcal{N} = 6\ell\left(\frac{\mathcal{L}}{c}-\frac{\bar{\mathcal{L}}}{\bar{c}}\right), \tag{51a}$$

$$\mathcal{V} = 12\left(\frac{\mathcal{W}}{c}+\frac{\bar{\mathcal{W}}}{\bar{c}}\right), \qquad \mathcal{Z} = 6\ell\left(\frac{\mathcal{W}}{c}-\frac{\bar{\mathcal{W}}}{\bar{c}}\right), \tag{51b}$$

and in addition defines:

$$C = \mathcal{R} + \frac{2}{\ell}D(\mathcal{R},\mathcal{P},\mathcal{M},\mathcal{N}), \qquad \bar{C} = \mathcal{R} - \frac{2}{\ell}D(\mathcal{R},\mathcal{P},\mathcal{M},\mathcal{N}). \tag{52}$$

[18] Alternatively, one can also use a closed Wilson loop wrapped around the horizon [153] in order to determine the thermal entropy.

with:
$$D(\mathcal{R},\mathcal{P},\mathcal{M},\mathcal{N}) = \frac{\mathcal{N}}{\mathcal{M}}\frac{\mathcal{R}\left(\mathcal{R}^{\frac{3}{2}}\mathcal{P}+3\mathcal{R}-3\right)}{(\mathcal{R}-3)}, \tag{53}$$

then it is straightforward to show that, indeed, one reproduces the entropy formula (48) in the limit $\ell \to \infty$.

Having an entropy formula like (49) at hand also allows one to study possible phase transitions of these higher spin cosmological solutions in flat space by looking at the free energy. Indeed, one finds the usual phase transition to hot flat space first described in [118] plus additional phase transitions because of the additional spin-3 charges. Interestingly and in contrast to the possible phase transitions in AdS [154], there can also be first order phase transitions between various thermodynamical phases in the flat space case.

3.3. Higher Spin Soft Hair in Flat Space

Soft hair excitations of black holes as possible solutions[19] to the black hole information paradox have attracted quite some research interest lately; see, e.g., [157,158]. Especially, three-dimensional gravity proved to be quite an active playground to study soft hair on (higher spin) black holes in AdS [159–162], higher-derivative theories of gravity [163], as well as flat space [112,164]. What is most intriguing about all these near horizon boundary conditions is that they all lead to a (number of) $\hat{\mathfrak{u}}(1)$ current algebra(s), but the entropy is always given in a very simple form:

$$S_{\mathrm{Th}} = 2\pi\left(\mathtt{J}_0^+ + \mathtt{J}_0^-\right), \tag{54}$$

where $\mathtt{J}_0^\pm$ are the spin-2 zero modes of the near horizon symmetry algebras. In the following, we give a brief overview on how to obtain this result for the entropy for spin-3 gravity in flat space.

The starting point is again a Chern–Simons formulation of gravity with a gauge algebra $\mathfrak{isl}(3,\mathbb{R})$ as in Section 3.1[20]. However, one is now interested in describing near horizon boundary conditions of flat space cosmologies with additional spin-3 hair in contrast to the examples previously that focused on the asymptotic symmetries of such configurations. These near horizon boundary conditions can be described by:

$$\mathcal{A} = b^{-1}(a+\mathrm{d})\,b \tag{55}$$

where the radial dependence is encoded in the group element b as [112]:

$$b = \exp\left(\frac{1}{\mu_{\mathcal{P}}}\,\mathtt{M}_1\right)\,\exp\left(\frac{r}{2}\,\mathtt{M}_{-1}\right), \tag{56}$$

and the connection a reads:

$$a = a_v\,\mathrm{d}v + a_\varphi\,\mathrm{d}\varphi\,, \tag{57}$$

with:

$$a_\varphi = \mathcal{J}\,\mathtt{L}_0 + \mathcal{P}\,\mathtt{M}_0 + \mathcal{J}^{(3)}\,\mathtt{U}_0 + \mathcal{P}^{(3)}\,\mathtt{V}_0\,, \tag{58a}$$
$$a_v = \mu_{\mathcal{P}}\,\mathtt{L}_0 + \mu_{\mathcal{J}}\,\mathtt{M}_0 + \mu_{\mathcal{P}}^{(3)}\,\mathtt{U}_0 + \mu_{\mathcal{J}}^{(3)}\,\mathtt{V}_0\,. \tag{58b}$$

All the functions appearing in (58) are in principle arbitrary functions of the advanced time v and the angular coordinate φ. Based on these boundary conditions, it is straightforward to determine the near horizon symmetry algebra as:

[19] For a contrasting view on the role of soft hair in solving the black hole paradox, see, e.g., [155,156] and the references therein.
[20] Please note that instead of the retarded time coordinate u it is more natural to use the advanced time coordinate v.

$$[\mathtt{J}_n, \mathtt{P}_m] = k\, n\, \delta_{n+m,0} \qquad [\mathtt{J}^{(3)}_n, \mathtt{P}^{(3)}_m] = \frac{4k}{3}\, n\, \delta_{n+m,0} \tag{59}$$

where $k = \frac{1}{4G}$, which can also be brought into a different form by:

$$\mathtt{J}^{\pm}_{\pm n} = \frac{1}{2}(\mathtt{P}_n \pm \mathtt{J}_n) \qquad \mathtt{J}^{(3)\pm}_{\pm n} = \frac{1}{2}(\mathtt{P}^{(3)}_n \pm \mathtt{J}^{(3)}_n)\,. \tag{60}$$

The generators $\mathtt{J}^{\pm}_n$ and $\mathtt{J}^{(3)\pm}_n$ then satisfy:

$$[\mathtt{J}^{\pm}_n, \mathtt{J}^{\pm}_m] = \frac{k}{2} n \delta_{n+m,0} \qquad [\mathtt{J}^{+}_n, \mathtt{J}^{-}_m] = 0\,, \tag{61a}$$

$$[\mathtt{J}^{(3)\pm}_n, \mathtt{J}^{(3)\pm}_m] = \frac{2k}{3} n \delta_{n+m,0}\,, \qquad [\mathtt{J}^{(3)+}_n, \mathtt{J}^{(3)-}_m] = 0\,. \tag{61b}$$

Thus, the near horizon symmetries are given by two pairs of $\hat{\mathfrak{u}}(1)$ current algebras. Calculating both the Hamiltonian (in order to check that these excitations are, indeed, soft), as well as the thermal entropy, which is given by (54), is a straightforward exercise, and we refer the interested reader to [164] for more details.

With a simple result like (54) for the thermal entropy of a spin-3 charged flat space cosmology and a rather complicated one like (48), a natural question to ask is: How exactly are these two related? Is there a way to construct the asymptotic state-dependent functions $\mathcal{M}, \mathcal{N}, \mathcal{V}$ and $\mathcal{Z}$ in terms of the near-horizon state-dependent functions $\mathcal{J}, \mathcal{P}, \mathcal{J}^{(3)}$ and $\mathcal{P}^{(3)}$?

In order to answer these questions, one has to find a gauge transformation that maps these two connections into each other without changing the canonical boundary charges. Such a gauge transformation can, indeed, be found and gives the relations:

$$\mathcal{M} = \mathcal{J}^2 + \frac{4}{3}\left(\mathcal{J}^{(3)}\right)^2 + 2\mathcal{J}', \tag{62a}$$

$$\mathcal{N} = \mathcal{J}\mathcal{P} + \frac{4}{3}\mathcal{J}^{(3)}\mathcal{P}^{(3)} + \mathcal{P}', \tag{62b}$$

$$\mathcal{V} = \frac{1}{54}\left(18\mathcal{J}^2\mathcal{J}^{(3)} - 8\left(\mathcal{J}^{(3)}\right)^3 + 9\mathcal{J}'\mathcal{J}^{(3)} + 27\mathcal{J}\mathcal{J}^{(3)\prime} + 9\mathcal{J}^{(3)\prime\prime}\right), \tag{62c}$$

$$\mathcal{Z} = \frac{1}{36}\left(6\mathcal{J}^2\mathcal{P}^{(3)} - 8\mathcal{P}^{(3)}\left(\mathcal{J}^{(3)}\right)^2 + 3\mathcal{P}^{(3)}\mathcal{J}' + 3\mathcal{J}^{(3)}\mathcal{P}' \right.$$
$$\left. + 9\mathcal{J}\mathcal{P}^{(3)\prime} + 9\mathcal{P}\mathcal{J}^{(3)\prime} + 12\mathcal{P}\mathcal{J}\mathcal{J}^{(3)} + 3\mathcal{P}^{(3)\prime\prime}\right) \tag{62d}$$

that are basically (twisted) Sugawara constructions for the spin-2 and spin-3 fields. One can use these relations and an appropriate Fourier expansion in order to solve for the zero modes $\mathtt{P}_0 = \mathtt{J}^+_0 + \mathtt{J}^-_0$, which gives:

$$P_0 = \mathtt{J}^+_0 + \mathtt{J}^-_0 = \frac{1}{4G}\frac{\mathcal{N}\left(4\mathcal{R} - 6 + 3\mathcal{P}\sqrt{\mathcal{R}}\right)}{4\sqrt{\mathcal{M}}(\mathcal{R}-3)\sqrt{1-\frac{3}{4\mathcal{R}}}} \tag{63}$$

and correctly reproduces (48). Thus, one sees that also for flat space cosmologies, there seems to be a much easier way to count the microscopical states contributing to the thermal entropy; that is, in terms of near horizon variables instead of asymptotic ones.

3.4. One Loop Higher Spin Partition Functions in Flat Space

One loop partition functions often provide very useful insights on the consistency of the spectrum for a possible interacting quantum field theory. On (A)dS backgrounds, this feature has been exploited quite successfully. In three bulk dimensions for example, the comparison between bulk and boundary partition functions [165–167] has been an important ingredient in defining the holographic correspondence between higher spin gauge theories and minimal model CFTs [45]. In spacetime dimensions higher than three, the analysis of one-loop partition functions of infinite sets of higher spin fields provided the first quantum checks [168–172] of analogous AdS/CFT dualities [1].

Since the study of one-loop higher spin partition functions has proven to be quite a fruitful line of research, a natural question to ask is whether or not one can extend such considerations also to higher spin theories in d-dimensional flat space. This venue has been successfully pursued in [173], where one-loop partition functions of (supersymmetric) higher spin fields in d-dimensional thermal flat space with angular potentials $\vec{\theta}$ and inverse temperature β have been computed for the first time using both a heat kernel, as well as a group theoretic approach. Also in this case, the three-dimensional case is of special interest since for $d = 3$, one can explicitly show that suitable products of massless one-loop partition functions:

$$Z[\beta,\vec{\theta}] = e^{\delta_{s,2}\frac{\beta c_M}{24}} \prod_{n=s}^{\infty} \frac{1}{|1-e^{in(\theta+i\epsilon)}|^2}, \quad c_M = 3/G, \tag{64}$$

coincide with vacuum characters of $\mathcal{FW}_N$ algebras:

$$\chi_{\mathcal{FW}_N} = e^{\beta c_M/24} \prod_{s=2}^{N} \left(\prod_{n=s}^{\infty} \frac{1}{|1-e^{in(\theta+i\epsilon)}|^2} \right). \tag{65}$$

3.5. Further Aspects of Higher Spins in 3D Flat Space

As some final remarks regarding higher spins in flat space, we want to describe a little bit more explicitly the content of the two works [174,175].

The first work [174] shows how higher spin symmetries could be used to get rid of the causal singularity in the Milne metric[21] in three dimensions [115,116]. The basic idea here is that one can reformulate the Milne metric equivalently in terms of a Chern–Simons connection and then enlarging the gauge algebra of the Chern–Simons connection from $\mathfrak{isl}(2,\mathbb{R})$ to $\mathfrak{isl}(3,\mathbb{R})$. Requiring that the holonomies of the higher spin connection match those of the original spin-2 connection does place some restrictions on the possible spin-3 extensions of the Chern–Simons gauge field; however, it still leaves enough freedom that can be used to get rid of the causal singularity that is present in the spin-2 case at the level of the Ricci scalar[22] and in addition have a non-singular spin-3 field supporting the geometry.

The second work in this context that we would like to mention explicitly is [175]. One very important ingredient in establishing a (higher-spin) holographic principle in asymptotically flat spacetimes is to find concrete theories that are invariant under the corresponding asymptotic symmetries. For Einstein gravity without a cosmological constant and Barnich–Compère boundary conditions, this would be the $\mathfrak{bms}_3$ algebra, and indeed, for this case, it has been suggested in [181] that a flat limit of Liouville theory would be a suitable candidate[23]. The work [175] extended the previous considerations accordingly to a two-dimensional action invariant under a spin-3 extension of the $\mathfrak{bms}_3$ algebra. The corresponding action can also be obtained as a suitable limit of $\mathfrak{sl}(3,\mathbb{R})$ Toda theory as expected.

4. Non-AdS through the Choice of Gauge Group

After the considerations of the preceding sections, it is natural to ask if there are higher spin theories based on Lie algebras beyond (A)dS and Poincaré. This is of interest because it became clear

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21 See [176] for a work along similar lines, however, for a null-orbifold of flat space [177–180].

22 It should be noted that there is still the possibility that a possible spin-3 generalization of the Ricci scalar is singular. However, there is at the moment no full geometric interpretation of higher spin symmetries that would be necessary in order to check this.

23 See also [131] for a more group theoretic approach to the problem.

that for nonrelativistic holography, also nonrelativistic geometries play a fundamental role; for a review, see, e.g., [182]. In Section 2.3, the focus was on obtaining geometries beyond (A)dS using different backgrounds and boundary conditions than (A)dS, while still working in a theory given by the gauge group of AdS and its higher spin generalizations. In this section (like in Section 3), we are going to make non-(A)dS geometries manifest by changing the gauge algebra.

Since the tools that were used in the derivation of kinematical algebras and their Chern–Simons theories are the same for the spin-2 case and their spin-3 generalization, we will first focus on the former and comment afterwards on the latter.

4.1. Kinematical Algebras

A classification of interesting kinematical algebras, consisting of generators of time and spatial translations $\mathtt{H}$ and $\mathtt{P}_a$[24] , rotations $\mathtt{J}$ and inertial transformations $\mathtt{G}_a$ has been given by Bacry and Levy-Leblond [183]. The classification was provided under the assumptions that:

1. Space is isotropic.
2. Parity and time-reversal are automorphisms of the kinematical groups.
3. Inertial transformations in any given direction form a non-compact subgroup.

This analysis led to other Lie algebras besides the already mentioned (A)dS and Poincaré algebras. Other prominent examples are the Galilei algebra and Carroll algebra and their cousins that appear in the context of spacetimes with non-vanishing cosmological constant. All of them can be conveniently summarized as a cube of İnönü–Wigner contractions[25] starting from the (A)dS algebras; see Figure 1. Since contractions are physically seen as approximations, they often automatically provide insights from the original to the contracted theory.

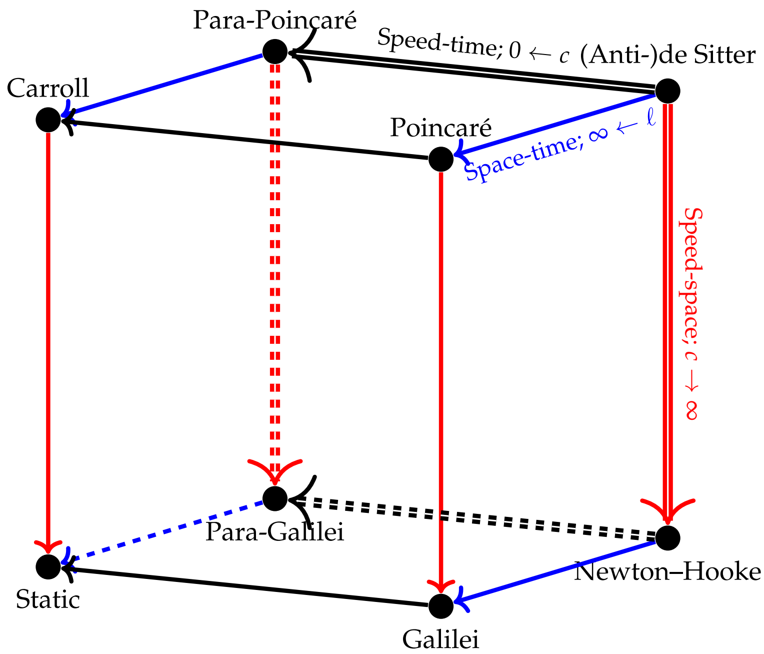

Figure 1. This cube summarizes the kinematical Lie algebras [183]. Each dot represents a kinematical Lie algebra, given explicitly in Appendix A, and each arrow represents an İnönü–Wigner contraction.

---

[24] The indices take now the values $a, b, m = (1, 2)$.

[25] We will use the term İnönü–Wigner contractions here to denote contractions of the form originally defined in [142], sometimes called simple İnönü–Wigner contractions. In contrast to some generalizations like generalized İnönü–Wigner contraction, they are linear in the contraction parameter.

The double arrows mean that there are two contractions since we start with two Lie algebras (AdS and dS). The algebras on the back surface and therefore of finite (A)dS radius ℓ can be considered as cosmological algebras. The top and bottom surfaces can be understood as relative and absolute time Lie algebras, connected by the nonrelativistic limit $c \to \infty$. Sending the speed of light c to zero, the ultrarelativistic limit, leads to absolute space Lie algebras. The parameters of the limits should not be taken too serious since, one of course cannot take the limit of $c \to 0$ and $c \to \infty$ simultaneously (there are actually three parameters involved). They should merely make intuitively clear that the light cone either closes in the ultrarelativistic limit or opens up as for the nonrelativistic one.

To define an İnönü–Wigner contraction, one starts with a Lie algebra $\mathfrak{g}$, which is a vector space direct sum of $\mathfrak{h}$ and $\mathfrak{i}$, i.e., $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{i}$. One then rescales $\mathfrak{i} \mapsto \frac{1}{\epsilon}\mathfrak{i}$. The commutation relations before and after the contraction are then explicitly given by:

$$[\mathfrak{h},\mathfrak{h}] = \quad \mathfrak{h} + \cancel{1/\epsilon}\,\mathfrak{i}, \qquad\qquad [\mathfrak{h},\mathfrak{h}] = \mathfrak{h}, \tag{66}$$

$$[\mathfrak{h},\mathfrak{i}] = \epsilon\mathfrak{h} + \quad \mathfrak{i}, \qquad \xrightarrow{\epsilon\to 0} \qquad [\mathfrak{h},\mathfrak{i}] = \mathfrak{i}, \tag{67}$$

$$[\mathfrak{i},\mathfrak{i}] = \epsilon\mathfrak{h} + \epsilon^2\mathfrak{i}, \qquad\qquad [\mathfrak{i},\mathfrak{i}] = 0\,. \tag{68}$$

The term on the right-hand side of the $[\mathfrak{h},\mathfrak{h}]$ commutator that has been crossed out basically shows that the contraction is convergent in the $\epsilon \to 0$ limes, and therefore well defined, if and only if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ [142]. Specifying $\mathfrak{h}$ completely determines the İnönü–Wigner contraction (up to an isomorphism) [184], and one can therefore enumerate possible contractions by specifying a subalgebra. With this knowledge, we start with the three-dimensional (anti)-de Sitter algebra (the upper sign is the AdS algebra, the lower for dS),

$$[\mathtt{J},\mathtt{G}_a] = \epsilon_{am}\mathtt{G}_m, \qquad [\mathtt{J},\mathtt{P}_a] = \epsilon_{am}\mathtt{P}_m, \tag{69}$$

$$[\mathtt{G}_a,\mathtt{G}_b] = -\epsilon_{ab}\mathtt{J}, \qquad [\mathtt{G}_a,\mathtt{H}] = -\epsilon_{am}\mathtt{P}_m, \tag{70}$$

$$[\mathtt{G}_a,\mathtt{P}_b] = -\epsilon_{ab}\mathtt{H}, \qquad [\mathtt{H},\mathtt{P}_a] = \pm\epsilon_{am}\mathtt{G}_m, \tag{71}$$

$$[\mathtt{P}_a,\mathtt{P}_b] = \mp\epsilon_{ab}\mathtt{J}, \tag{72}$$

and specify the contractions according to Table 1. Consecutive İnönü–Wigner contractions then lead to the cube of Figure 1. For completeness, all the Lie algebras are explicitly given in Table A1 and A2 in Appendix A. For nontrivial contractions, i.e., $\mathfrak{g} \neq \mathfrak{h}$, the resulting algebra is always non-semisimple due to the abelian ideal spanned by $\mathfrak{i}$.

Table 1. The four different İnönü–Wigner contractions classified in [183].

Contraction	$\mathfrak{h}$	$\mathfrak{i}$
Space-time	$\{\mathtt{J},\mathtt{G}_a\}$	$\{\mathtt{H},\mathtt{P}_a\}$
Speed-space	$\{\mathtt{J},\mathtt{H}\}$	$\{\mathtt{G}_a,\mathtt{P}_a\}$
Speed-time	$\{\mathtt{J},\mathtt{P}_a\}$	$\{\mathtt{G}_a,\mathtt{H}\}$
General	$\{\mathtt{J}\}$	$\{\mathtt{H},\mathtt{P}_a,\mathtt{G}_a\}$

4.2. Carroll Gravity

As already discussed in Section 2.1, if one wants to write a Chern–Simons theory for the Lie algebras at hand, it is important for the Lie algebra to admit an invariant metric. While the three-dimensional Carroll algebra automatically admits an invariant metric, others like the Galilei algebra do not. We will discuss in the next section why this is no surprise, but first, we want to show how to construct a Chern–Simons theory with the Carroll algebra and impose boundary conditions (we will follow closely [185], to which we also refer to for more details). Carroll geometries were recently studied because of their relation to asymptotically flat spacetimes [186,187].

The Carroll algebra:

$$[\mathtt{J}, \mathtt{G}_a] = \epsilon_{am}\mathtt{G}_m, \qquad [\mathtt{J}, \mathtt{P}_a] = \epsilon_{am}\mathtt{P}_m, \qquad [\mathtt{G}_a, \mathtt{P}_b] = -\epsilon_{ab}\mathtt{H}, \tag{73}$$

has the invariant metric:

$$\langle \mathtt{H}, \mathtt{J} \rangle = -1, \qquad \langle \mathtt{P}_a, \mathtt{G}_b \rangle = \delta_{ab}\,. \tag{74}$$

The connection:

$$A = \tau\,\mathtt{H} + e^a\,\mathtt{P}_a + \omega\,\mathtt{J} + B^a\,\mathtt{G}_a, \tag{75}$$

is the one-form that takes values in the Carroll algebra, and the action is the usual Chern–Simons action (3). The next step is to construct Brown–Henneaux-like boundary conditions around the Carroll vacuum configuration:

$$e^1_\varphi = \rho, \qquad e^2_\rho = 1, \qquad e^1_\rho = e^2_\varphi = 0, \tag{76}$$

which one can also write as

$$ds^2_{(2)} = e^a e^b \delta_{ab} = \rho^2 d\varphi^2 + d\rho^2\,. \tag{77}$$

We assume that ρ is a radial coordinate and φ is an angular coordinate that is periodically identified by $\varphi \sim \varphi + 2\pi$. Moreover, on the background, the time-component should be fixed as:

$$\tau = dt\,. \tag{78}$$

This can be accomplished by the gauge transformation:

$$A = b^{-1}(\rho)\,\big(d + a(t,\,\varphi)\big)\,b(\rho), \qquad b(\rho) = e^{\rho \mathtt{P}_2}, \tag{79}$$

and using [185]:

$$a_\varphi = -\mathtt{J} + h(t,\,\varphi)\,\mathtt{H} + p_a(t,\,\varphi)\,\mathtt{P}_a + g_a(t,\,\varphi)\,\mathtt{G}_a \tag{80}$$

$$a_t = \mu(t,\,\varphi)\,\mathtt{H}\,. \tag{81}$$

These off-shell boundary conditions lead to:

$$ds^2_{(2)} = \left[\big(\rho + p_1(t,\,\varphi)\big)^2 + p_2(t,\,\varphi)^2\right] d\varphi^2 + 2p_2(t,\,\varphi)\,d\varphi d\rho + d\rho^2, \tag{82}$$

$$= \big(\rho^2 + \mathcal{O}(\rho)\big)\,d\varphi^2 + \mathcal{O}(1)\;\mathrm{d}\rho d\varphi + d\rho^2, \tag{83}$$

and:

$$\tau = \mu(t,\,\varphi)\,dt + \big(h(t,\,\varphi) - \rho\,g_1(t,\,\varphi)\big)\,d\varphi\,. \tag{84}$$

$$= \mu(t,\,\varphi)\,dt + \mathcal{O}(\rho)\,d\varphi\,. \tag{85}$$

The analysis of the asymptotic symmetries leads to conserved, integrable and finite charges, and using a suitable Fourier decomposition, these lead to the asymptotic symmetry algebra:

$$[\mathtt{J}, \mathtt{P}^a_n] = \epsilon_{ab}\, \mathtt{P}^b_n \tag{86}$$
$$[\mathtt{J}, \mathtt{G}^a_n] = \epsilon_{ab}\, \mathtt{G}^b_n \tag{87}$$
$$[\mathtt{P}^a_n, \mathtt{G}^b_m] = -\big(\epsilon_{ab} + in\delta_{ab}\big)\, \mathtt{H}\, \delta_{n+m,0}\,. \tag{88}$$

It is interesting to note that the zero-mode generators $\mathtt{J}$,$\mathtt{H}$, $\mathtt{P}^a_0$ and $\mathtt{G}^a_0$ span a subalgebra that equals again the original Carroll algebra. Thus, this asymptotic symmetry algebra mirrors constructions from three-dimensional asymptotically flat and anti-de Sitter spacetimes. Two further sets of boundary conditions, which can be seen as a limit from AdS, were proposed in [113], one of which we will briefly describe in the following.

Assume again that Carroll gravity can be described by a Chern–Simons action with gauge algebra (73) and invariant metric (74). Choosing a connection like:

$$a_\varphi = \mathcal{K}(t,\varphi)\mathtt{J} + \mathcal{J}(t,\varphi)\mathtt{H} + \mathcal{G}^a(t,\varphi)(\mathtt{G}_a + \mathtt{P}_a) \qquad a_t = \mu(t,\,\varphi)\,\mathtt{H} \tag{89}$$

it is straightforward to determine the asymptotic symmetry algebra that reads in terms of the Fourier modes of the state dependent functions $\mathcal{K}$, $\mathcal{J}$ and $\mathcal{G}^a$:

$$[\mathtt{K}_n, \mathtt{J}_m] = kn\delta_{m+n,0}, \tag{90a}$$
$$[\mathtt{J}_n, \mathtt{G}^a_m] = \epsilon^a{}_b \mathtt{G}^b_{n+m}, \tag{90b}$$
$$[\mathtt{G}^a_n, \mathtt{G}^b_m] = -2\epsilon^{ab}\mathtt{K}_{n+m} - 2nk\,\delta^{ab}\delta_{n+m,0}\,. \tag{90c}$$

One puzzling aspect of these boundary conditions is that they appear to describe solutions that carry entropy despite (seemingly) having no horizon.

The spin-3 Carroll algebras (see Section 4.4), like their spin-2 subalgebras, admit an invariant metric and thus can be written, without obstructions, as a Chern–Simons theory. While they have been analyzed at a linear level, no boundary conditions were proposed so far.

4.3. Invariant Metrics and Double Extensions

The connection between Lie algebras and their invariant metrics has been greatly clarified by Mediny and Revoy [188] (here, we will follow [189]). They proved a structure theorem that explains how all Lie algebras permitting such an invariant nondegenerate symmetric bilinear form[26] can be constructed. This provides a useful guiding principle for the construction of Lie algebras with invariant metrics and, as will be shown later, also explains why the Carroll algebras inherit their invariant metric from the Poincaré algebra.

For that, one first restricts to indecomposable Lie algebras. We call a Lie algebra indecomposable if it cannot be decomposed as an orthogonal direct sum of two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, i.e., it cannot be written in such a way that the two algebras commute $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$ and that they are orthogonal $\langle\mathfrak{g}_1, \mathfrak{g}_2\rangle = 0$. Additionally, one has to define double extensions [188]. The Lie algebra $\mathfrak{d} = D(\mathfrak{g}, \mathfrak{h})$ defined on the vector space direct sum $\mathfrak{g} \oplus \mathfrak{h} \oplus \mathfrak{h}^*$ (spanned by $\mathtt{G}_i$, $\mathtt{H}_\alpha$ and $\mathtt{H}^\alpha$, respectively) by:

---

[26] These Lie algebras are sometimes called symmetric self-dual or quadratic.

$$
\begin{aligned}
[\mathtt{G}_i, \mathtt{G}_j] &= f_{ij}{}^k \mathtt{G}_k + f_{\alpha i}{}^k \Omega^{\mathfrak{g}}_{kj} \mathtt{H}^\alpha, && (91)\\
[\mathtt{H}_\alpha, \mathtt{G}_i] &= f_{\alpha i}{}^j \mathtt{G}_j, && (92)\\
[\mathtt{H}_\alpha, \mathtt{H}_\beta] &= f_{\alpha\beta}{}^\gamma \mathtt{H}_\gamma, && (93)\\
[\mathtt{H}_\alpha, \mathtt{H}^\beta] &= -f_{\alpha\gamma}{}^\beta \mathtt{H}^\gamma, && (94)\\
[\mathtt{H}^\alpha, \mathtt{G}_j] &= 0, && (95)\\
[\mathtt{H}^\alpha, \mathtt{H}^\beta] &= 0, && (96)
\end{aligned}
$$

is a double extension of $\mathfrak{g}$ by $\mathfrak{h}$. It has the invariant metric:

$$
\Omega^{\mathfrak{d}}_{ab} = \begin{array}{c|ccc}
 & \mathtt{G}_j & \mathtt{H}_\beta & \mathtt{H}^\beta \\
\hline
\mathtt{G}_i & \Omega^{\mathfrak{g}}_{ij} & 0 & 0 \\
\mathtt{H}_\alpha & 0 & h_{\alpha\beta} & \delta_\alpha{}^\beta \\
\mathtt{H}^\alpha & 0 & \delta^\alpha{}_\beta & 0
\end{array}, \tag{97}
$$

where $\Omega^{\mathfrak{g}}_{ij}$ is an invariant metric on $\mathfrak{g}$ and $h_{\alpha\beta}$ is some arbitrary (possibly degenerate) symmetric invariant bilinear form on $\mathfrak{h}$.

An example is the Poincaré algebra; see Equation (2) with $\ell \to \infty$. In this case, $\mathfrak{g}$ is trivial, and $\mathfrak{h}$ and $\mathfrak{h}^*$ is spanned by $\mathtt{J}_a$ and $\mathtt{P}_a$, respectively. The invariant metric is then given by (4). Similar considerations apply to the Carroll algebra of Section 4.2. These two algebra are actually related by a natural generalization of the İnönü–Wigner contractions to double extensions; see Section 5.3 in [75]. For that, one needs to apply the "dual contraction" on the dual part of a subspace of $\mathfrak{h}$. Explicitly, this means that one takes the Poincaré algebra (see Table A1) and rescales $\mathtt{G}_a \mapsto \frac{1}{c}\mathtt{G}_a$. Since $\mathtt{P}_a$ is in the dual part, this can be read off of Equation (4). One then applies the dual contraction $\mathtt{P}_a \mapsto c\mathtt{P}_a$. Using these rescalings leads to:

$$
\begin{aligned}
&[\mathtt{J}, \mathtt{G}_a] = \epsilon_{am}\mathtt{G}_m, && [\mathtt{J}, \mathtt{P}_a] = \epsilon_{am}\mathtt{P}_m, && [\mathtt{G}_a, \mathtt{P}_b] = -\epsilon_{ab}\mathtt{H}, && (98)\\
&[\mathtt{G}_a, \mathtt{G}_b] = -c^2\epsilon_{ab}\mathtt{J}, && [\mathtt{G}_a, \mathtt{H}] = -c^2\epsilon_{am}\mathtt{P}_m, && && (99)
\end{aligned}
$$

and therefore to the Carroll algebra for $c \to 0$. The part that makes this new interpretation interesting is that it automatically leaves the invariant metric untouched since $\langle \mathtt{G}_a, \mathtt{P}_b \rangle \mapsto \langle \frac{1}{c}\mathtt{G}_a, c\mathtt{P}_b \rangle = \langle \mathtt{G}_a, \mathtt{P}_b \rangle$. That this is not just a coincidence, but that these non-simple Lie algebras actually have to be a double extension is explained by the following theorem.

Every indecomposable Lie algebra that permits an invariant metric, i.e., every indecomposable symmetric self-dual Lie algebra, is either [188,189]:

1. A simple Lie algebra.
2. A one-dimensional Lie algebra.
3. A double extended Lie algebra $D(\mathfrak{g}, \mathfrak{h})$ where:
   (a) $\mathfrak{g}$ has no factor $\mathfrak{p}$ for which the first and second cohomology group vanishes $H^1(\mathfrak{p}, \mathbb{R}) = H^2(\mathfrak{p}, \mathbb{R}) = 0$. This includes semisimple Lie algebra factors.
   (b) $\mathfrak{h}$ is either simple or one-dimensional.
   (c) $\mathfrak{h}$ acts on $\mathfrak{g}$ via outer derivations.

Since every decomposable Lie algebra can be obtained from the indecomposable ones, this theorem describes how all of them can be generated; see Figure 2 [75].

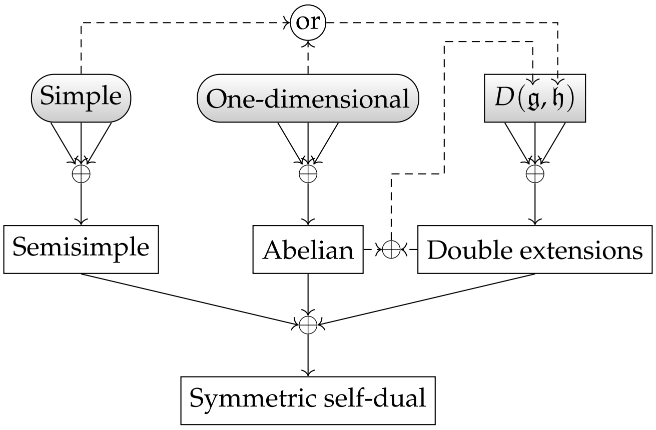

Figure 2. This diagram shows how all Lie algebras with an invariant metric, i.e., symmetric self-dual Lie algebras, are constructed. The fundamental indecomposable building blocks are the simple and the one-dimensional Lie algebras, and they need to be accompanied by the operations of direct sums ($\oplus$) and double extensions ($D(\mathfrak{g}, \mathfrak{h})$). Direct sums of simple and one-dimensional Lie algebras lead to semisimple and abelian ones, respectively. To construct new indecomposable Lie algebras that admit an invariant metric, one needs to double extend an abelian or an already double extended Lie algebra (as explained, it should not have a simple factor).

4.4. Kinematical Higher Spin Algebras

We now want to investigate in which sense the kinematical Lie algebras can be generalized to higher spins; specifically, we will focus on spin-two and three fields. For that, it is again useful to start with the (semi)simple, (A)dS algebras[27] explicitly given in Table 2.

The spin-2 part is a subalgebra and is extended by the spin-3 generators $\mathtt{J}_a, \mathtt{H}_a, \mathtt{G}_{ab}, \mathtt{P}_{ab}$. For the generalization of the contractions to the higher spin algebra, the following restrictions are imposed [185]:

- The İnönü–Wigner contractions are restricted such that the contracted spin-2 Lie subalgebra of the contracted one coincides with the kinematical ones of Bacry and Levy-Leblond [183] (see Table 1 and Appendix A).
- The commutator of the spin-3 fields should be non-vanishing. This ensures that the spin-3 field also interacts with the spin-2 field.

Using these restrictions, one can systematically examine the possible contractions summarized in Table 3 [185][28]. These contractions can then be performed leading to the kinematical higher spin algebras. Consecutive contractions span the (higher spin) cube of Figure 3.

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[27] Semisimple Lie algebras are a natural starting point for these kinds of considerations since no (nontrivial) contraction can lead to a semisimple Lie algebra.

[28] We ignore the traceless contractions in this review.

Table 2. Higher spin versions of the (A)dS algebras. The upper sign is for AdS and the lower sign for dS.

	$\mathfrak{hs}_3(\mathfrak{A})\mathfrak{dS}_{(\mp)}$
Equations (69) to (72)	
$[\mathtt{J}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{J}_m$
$[\mathtt{J}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\mathtt{J}, \mathtt{H}_a]$	$\epsilon_{am}\mathtt{H}_m$
$[\mathtt{J}, \mathtt{P}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{G}_a, \mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$
$[\mathtt{G}_a, \mathtt{G}_{bc}]$	$-\epsilon_{a(b}\mathtt{J}_{c)}$
$[\mathtt{G}_a, \mathtt{H}_b]$	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$
$[\mathtt{G}_a, \mathtt{P}_{bc}]$	$-\epsilon_{a(b}\mathtt{H}_{c)}$
$[\mathtt{H}, \mathtt{J}_a]$	$\epsilon_{am}\mathtt{H}_m$
$[\mathtt{H}, \mathtt{G}_{ab}]$	$-\epsilon_{m(a}\mathtt{P}_{b)m}$
$[\mathtt{H}, \mathtt{H}_a]$	$\pm\epsilon_{am}\mathtt{J}_m$
$[\mathtt{H}, \mathtt{P}_{ab}]$	$\mp\epsilon_{m(a}\mathtt{G}_{b)m}$
$[\mathtt{P}_a, \mathtt{J}_b]$	$-(\epsilon_{am}\mathtt{P}_{bm} + \epsilon_{ab}\mathtt{P}_{mm})$
$[\mathtt{P}_a, \mathtt{G}_{bc}]$	$-\epsilon_{a(b}\mathtt{H}_{c)}$
$[\mathtt{P}_a, \mathtt{H}_b]$	$\mp(\epsilon_{am}\mathtt{G}_{bm} + \epsilon_{ab}\mathtt{G}_{mm})$
$[\mathtt{P}_a, \mathtt{P}_{bc}]$	$\mp\epsilon_{a(b}\mathtt{J}_{c)}$
$[\mathtt{J}_a, \mathtt{J}_b]$	$\epsilon_{ab}\mathtt{J}$
$[\mathtt{J}_a, \mathtt{G}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\mathtt{J}_a, \mathtt{H}_b]$	$\epsilon_{ab}\mathtt{H}$
$[\mathtt{J}_a, \mathtt{P}_{bc}]$	$\delta_{a(b}\epsilon_{c)m}\mathtt{P}_m$
$[\mathtt{G}_{ab}, \mathtt{G}_{cd}]$	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{J}$
$[\mathtt{G}_{ab}, \mathtt{H}_c]$	$-\delta_{c(a}\epsilon_{b)m}\mathtt{P}_m$
$[\mathtt{G}_{ab}, \mathtt{P}_{cd}]$	$\delta_{(a(c}\epsilon_{d)b)}\mathtt{H}$
$[\mathtt{H}_a, \mathtt{H}_b]$	$\pm\epsilon_{ab}\mathtt{J}$
$[\mathtt{H}_a, \mathtt{P}_{bc}]$	$\pm\delta_{a(b}\epsilon_{c)m}\mathtt{G}_m$
$[\mathtt{P}_{ab}, \mathtt{P}_{cd}]$	$\pm\delta_{(a(c}\epsilon_{d)b)}\mathtt{J}$

Table 3. The contractions to the kinematical higher spin algebras. They can be summarized again as a (higher spin) cube; see Figure 3.

Contraction	**#**	$\mathfrak{h}$	$\mathfrak{i}$
Space-time	1	$\{\mathtt{J}, \mathtt{G}_a, \mathtt{J}_a, \mathtt{G}_{ab}\}$	$\{\mathtt{H}, \mathtt{P}_a, \mathtt{H}_a, \mathtt{P}_{ab}\}$
	2	$\{\mathtt{J}, \mathtt{G}_a, \mathtt{H}_a, \mathtt{P}_{ab}\}$	$\{\mathtt{H}, \mathtt{P}_a, \mathtt{J}_a, \mathtt{G}_{ab}\}$
Speed-space	3	$\{\mathtt{J}, \mathtt{H}, \mathtt{J}_a, \mathtt{H}_a\}$	$\{\mathtt{G}_a, \mathtt{P}_a, \mathtt{G}_{ab}, \mathtt{P}_{ab}\}$
	4	$\{\mathtt{J}, \mathtt{H}, \mathtt{G}_{ab}, \mathtt{P}_{ab}\}$	$\{\mathtt{G}_a, \mathtt{P}_a, \mathtt{J}_a, \mathtt{H}_a\}$
Speed-time	5	$\{\mathtt{J}, \mathtt{P}_a, \mathtt{J}_a, \mathtt{P}_{ab}\}$	$\{\mathtt{G}_a, \mathtt{H}, \mathtt{H}_a, \mathtt{G}_{ab}\}$
	6	$\{\mathtt{J}, \mathtt{P}_a, \mathtt{H}_a, \mathtt{G}_{ab}\}$	$\{\mathtt{G}_a, \mathtt{H}, \mathtt{J}_a, \mathtt{P}_{ab}\}$
General	7	$\{\mathtt{J}, \mathtt{J}_a\}$	$\{\mathtt{H}, \mathtt{P}_a, \mathtt{G}_a, \mathtt{H}_a, \mathtt{G}_{ab}, \mathtt{P}_{ab}\}$
	8	$\{\mathtt{J}, \mathtt{G}_{ab}\}$	$\{\mathtt{H}, \mathtt{P}_a, \mathtt{G}_a, \mathtt{J}_a, \mathtt{H}_a, \mathtt{P}_{ab}\}$
	9	$\{\mathtt{J}, \mathtt{H}_a\}$	$\{\mathtt{H}, \mathtt{P}_a, \mathtt{G}_a, \mathtt{J}_a, \mathtt{G}_{ab}, \mathtt{P}_{ab}\}$
	10	$\{\mathtt{J}, \mathtt{P}_{ab}\}$	$\{\mathtt{H}, \mathtt{P}_a, \mathtt{G}_a, \mathtt{J}_a, \mathtt{H}_a, \mathtt{G}_{ab}\}$

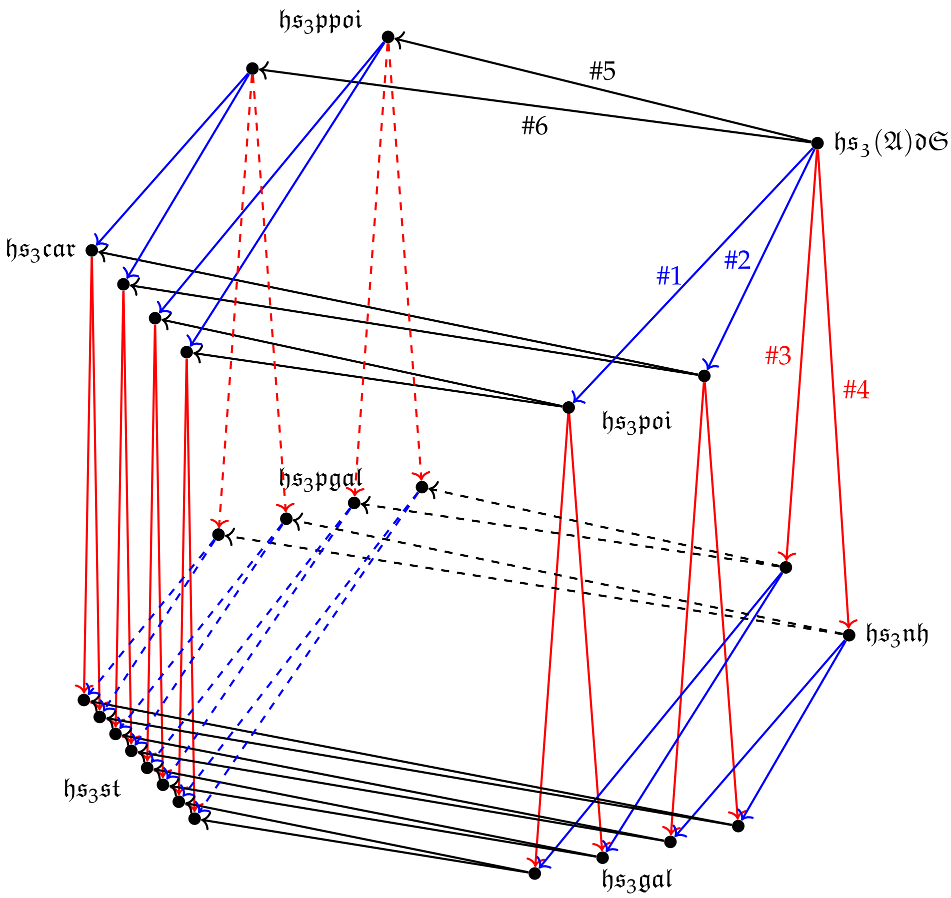

Figure 3. This figure [185] summarizes the contractions of Table 3. There are 2 space-time (blue; #1,#2), 2 speed-space (red; #3,#4) and 2 speed-time (black; #5,#6) contractions, and combining them leads to the full cube. The explicit commutators of all algebras can be found in the Appendix of [185]. In comparison to Figure 1, we have for clarity omitted the double lines.

Given the zoo of higher spin algebras, each corner of the higher spin cube representing one, the question arises if they permit an invariant metric. For the (semi)simple (A)dS algebras, this is obvious, but not so much for the other ones. For the case of the higher spin Poincaré algebras, the considerations of Section 2 generalize, and for the higher spin Carroll algebras, there are again invariant metric preserving contractions [75] analogous to the ones discussed in Section 4.4.

For the Galilei algebras, the situation is different, and the knowledge of double extensions proves to be useful. Already in the case of spin-2, the three-dimensional Galilei algebra has no invariant metric [190]. However, it is possible to centrally extend the Galilei algebra by two nontrivial central extensions (out of three possible ones [191]) to obtain a Lie algebra with an invariant metric. One of these central extensions is possible in any dimension and corresponds to the mass in the so-called Bargmann algebra. Due to the second extension that is a peculiarity of three spacetime dimensions, this algebra is called extended Bargmann algebra and makes a Chern–Simons formulation possible [190].

The higher spin Galilei generalizations also do not permit any invariant metric [185]. In contrast to the spin-2 case, central extensions are not sufficient to provide an invariant metric, but double extensions provide guidance. Interestingly, the double extension of the spin-3 Galilei algebras leads naturally to Lie algebras where the spin-2 part is exactly the just mentioned extended Bargmann algebra. Therefore, the higher spin generalization of the Galilei algebra can be considered the spin-3 extended Bargmann algebra. Furthermore, the properties of the higher spin Carroll and extended Bargmann theories have been studied in detail [185].

5. Conclusions and Outlook

Three-dimensional higher spin theories beyond (anti)-de Sitter can be roughly separated by the gauge algebra that is used in their Chern–Simons formulation. As reviewed in Section 2, using the higher spin (A)dS gauge algebras, one is able to construct backgrounds and boundary conditions for Lobachevsky, Lifshitz, null warped and Schrödinger spacetimes. Going beyond the (A)dS gauge algebra, the best understood cases so far are flat space higher spin theories that were discussed in Section 3. The interesting question if there are higher spin algebras beyond these cases has been answered by the construction of the kinematical higher spin algebras reviewed in Section 4.

While some progress has been made up until now, there are certainly interesting open problems that demand further investigation.

5.1. Boundary Conditions and Boundary Theories

While it was shown that AdS higher spin gauge algebras permit backgrounds and boundary conditions beyond the standard AdS choices, their asymptotic symmetry algebras often turned out to be related to already known ones. It would be interesting to further investigate if these boundary conditions and their asymptotic symmetry algebras can be further specialized in order to yield, e.g., Lifshitz-like asymptotic symmetries.

While for the Carroll case, boundary conditions have been proposed [113,185], for most of the other kinematical algebras, and especially the higher spin generalizations, no consistent boundary conditions have been established yet. Further examination is also needed for the mysterious result that it seems that one can assign entropy to Carroll geometries [113]. It would be interesting to see if this result can be generalized to the higher spin case. A generalization and interpretation of higher spin entanglement entropy [50,51] in these setups would be another intriguing option.

Another interesting generalization would be the calculation of one loop partition functions. Here, Newton–Hooke and Para-Poincaré seem to be intriguing options. This is due to the still non-vanishing cosmological constant, and one might therefore hope that they exhibit the "box-like" behavior of AdS.

For the higher spin cases, it would be interesting to see if the asymptotic symmetry algebras lead to nonlinear generalizations similar to the $\mathcal{W}_N$ algebras for AdS (some of them might be related to the ones discussed in [192]).

One interesting observation is related to possible dual theories of the Chern–Simons theories treated in this review to the Wess–Zumino–Novikov–Witten (WZNW) models [193,194]. Here, again, WZNW models based on a Lie algebra that admits an invariant metric play a distinguished role since they admit a (generalized) Sugawara construction [195]. Double extensions and the Medina–Revoy theorem are fundamental for the proof that the Sugawara construction factorizes in a semisimple and a non-semisimple one [196].

5.2. Kinematical (Higher Spin) Algebras

For the spin-2 extended Bargmann algebras, it was shown that they emerge as contractions of (anti)-de Sitter algebras that have been (trivially) centrally extended by two one-dimensional algebras [190][29]. It is also not clear as of yet which (semisimple) algebra can be naturally contracted to yield the double extended higher spin versions of the Bargmann algebra. This is interesting, because the deformed theories are often seen as more fundamental. We are not aware of a systematic discussion of contractions and double extensions; see however Section 5 in [75] for a start. Furthermore, it might be interesting to also look at the Chern–Simons theories based on the Lie algebras that have not been double extended.

[29] See also Section 9.2 in [75].

For many considerations, a generalization to the supersymmetric case seems possible; especially since the supersymmetric analog of double extensions exists [197] and an analog of the structure theorem of Section 4.3 has been proposed [198]. A Chern–Simons theory based on a supersymmetric version of the extended Bargmann algebra has already been investigated in [199].

Since the Chern–Simons theory based on the extended Bargmann algebra has been shown to be related to a specific version of Hořava-Lifshitz gravity [200], it would be interesting to see if the higher spin extended Bargmann theories lead to a spin-3 Hořava-Lifshitz theory.

Last, but not least, two obvious generalizations are to higher spins ($s > 3$), as well as to higher dimensions.

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Appendix A. Explicit Kinematical Algebra Relations

Table A1. (Anti-)de Sitter, Poincaré, Newton–Hooke and para-Poincaré algebras. The upper sign is for AdS (and contractions thereof) and the lower sign for dS (and contractions thereof).

	$(\mathfrak{A})\mathfrak{dS}_{(\mp)}$	$\mathfrak{poi}$	$\mathfrak{nh}$	$\mathfrak{ppoi}$
$[\mathtt{J},\mathtt{J}]$	0	0	0	0
$[\mathtt{J},\mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\mathtt{J},\mathtt{H}]$	0	0	0	0
$[\mathtt{J},\mathtt{P}_a]$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a,\mathtt{G}_b]$	$-\epsilon_{ab}\mathtt{J}$	$-\epsilon_{ab}\mathtt{J}$	0	0
$[\mathtt{G}_a,\mathtt{H}]$	$-\epsilon_{am}\mathtt{P}_m$	$-\epsilon_{am}\mathtt{P}_m$	$-\epsilon_{am}\mathtt{P}_m$	0
$[\mathtt{G}_a,\mathtt{P}_b]$	$-\epsilon_{ab}\mathtt{H}$	$-\epsilon_{ab}\mathtt{H}$	0	$-\epsilon_{ab}\mathtt{H}$
$[\mathtt{H},\mathtt{P}_a]$	$\pm\epsilon_{am}\mathtt{G}_m$	0	$\pm\epsilon_{am}\mathtt{G}_m$	$\pm\epsilon_{am}\mathtt{G}_m$
$[\mathtt{P}_a,\mathtt{P}_b]$	$\mp\epsilon_{ab}\mathtt{J}$	0	0	$\mp\epsilon_{ab}\mathtt{J}$

Table A2. Carroll, Galilei, para-Galilei and static algebra. The upper sign is for AdS (and contractions thereof) and the lower sign for dS (and contractions thereof).

	$\mathfrak{car}$	$\mathfrak{gal}$	$\mathfrak{pgal}$	$\mathfrak{st}$
$[\mathtt{J},\mathtt{J}]$	0	0	0	0
$[\mathtt{J},\mathtt{G}_a]$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$	$\epsilon_{am}\mathtt{G}_m$
$[\mathtt{J},\mathtt{H}]$	0	0	0	0
$[\mathtt{J},\mathtt{P}_a]$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$	$\epsilon_{am}\mathtt{P}_m$
$[\mathtt{G}_a,\mathtt{G}_b]$	0	0	0	0
$[\mathtt{G}_a,\mathtt{H}]$	0	$-\epsilon_{am}\mathtt{P}_m$	0	0
$[\mathtt{G}_a,\mathtt{P}_b]$	$-\epsilon_{ab}\mathtt{H}$	0	0	0
$[\mathtt{H},\mathtt{P}_a]$	0	0	$\pm\epsilon_{am}\mathtt{G}_m$	0
$[\mathtt{P}_a,\mathtt{P}_b]$	0	0	0	0

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