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*Article*

# Plea for Diagonals and Telescopers of Rational Functions

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**Abstract:** This paper is a plea for diagonals and telescopers of rational or algebraic functions using creative telescoping, using a computer algebra experimental mathematics learn-by-examples approach. We show that diagonals of rational functions (and this is also the case with diagonals of algebraic functions) are left-invariant when one performs an infinite set of birational transformations on the rational functions. These invariance results generalize to telescopers. We cast light on the almost systematic property of homomorphism to their adjoint of the telescopers of rational or algebraic functions. We shed some light on the reason why the telescopers, annihilating the diagonals of rational functions of the form $P/Q^k$ and $1/Q$, are homomorphic. For telescopers with solutions (periods) corresponding to integration over non-vanishing cycles, we have a slight generalization of this result. We introduce some challenging examples of the generalization of diagonals of rational functions, like diagonals of transcendental functions, yielding simple ${}_2F_1$ hypergeometric functions associated with elliptic curves, or the (differentially algebraic) lambda-extension of correlation of the Ising model.

**Keywords:** diagonals of rational and algebraic functions; creative telescoping; globally bounded series; modular forms; multi-Taylor expansions; multivariate series expansions; magnetic susceptibility of the Ising model; lattice Green functions; fuchsian linear differential equations; homomorphisms of differential operators; self-adjoint operators; Poincaré duality; differential Galois groups

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## 1. Introduction: Plea for a Computer Algebra Experimental Mathematics Learn by Example Approach

A paper in honor of Professor Richard Kerner must be a paper on theoretical physics, mathematical physics, physical mathematics, applied mathematics, applicable mathematics or even experimental mathematics [1]. These different domains have large overlaps and, quite often, their differences or shades are slightly irrelevant, only corresponding to social membership to different "mathematical tribes". This computer algebra paper will actually be a plea for *diagonals and telescopers of rational (or algebraic) functions* and for *creative telescoping*, with a computer algebra experimental mathematics learn-by-examples approach.

### 1.1. Honor, Pride, and Prejudice

The "Journal of Mathematical Physics" defines mathematical physics as "the application of mathematics to problems in physics and the development of mathematical methods suitable for such applications and for the formulation of physical theories". An alternative definition would also include those mathematics that are inspired by physics (also known as physical mathematics). Mathematical physics clearly raises the question of the watershed between mathematics and physics (especially in France…). Does "Mirror Symmetry" [2–5], which is a relationship between geometric objects called Calabi–Yau manifolds, belong to algebraic geometry or theoretical physics? Does "Special Relativity" belong to physics or mathematics, Einstein or Poincaré? "Einstein was reluctant to acknowledge that the Michelson-Morley experiment had a significant influence on his road to special relativity" [6]. In fact, "once Maxwell's equations are properly understood mathematically, special relativity is an inevitable consequence" [6]. Physical mathematics is sometimes viewed with suspicion by both physicists *and* mathematicians. On the one hand, mathematicians regard it as deficient, for lack of proper mathematical rigor. In the years since this "mathematical physics debate" erupted [7], there have been many spectacular successes scored by physical mathematics, thanks to the "unreasonable effectiveness" of physics in the mathematical sciences. Dyson famously proclaimed: "As a working physicist, I am actualy aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce". This "divorce" is particularly serious in France because of the overwhelmingly leading figure of Alexander Grothendieck and the huge influence of the Bourbaki group, which raises the question of rigor versus creativity ("We should not confuse rigor with rigor mortis", Isadore Singer, see [6]). Recalling Pierre Cartier [8], the Bourbaki group has been criticized by several mathematicians, including its own former members, for a variety of reasons. " Criticisms have included the choice of presentation of certain topics within the Éléments [9] at the expense of others, *dislike of the method of presentation* for given topics, dislike of the group's working style, and a perceived elitist mentality around Bourbaki's project and its books, especially during the collective's most productive years in the 1950s and 1960s. There is essentially *no analysis* beyond the foundations: *nothing about partial differential equations*, nothing about probability. There is also *nothing about combinatorics*, nothing about algebraic topology [10], *nothing about concrete geometry*. Anything connected with *mathematical physics is totally absent* from Bourbaki's text." Dieudonné (founding member), later, regretted that Bourbaki's success had contributed to snobbery regarding pure mathematics in France, *at the expense of applied mathematics* [11,12]. In an interview (to Marian Schmidt in 1990), he said: "It is possible to say that there was no serious applied mathematics in France for forty years after Poincaré. There was even a snobbery for pure mathematics. When one noticed a talented student, one would tell him "You should do pure math". On the other hand, one would advise a mediocre student to do applied mathematics while thinking, "It's all that he can do! ". Apart from french mathematicians (when in doubt, blame the French), this snobbery regarding pure mathematics met with harsh criticism from Vladimir Arnold in his deliciously polemical paper [13] "Sur l'éducation mathématique".

Quantum groups emerged from one (Yang–Baxter integrable) explicit example, namely, Quantum Toda, and *not* from an ex-nihilo abstract, formal construction of a noncommutative algebra formalism, and other $C^\star$-algebras, dressed with coassociative coproducts. In theoretical physics, we get used to the emergence of *modular forms* [14] and sometimes automorphic forms [15] like *Shimura forms* [16]. If a physicist asks a mathematician for more information on these structures he will probably only receive the academical Poincaré upper half-plane definition and formalism, which will be totally and utterly useless to him, and he will not recognize the representation of *modular forms* and *Shimura forms*, which naturally emerges in physics [16,17] in terms of pullbacked ${}_2F_1$ hypergeometric functions. In theoretical physics, we are flooded by *elliptic curves*, K3 surfaces, and Calabi–Yau manifolds [3,18–23]. If a physicist tries to discuss with a mathematician the elliptic curve he has just discovered (when he has even calculated the j-invariant, or the Hauptmodul, of this elliptic curve . . . ), he might be severely rebuked that he has absolutely no right to talk of an elliptic curve because an elliptic curve must have a "specified point", or will be seen with suspicion because his elliptic curve does not correspond to the complete intersection of quadrics [24] framework mathematicians like to consider in their theorems. Along this (slightly polemical . . . ) line, pure mathematicians will, often, refuse to provide a *representation* of their formalism; in particular, they will refuse to provide *examples*. If a physicist, eager to understand a mathematical concept, asks for an example of an algebraic variety, an example of a holonomic function, or an example of functor, some mathematicians will, maliciously, reply: a point, the constant function and the oblivion functor. In such a frustrating "dialogue of the deaf" between physicists and mathematicians, mathematical physics is probably the perfect for being criticized by physicists as too abstract, or too mathematical, and also by mathematicians for a lack of rigor and a lack of mathematical proofs.

Jean-Louis Verdier performed his thesis under the direction of Alexandre Grothendieck. He was a member of the Bourbaki group. He passed away in August 1989. At this stage, one of us (JMM) would like to seize the opportunity of this experimental mathematics paper in honor of Professor Richard Kerner, to express his deep regrets for the numerous fruitful conversations with Jean-Louis Verdier and his very generous pedagogical explanations. A discussion with him was not flooded with "Derived Categories" or "p-adic cohomology" but with simple examples and representations of the mathematical concepts. A really good mathematician can provide examples; he is not afraid, or ashamed, to provide examples and representations. For Jean-Louis Verdier, mathematics was not an obfuscation contest.

This paper is an experimental mathematics [1] paper with a learn-by-example approach: we obtain puzzling *exact* results from computer algebra (Maple, Mathematica), and we hope mathematicians will be interested to provide proofs of these results, in a proper framework. Furthermore, these exact results, which are useful for physics, raise a lot of fascinating new questions at the crossroad of different domains of mathematics.

### *1.2. Diagonal of Rational Functions, Creative Telescoping, Birational Transformations, and Effective Algebtraic Geometry*

Diagonals of rational functions (or diagonals of algebraic functions) have been shown to emerge naturally [25] for $n$-fold integrals in physics (corresponding to solutions of linear differential operators of quite high order [26]); field theory; and enumerative combinatorics [27,28], and have been seen as "Periods" [29–31] of algebraic varieties (corresponding to the denominators of these rational functions). The fact that diagonals of rational or algebraic functions occur frequently in physics explains many unexpected mathematical properties encountered in physics that are far more obvious from a physics viewpoint. Physicists are clearly very interested to see if the critical exponents of the *three-dimensional* Ising model are or are not *rational* numbers. In contrast, since many lattice Green functions in any dimension [32] are diagonals of rational functions, their critical exponents are necessarily rational numbers in *any* dimension. Accordingly, the linear differential operators, annihilating these "Periods", are *globally nilpotent* [33], and, consequently, the *critical exponents* of *all* the (regular) singular points of these operators are necessarily *rational numbers* (Katz theorem states that *globally nilpotent* linear differential operators are *fuchsian* with *rational exponents*; see, for instance, [34]). These $n$-fold integrals are also a *globally bounded* [25,35] series, which means that they can be recast into a (finite radius of convergence) series with *integer coefficients*. Furthermore, these series, with integer coefficients, reduce *modulo every prime* to *algebraic functions*. The calculation of the linear differential operators annihilating these $n$-fold integrals of algebraic functions can be systematically performed using the *creative telescoping method* [36–38], which corresponds, essentially, to successive *differential algebra eliminations, which are blind to the cycles* over which one performs the $n$-fold integrals. At first sight one expects the analysis of these $n$-fold integrals to require, as in the S-matrix theory [39], a lot of complex analysis of *several complex variables*, but one quickly discovers, with creative telescoping, that one needs *differential algebra*, possibly *algebraic geometry* [40], because of the crucial role of an algebraic variety, and, surprisingly, one finds out almost "arithmetical" properties (like in the Grothendieck–Katz p-curvature conjecture, which is a local–global principle for linear ordinary differential equations, related to differential Galois theory). More experimentally, this time, one finds out that *almost all* the diagonals of rational or algebraic functions, corresponding or not to physics, are annihilated by linear differential operators that are *homomorphic to their adjoint*, and, consequently, their differential Galois groups are (or are a subgroup of) selected $Sp(n, \mathbb{C})$ symplectic or $SO(n, \mathbb{C})$ orthogonal groups [41,42]. More generally, one finds out that the *telescopers of almost all* the rational or algebraic functions are *also* homomorphic to their adjoint [41]. A physicist, already surprised to see the emergence of all these mathematical concepts in his backyard, will have the prejudice that these *selected differential Galois groups* are probably a consequence of some "sampling bias", these diagonals and telescopers being, in fact, related to (Yang–Baxter) integrable models, like the $\chi^{(n)}$ components of the susceptibility of the Ising model [26], or beyond, Calabi–Yau manifolds, Mirror Symmetries, Picard–Fuchs systems, and other theory "integrable" in some way (Yang-Mills ... ). In contrast, a mathematician will have the prejudice that this is nothing but the *Poincaré duality* [43] since we have a canonical algebraic variety for all these diagonals or telescopers [40]. Experimentally, it is quite hard to find telescopers or linear differential operators that are not homomorphic to their adjoint, i.e., that do not have selected symplectic, or orthogonal, differential Galois groups [41,42]. Christol conjectured [44,45] that every D-finite globally bounded series is the diagonal of a rational function. If one considers Christol's conjecture [44–48], one can seek for ${}_nF_{n-1}$ hypergeometric series with integer coefficients that are candidates to be counter-examples to Christol's conjecture [44–47]. Among these candidates a sub-set has actually been seen [48] to be diagonals of rational or algebraic functions like ${}_3F_2([2/9, 5/9, 8/9], [2/3, 1], x)$ or ${}_3F_2([1/9, 4/9, 7/9], [1/3, 1], x)$. The fact that the others, like the original example of G. Christol, ${}_3F_2([1/9, 4/9, 5/9], [1/3, 1], 3^6\, x)$, are or are not diagonals of rational or algebraic functions remains an open question. It turns out that the linear differential operators of these ${}_nF_{n-1}$ candidates precisely provide such rare examples of linear differential operators (annihilating diagonals of rational or algebraic functions) that are *not* homomorphic to their adjoint. The existence of such examples (curiously related to Christol's conjecture ... ) shows that seeing the emergence of such selected differential Galois groups [41] for diagonals of rational or algebraic functions cannot simply be seen as some consequence of the *Poincaré duality*. The Poincaré duality works for *any* algebraic variety: the diagonal of *any* rational or algebraic function should always yield "self-dual" linear differential operators in the sense that they are homomorphic to their adjoint. This is *not* the case. Could it be that the physicist's prejudice is right and that, trying to be generic in our computer algebra experiments, we were, in fact, just exploring diagonals of selected subsets of rational or algebraic functions related to some kind of "integrable" physics?

Like Monsieur Jourdain (in "Le Bourgeois Gentilhomme", Molière) speaking "prose" without noticing himself, physicists often perform some fundamental mathematics when they work on their $n$-fold integrals without noticing these $n$-fold integrals are, in fact, diagonals of rational or algebraic functions. In fact, diagonals of rational or algebraic functions, and more generally telescopers, are a perfect subject of analysis in mathematical physics: they are, essentially, not well-known by mathematicians and by physicists (even if physicists speak "diagonal" without noticing ... ), and even when these concepts are superficially known, they are not taken seriously by mathematicians, probably because the definition is so simple, and the calculations are just "computer algebra", which is not highly regarded in the "mathematical food chain". This is in contrast to the fact that almost every calculation of a diagonal of a rational or algebraic function or of the calculation of a telescoper yields interesting, remarkable, and sometimes even puzzling exact results, providing answers in physics and mathematics but also raising new interesting questions that could be called "speculative mathematics".

In a learn-by-example approach, we are going to address the previous questions of "duality-breaking" of some telescopers of rational or algebraic functions, and we will also sketch some remarkable *birational symmetries* [24,49] of the diagonals and telescopers of rational or algebraic functions.

## 2. Definition of the Diagonals of Rational or Algebraic functions: Definition of Telescopers

The purpose of this paper is not to provide an introduction to *creative telescoping* [36–38,50–53]. The purpose of this paper is, rather, to provide many (non-trivial) pedagogical examples of "telescopers" by extensively using Chyzak's algorithm [53] or Koutschan's semi-algorithm [54] "HolonomicFunctions" package [54]. Koutschan's package [54] corresponds to a semi-algorithm because the termination is not proven. For the examples displayed in this paper, Koutschan's package [54] turns out to be more user-friendly and also more efficient.

Creative telescoping [36–38,50–53] has become popular in computer algebra in the last twenty years. It is a methodology to deal with (parametrized) $n$-fold integrals, or with symbolic sums, yielding linear differential/recurrence equations. By the "telescoper" of a rational function, say $R(x,y,z)$, we here refer to one of the output of the creative telescoping program [54] (the other outputs being the so-called "certificates"). The telescoper $T$ is a linear differential operator that is satisfied by the diagonal $Diag(R)$, as well as other solutions. These other solutions correspond to "periods" [29–31] of algebraic varieties over *non-vanishing cycles* [55].

The reader interested in the connection between all these notions can read the thesis of Pierre Lairez [56] (see also [31]).

### 2.1. Definition

The diagonal of the rational function $R$ dependent on (for example) three variables is obtained by expanding $R$ around the origin

$$R(x,y,z) \;=\; \sum_m \sum_n \sum_l a_{m,n,l} \cdot x^m\, y^n\, z^l, \tag{1}$$

and keeping only the terms such that $m = n = l$. The diagonal reads, with $p = xyz$,

$$Diag\Big(R(x,y,z)\Big) \;=\; \sum_m a_{m,m,m} \cdot p^m. \tag{2}$$

In order to avoid a proliferation of variables, the variable $p$, which the diagonal (2) depends on, is, in the following, simply denoted as $x$ (see below (3)). Extracting these diagonal terms essentially amounts to finding constant terms [57] in several complex variable expansions, i.e., it amounts to performing a residue calculation in several complex variable expansions

$$\begin{aligned} Diag\Big(R(x,y,z)\Big) \;&=\; \int_{\mathcal{C}} \frac{1}{y\,z} \cdot R\Big(\frac{x}{y\,z},\, y,\, z\Big) \cdot dy\; dz \\ &=\; \frac{1}{2\,i\,\pi}\int \frac{1}{2\,i\,\pi}\int \; \sum_m \sum_n \sum_l a_{m,n,l} \cdot x^m\, y^{n-m}\, z^{l-m} \cdot \frac{dy}{y}\,\frac{dz}{z} \;=\; \sum_m a_{m,m,m} \cdot x^m, \end{aligned} \tag{3}$$

or equivalently

$$Diag\Big(R(x,y,z)\Big) \;=\; \int_{\mathcal{C}} \frac{1}{y\,z} \cdot R\Big(\frac{x}{y},\, \frac{y}{z},\, z\Big) \cdot dy\, dz, \tag{4}$$

where $\mathcal{C}$ denotes a vanishing cycle [55], where $\int_{\mathcal{C}}$ is a symbolic notation for the $n$-fold integral with the well-suited pre-factors and where the diagonal (4) is seen as a function of the remaining variable $x$. This is the very reason why diagonals of rational or algebraic functions can be interpreted as $n$-fold integrals [25]. More generally, with $n$ variables, one can write the diagonal of a rational function of $n$-variables as the residue in $n-1$ variables $x_2, \, \cdots, \, x_n$:

$$\begin{aligned} &Diag\Big(R(x_1,\, x_2,\, \cdots,\, x_n)\Big) \\ &\quad=\; \frac{1}{2\,i\,\pi}\int \cdots \frac{1}{2\,i\,\pi}\int \frac{1}{x_2\,\cdots\,x_n} \cdot R\Big(\frac{x_1}{x_2\;\cdots\;x_n},\, x_2,\, \cdots,\, x_n\Big) \cdot dx_2\,\cdots\,dx_n. \end{aligned} \tag{5}$$

If the definition of the diagonal of a rational or algebraic function is very simple, it *does not mean* that calculating such a diagonal is simple. By "calculating", we mean finding that the series, corresponding to the diagonal, is the series expansion of some known special function [58–61] (an algebraic function [62], a pullbacked ${}_2F_1$ hypergeometric function that turns out to be a *modular form* [16,63], a ${}_nF_{n-1}$ hypergeometric function, a Heun function [64], etc.). Most of the time, this involves, since diagonals of rational or algebraic functions are selected (Fuchsian [26], G-nilpotent operators, globally bounded series [35]) D-finite functions, finding the linear differential operator annihilating the diagonal series, even if we are not able to "solve" this linear differential equation. Finding this linear differential operator can be performed by first obtaining the large series expansion of the diagonal and then finding, by a "guessing" approach, the linear differential operator or obtaining the linear differential operator from a more global differential algebra approach, called creative telescoping.

### 2.2. *Telescopers*

For pedagogical reasons, let us sketch creative telescoping [36–38,50–53] in the case of a rational function of three variables. The "telescoper" of a rational function, say $R(x,y,z)$, applied not to the rational function $R(x,y,z)$ but to the transformed rational function $\hat{R} \, = \, R(x/y,y/z,z)/(yz)$, is a linear differential operator $T$ in $x$ and $\partial x$, such that

$$T \cdot \Big( \frac{1}{y\,z} \cdot R\Big(\frac{x}{y}, \frac{y}{z}, z\Big)\Big) \,\, + \frac{\partial U}{\partial y} + \frac{\partial V}{\partial z} \,\, = \,\, 0, \tag{6}$$

where $U$, $V$ are rational functions in $x$, $y$, $z$ called "certificates". These rational functions are often quite large rational functions. This equation is called the *telescoping equation*. Extracting the diagonal of a rational function amounts to calculating residues in several complex variables, namely,

$$Diag\Big(R(x,\, y,\, z)\Big) \,\, = \,\, \int_{\mathcal{C}} \frac{1}{y\,z} \cdot R\Big(\frac{x}{y}, \frac{y}{z}, z\Big), \tag{7}$$

where the cycle $\mathcal{C}$ is a vanishing cycle [55]. By performing the previous integration over a cycle $\mathcal{C}$ on the LHS of the telescoping Equation (6), one will obtain (with the reasonable assumption that the linear differential operator $T$ commutes with the integration)

$$T \cdot \, Diag\Big(R(x,\, y,\, z)\Big) \,\, + \int_{\mathcal{C}} \Big(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial z}\Big) \,\, = \,\, 0. \tag{8}$$

Again, (with reasonable assumptions) one can expect the second term in (8) to be equal to zero, thus yielding the equation

$$T \cdot \, Diag\Big(R(x,\, y,\, z)\Big) \,\, = \,\, T \cdot \int_{\mathcal{C}} \frac{1}{y\,z} \cdot R\Big(\frac{x}{y}, \frac{y}{z}, z\Big) \,\, = \,\, 0. \tag{9}$$

In other words, the telescoper $T$ represents a linear differential operator annihilating the diagonal $Diag(R)$. For the calculation of a diagonal, the cycle $\mathcal{C}$ has to be a *vanishing cycle* (residue calculation). Note that the creative telescoping calculations giving as an output the telescoper $T$ and the two "certificates" $U$ and $V$ essentially amount to performing differential algebra calculations (similar to integration by part for several complex variables). Since these creative telescoping calculations are differential algebra eliminations, *they are totally and utterly blind to the cycle* $\mathcal{C}$. Consequently, even if one performs an integration over a *non-vanishing cycle*, the telescoper $T$ will also be such that

$$T \cdot \mathcal{P} \, = \, 0 \qquad \text{where:} \qquad \mathcal{P} \,\, = \,\, \int_{\mathcal{C}} \frac{1}{y\,z} \cdot R\Big(\frac{x}{y}, \frac{y}{z}, z\Big), \tag{10}$$

this integral being *not necessarily equal* to the diagonal $Diag(R(x,\, y,\, z))$ (which could be, for instance, equal to zero). Equation (10) means that the telescoper annihilates *all the periods* $\mathcal{P}$.

**To sum-up:** In order to calculate the diagonal of a rational function one can try, in a very down-to-earth way, to obtain large enough series expansions of this diagonal from multi-series expansion and then try a guessing approach to obtain the linear differential operator annihilating the diagonal of a rational function, or one can perform the creative telescoping approach that will provide this telescoper even if the diagonal is zero, or even if it cannot be nicely defined because the rational function does not have a multi-Taylor expansion: in this case, the telescoper annihilates periods corresponding to *all* the cycles, in particular non-vanishing cycles.

### *2.3. Diagonals versus Telescopers: Vanishing Cycles versus Non-Vanishing Cycles*

#### 2.3.1. Diagonals versus Telescopers: A First Example

Let us first consider the following rational function of three variables:

$$R(x,y,z) \quad = \quad \frac{1}{-x \, -y \, -z^2}. \tag{11}$$

This rational function does not have a multi-Taylor expansion; thus, we cannot define the diagonal of the rational function. This rational function has, however, a telescoper that is a linear differential operator of order one, namely, $5\,\theta\,+2$, where $\theta \, = x\,D_x \, = x\,d/dx$ is the homogeneous derivative. Let us now consider a slightly more general rational function

$$R(x,y,z) \quad = \quad \frac{1}{\alpha \, \, -x \, -y \, \, -z^2}. \tag{12}$$

This rational function (12) has a multi-Taylor expansion, and one can, thus, obtain the first terms of the diagonal of this rational function (12):

$$Diag\Big(R(x,y,z)\Big) \quad = \quad \frac{1}{\alpha} \, + \frac{30}{\alpha^6} \cdot x^2 \, + \frac{3150}{\alpha^{11}} \cdot x^4 \, + \frac{420420}{\alpha^{16}} \cdot x^6 \quad + \, \cdots \tag{13}$$

The $\alpha$-dependent rational function (12) has an order-four $\alpha$-dependent telescoper $L_4(\alpha)$

$$\begin{aligned} x^2 \cdot L_4(\alpha) \quad = \quad & -5 \cdot x^2 \cdot (5\,\theta\,+2) \cdot (5\,\theta\,+4) \cdot (5\,\theta\,+6) \cdot (5\,\theta\,+8) \\ & +16 \cdot \alpha^5 \cdot \theta^2 \cdot (\theta\,-1)^2, \end{aligned} \tag{14}$$

which has the following ${}_4F_3$ hypergeometric function solution

$$\frac{1}{\alpha} \cdot {}_4F_3\Big([\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}], \, [\frac{1}{2}, \frac{1}{2}, 1], \, \frac{3125}{16\,\alpha^5} \cdot x^2\Big). \tag{15}$$

The series expansion of this ${}_4F_3$ hypergeometric function (15) is in agreement with the series expansion (13). In the $\alpha \, \rightarrow \, 0$ limit, the order-four $\alpha$-dependent telescoper $L_4(\alpha)$ becomes the direct-sum:

$$-5 \cdot x^4 \cdot \Big((5\,\theta\,+2) \oplus (5\,\theta\,+4) \oplus (5\,\theta\,+6) \oplus (5\,\theta\,+8)\Big). \tag{16}$$

We thus see, in this $\alpha \, \rightarrow \, 0$ limit, that one recovers, among the different factors in (16), the order-one telescoper of the rational function (11), namely, $5\,\theta\,+2$. This first example is a bit too simple, so let us consider another example.

#### 2.3.2. Diagonals versus Telescopers: A Second Example

Let us now consider the rational function of three variables:

$$R(x,y,z) \quad = \quad \frac{1}{-x \, -y \, -z \, \, -x^5\,y}. \tag{17}$$

This rational function has a telescoper $L_4$, which is a linear differential operator of order four, which reads:

$$\begin{aligned} L_4 \;=\; & -(800000\, x^5 \, - 27) \cdot x^4 \, D_x^4 \; - (11200000\, x^5 \, + 27) \cdot x^3 \, D_x^3 \\ & -15 \cdot (2800000\, x^5 \, - 1) \cdot x^2 \, D_x^2 \; - 60 \cdot (700000\, x^5 \, - 1) \cdot x \, D_x \\ & -12 \cdot (437500\, x^5 \, + 9), \end{aligned} \tag{18}$$

or, introducing the homogeneous derivative $\theta \, = \, x \, D_x$,

$$\begin{aligned} L_4 \;=\; & -50000 \cdot x^5 \cdot (2\,\theta \, + 7)\,(2\,\theta \, + 5)\,(2\,\theta \, + 3)\,(2\,\theta \, + 1) \\ & +3 \cdot (3\,\theta \, + 1)\,(3\,\theta \, - 4)\,(\theta \, - 3)^2. \end{aligned} \tag{19}$$

The rational function (17) *does not* have a multi-Taylor expansion. We have a problem to define the diagonal of the rational function (17). The analytic solutions of (18) or (19) are thus just "Periods" of the rational function (17), i.e., integrals over a non-vanishing cycle of the rational function (17). A solution of (18) or (19) is, for instance, the hypergeometric function:

$$x^3 \cdot {}_4F_3\Big([\frac{7}{10}, \frac{9}{10}, \frac{11}{10}, \frac{13}{10}], \; [1, \frac{4}{3}, \frac{5}{3}], \; \frac{800000}{27} \cdot x^5\Big). \tag{20}$$

If one finds that the concept of diagonal is easier to understand, compared to"Periods" over non-vanishing cycles that are not really defined (we just know they exist), such a result may look a bit too abstract and thus slightly frustrating. In fact, one can recover some contact with the easier concept of diagonals, performing some kind of "desingularization". Let us consider the more general $\alpha$-dependent rational function of three variables:

$$R(x, y, z) \;=\; \frac{1}{\alpha \, - x \, - y \, - z \, - x^5 \, y}. \tag{21}$$

It has a telescoper that is a linear differential operator of order four $M_4(\alpha)$. The first terms of the diagonal of that rational function (21) can easily be calculated. We have calculated this order-four linear differential operator $M_4(\alpha)$. It is a bit too large to be given here. However, one remarks that this $\alpha$-dependent order-four linear differential operator $M_4(\alpha)$ is actually related to the previous order-four linear differential operator $L_4$, in the $\alpha \, \rightarrow \, 0$ limit:

$$M_4(0) \;=\; -675000000\, x^{11} \cdot L_4. \tag{22}$$

**To sum-up:** The telescoper corresponding to "Periods" over a non-vanishing cycles can be obtained from a one-parameter telescoper having clear-cut diagonal solutions ("Periods" over a vanishing cycle).

*2.4. The Devil Is in the Detail: The Number of Variables*

Let us consider the diagonal of the following rational function of four variables

$$\frac{1}{1 \, - \alpha\, x - y - z \, - \beta \cdot x\, u}. \tag{23}$$

Its telescoper is, *for any value of* $\alpha$, and for $\beta \, \neq \, 0$, the order-two linear differential operator

$$L_2 \;=\; (1 \, - 27\,\beta \cdot x) \cdot x\, D_x^2 \; + (1 \, - 54\,\beta \cdot x) \cdot D_x \; - 6\,\beta, \tag{24}$$

which has the following hypergeometric ${}_2F_1$ solution:

$$
{}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \ 27\beta \cdot x\Big). \tag{25}
$$

Recalling the definition of the diagonal of a rational function based on multi-Taylor expansion, it is easy to see, based on this almost trivial example, that the various powers of the product $t = xyzu$ that the diagonal extracts require the occurrence of the variable $u$, which only occurs in the denominator of (23), through the product $xu$, automatically yielding the occurrence of the variable $x$. Consequently, any further occurrence of the variable $x$, from the $-\alpha x$ monomial in the denominator of (23), is excluded. This explains why the diagonal of (23) is actually blind to the $-\alpha x$ term. In other words, the diagonal of the four variables rational function (23) is in fact the diagonal of a rational function of *three variables:* $y$, $z$, and the product $xu$.

**Remark 1.** *To take into account this problem, we will introduce the concept of an "effective number" of variables. In the previous example, the number of variables is four but the "effective number" of variables is three.*

### *2.5. Understanding the Complexity of the Diagonal of a Rational Function*

#### 2.5.1. Order of the Linear Differential Operator and Number of Variables

The simplest example of a diagonal of rational function of $n$ variables corresponds to the diagonal of the rational function

$$
\frac{1}{1 \ - x_1 \ - x_2 \ - x_3 \ \ \cdots \ - x_n}. \tag{26}
$$

The diagonal of (26) is annihilated by an order-$(n-1)$ linear differential operator with a ${}_{n-1}F_{n-2}$ hypergeometric solution

$$
{}_{n-1}F_{n-2}\Big([\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \cdots, \frac{n-1}{n}], [1, 1, \cdots, 1], \ n^n \cdot x\Big). \tag{27}
$$

This simple example may provide the prejudice that, for a given globally bounded series (36), the number of variables of the rational function is related to the (minimal) order of the linear differential operator annihilating the series. One should note, however, for the class of the above example, that the corresponding linear differential operator has the Maximally Unipotent Monodromy property (MUM), which means that all its indicial exponents (at the origin) of the operator are equal (see, for instance, [22,32]).

This result is reminiscent of the well-known ${}_4F_3([1/5, 2/5, 3/5, 4/5], [1, 1, 1], x)$ Candelas et al. hypergeometric series emerging in [3] for a particular Calabi–Yau manifold. The simplest Calabi–Yau series (see, for instance, [18]) are ${}_4F_3$ hypergeometric series like ${}_4F_2([1/2, 1/2, 1/2, 1/2], [1, 1, 1], x)$ or ${}_4F_2([1/5, 2/5, 3/5, 4/5], [1, 1, 1], x)$ (see equation 3.11 in [3]).

Let us recall that Calabi-operators [22], annihilating the Calabi–Yau series [18], are (self-adjoint) order-four linear differential operators that have the Maximally Unipotent Monodromy property (MUM) at $x = 0$: if one considers their formal series expansions at $x = 0$, among the four formal series expansions, one is analytic (it actually corresponds to our diagonals of rational functions); another one is a formal series with a $\ln(x)^1$, another one is a formal series with a $\ln(x)^2$, and the last one is a formal series with a $\ln(x)^3$. Accordingly, with ((26) yielding (27)), one would expect that the diagonal of rational function representation of a Calabi–Yau series (the solution of an order-four linear differential operator) should require at least *five* variables for the rational function.

2.5.2. Order of the Linear Differential Operator and Degree in the Variables

Let us now consider the diagonal of the following rational function of three variables

$$\frac{1}{1 \; - x - \alpha\, y \; - z^2}, \tag{28}$$

whose diagonal writes as a simple $_4F_3$ hypergeometric solution:

$$_4F_3\Big([\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}], [1, \frac{1}{2}, \frac{1}{2}], \frac{5^5}{2^4} \cdot \alpha^2 \cdot x^2\Big). \tag{29}$$

In contrast with example (26), here, we just need, for the rational function, *three* variables, instead of the expected *five* variables. Note, however, that the order-four linear differential operator $L_4$, annihilating this hypergeometric solution (29), *does not* have MUM. As usual, this order-four linear differential operator is homomorphic to it adjoint with a very simple order-two intertwiner:

$$L_4 \cdot \Big(x\, D_x^2 \; + D_x\Big) \;\; = \;\; \Big(x\, D_x^2 \; + D_x\Big) \cdot adjoint(L_4). \tag{30}$$

One thus expects [42] this order-four linear differential operator $L_4$ to have a symplectic differential Galois group included in $Sp(4, \mathbb{C})$. Actually, the exterior square of this order-four operator $L_4$ has a simple rational function solution [42], namely, $1/x/(5^5 \cdot x^2 \; - 2^4)$.

Let us now consider the diagonal of the following rational function of three variables:

$$\frac{1}{1 \; - x - \alpha\, y \; - z^3}. \tag{31}$$

The linear differential operator annihilating this diagonal is an order-six linear differential operator with a quite simple $_6F_5$ hypergeometric solution:

$$_6F_5\Big([\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}], [1, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}], \frac{7^7}{3^6} \cdot \alpha^3 \cdot x^3\Big). \tag{32}$$

Let us restrict ourselves to $\alpha \, = \, 1$. The order-six linear differential operator, annihilating the diagonal of (31), does not have MUM. One has *three* series analytic at $x \, = \, 0$: one of the form $x \cdot (1 \, + 2377375/6561\, x^3 \, + \cdots)$, one of the form $x^2 \cdot (1 \, + 16509584/32805\, x^3 \, + \cdots)$, and the third one being the diagonal of the rational function which is the expansion of (32):

$$\begin{aligned} &1 \; + 140\, x^3 \; + 84084\, x^6 \; + 64664600\, x^9 \; + 55367594100\, x^{12} \; + 50356110752640\, x^{15} \\ &\quad +47606217704845800\, x^{18} \; + 46236665756994672960\, x^{21} \quad + \cdots \end{aligned} \tag{33}$$

One also has three other formal series solutions with a $\ln(x)^1$ but no $\ln(x)^2$ or $\ln(x)^3$.

As usual, this order-six linear differential operator is homomorphic to its adjoint with a very simple order-four intertwiner:

$$L_6 \cdot \Big(x^2\, D_x^4 + 4\, x\, D_x^3 + 2\, D_x^2\Big) \;\; = \;\; \Big(x^2\, D_x^4 + 4\, x\, D_x^3 + 2\, D_x^2\Big) \cdot adjoint(L_6). \tag{34}$$

One expects [42] this order-six linear differential operator $L_6$ to have a symplectic differential Galois group included in $Sp(6, \mathbb{C})$. Actually, the exterior square of this order-six linear differential operator $L_6$ has a simple rational function solution [42], namely, $1/x/(7^7 \cdot x^3 \; - 3^6)$.

**Remark 2.** *This result can be generalized. Let us consider the rational function:*

$$\frac{1}{1 \; - x - y \; - z^n}. \tag{35}$$

*The linear differential operator $L_{2n}^{(1)}$, annihilating this diagonal, is an order-$(2\,n)$ linear differential operator with a quite simple ${}_{2n}F_{2n\,-1}$ hypergeometric solution:*

$$
{}_{2n}F_{2n\,-1}\Big([\frac{1}{2\,n\,+1}, \, \frac{2}{2\,n\,+1}, \, \frac{3}{2\,n\,+1}, \, \cdots, \, \frac{2\,n}{2\,n\,+1}], \quad [1, \, \frac{1}{n}, \, \frac{1}{n}, \, \frac{2}{n}, \, \frac{2}{n}, \, \cdots, \, \frac{n-1}{n}, \, \frac{n-1}{n}], \, \frac{(2\,n\,+1)^{(2\,n+1)}}{n^{2\,n}} \cdot x^n\Big). \tag{36}
$$

*Let us also consider the linear differential operators $L_{2n}^{(m)}$ annihilating the diagonal of the rational function:*

$$
\Big(\frac{1}{1\,-x\,-\,y\,\,-z^n}\Big)^m. \tag{37}
$$

*One finds (using the Homomorphisms command in Maple) the following homomorphisms between successive linear differential operators $L_{2n}^{(m)}$:*

$$
Homomorphisms\Big(L_{2n}^{(m)}, \, L_{2n}^{(m+1)}\Big) \quad = \quad (2\,n\,+1) \cdot x \cdot D_x \,\, + m \cdot n. \tag{38}
$$

*In other words, one has the relations:*

$$
L_{2n}^{(m+1)} \cdot \Big((2\,n\,+1) \cdot \theta \,\, + m \cdot n\Big) \quad = \quad Z_1(m) \cdot L_{2n}^{(m)}, \tag{39}
$$

*where $Z_1(m)$ is an order-one linear differential operator. The linear differential operator $L_{2n}^{(1)}$ is simply homomorphic to its adjoint:*

$$
Homomorphisms\Big(adjoint(L_{2n}^{(1)}), \, L_{2n}^{(1)}\Big) \,\, = \,\, \frac{1}{x^{n-1}} \cdot \theta^2 \cdot (\theta\,-1)^2 \cdot (\theta\,-2)^2 \cdot (\theta\,-3)^2 \,\, \cdots \, \Big(\theta - (n\,-2)\Big)^2. \tag{40}
$$

**Remark 3.** *With the previous, rather simple, examples we see that the order of the linear differential operator annihilating the diagonal of a rational function is not related to the number of variables of the rational function (or even to the number of "effective" variables see Section 2.4). Furthermore, a given globally bounded series can be seen to be the diagonals of an infinite number of rational functions of a certain number of variables, but also, in the same time, of other infinite number of rational functions with a different number of variables. For a given globally bounded series, we can find the (minimal order) linear differential operator annihilating this series. Having this (minimal order) linear differential operator, the question is: can we find the minimal number of variables necessary to see this globally bounded series as the diagonal of a rational function of that number of variables? We will address these questions in a forthcoming paper.*

## 3. Diagonals of Rational Functions: Should We Only Consider Rational Functions of the Form $1/Q$?

With $P$ and $Q$ multivariate polynomials (with $Q(0) \neq 0$), the diagonals of the rational functions $P/Q^k$ are, for fixed polynomial $Q$, and for arbitrary integer $k$, a *finite dimensional* vectorial space related, as shown by Christol [44,45], to the de Rham cohomology (we are thankful to P. Lairez for having clarified this point). There are so many cohomologies in mathematics. For non-mathematicians, let us just say that the introduction of a cohomology often amounts to seeing that "something" you expect, at first sight, to be infinite, for instance, the number of solutions of a system of PDE's (partial differential equations), is in fact a *finite* set (for instance for D-finite systems of PDE's). For physicists, not familiar with de Rham cohomology, let us just say that this can be seen as a consequence of the fact that these $P/Q^k$ rational functions are solutions of *D-finite systems*, which means that these systems of PDE's have a *finite set of solutions* of the form $P/Q^k$. Being in such a "finite box" will force the telescopers of the diagonals of $P/Q^k$ and $1/Q$ to be related (by homomorphisms). This requires one to find a "cyclic vector" in mathematicians wording.

Experimentally, if one considers the (minimal order) linear differential operators for the diagonal of $P/Q^k$ and for the diagonal of $1/Q$, these two linear differential operators are *actually homomorphic*. Note that this experimental result, valid for diagonals (i.e., integrals over vanishing cycles), is *no longer* valid for telescopers of rational functions with analytic solutions corresponding to "periods", $n$-fold integrals, over non-vanishing cycles. In this case, we have a slight generalization of that homomorphism between telescopers $P/Q^k$ and telescopers $1/Q$, which will be described in the sequel (see Section 5.2 below).

It is true that the analysis of lattice Green functions (LGF) [65–68] in physics naturally yields to diagonals of rational functions in the form $R = 1/Q$, where $Q$ is a polynomial. However, the other $n$-fold integrals, emerging in physics, are much more complex (for instance, the $\chi^{(n)}$ terms of the susceptibility of the two-dimensional Ising model [26]). The lattice Green functions [32,65–69] and some Occam's razor simplicity argument are not sufficient to justify a bias of studying, quite systematically, rational functions of the form $R = 1/Q$ (as we often do). In fact, these de Rham cohomology arguments are the reason why, for diagonals (and diagonals only), one can focus only on rational functions in the form $R = 1/Q$, but since these arguments may look too esoteric for physicists, let us, in a learn-by-example, pedagogical approach, provide examples showing that telescopers of rational functions in the form $R = 1/Q^k$ are homomorphic to telescopers of rational functions in the form $R = 1/Q$ and then that telescopers of rational functions in the form $R = P/Q$ are homomorphic to telescopers of rational functions in the form $R = 1/Q$.

### *3.1. Diagonals of Rational Functions: $R = 1/Q^k$ Reducing to $1/Q$*

Let us denote $Q$ the polynomial:

$$Q \,\,=\,\, 1 \,+xy+yz+zx \,+3\cdot(x^2+y^2+z^2). \tag{41}$$

Let us denote $L_4^{(n)}$ the telescopers of $Diag(1/Q^n)$

$$L_4^{(n)}\cdot Diag\Big(\frac{1}{Q^n}\Big) \,=\, 0. \tag{42}$$

One remarks that these telescopers are all of order four. One actually finds the following homomorphisms between successive telescopers (42)

$$Homomorphisms\Big(L_4^{(n)},\, L_4^{(n+1)}\Big) \,\,=\,\, 3x\cdot D_x \,+2n. \tag{43}$$

In other words, one has the relations:

$$L_4^{(n+1)}\cdot(3\theta+2n) \,\,=\,\, Z_1(n)\cdot L_4^{(n)}, \tag{44}$$

where $Z_1(n)$ is an order-one linear differential operator, with the intertwining relation (44) yielding:

$$\begin{aligned} &L_4^{(n+1)}\cdot(3\theta+2n)\,\cdots\,(3\theta+6)\cdot(3\theta+4)\cdot(3\theta+2)\\ &\quad=\,\, Z_1(n)\,\cdots\, Z_1(3)\cdot Z_1(2)\cdot Z_1(1)\cdot L_4^{(1)}. \end{aligned} \tag{45}$$

One deduces:

$$\begin{aligned} &2^n\cdot n!\cdot Diag\Big(\frac{1}{Q^{n+1}}\Big)\\ &\quad=\,\,(3\theta+2n)\,\cdots\,(3\theta+6)\cdot(3\theta+4)\cdot(3\theta+2)\cdot Diag\Big(\frac{1}{Q}\Big). \end{aligned} \tag{46}$$

In other words, the diagonal of $1/Q^{n+1}$ can be simply deduced from the diagonal of $1/Q$.

**Remark 4.** *The product* $(3\theta + 2n) \cdots (3\theta+6)\cdot(3\theta+4)\cdot(3\theta+2)$ *in the intertwining relation (45)* is in fact a direct sum*:*

$$\begin{aligned}&(3\theta+6)\cdot(3\theta+4)\cdot(3\theta+2)\\ &\qquad = \; 27\,x^3 \cdot LCLM\Big((3\theta+6),\; (3\theta+4),\; (3\theta+2)\Big).\end{aligned} \tag{47}$$

*One has, for instance, the relations:*

$$\begin{aligned} 2\cdot Diag\Big(\frac{1}{Q^2}\Big) \; &= \; (3\theta+2)\cdot Diag\Big(\frac{1}{Q}\Big) \\ 8\cdot Diag\Big(\frac{1}{Q^3}\Big) \; &= \; (3\theta+4)\cdot(3\theta+2)\cdot Diag\Big(\frac{1}{Q}\Big) \\ 48\cdot Diag\Big(\frac{1}{Q^4}\Big) \; &= \; (3\theta+6)\cdot(3\theta+4)\cdot(3\theta+2)\cdot Diag\Big(\frac{1}{Q}\Big) \\ 384\cdot Diag\Big(\frac{1}{Q^5}\Big) \; &= \; (3\theta+8)\cdot(3\theta+6)\cdot(3\theta+4)\cdot(3\theta+2)\cdot Diag\Big(\frac{1}{Q}\Big). \end{aligned} \tag{48}$$

*Of course, since the telescoper of* $Diag\Big(\frac{1}{Q}\Big)$ *is an order-four linear differential operator, the order-*$(k-1)$ *product in front of* $Diag\Big(\frac{1}{Q}\Big)$ *in (48) can be, for* $Diag\Big(\frac{1}{Q^k}\Big)$*, reduced to an order-three linear differential operator (the simple products* $(3\theta + 2\cdot(k-1))\cdots(3\theta+4)\cdot(3\theta+2)$ *in (48) being taken "modulo"* $L_4$*, for* $k \geq 5$*).*

### *3.2. Diagonals of Rational Functions:* $R = P/Q$ *Reducing to* $1/Q$

Experimentally, one finds, quite often, that the telescoper of a rational function of the form $R = P/Q$ and the telescoper of the simple rational function $1/Q$ with its numerator normalized to $1$ are homomorphic. The intertwiner $M$ occurring in the homomorphisms of these two telescopers yields a relation of the form

$$Diag\Big(\frac{P}{Q}\Big) \; = \; M\cdot Diag\Big(\frac{1}{Q}\Big), \tag{49}$$

yielding the prejudice that the diagonals of the rational functions of the form $P/Q$ should reduce to the "simplest" diagonal, namely, $Diag(1/Q)$. In fact, things are slightly more subtle, as will be seen below. In fact, one is looking for a *cyclic vector*, and the cyclic vector is not necessarily $Diag(1/Q)$ (see relation (58) and (59) below).

Sticking with the polynomial (41), one has

$$L_4^{(1)}\cdot Diag\Big(\frac{1}{Q}\Big) \; = \; 0, \tag{50}$$

and considering the diagonal of $x\,y/Q$, one obtains an order-five differential operator with unique factorization:

$$L_4^{(xy)}\cdot D_x\cdot Diag\Big(\frac{x\,y}{Q}\Big) \; = \; 0. \tag{51}$$

The homomorphisms between $L_4^{(1)}$ and $L_4^{(xy)}$ amount to seeking for linear differential operators that map the solutions of one differential operator into the other. These relations are

$$L_4^{(xy)}\cdot Q_3 \; = \; K_3\cdot L_4^{(1)}, \tag{52}$$

and

$$L_4^{(1)} \cdot J_3 \; = \; P_3 \cdot L_4^{(xy)}, \tag{53}$$

where the intertwiners $Q_3$, $K_3$, $J_3$, and $P_3$ are linear differential operators of order three.

Note that the above two relations show [23] that the linear differential operator $J_3 \cdot Q_3$ (resp. $Q_3 \cdot J_3$) leaves the solutions of $L_4^{(1)}$ (resp. $L_4^{(xy)}$) unchanged,

$$J_3 \cdot Q_3 \cdot Diag\Big(\frac{1}{Q}\Big) \; = \; Diag\Big(\frac{1}{Q}\Big) \tag{54}$$

$$= \; 1 \; - 195\, x^2 \; + 135225\, x^4 \; - 143647728\, x^6 \; + 182699446545\, x^8 \; -252437965534755 \; x^{10} \; + 364803972334074000 \; x^{12} \; + \; \cdots$$

and:

$$Q_3 \; \cdot J_3 \cdot D_x \cdot Diag\Big(\frac{x\,y}{Q}\Big) \; = \; D_x \cdot Diag\Big(\frac{x\,y}{Q}\Big) \tag{55}$$

$$= \; 16\, x \; - 38400\, x^3 \; + 71593536\, x^5 \; - 126120445440\, x^7 \; +218901889206000\, x^9 \; - 378463218115207680\, x^{11} \; + \; \cdots \tag{56}$$

Equivalently, the adjoint of $P_3 \cdot K_3$ (resp. the adjoint of $K_3 \cdot P_3$) leaves the solutions of the adjoint of $L_4$ (resp. the adjoint of $L_4^{(xy)}$) unchanged.

Introducing the differential operator $D_x$ on both sides of (53), and using (51), one obtains:

$$L_4^{(1)} \cdot J_3 \cdot D_x \cdot Diag\Big(\frac{x\,y}{Q}\Big) \; = \; P_3 \cdot (L_4^{(xy)} \cdot D_x) \cdot Diag\Big(\frac{x\,y}{Q}\Big). \tag{57}$$

The RHS of (57) cancels and, therefore, the LHS of (57), according to (50), leads to

$$Diag\Big(\frac{1}{Q}\Big) \; = \; J_3 \cdot D_x \cdot Diag\Big(\frac{x\,y}{Q}\Big). \tag{58}$$

Also, acting by both sides of (52) on $Diag(1/Q)$, using (50) and keeping (51) in mind leads to:

$$D_x \cdot Diag\Big(\frac{x\,y}{Q}\Big) \; = \; Q_3 \cdot Diag\Big(\frac{1}{Q}\Big). \tag{59}$$

With these relations, we see that the derivative of the diagonal of $xy/Q$ simply reduces to the diagonal of $1/Q$, but the diagonal of $xy/Q$ does not simply reduce to the diagonal of $1/Q$. Here, $1/Q$ is not the "cyclic vector".

## 4. Diagonals of Algebraic Functions

### *4.1. Diagonals of Algebraic Functions: A First Example*

Let us consider the algebraic functions:

$$A(x,y) \; = \; \frac{1}{\Big(1 \, - \alpha \cdot \, (x \, + y)\Big)^{1/n}} \qquad\qquad n \; = \; 2, 3, \; \cdots \tag{60}$$

The telescopers of these algebraic functions are order-two linear differential operators with the simple ${}_2F_1$ hypergeometric solution:

$${}_2F_1\Big([\frac{1}{2\,n}, \, \frac{n\,+1}{2\,n}], \, [1], \, 4 \cdot \alpha^2 \cdot x\Big)$$

$$= \; 1 \; + \frac{n+1}{n^2}\, \alpha^2\, x \; + \frac{(1\,+n) \cdot (1\,+2\,n) \cdot (1\,+3\,n)}{4 \cdot n^4}\, \alpha^4\, x^2 \; + \; \cdots \tag{61}$$

Note that, among these ${}_2F_1$ hypergeometric functions, the $n = 2$, $n = 3$, $n = 4$, and $n = 6$ cases correspond to *modular forms* (see Appendix B in [16]).

These hypergeometric series can be seen to be, as they should, the diagonals of the algebraic functions (60). In particular, for $n = 2$, one obtains:

$$
\begin{aligned}
&{}_2F_1\Big([\frac{1}{4}, \frac{3}{4}], [1], 4 \cdot \alpha^2 \cdot x\Big) \\
&\quad = \Big(\frac{1}{1 - 3\alpha^2 x}\Big)^{1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{27}{4} \cdot \frac{\alpha^4 \cdot x^2 \cdot (1 - 4\alpha^2 x)}{(1 - 3\alpha^2 x)^3}\Big) \\
&\quad = 1 + \frac{3}{4}\alpha^2 x + \frac{105}{64}\alpha^4 x^2 + \frac{1155}{256}\alpha^6 x^3 + \frac{225225}{16384}\alpha^8 x^4 + \cdots
\end{aligned}
\tag{62}
$$

For $n = 2$, it is natural to associate the denominator of (60) with the algebraic surface

$$z^2 = 1 - \alpha \cdot (x + y), \tag{63}$$

and, following ideas developed in [40], since calculating the diagonal of the function (60) for $n = 2$ amounts, in the multi-Taylor expansion, to extracting the terms depending only on the product $p = xy$, take the intersection of the algebraic surface (63) with the surface $p = xy$. This amounts, for instance, to eliminating $y = p/x$ in (63), thus obtaining the algebraic curve

$$-\alpha \cdot x^2 - x z^2 - \alpha \cdot p + x = 0, \tag{64}$$

which turns out to be an *elliptic curve* (genus-one). Calculating the j-invariant of the elliptic curve (64), one deduces the following Hauptmodul

$$\mathcal{H} = \frac{1728}{j} = \frac{27}{4} \cdot \frac{\alpha^4 \cdot p^2 \cdot (1 - 4\alpha^2 p)}{(1 - 3\alpha^2 p)^3}, \tag{65}$$

which is actually the Hauptmodul pullback in (62). This example gives some hope that the effective algebraic geometry approach of diagonals of rational functions, detailed in [40], could also work with diagonals of algebraic functions.

For $n \neq 2$, it is tempting to associate the denominator of (60) with the algebraic surface

$$z^n = 1 - \alpha \cdot (x + y), \tag{66}$$

and after the elimination $y = p/x$ in (63), the algebraic curve

$$-\alpha \cdot x^2 - x z^n - \alpha \cdot p + x = 0, \tag{67}$$

but such algebraic curves turn out to be of the genus $g = n - 1$. Understanding the emergence of *modular forms* for the $n = 3$, $n = 4$, and $n = 6$ subcases of (61) from, respectively, genus 2, 3, and 5 algebraic curves is an open (and challenging) problem.

**Remark 5.** *From the definition of the diagonals of rational or algebraic functions, it is straightforward to see that the diagonals of the algebraic functions (60) are series of the variable $\alpha^2 x$. Consequently, the previous calculations for a particular value of $\alpha$ are sufficient to recover the previous results valid for arbitrary $\alpha$. For that reason, we will, in the next example, take specific values of the parameters.*

*4.2. Diagonals of Algebraic Functions: A Second Example*

Let us consider the algebraic functions:

$$A(x,y) \;=\; \frac{1}{\Big(1 - 3\cdot\,(x+y) + 5\cdot\,(x^2+y^2)\Big)^{1/n}}, \qquad n \,=\, 2, 3, \;\cdots \tag{68}$$

For $n = 2$, the telescoper of the algebraic function (68) is an order-two linear differential operator with the pullbacked ${}_2F_1$ hypergeometric solution:

$$\begin{aligned} &\frac{1}{(1-30\,x)^{1/2}} \cdot {}_2F_1\Big([\frac{1}{4}, \frac{3}{4}], [1], \, -\frac{4\cdot\,(11-200\,x)\cdot\,x}{(1-30\,x)^2}\Big) \\ &\quad= \frac{1}{(1-27\,x+300\,x^2)^{1/4}} \\ &\qquad \times {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \, \frac{27}{4}\cdot\frac{x^2\cdot\,(11-200\,x)^2\cdot\,(1-16\,x+100\,x^2)}{(1-27\,x+300\,x^2)^3}\Big) \\ &\quad= 1 + \frac{27}{4}\,x + \frac{4305}{64}\,x^2 + \frac{199395}{256}\,x^3 + \frac{167040825}{16384}\,x^4 + \cdots \end{aligned} \tag{69}$$

With multi-Taylor series expansion, it is straightforward to see that the hypergeometric series is actually the diagonal of the algebraic function (68) for $n = 2$.

As in the previous subsection, we introduce the algebraic surface

$$z^2 \;=\; 1 - 3\cdot\,(x+y) + 5\cdot\,(x^2+y^2), \tag{70}$$

and, again, eliminate $y \,=\, p/x$ in (70), thus obtaining the algebraic curve

$$5\,x^4 - x^2\,z^2 - 3\,x^3 + 5\,p^2 - 3\,p\,x + x^2 \;=\; 0, \tag{71}$$

which turns out to be an *elliptic curve* (genus-one). Calculating the j-invariant of the elliptic curve (71), one deduces the following Hauptmodul:

$$\mathcal{H} \;=\; \frac{1728}{j} \;=\; \frac{27}{4}\cdot\frac{p^2\cdot\,(11-200\,p)^2\cdot\,(1-16\,p+100\,p^2)}{(1-27\,p+300\,p^2)^3}, \tag{72}$$

*which is actually the Hauptmodul* pullback in (69). Again, this last example gives some hope that the effective algebraic geometry approach of diagonals of rational functions, detailed in [40], could also work with diagonals of algebraic functions. For $n \neq 2$, it is tempting to introduce the algebraic surface

$$z^n \;=\; 1 - 3\cdot\,(x+y) + 5\cdot\,(x^2+y^2), \tag{73}$$

and, again, eliminate $y \,=\, p/x$ in (70), thus obtaining the algebraic curve

$$5\,x^4 - x^2\,z^n - 3\,x^3 + 5\,p^2 - 3\,p\,x + x^2 \;=\; 0, \tag{74}$$

which is an algebraic curve of genus $g \,=\, 2\,n - 3$ for $n$ even, and $g \,=\, 2\,n - 2$ for $n$ odd. For $n \,=\, 3$ (genus 4), the telescoper of the algebraic function (68) is an (irreducible) order-three linear differential operator *that is not homomorphic to its adjoint*. The interpretation of such non-self-dual order-three linear differential operators from these higher genus algebraic curves is a totally open problem.

## 5. Understanding the Emergence of Selected Differential Galois Groups for Diagonals of Rational Functions

Experimentally, one finds that almost all the linear differential operators annihilating the diagonal of a rational or algebraic function are homomorphic to their adjoint [41]. For instance, recalling an example in [41]

$$ {}_4F_3\Big([\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}], [\frac{1}{2}, 1, 1], \frac{729}{4} \cdot x\Big) \; = \; Diag\Big(\frac{1}{1 \, - (1 \, + u) \cdot \, (x \, + y \, + z)}\Big) $$
$$ = \; 1 \; + 18\, x \; + 1350\, x^2 \; + \cdots \qquad (75) $$

we find the corresponding order-four linear differential operator

$$ x \cdot L_4 \;\; = \;\; 2 \cdot x \cdot (3\,\theta \, + 2)^2 \cdot (3\,\theta \, + 1)^2 \;\; - 81 \cdot \theta^3 \cdot (2\,\theta \, - 1), \qquad (76) $$

which can be seen to be non-trivially homomorphic to its adjoint:

$$ L_4 \cdot \Big(\theta + \frac{1}{2}\Big) \;\; = \;\; \Big(\theta + \frac{1}{2}\Big) \cdot adjoint(L_4). \qquad (77) $$

Beyond diagonals of rational or algebraic functions, one also finds experimentally that almost all the telescopers of rational or algebraic functions are homomorphic to their adjoint. This homomorphism to the adjoint property is so systematic that, following a mathematician's prejudice, one can imagine that this is nothing but the *Poincaré duality*. The Poincaré duality [43] works for *any* algebraic variety: the diagonal of *any* rational or algebraic function should yield self-dual linear differential operators in the sense that they are homomorophic to their adjoint. This is not the case. It turns out that the linear differential operators of some ${}_nF_{n-1}$, candidates to rule-out Christol's conjecture [44,45,48], precisely provide such rare examples of linear differential operators annihilating the diagonal of rational or algebraic functions that are not homomorphic to their adjoint. Among these candidates, a large set has been seen to actually be composed of diagonals of rational or algebraic functions [48,70].

### 5.1. A Recall on Christol's Conjecture

Let us recall one of the ${}_3F_2$ hypergeometric candidates introduced to rule out Christol's conjecture:

$$ {}_3F_2\Big([\frac{2}{9}, \frac{5}{9}, \frac{8}{9}], [\frac{2}{3}, 1], \; 27 \cdot x\Big) \qquad (78) $$
$$ = \; 1 \; + \; \frac{40}{9} \cdot x \; + \; \frac{5236}{81} \cdot x^2 \; + \; \frac{7827820}{6561} \cdot x^3 \; + \; \frac{1444588600}{59049} \cdot x^4 \;\; + \cdots $$

It is a globally bounded series (change $x \, \rightarrow \, 3^3 \cdot x$ to obtain a series with integer coefficients). In fact, it actually corresponds [48] to the diagonal of the algebraic function:

$$ \frac{(1 \; - y \; - z)^{1/3}}{1 \, - x \, - \, y \, - z}. \qquad (79) $$

The telescoper of the algebraic function (79) is the order-three linear differential operator, which has (78) as a solution. This order-three linear differential operator is *not* homomorphic to its adjoint. We have a $SL(3, \mathbb{C})$ differential Galois group.

Other similar examples are, for instance,

$$ {}_3F_2\Big([\frac{1}{9}, \frac{4}{9}, \frac{7}{9}], [\frac{2}{3}, 1], \; 27 \cdot x\Big) \;\; = \;\; Diag\Big(\frac{(1 \; - y \; - 2\,z)^{2/3}}{1 \, - x \, - \, y \, - z}\Big), \qquad (80) $$

or

$$ {}_3F_2\Big([\frac{2}{9}, \frac{5}{9}, \frac{8}{9}], [\frac{5}{6}, 1], 27 \cdot x\Big) \;=\; Diag\Big(\frac{(1 - y - 2z)^{1/3}}{1 - x - y - z}\Big), \tag{81} $$

or even the ${}_4F_3$ hypergeometric function:

$$ {}_4F_3\Big([\frac{2}{9}, \frac{5}{9}, \frac{8}{9}, \frac{1}{2}], [\frac{1}{3}, \frac{5}{6}, 1], 27 \cdot x\Big) \;=\; Diag\Big(\frac{(1 - x)^{1/3}}{1 - x - y - z}\Big). \tag{82} $$

Again, these three diagonals, (80), (81), and (82), are solutions of telescopers that *are not* homomorphic to their adjoint.

These examples are taken in a list of 116 potential counter-examples constructed in 2011 by Bostan et al. [25]. Note that, more recently, 38 cases in that list of 116 have actually been found to be diagonals of algebraic functions [70]. The two relations (80) and (81) can be generalized [70,71] as follows:

$$ \begin{aligned} &{}_4F_3\Big([\frac{1-(R+S)}{3}, \frac{2-(R+S)}{3}, \frac{3-(R+S)}{3}, \frac{1-S}{2}], \\ &\qquad [\frac{1-(R+S)}{2}, \frac{2-(R+S)}{2}, 1], 27 \cdot x\Big) \\ &\qquad = \; Diag\Big(\frac{(1-x)^R \cdot (1-x-2y)^S}{1-x-y-z}\Big), \end{aligned} \tag{83} $$

where $R$ and $S$ are rational numbers. These diagonals are annihilated by the order-four linear differential operator:

$$ \begin{aligned} &2 \cdot x \cdot (S-1-2\theta) \cdot (S+R-3\theta) \cdot (S+R-1-3\theta) \cdot (S+R-2-3\theta) \\ &\qquad -\theta^2 \cdot (S+R+1-2\theta) \cdot (S+R-2\theta). \end{aligned} \tag{84} $$

This order-four linear differential operator is *not* homomorphic to its adjoint. Other more involved similar relations can be found in section 2.1 of chapter 2 of [70].

Experimentally, we found, after quite systematic calculations of thousands of telescopers of rational or algebraic functions, that the telescopers are (almost always) homomorphic to their adjoint, or if they are not irreducible, that each of the factors of these telescopers are homomorphic to their adjoint. Such previous examples like (78), (79), or (80) and (81), curiously related to Christol's conjecture, provide *rare* examples of diagonals of algebraic functions such that their corresponding telescopers are *not* homomorphic to their adjoint. We have similar results with the algebraic function:

$$ \frac{x^{1/3}}{1 - x - y - z}. \tag{85} $$

In order to understand this "duality-breaking" (the telescoper is not self-adjoint up to homomorphisms), it is tempting to introduce the (algebraic) function:

$$ \frac{1}{1 - x - y - z - \alpha \cdot x^{1/3}}. \tag{86} $$

However, in order to avoid the introduction of rational functions of $n$-th roots of variables, we will introduce the diagonal of the following rational function:

$$ \frac{1}{1 - x^3 - y^3 - z^3 - \alpha \cdot x}. \tag{87} $$

*5.2. Understanding the Emergence of Selected Differential Galois Groups for Almost All the Diagonal of Rational Functions*

The linear differential operator annihilating the diagonal of the rational function (87) is a (quite large) order-eight linear differential operator $L_8(\alpha)$, depending on the parameter $\alpha$, which is homomorphic to its adjoint with an order-six intertwiner. This order-eight linear differential operator $L_8(\alpha)$ is irreducible except at $\alpha = 0$. For $\alpha = 1$, $\alpha = 2$, and $\alpha = 3$ the order-eight linear differential operator $L_8(\alpha)$ is homomorphic to its adjoint with an order-six intertwiner. The differential Galois group should, thus, be included in $Sp(8, \mathbb{C})$. This is confirmed when calculating [42] the exterior square of $L_8(\alpha)$. This exterior square has a rational function solution $P_a/x/Q_a$, where the polynomials $P_a$ and $Q_a$ read:

$$\begin{aligned} P_a \,\, &= \,\, (4\,\alpha^3 - 27) \cdot \, (20\,\alpha^3 - 81) \, + 18 \cdot \, (-6561 \, - 891\,\alpha^3 \, + 500\,\alpha^6) \cdot \, x^3 \, + 1594323\, x^6, \\ Q_a \,\, &= \,\, 387420489\, x^9 \, - 531441 \cdot \, (81 + 100\,\alpha^3) \cdot \, x^6 \\ & \quad +(1594323 \, - 2972133\,\alpha^3 \, + 729000\,\alpha^6 \, - 50000\,\alpha^9) \cdot \, x^3 \, - 27 \cdot \, (4\,\alpha^3 - 27)^2. \end{aligned} \tag{88}$$

Let us now take the $\alpha \rightarrow 0$ limit of the order-eight linear differential operator $L_8(\alpha)$. In this limit, the order-eight linear differential operator just becomes the direct-sum

$$L_2 \oplus L_3 \oplus M_3, \tag{89}$$

where the order-two linear differential operator $L_2$ has the ${}_2F_1$ hypergeometric solution

$${}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], 27\,x^3\Big), \tag{90}$$

where the order-three linear differential operator $L_3$ has the ${}_3F_2$ hypergeometric function solution

$${}_3F_2\Big([\frac{5}{9}, \frac{8}{9}, \frac{11}{9}], \; [\frac{2}{3}, 1], 27\,x^3\Big), \tag{91}$$

and where the order-three linear differential operator $M_3$ has the ${}_3F_2$ hypergeometric function solution:

$${}_3F_2\Big([\frac{7}{9}, \frac{10}{9}, \frac{13}{9}], \; [\frac{1}{3}, 1], 27\,x^3\Big). \tag{92}$$

These two order-three linear differential operators, similarly to the previous example (78), are not homomorphic to their adjoint: they break the self-adjoint duality (up to homomorphisms of operators) and thus have a $SL(3, \mathbb{C})$ differential Galois group.

These two hypergeometric series are exactly on the same footing as (78): they are *globally bounded series* (just change $x^3 \rightarrow 3^3\, x^3$ in order to obtain a series with integer coefficients), and their respective order-three linear differential operators are not homomorphic to their adjoint, their differential Galois group being $SL(3, \mathbb{C})$. Let us note, however, that the order-three linear differential operator $L_3$ *is actually homomorphic to the adjoint of* $M_3$, and of course the order-three linear differential operators $M_3$ is homomorphic to the adjoint of $L_3$.

If, from an algebraic geometry perspective [40], one sees the fact that all our linear differential operators, annihilating diagonals of rational functions, are homomorphic to their adjoint as the differential algebra expression of the Poincaré duality on the algebraic varieties corresponding to the denominators of our rational functions [40], the fact that this Poincaré duality is broken for $L_3$ or $M_3$ *is, in fact, restored in the bigger picture* (87) with the linear differential order-eight operator. In the $\alpha \rightarrow 0$ limit, we see that these two linear differential operators breaking the duality actually emerge in a dual pair, thus *restoring the duality*. For instance, if one focuses on $L_6 = L_3 \oplus M_3$ in (90), one finds easily that this order-six linear differential operator is homomorphic to its adjoint. Its exterior square has the following rational function solution:

$$\frac{4 \, + 621 \, x^3}{(1 \, - 27 \, x^3)^3 \cdot \, x}. \tag{93}$$

Since these calculations are in the $\alpha \, \to \, 0$ limit, let us expand, in $\alpha$, the rational function (87):

$$\begin{aligned} \frac{1}{1 \, - x^3 - y^3 - z^3 \, - \alpha \cdot \, x} \, &= \, \frac{1}{1 \, - x^3 - y^3 - z^3} \, + \frac{x}{(1 \, - x^3 - y^3 - z^3)^2} \cdot \, \alpha \\ &+ \frac{x^2}{(1 \, - x^3 - y^3 - z^3)^3} \cdot \, \alpha^2 \, + \frac{x^3}{(1 \, - x^3 - y^3 - z^3)^4} \cdot \, \alpha^3 \\ &+ \frac{x^4}{(1 \, - x^3 - y^3 - z^3)^5} \cdot \, \alpha^4 \, + \, \cdots \end{aligned} \tag{94}$$

The diagonal of a sum is clearly the sum of the diagonals. Thus, the diagonal of the LHS of (94) will be the sum of the various rational function terms in $\alpha^n$ in the RHS of (94). The diagonal of the $\alpha^1$ term in the $\alpha$-expansion (94)

$$\frac{x}{(1 \, - x^3 - y^3 - z^3)^2}, \tag{95}$$

is clearly equal to zero since the diagonal extracts, in the multi-Taylor series, the terms in the product $p \, = \, x \, y \, z$, or, in this case, the terms in the product $x^3 \, y^3 \, z^3$. Similarly, the diagonal of the $\alpha^2$ term in the $\alpha$-expansion (94)

$$\frac{x^2}{(1 \, - x^3 - y^3 - z^3)^3}, \tag{96}$$

is also zero, but the diagonal of the $\alpha^3$ term

$$\frac{x^3}{(1 \, - x^3 - y^3 - z^3)^4}, \tag{97}$$

is not zero. Actually, this last diagonal reads:

$$\begin{aligned} &-\frac{1}{9} \cdot \, \frac{1 \, + 216 \, x^3}{(1 - 27 \, x^3)^3} \cdot \, x \cdot \, \frac{d}{dx} \, {}_2F_1\Big( [\frac{1}{3}, \, \frac{2}{3}], \, [1], \, 27 \, x^3 \Big) \\ &\qquad -18 \cdot \, \frac{x^3}{(1 - 27 \, x^3)^2} \cdot \, {}_2F_1\Big( [\frac{1}{3}, \, \frac{2}{3}], \, [1], \, 27 \, x^3 \Big) \\ &= \, -20 \, x^3 \, - 1680 \, x^6 \, - 92400 \, x^9 \, - 4204200 \, x^{12} \, - 171531360 \, x^{15} \, + \, \cdots \end{aligned} \tag{98}$$

and is annihilated by an order-two operator $M_2$.

We have a different story with telescopers. Since the telescoper of a sum of rational functions is the direct sum (LCLM) of the telescopers of these rational functions (or at least is a rightdivisor of the LCLM of the telescopers), let us consider the telescopers of the first five terms in the RHS of (94). The telescoper of the first term is, of course, the order-two linear differential operator $L_2$ annihilating the diagonal of this rational function. The telescoper of the second term (in $\alpha^1$) is the previous order-*three* linear differential operator $L_3$. The telescoper of the third term (in $\alpha^2$) is exactly the previous $M_3$. The telescoper of the fourth term (in $\alpha^3$) is the order-two linear differential operator $M_2$. The telescoper of the sum of the first orders in $\alpha$ in the expansion (94)

$$\frac{1}{1 \, - x^3 - y^3 - z^3} \, + \frac{x}{(1 \, - x^3 - y^3 - z^3)^2} \cdot \, \alpha \, + \frac{x^2}{(1 \, - x^3 - y^3 - z^3)^3} \cdot \, \alpha^2, \tag{99}$$

is actually the LCLM of the three telescopers $L_2$, $L_3$, and $M_3$ , *which is precisely the* $\alpha \rightarrow 0$ *limit of the order-eight linear differential operator.*

### 5.3. Revisiting $1/Q \rightarrow P/Q^k$ for Telescopers

The next terms in the $\alpha$-expansion (94), namely, the terms in $\alpha^{4+3n}$ with $n = 0, 1, \cdots$

$$\frac{x^{4+3n}}{(1 - x^3 - y^3 - z^3)^{5+3n}}, \tag{100}$$

have telescopers homomorphic to the telescoper $L_3$ for (95). Similarly, considering the $\alpha$-expansion (94), namely, the terms in $\alpha^{5+3n}$ with $n = 0, 1, \cdots$

$$\frac{x^{5+3n}}{(1 - x^3 - y^3 - z^3)^{6+3n}}, \tag{101}$$

have telescopers homomorphic to the telescoper $M_3$ for (96). Finally, the terms in $\alpha^{3+3n}$ with $n = 0, 1, \cdots$

$$\frac{x^{3+3n}}{(1 - x^3 - y^3 - z^3)^{4+3n}}, \tag{102}$$

have telescopers homomorphic to the telescoper $L_2$, generalizing the result (98) for $n = 0$. This last sequence of telescopers can be understood from the ideas sketched in Sections 3.1 and 3.2 for diagonals (changing, for instance, $(x,y,z)$ into $(x^3,y^3,z^3)$). However, we see that these ideas do not work anymore when we compare the telescopers for (100) (resp. the telescopers for (101)) with the telescopers for (102). These different telescopers are not homomorphic. They correspond to *three different sequences* of telescopers of a different nature, corresponding to three hypergeometric functions of quite a different nature:

$${}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], 27\,x^3\Big), \quad {}_3F_2\Big([\frac{7}{9}, \frac{10}{9}, \frac{13}{9}], [\frac{1}{3}, 1], 27\,x^3\Big), \quad {}_3F_2\Big([\frac{5}{9}, \frac{8}{9}, \frac{11}{9}], [\frac{2}{3}, 1], 27\,x^3\Big).$$

Accordingly, similar $\alpha$-dependent examples are sketched in Appendix A.

**To sum-up:** The ideas sketched in Sections 3.1 and 3.2 for diagonals can be generalized to telescopers (which may correspond to vanishing cycles, i.e., diagonals), with the caveat that the unique "root" rational function $1/Q$ has to be replaced by a finite set of rational functions ($1/Q_1$, $1/Q_2$, $1/Q_3$ in our previous example).

## 6. An Infinite Number of Birational Symmetries of the Diagonals and Telescopers

Let us consider the simplest example of the non-trivial diagonal of rational function, namely, the diagonal of the rational function of three variables:

$$R(x,y,z) \;=\; \frac{1}{1 - x - y - z}. \tag{103}$$

Let us consider the *birational transformation* $B$:

$$B: \quad (x,\, y,\, z) \quad \longrightarrow \quad \Big(x,\;\; y \cdot (1 + 3\,x + 7\,x^2),\;\; \frac{z}{1 + 3\,x + 7\,x^2}\Big). \tag{104}$$

It is birational because its compositional inverse is also a rational function:

$$(x,\, y,\, z) \quad \longrightarrow \quad \Big(x,\;\; \frac{y}{1 + 3\,x + 7\,x^2},\;\; z \cdot (1 + 3\,x + 7\,x^2)\Big). \tag{105}$$

Note that this birational transformation preserves the product $p = x\,y\,z$, as well as the neighbourhood of the point $(x, y, z) = (0, 0, 0)$. This birational transformation is an *infinite order* transformation. The composition of this transformation $n$ times gives:

$$(x, y, z) \quad \longrightarrow \quad \Big(x, \;\; y \cdot (1 + 3\,x + 7\,x^2)^n, \;\; \frac{z}{(1 + 3\,x + 7\,x^2)^n}\Big). \tag{106}$$

The rational function (103), transformed by the (infinite order) birational transformation (104), reads:

$$\begin{aligned} R_B(x,y,z) \;\; &= \;\; R\Big(x, \;\; y \cdot (1 + 3\,x + 7\,x^2), \;\; \frac{z}{1 + 3\,x + 7\,x^2}\Big) \\ &= \;\; \frac{1}{1 \; - x \; - y \cdot (1 + 3\,x + 7\,x^2) \; - z/(1 + 3\,x + 7\,x^2)}. \end{aligned} \tag{107}$$

With the multi-Taylor expansion of (107), one finds easily that the diagonals of (103) and (107) are *actually identical*.

More generally, let us consider

$$B_x : \quad (x, y, z) \quad \longrightarrow \quad \Big(x, \;\; y \cdot Q_1(x), \;\; \frac{z}{Q_1(x)}\Big), \tag{108}$$

where $Q_1(x)$ is a rational function (see, however, Section 6.4) with a Taylor expansion such that $Q_1(0) \neq 0$. One also finds for any such rational function $Q_1(x)$ that the diagonals of (103) and (107) are actually identical. This can be seen from the multi-Taylor expansion of (107):

$$\begin{aligned} R_B(x,y,z) \;\; &= \;\; \sum_m \sum_n \sum_l a_{m,n,l} \cdot x^m \cdot y^n \cdot Q_1(x)^n \cdot z^l \cdot Q_1(x)^{-l} \\ &= \;\; \sum_m a_{m,m,m} \cdot (x\,y\,z)^m \;\; + \sum_{(m,n,l) \neq (m,m,m)} a_{m,n,l} \cdot x^m \cdot y^n \cdot z^l \cdot Q_1(x)^{n-l}. \end{aligned} \tag{109}$$

The second triple sum can be decomposed into the terms such that $n \neq l$, which cannot contribute to the diagonal (which extracts terms in $p = x\,y\,z$ and thus terms in the product $y\,z$) and the $n = l$ terms (such that the $Q_1(x)^{n-l}$ factor in (109) disappear):

$$\sum_{m \neq n} a_{m,n,n} \cdot x^m \cdot y^n \cdot z^n. \tag{110}$$

This last sum (110), which excludes the power of $x$ to be equal to the power of the product $y\,z$, cannot contribute to the diagonal. We have thus proved that the diagonals of (103) and (107) are equal.

Of course, there is nothing particular with the variable $x$. We can also introduce other birational transformations that single out, respectively, $y$ and $z$:

$$B_y : \quad (x, y, z) \quad \longrightarrow \quad \Big(x \cdot Q_2(y), \;\; y, \;\; \frac{z}{Q_2(y)}\Big), \tag{111}$$

and

$$B_z : \quad (x, y, z) \quad \longrightarrow \quad \Big(x \cdot Q_3(z), \;\; \frac{y}{Q_3(z)}, \;\; z\Big), \tag{112}$$

for any rational functions $Q_2(x)$ and $Q_3(x)$ with a Taylor expansion such that $Q_2(0) \neq 0$ and $Q_3(0) \neq 0$. We can compose these birational transformations (108), (111) and (112), in any order and change the various $Q_1(x)$, $Q_2(x)$, and $Q_3(x)$ at each step. Thus, we obtain quite a large *infinite* set of birational transformations preserving the product $p = x\,y\,z$ and the neighbourhood of the point $(x, y, z) = (0, 0, 0)$. Since the product $p = x\,y\,z$ is preserved, let us eliminate (for instance) the variable $z = p/x/y$. The three previous birational transformations (108), (111), and (112), on the three variables $x, y, z$ become birational transformations depending on a parameter $p$, of only two variables $x, y$:

$$\tilde{B}_x : \quad (x, y) \quad \longrightarrow \quad \Big(x, \ y \cdot Q_1(x)\Big), \tag{113}$$

$$\tilde{B}_y : \quad (x, y) \quad \longrightarrow \quad \Big(x \cdot Q_2(y), \ y\Big), \tag{114}$$

and

$$\tilde{B}_z : \quad (x, y) \quad \longrightarrow \quad \Big(x \cdot Q_3\Big(\frac{p}{x\,y}\Big), \quad y/Q_3\Big(\frac{p}{x\,y}\Big)\Big). \tag{115}$$

Composing these birational transformations of two variables (113), (114), and (115), in any order and changing the various $Q_1(x)$, $Q_2(x)$, and $Q_3(x)$ at each step, one obtains that way a quite large subset of the (huge set of) *Cremona transformations* [49,72].

**Remark 6.** *Of course there is nothing specific about the particular simple example (103) of the rational function. The previous birational transformations (113), (114), and (115) are symmetries of the diagonals of any rational function of three variables. Furthermore, there is nothing specific about rational function of three variables. We can generalize such birational transformations for the diagonal of the rational function of $n$ variables, for any number of variables $n$.*

### *6.1. Non-Birational Symmetries for Diagonals*

#### 6.1.1. Monomial Transformation

Let us consider the (non-birational) *monomial* transformation:

$$M : \quad (x, y, z) \quad \longrightarrow \quad \Big(x, \, x^2\,y^2, \, y\,z^3\Big). \tag{116}$$

By performing this monomial transformation (116) on the rational function (103), one obtains the new rational function:

$$R_M(x, y, z) \; = \; R\Big(x, \, x^2\,y^2, \, y\,z^3\Big) \; = \; \frac{1}{1 \; - x \; - x^2\,y^2 \; - y\,z^3}. \tag{117}$$

The calculation of the telescoper of (117) gives an order-two linear differentizal operator that has the ${}_2F_1$ hypergeometric series solution:

$$\begin{aligned} {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], \, 27\,x^3\Big) \; &= \; 1 \; + 6\,x^3 \; + 90\,x^6 \; + 1680\,x^9 \; + 34650\,x^{12} \\ &+ 756756\,x^{15} \; + 17153136\,x^{18} \; + \; \cdots \end{aligned} \tag{118}$$

One verifies easily, that the diagonal of the multi-Taylor expansion of (117), is actually the ${}_2F_1$ hypergeometric series (118). This series identifies with the diagonal of (103), where $x$ is changed into $x^3$, by the monomial transformation (116).

#### 6.1.2. Non-Birational Transformation

Let us now consider the non-birational "monomial-like" transformation

$$B : \quad (x, y, z) \quad \longrightarrow \quad \Big(x, \ x^2\,y^2 \cdot (1 + 3\,x), \ \frac{y\,z^3}{1 + 3\,x}\Big). \tag{119}$$

By performing this non-birational monomial transformation (119) on the rational function (103), one obtains the new rational function

$$R_B(x, y, z) \;=\; R\Big(x,\; x^2 y^2 \cdot (1 + 3x),\; \frac{y z^3}{1 + 3x}\Big)$$
$$= \frac{1}{1 - x - x^2 y^2 \cdot (1 + 3x) - y z^3/(1 + 3x)}. \tag{120}$$

The calculation of the telescoper of (120) gives an order-two linear differential operator that has, again, the ${}_2F_1$ hypergeometric series solution:

$${}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1]\; 27 x^3\Big) \;=\; 1 + 6x^3 + 90x^6 + 1680x^9$$
$$+34650 x^{12} + 756756 x^{15} + 17153136 x^{18} + \cdots \tag{121}$$

For the multi-Taylor expansion of (120), one verifies easily that its diagonal is the ${}_2F_1$ hypergeometric series (121). This result can be understood from the results on (117) and the diagonal-preservation results on the birational transformations (108), (111), and (112).

Consequently, we have another infinite set of (non-birational) transformations such that the diagonal of a rational function is changed into the diagonal of that rational function where $x$ is changed into $x^N$.

### 6.2. *Birational Symmetries for Telescopers*

Recalling the creative telescoping Equation (6) and (9), we have verified experimentally, on thousands of examples, that the previous birational transformations generated by (108), (111) and (112), are actually compatible with the creative telescoping Equations (6) and (9). Note however, in the birationally transformed creative telescoping equations, that if the telescoper does remain invariant (*even if the rational function has not a multi-Taylor expansion*), the two "certificates" $U$ and $V$ are transformed in a very involved way (they become quite large rational functions).

#### 6.2.1. Birational Symmetries not Preserving $(x, y, z) = (0, 0, 0)$

Let us consider the involutive birational transformation:

$$I: \quad (x, y, z) \;\longrightarrow\; \Big(\frac{1}{x}, \frac{1}{y}, x^2 y^2 z\Big). \tag{122}$$

This involutive birational transformation transforms the rational function (103) into:

$$R_I(x, y, z) \;=\; -\frac{x y}{x^2 y^3 z - x y + x + y}. \tag{123}$$

The calculation of the telescoper of (123) gives the same telescoper as the telescoper of (103), whose diagonal is the hypergeometric series:

$${}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], 27x\Big)$$
$$= (1 - 24x)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{12}, \frac{5}{12}], [1], \frac{1728 x^3 \cdot (1 - 327x)}{(1 - 24x)^3}\Big) \tag{124}$$
$$= 1 + 6x + 90x^2 + 1680x^3 + 34650x^4 + 756756x^5 + 17153136x^6 + \cdots$$

The hypergeometric series (124) (which is equal to the diagonal of (103)), is, here, just an analytical solution of the telescoper of (123), that is, a "Period" of (123) but corresponding to a non-vanishing cycle since (123) does not have a multi-Taylor expansion.

#### 6.2.2. Birational Symmetries from Collineations

Let us recall Noether's theorem [49,73,74] on the decomposition [75] of Cremona transformations. Noether's theorem shows that any Cremona transformation can be seen as the composition [49,75] of *collineation transformations* and of the Hadamard inverse transformation:

$$(x,\, y) \quad \longrightarrow \quad \Big(\frac{1}{x},\, \frac{1}{y}\Big). \tag{125}$$

Let us consider Cremona transformations preserving $(x,\, y) \,=\, (0,\, 0)$:

$$(x,\, y) \quad \longrightarrow \quad \Big(\frac{x}{1\,-\,x\,+2\,y},\, \frac{y}{1\,-\,x\,+2\,y}\Big). \tag{126}$$

With this theorem in mind, since we have already considered the involutive transformation (122) corresponding to the Hadamard inverse (125), let us just introduce the following birational transformation associated with the *collineation* (126):

$$(x,\, y,\, z) \quad \longrightarrow \quad \Big(\frac{x}{1\,-\,x\,+2\,y},\, \frac{y}{1\,-\,x\,+2\,y},\, z \cdot\, (1\,-x\,+2\,y)^2\Big). \tag{127}$$

This birational transformation (associated with collineations) is an (infinite order) transformation. It preserves $(x,\, y,\, z) \,=\, (0,\, 0,\, 0)$ and the product $p \,=\, x\,y\,z$. Let us perform this birational transformation (127) on the rational function (103). One obtains a new rational function whose telescoper is an order-four linear differential operator $L_4$, which is the product of two order-two linear differential operator $M_2$ and $N_2$: $L_4 \,=\, M_2 \cdot N_2$. The order-two linear differential operator $M_2$ is (non-trivially) homomorphic to the order-two telescoper of the rational function (103). The second order-two linear differential operator $N_2$ corresponds to algebraic functions. For such transformations, associated with collineations, we see that *the telescoper is not preserved*, we just have a (non-trivial) homomorphism property. The example (127) is revisited in detail in Appendix B.4. More examples of birational symmetries for telescopers, associated with collineations, are given in Appendix B. These examples illustrate the complexity of the homomorphism.

### *6.3. Algebraic Geometry Comments on These Birational Symmetries*

The diagonal of the rational function (103) is the hypergeometric series:

$$\begin{aligned}
&{}_2F_1\Big([\frac{1}{3},\, \frac{2}{3}],\, [1],\, 27\,x\Big) \\
&\quad=\; (1\,-24\,x)^{-1/4} \cdot {}_2F_1\Big([\frac{1}{12},\, \frac{5}{12}],\, [1],\, \frac{1728\,x^3 \cdot\, (1\,-327\,x)}{(1\,-24\,x)^3}\Big) \\
&\quad=\; 1\, +6\,x\, +90\,x^2\, +1680\,x^3\, +34650\,x^4\, +756756\,x^5\, +17153136\,x^6\; +\; \cdots
\end{aligned} \tag{128}$$

The algebraic curve, associated with the denominator of the rational function (103), is the *genus-one* algebraic curve (elliptic curve):

$$1 - x\, - y\, - \frac{p}{x\,y} \;=\; 0 \qquad \text{or:} \qquad -\,x^2\,y\, - x\,y^2\, + x\,y\, - p \;=\; 0. \tag{129}$$

The calculation of its j-invariant gives the following Hauptmodul:

$$\mathcal{H} \;=\; \frac{1728}{j} \;=\; \frac{1728\,p^3 \cdot\, (1\,-27\,p)}{(1\,-24\,p)^3}, \tag{130}$$

which is exactly the Hauptmodul pullback in (128).

Let us consider the rational function (107); the algebraic curve corresponding to eliminate $z = p/x/y$ in the denominator of (107) reads:

$$-49\,x^5\,y^2\; - 42\,x^4\,y^2\; - 7\,x^4\,y\; - 23\,x^3\,y^2\; + 4\,x^3\,y\; - 6\,x^2\,y^2 \\ +2\,x^2\,y\; - x\,y^2\; + x\,y\; - p \;\;=\;\; 0. \tag{131}$$

This algebraic curve is a *genus-one* algebraic curve (elliptic curve), and the calculation of its j-invariant gives the same Hauptmodul pullback in (128) as the Hauptmodul (130) for (129). This is in agreement with the fact that the diagonals of (103) and (107) are equal. At first sight, the fact that (131) is an elliptic curve is not totally obvious; however, it is a consequence of the fact that (129) and (131) are *birationally equivalent elliptic curves* (since one obtains one from the other one from a birational transformation). Consequently, they should have the same j-invariant.

This kind of remark will be seen as obvious, or slightly tautological, for an algebraic geometer; however, as far as down-to-earth computer algebra calculations of diagonals of rational functions or telescopers of rational functions are concerned, it becomes more and more spectacular for more complicated birational transformations generated by the composition of birational transformations like (108), (111), and (112).

More generally, the previous birational transformations preserving the product $p = xyz,\; p = xyzu, \ldots$ occurring in the diagonals will preserve the algebraic geometry description of the diagonal of rational functions [40]. For instance, the genus-two curves associated with the split Jacobians situation we have encountered in [40] (which corresponds to products of elliptic curves) will be preserved by such birational transformations.

### 6.4. Diagonal of Transcendental Functions

Generalizing the rationals functions

$$R_B(x,y,z) \;=\; R\Big(x,\; y\cdot Q_1(x),\; \frac{z}{Q_1(x)}\Big) \;=\; \frac{1}{1 - x - y\cdot Q_1(x) - z/Q_1(x)}, \tag{132}$$

deduced from (107), using birational transformations like (108), one can consider beyond *transcendental* functions like

$$R_T(x,y,z) \;=\; R\Big(x,\; y\cdot\cos(x),\; \frac{z}{\cos(x)}\Big) \;=\; \frac{1}{1\; - x\; - y\cdot\cos(x)\; - z/\cos(x)}. \tag{133}$$

One verifies easily, from the multi-Taylor expansion of the (simple) transcendental function (133), that its diagonal *is actually the same as the one* of (103), namely, (128). This is not a surprise since the demonstration of the invariance of the diagonal by birational transformation sketched in section 6 (see (109)) just requires that $Q_1(0) \neq 0$ with $Q_1(x)$ behaving at the origin as a polynomial.

## 7. Conclusions

Diagonals of rational functions have been shown to emerge naturally for $n$-fold integrals in physics, field theory, and enumerative combinatorics, seen as "Periods" of algebraic varieties (corresponding to the denominators of these rational functions). Of the thousands of examples we have analyzed, corresponding to $n$-fold integrals of theoretical physics (in particular the $\chi^{(n)}$'s of the susceptibility of the Ising model, ... ) or corresponding to the rather academic diagonal of rational functions, we have seen the emergence of many striking properties, and we want to understand if these remarkable properties are inherited from the "physics", and, more precisely, the rather "integrable" framework of these examples (Yang–Baxter integrability, 2D Ising models, Calabi–Yau and other mirror symmetries, ... ), or, on the contrary, if they are a consequence of the remarkable nature of diagonals of rational functions in the most general framework.

This paper is a plea for diagonals of rational or algebraic functions and more generally telescopers of rational or algebraic functions.

- We show that "periods" corresponding to non-vanishing cycles, obtained as solutions of telescopers of rational functions, can sometimes be recovered from diagonals of rational functions corresponding to vanishing cycles, introducing an extra parameter. These two concepts are not that compartmentalized.
- When considering the diagonals of rational functions we have shown that the number of variables of a rational function must, from time to time, be replaced by a notion of "effective number" of variables.
- We have shown that the "complexity" of the diagonals of a rational function, and, for instance, the order of the (minimal order) linear differential operator annihilating this diagonal, is not related to the number of variables or the "effective number" of variables of the rational function. In a forthcoming publication, we will try to understand what is the minimal number of variables necessary to represent a given D-finite globally bounded series as a diagonal of a rational function.
- We have shown that the algebraic geometry approach of the diagonals of rational functions, or of the telescopers of these rational functions, described in [40], can, probably, be generalized to diagonals of algebraic functions, or the telescoper of algebraic functions. These are just preliminary studies, and almost everything remains to be done.
- When studying diagonals of rational functions, our explicit examples enable one to understand why one can actually be restricted to rational functions of the form $1/Q$ provided the polynomial at the denominator is irreducible. The situation where the denominator $Q$ factorizes clearly needs further analysis, which will be displayed in a forthcoming paper. The case of the calculations of telescopers is slightly different: one can (probably), again, be restricted to rational functions of the form $1/Q$ but with a *finite set* of polynomials $Q$.
- We have shown that diagonals of rational functions (and this is also the case with diagonals of algebraic functions) are left-invariant when one performs an *infinite set of birational transformations* on the rational functions. This remarkable result can, in fact, be generalized to an *infinite set of rational transformations*, with the diagonals of the transformed rational functions becoming the diagonal of the original rational function where the variable $x$ is changed into $x^n$. These invariance results generalize to telescopers. A more general (infinite) set of birational transformations is shown to correspond to a more convoluted "covariance" property of the telescopers (see Appendix B).
- We provide some examples of diagonals of transcendental functions that can also yield simple $_2F_1$ hypergeometric functions associated with elliptic curves. The analysis of diagonal of transcendental functions is clearly an interesting new domain to study. Accordingly, we thank one of the referees for his remark of a link to recent preprints of Golyshev et al. [76], where the classical Clausen–Sonin–Gegenbauer formulae are interpreted as special degenerated cases of the more general "multiplication kernel" setting developed by Kontsevich and Odesskii [77] (these formulae can be seen as examples of "diagonal" forms of generating functions for the multiplication kernels).
- Finally, when trying to understand the puzzling fact that telescopers of rational functions are almost always homomorphic to their adjoint and thus have selected symplectic or orthogonal differential Galois groups, we understand a bit better the emergence of curious examples of telescopers that are not homomorphic to their adjoint; this (up to homomorphisms) *self-duality-breaking* rules out a Poincaré duality interpretation of this quite systematic emergence of operators homomorphic to their adjoint. A "desingularization" of such puzzling cases, corresponding to the introduction of an extra parameter, shows that such operators now occur in dual (adjoint) pairs, thus restoring the duality (homomorphism to the adjoint). The limit when the extra parameter goes to zero is the *direct sum* of different telescopers corresponding to the

first rational function terms of the expansion of the extended rational function in term of this extra parameter. With Section 5.2, we see that the puzzling (non self-adjoint up to homomorphism) order-three linear differential operator $L_3$ with $SL(3, \mathbb{C})$ differential Galois group is better understood as a member of a triplet of three "quarks" (90), (91), and (92), which restores the duality. This may suggest that the quite strange ${}_3F_2$ hypergeometric functions (91) or (92) could be related to (90), which has a clear elliptic curve origin. After all, these functions are three periods of the same algebraic variety. The existence of such a relation between hypergeometric functions of a totally and utterly different nature is a challenging open question.

- In Appendix B, the calculations of telescopers of rational functions, associated with very simple collineations, yield quite massive linear differential operators, which factor into an order-two operator associated with an elliptic curve, and a "dressing" of products of factors, which turn out to be direct sums of operators with algebraic function solutions. This occurrence of this "mix" between products and direct sums of a large number of operators (occurring, for instance, for the linear differential operators annihilating the $\chi^{(n)}$ components of the susceptibility of the Ising model [1,26]) will be revisited in a forthcoming paper.

Instead of pursuing one specific mathematical problem, this paper can be seen as a journey into the amazing world of integer sequences and differential equations. With all the examples displayed in this paper, we provide some answers, sometimes involving some plausible scenarios, to many important questions naturally emerging when working on diagonals of rational or algebraic functions, or on telescopers of rational or algebraic functions related, or not related, to problems of physics or enumerative combinatorics. Like any fruitful concept, every answered questions does not "close" the subject but, on the contrary, often raises more new questions than the number of answered questions.

Diagonals of rational or algebraic functions correspond to (globally bounded) series that can be recast into series with integer coefficients and are solutions of *linear* differential operators. When studying the two dimensional Ising model and its related *Painlevé equations*, one finds that the $\lambda$-extensions of the correlation functions [78,79] can also produce series with integer coefficients that are differentially algebraic [80] solutions of *non-linear* differential equations of the Painlevé type. These series are, also, such that their reduction modulo primes give algebraic functions, just like diagonals of rational or algebraic functions (for other examples of differentially algebraic series with integer coefficients see, for instance, [81]).

This paper tries to show that the concept of diagonals of rational or algebraic functions is a remarkably rich and fruitful concept not only providing tools for physics but also bridging, in a quite fascinating way, different domains of mathematics. The case of the diagonal of transcendental functions, or of these $\lambda$-extensions, seems to show that the "unreasonable richness" of diagonals and telescopers may just be the tip of an even more fascinating mathematical "iceberg" of mathematical physics.

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## Appendix A. Other $\alpha$-Dependent Example

### *Appendix A.1. A First very Simple Example*

Another example, similar to the rational function (87) studied in Section 5.2, is

$$\frac{1}{1 \;\; - x^2 - y^2 - z^2 \;\; - \alpha \cdot \, x\, y^2}. \tag{A1}$$

The telescoper is an order-four linear differential operator that becomes, in the $\alpha \, \rightarrow \, 0$ limit, the LCLM of two order-two linear differential operators, with one, $L_2$, corresponding to the hypergeometric solution (which is actually the $\alpha \, = \, 0$ diagonal)

$${}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], \, [1], \; 27\, x^2\Big), \tag{A2}$$

and an order-two linear differential operator $M_2$ having the solution

$$\frac{d}{dx}\, {}_2F_1\Big([\frac{1}{6}, \frac{5}{6}], \, [1], \; 27\, x^2\Big). \tag{A3}$$

The order-two operator $M_2$ is not homomorphic to the order-two operator $L_2$. Let us consider the $\alpha$ expansion of (A1)

$$\begin{aligned} \frac{1}{1 \; - x^2 - y^2 - z^2 \; - \alpha \cdot \, x\, y^2} \;\; &= \;\; \frac{1}{1 \; - x^2 - y^2 - z^2} \;\; + \frac{x\, y^2}{(1 \; - x^2 - y^2 - z^2)^2} \cdot \alpha \\ &+ \frac{x^2\, y^4}{(1 \; - x^2 - y^2 - z^2)^3} \cdot \alpha^2 \;\;\; + \; \cdots \end{aligned} \tag{A4}$$

The diagonal of the term in $\alpha^1$ in (A4) is trivial: it is equal to zero. In contrast, the telescoper of the term in $\alpha^1$ in (A4) is actually nothing but the order-two linear differential operator $M_2$. The telescoper of the term in $\alpha^2$ in (A4) is an order-two linear differential operator homomorphic to the previous order-two linear differential operator $L_2$. Similarly to the calculations displayed in (87), the telescopers for the terms in $\alpha^{2\,n}$ in the expansion (A4) yield order-two linear differential operators, homomorphic to $L_2$, and the telescopers for the terms in $\alpha^{2\,n\,+1}$ yield order-two operators, homomorphic to $M_2$.

### *Appendix A.2. Christol: Breaking the Duality Symmetry*

These results can be compared with ones for the diagonal of the rational function

$$\frac{1}{1 \;\; - x^4 - y^4 - z^4 \;\; - \alpha \cdot \, x}. \tag{A5}$$

The linear differential operator annihilating the diagonal of the rational function (A5) is an order-ten linear differential operator $L_{10}(\alpha)$ depending on the parameter $\alpha$, and is homomorphic to its adjoint with an order-eight intertwiner. Consequently, its differential Galois group is included in $Sp(10, \, \mathbb{C})$. The order-ten linear differential operator $L_{10}(\alpha)$ is irreducible except at $\alpha \, = \, 0$.

At $\alpha = 0$, it is the direct sum $LCLM(L_2, M_2, L_3, M_3)$, of two order-three linear differential operators and two order-two linear differential operators, namely, $L_2$ corresponding to the solution

$$
{}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], 27\,x^4\Big) \\
= \quad 1 \ + 6\,x^4 \ + 90\,x^8 \ + 1680\,x^{12} \ + 34650\,x^{16} \ + 756756\,x^{20} \ + \ \cdots \tag{A6}
$$

as it should (this is the diagonal of (A5) at $\alpha = 0$), and the other one, $M_2$, corresponding to the globally bounded series solution expressed in terms of HeunG functions (use Table page 24 of [64]):

$$
\frac{(1 - 24\,x^4)^2}{(1 - 27\,x^4)^2} \cdot HeunG\Big(\frac{9}{8}, \frac{97}{32}, \frac{7}{6}, \frac{5}{6}, 1, -1;\ 27 \cdot x^4\Big). \tag{A7}
$$

The linear differential operator $M_2$ is homomorphic to the order-two linear differential operator corresponding to the modular form (see Appendix B in [16])

$$
{}_2F_1\Big([\frac{1}{6}, \frac{5}{6}], [1], 27\,x^4\Big). \tag{A8}
$$

Using the identity

$$
HeunG\Big(\frac{9}{8}, \frac{97}{32}, \frac{7}{6}, \frac{5}{6}, 1, -1;\ 27 \cdot x\Big) \ = \\
4 \cdot (1 - 27\,x) \cdot \frac{(27\,x + 2)}{(1 - 24\,x)^2} \cdot x \cdot \frac{d}{dx}\, {}_2F_1\Big([\frac{1}{6}, \frac{5}{6}], [1], 27\,x\Big) \\
+ \frac{1\,9\,x - 486\,x^2}{(1 - 24\,x)^2} \cdot {}_2F_1\Big([\frac{1}{6}, \frac{5}{6}], [1], 27\,x\Big), \tag{A9}
$$

the solution (A7) is rewritten in terms of the modular form (A8). The solution of $M_2$, thus, reads

$$
\frac{2 + 27\,x^4}{1 - 27\,x^4} \cdot x \cdot \frac{d}{dx}\, {}_2F_1\Big([\frac{1}{6}, \frac{5}{6}], [1], 27\,x^4\Big) \ + \frac{1 + 18\,x^4}{1 - 27\,x^4} \cdot x^4 \cdot {}_2F_1\Big([\frac{1}{6}, \frac{5}{6}], [1], 27\,x^4\Big) \\
= \quad 1 \ + \frac{315}{4}\,x^4 \ + \frac{225225}{64}\,x^8 \ + \frac{33948915}{256}\,x^{12} \ + \frac{75293843625}{16384}\,x^{16} \\
+ \frac{9927744261435}{65536}\,x^{20} \ + \ \cdots \tag{A10}
$$

The solution of the order-three linear differential operator $L_3$ is

$$
{}_3F_2\Big([\frac{7}{12}, \frac{11}{12}, \frac{15}{12}], [\frac{3}{4}, 1], 27\,x^4\Big), \tag{A11}
$$

while the order-three linear differential operator $M_3$ has as solution

$$
{}_3F_2\Big([\frac{13}{12}, \frac{17}{12}, \frac{21}{12}], [\frac{1}{4}, 1], 27\,x^4\Big). \tag{A12}
$$

These two linear differential operators are such that $L_3$ is actually homomorphic to the adjoint of $M_3$, and, of course, $M_3$ is homomorphic to the adjoint of $L_3$, but $L_3$ *is not* homomorphic to the adjoint of $L_3$ (and $M_3$ is not homomorphic to the adjoint of $M_3$). We have, again, a pair of dual linear differential operators.

Since these calculations are in the $\alpha \rightarrow 0$ limit, let us expand in $\alpha$ the rational function (A5):

$$\frac{1}{1 - x^4 - y^4 - z^4 - \alpha \cdot x} = \frac{1}{1 - x^4 - y^4 - z^4} + \frac{x}{(1 - x^4 - y^4 - z^4)^2} \cdot \alpha$$
$$+ \frac{x^2}{(1 - x^4 - y^4 - z^4)^3} \cdot \alpha^2 + \frac{x^3}{(1 - x^4 - y^4 - z^4)^4} \cdot \alpha^3$$
$$+ \frac{x^4}{(1 - x^4 - y^4 - z^4)^5} \cdot \alpha^4 + \cdots \tag{A13}$$

Since the telescoper of a sum of rational functions is the direct sum (LCLM) of the telescopers of these rational functions, let us consider the telescopers of the first five terms in the RHS of (A13). The telescoper of the first term is of course the order-two linear differential operator $L_2$ annihilating the diagonal of this rational function. The telescoper of the second term (in $\alpha^1$) is the order-three linear differential operator $L_3$. The telescoper of the third term (in $\alpha^2$) is the order-two linear differential operator $M_2$. The telescoper of the fourth term (in $\alpha^3$) is exactly $M_3$. The telescoper of the sum of the first orders in $\alpha$ in the expansion (A13)

$$\frac{1}{1 - x^4 - y^4 - z^4} + \frac{x}{(1 - x^4 - y^4 - z^4)^2} \cdot \alpha$$
$$+ \frac{x^2}{(1 - x^4 - y^4 - z^4)^3} \cdot \alpha^2 + \frac{x^3}{(1 - x^4 - y^4 - z^4)^4} \cdot \alpha^3, \tag{A14}$$

is actually the LCLM of the four telescopers $L_2$, $M_2$, $L_3$, and $M_3$, *and is precisely the $\alpha \rightarrow 0$ limit of the order-ten linear differential operator.*

Let us now consider the telescopers of the next $\alpha$ orders in the expansion (A13). The telescoper of the last rational function in (A13), namely, $x^4/(1 - x^4 - y^4 - z^4)^5$, is an order-two linear differential operator $N_2$. One can thus write the solution of $N_2$ as:

$$\mathcal{D}_1 = \frac{3}{48} \cdot \frac{1 + 540\,x^4 + 4374\,x^8}{(1 - 27\,x^4)^3} \cdot x \cdot \frac{d}{dx}\, {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], 27\,x^4\Big)$$
$$+ \frac{3}{2} \cdot \frac{(19 + 216\,x^4)}{(1 - 27\,x^4)^3} \cdot x^4 \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], 27\,x^4\Big) \tag{A15}$$
$$= 30\,x^4 + 3780\,x^8 + 277200\,x^{12} + 15765750\,x^{16} + 771891120\,x^{20} + \cdots$$

The telescoper of

$$\frac{x^8}{(1 - x^4 - y^4 - z^4)^9}, \tag{A16}$$

is an order-two linear differential operator whose analytic solution reads:

$$\mathcal{D}_2 = -\frac{3}{672} \cdot \frac{p_1}{(1 - 27\,x^4)^7} \cdot x \cdot \frac{d}{dx}\, {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], 27\,x^4\Big)$$
$$+ \frac{3}{28} \cdot \frac{p_2}{(1 - 27\,x^4)^7} \cdot x^4 \cdot {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], 27\,x^4\Big) \tag{A17}$$
$$= 2970\,x^8 + 900900\,x^{12} + 137837700\,x^{16} + 14665931280\,x^{20}$$
$$+ 1236826871280\,x^{24} + 88597190167200\,x^{28} + \cdots$$

where:

$$p_1 = 1 - 714\,x^4 - 924372\,x^8 - 54587520\,x^{12} - 530141922\,x^{16} - 554824404\,x^{20},$$
$$p_2 = 1 + 27030\,x^4 + 2062098\,x^8 + 23960772\,x^{12} + 29170206\,x^{16}. \tag{A18}$$

If we consider, instead of the telescoper, the diagonal of the rational function (A13), only the terms in $\alpha^{4n}$ $n = 0, 1, 2, \cdots$ will contribute; the other ones, corresponding to *non-vanishing cycles* [55], will make zero contribution. Consequently, we obtain for the diagonal of the rational function (A13):

$$\mathrm{Diag}\Big(\frac{1}{1 - x^4 - y^4 - z^4 - \alpha \cdot x}\Big) = {}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], 27\,x^4\Big) + \mathcal{D}_1 \cdot \alpha^4 + \mathcal{D}_2 \cdot \alpha^8 + \cdots \tag{A19}$$

$$\begin{aligned} = \; & 1 + (30\,\alpha^4 + 6) \cdot x^4 + (2970\,\alpha^8 + 3780\,\alpha^4 + 90) \cdot x^8 \\ & + (371280\,\alpha^{12} + 900900\,\alpha^8 + 277200\,\alpha^4 + 1680) \cdot x^{12} \\ & + (51482970\,\alpha^{16} + 185175900\,\alpha^{12} + 137837700\,\alpha^8 + 15765750\,\alpha^4 + 34650) \cdot x^{16} \\ & + (7571343780\,\alpha^{20} + 36141044940\,\alpha^{16} + 44975522592\,\alpha^{12} \\ & \quad + 14665931280\,\alpha^8 + 771891120\,\alpha^4 + 756756) \cdot x^{20} + \cdots \\ = \; & 1 + 6\,x^4 + 90\,x^8 + 1680\,x^{12} + 34650\,x^{16} + 756756\,x^{20} + \cdots \\ & + \Big(30\,x^4 + 3780\,x^8 + 277200\,x^{12} + 15765750\,x^{16} + 771891120\,x^{20} + \cdots\Big) \cdot \alpha^4 \\ & + \Big(2970\,x^8 + 900900\,x^{12} + 137837700\,x^{16} + 14665931280\,x^{20} + \cdots\Big) \cdot \alpha^8 + \cdots \end{aligned}$$

## Appendix B. Birational Symmetries from Collineations

### *Appendix B.1. Birational Symmetries from Collineations: A First Example*

Let us consider a collineation transformation not preserving $(x, y) = (0, 0)$:

$$(x, y) \quad \longrightarrow \quad \Big(\frac{2 + x + 3\,y}{1 - x + 2\,y}, \; \frac{1 + 5\,x + 7\,y}{1 - x + 2\,y}\Big), \tag{A20}$$

and let us now introduce the following *birational transformation* associated with the collineation (A20):

$$(x, y, z) \quad \longrightarrow \quad \Big(\frac{2 + x + 3\,y}{1 - x + 2\,y}, \; \frac{1 + 5\,x + 7\,y}{1 - x + 2\,y}, \; \frac{x\,y\,z \cdot (1 - x + 2\,y)^2}{(2 + x + 3\,y) \cdot (1 + 5\,x + 7\,y)}\Big), \tag{A21}$$

which preserves the product $p = x\,y\,z$.

Let us transform the simple rational function (103) with the birational transformation (A21). It becomes the rational function:

$$\mathcal{R} = \frac{(1 - x + 2\,y) \cdot (2 + x + 3\,y) \cdot (1 + 5\,x + 7\,y)}{\mathcal{D}}, \tag{A22}$$

where the denominator $\mathcal{D}$ reads:

$$\begin{aligned} \mathcal{D} = \; & x^4\,y\,z - 6\,x^3\,y^2\,z + 12\,x^2\,y^3\,z - 8\,x\,y^4\,z - 3\,x^3\,y\,z + 12\,x^2\,y^2\,z - 12\,x\,y^3\,z \\ & + 3\,x^2\,y\,z - 6\,x\,y^2\,z - 35\,x^3 - 194\,x^2\,y - 323\,x\,y^2 - x\,y\,z - 168\,y^3 - 87\,x^2 \\ & - 251\,x\,y - 178\,y^2 - 36\,x - 50\,y - 4. \end{aligned} \tag{A23}$$

The intersection of the algebraic surface $\mathcal{D} = 0$ with the algebraic surface $p = x\,y\,z$ is an elliptic curve. One obtains, almost instantaneously (using the *j_invariant* command in Maple with(algcurves)), the Hauptmodul of this elliptic curve:

$$\mathcal{H} = \frac{1728\,p^3 \cdot (1 - 27\,p)}{(1 - 24\,p)^3}. \tag{A24}$$

If one expects an *algebraic geometry interpretation* of the calculation of the diagonal of rational functions or telescopers [40], this Hauptmodul must be the same as the Hauptmodul (130) of the elliptic curve (129) since the two algebraic curves are birationaly equivalent, being related by a birational transformation, namely, (A20). The calculation of the telescoper of (A22) is really massive: it gives, after one month of computation, an order-eleven linear differential operator (we thank C. Koutschan for performing these slightly "extreme" computations). The result being too massive, let us consider other examples of birational transformations associated with collineations simpler than (A21).

**Remark A1.** *The diagonal of the rational function (A22) is a very simple series:*

$$Diag\Big(\mathcal{R}\Big) \;=\; -\frac{1}{2}\cdot\frac{1}{1+x/4}$$
$$= \; -\frac{1}{2}+\frac{1}{8}\cdot x-\frac{1}{32}\cdot x^2+\frac{1}{128}\cdot x^3-\frac{1}{512}\cdot x^4 \;+\;\cdots \tag{A25}$$

**Remark A2.** *If one considers, instead of (A22), the rational function with the same denominator (A23) but where the numerator is normalized to* $1$,

$$\mathcal{R} \;=\; \frac{1}{\mathcal{D}}. \tag{A26}$$

*The diagonal of (A26) is the same as (A25) up to factor two:*

$$Diag\Big(\mathcal{R}\Big) \;=\; -\frac{1}{4}\cdot\frac{1}{1+x/4}. \tag{A27}$$

*The telescoper of (A26) is an order-seven linear differential operator that factorizes as follows:*

$$L_7 \;=\; F_2\cdot G_2\cdot H_2\cdot H_1 \qquad \textit{with:} \qquad H_1 \;=\; D_x+\frac{1}{4+x}, \tag{A28}$$

*where the order-two linear differential operator* $F_2$ *is quite large and is (non-trivially) homomorphic to the order-two linear differential operator* $L_2$*, which is the telescoper of the rational function (103), and where the order-two linear differential operators* $G_2$ *and* $H_2$ *have algebraic solutions. The diagonal (A27) is solution of the order-one operator* $H_1$*. The homomorphism between* $F_2$ *and* $L_2$ *gives*

$$F_2\cdot X_1 \;=\; Y_1\cdot L_2 \qquad \textit{where:} \qquad X_1 \;=\; A(x)\cdot D_x\;+B(x), \tag{A29}$$

*where* $A(x)$ *and* $B(x)$ *are rational functions. Consequently, a solution* $\mathcal{S}$ *of the telescoper* $L_7$ *(but not of the product* $G_2\,H_2\,H_1$ *in (A28)) will be related to the hypergeometric solution* ${}_2F_1([1/3,2/3],[1],27\,x)$ *of the order-two linear differential operator* $L_2$*, as follows:*

$$X_1\Big({}_2F_1\Big([\frac{1}{3},\frac{2}{3}],[1],27\,x\Big)\Big) \;=\; G_2\cdot H_2\cdot H_1\cdot\mathcal{S}. \tag{A30}$$

**Remark A3.** *Note that the diagonal of the rational function (A22) is a very simple series (A25). Therefore, the solution* $\mathcal{S}$ *of the telescoper, associated with an elliptic curve of Hauptmodul (A24) (see equation (A30)), corresponds to a "period", an integral over a non-vanishing cycle, and is different from the integral over a vanishing cycle, namely, the diagonal (A25).*

**Remark A4.** *The factorization (A28) is far from being unique. The product of the last three factors can be seen to be a direct sum:*

$$G_2\cdot H_2\cdot H_1 \;=\; \tilde{G}_2\oplus\tilde{H}_2\oplus H_1, \tag{A31}$$

*where the two new order-two operators* $\tilde{G}_2$ *and* $\tilde{H}_2$ *are simpler, with, again, algebraic function solutions.*

*Appendix B.2. Birational Symmetries from Collineations. A Simpler Example*

Let us consider the following birational transformation associated with a collineation:

$$(x,\, y,\, z) \quad \longrightarrow \quad \Big(\frac{x\,+3\,y}{1\,-\,x\,+2\,y},\ \frac{1\,+5\,x\,+y}{1\,-\,x\,+2\,y},\ \frac{x\,y\,z\cdot\,(1\,-x\,+2\,y)^2}{(x\,+3\,y)\cdot\,(1\,+5\,x\,+7\,y)}\Big), \tag{A32}$$

which preserves the product $p \,=\, x\,y\,z$. Again, if one transforms the simple rational function (103) with the birational transformation (A32), one obtains the rational function of the form

$$\mathcal{R} \;=\; \frac{(1\,-\,x\,+2\,y)\cdot\,(x\,+3\,y)\cdot\,(1\,+5\,x\,+y)}{\mathcal{D}}, \tag{A33}$$

and, again, the intersection of the algebraic surface $\mathcal{D} \,=\, 0$ with the algebraic surface $p \,=\, x\,y\,z$ is an elliptic curve, corresponding to eliminate $z \,=\, p/x/y$ in $\mathcal{D} \,=\, 0$. One immediately obtains the same Hauptmodul (A24) for this new elliptic curve.

The telescoper of the rational function (A33) is an order-ten linear differential operator (we thank C. Koutschan for providing this order-ten linear differential operator). This telescoper is obtained using about nine days of computation time. It uses 286 evaluation points (in contrast with the 462 evaluation points required for (A23)), and one uses in total 38 primes (of size $9 \cdot 22 \cdot 10^{18}$) to reconstruct the solution with Chinese remaindering. The telescoper of the rational function (A33) factors as follows:

$$L_{10} \;=\; F_2 \cdot\, G_2 \cdot\, H_1, \cdot\, I_1 \,\cdot\, J_2 \cdot\, K_2, \tag{A34}$$

The order-two linear differential operator $F_2$ in (A34) is homomorphic to the order-two linear differential operator $L_2$, which is the telescoper of the rational function (103). The the order-two linear differential operators $G_2$, $J_2$, and $K_2$ have algebraic solutions.

**Remark A5.** *The factorization of (A34) is far from being unique. As usual, we have a mix between product and the direct-sum of factors. Since the order-ten operator is quite large, it is difficult to obtain the direct-sum factorization of* $L_{10}$ *in (A34). One finds, however, quite easily that* $L_{10}$ *has two simple rational function solutions*

$$\frac{1}{(x-35)\cdot\,(4\,x\,+3)}, \qquad\qquad \frac{x}{(x-35)\cdot\,(4\,x\,+3)}, \tag{A35}$$

*corresponding to two order-one operators* $L_1 \;=\; D_x \,+ (8\,x - 137)/(4\,x+3)/(x-35)$ *and* $M_1 \,=\, D_x \,+ (4\,x+3)/(x+21)/(x-35) \,-1/x$ *and, thus, can be rightdivided by the LCLM of* $L_1$ *and* $M_1$*. In fact, the product of the last factors at the right of the factorization of* $L_{10}$ *can be seen to be a direct sum:*

$$G_2 \cdot\, H_1 \,\cdot\, I_1 \cdot\, J_2 \cdot\, K_2 \;=\; L_1 \oplus M_1 \oplus \tilde{G}_2 \oplus \tilde{J}_2 \oplus K_2. \tag{A36}$$

*In contrast, the product* $F_2 \cdot\, G_2$ *is* not *a direct sum. The order-two operators* $\tilde{G}_2$ *and* $\tilde{J}_2$ *are (much) simpler than* $G_2$ *and* $J_2$*, again with algebraic function solutions.*

*The result remaining still too large, let us consider another example of birational transformation associated with collineations, simpler than (A21) or (A32).*

**Remark A6.** *If one considers, instead of (A33), the rational function with the same denominator $\mathcal{D}$ but where the numerator is normalized to* $1$,

$$\mathcal{R} \, = \, \frac{1}{\mathcal{D}}. \tag{A37}$$

*The telescoper of the rational function (A37) is an order-seven linear differential operator*

$$L_7 \, = \, F_2 \cdot G_1 \cdot G_2 \cdot H_2, \tag{A38}$$

*where the order-two linear differential operator $F_2$ is (non-trivially) homomorphic to the order-two linear differential operator $L_2$, which is the telescoper of the rational function (103), and where the order-two linear differential operators $G_2$ and $H_2$ have simple algebraic solutions. This factorization (A38) is not unique. Introducing the order-one operator $\tilde{G}_1 \, = \, D_x \, + 1/x$, one can see that $\tilde{G}_1$ rightdivides $L_7$ and that the product of the three factors, at the right of the decomposition (A38), can be written as a direct sum*

$$G_1 \cdot G_2 \cdot H_2 \, = \, \tilde{G}_1 \, \oplus \, \tilde{G}_2 \oplus \, H_2, \tag{A39}$$

*where the solutions of $\tilde{G}_2$ are algebraic.*

**Remark A7.** *In Appendix B, we encounter many order-two linear differential operators with algebraic solutions. Even for large order-two linear differential operators, one can see quite easily (using hypergeometricsols in DEtools of Maple) that the log-derivatives of these solutions are algebraic functions, but finding the algebraic expression (minimal polynomial) of the solutions is much harder. Just showing that the solutions are algebraic without having their exact expressions can be achieved by showing that their $p$-curvatures are zero, recalling the André–Christol conjecture that one must have a basis of globally bounded solutions or looking for rational solutions of symmetric powers of the operators. In principle, these algebraic functions solutions of order-two linear differential operators can be written as pullbacked ${}_2F_1$ hypergeometric functions, but again it is a difficult task [82].*

*Appendix B.3. Birational Symmetries from Collineations: An Even Simpler Example*

Let us consider the following birational transformation associated with a collineation:

$$(x, \, y, \, z) \quad \longrightarrow \quad \Big(\frac{x\, +3\,y}{1\, -\, x\, +2\,y}, \, \frac{5\,x\, +7\,y}{1\, -\, x\, +2\,y}, \, \frac{x\,y\,z \cdot \, (1\, -x\, +2\,y)^2}{(x\, +3\,y) \cdot \, (5\,x\, +7\,y)}\Big), \tag{A40}$$

which preserves the product $p \, = \, x\,y\,z$ and the origin $(x, \, y, \, z) \, = \, (0, \, 0, \, 0)$. Again, if one transforms the simple rational function (103) with the birational transformation (A40), one obtains the rational function of the form:

$$\mathcal{R} \, = \, \frac{(1\, -\, x\, +2\,y) \cdot \, (x\, +3\,y) \cdot \, (5\,x\, +7\,y)}{\mathcal{D}}, \tag{A41}$$

and, again, the intersection of the algebraic surface $\mathcal{D} \, = \, 0$ with the algebraic surface $p \, = \, x\,y\,z$ is an elliptic curve, corresponding to eliminate $z \, = \, p/x/y$ in $\mathcal{D} \, = \, 0$. One immediately obtains the same Hauptmodul (A24) for this new elliptic curve. The telescoper of the rational function (A41) is an order-ten linear differential operator

$$L_{10} \, = \, F_2 \cdot G_2 \cdot H_1 \, \cdot I_1 \cdot J_2 \cdot K_2, \tag{A42}$$

where the order-two linear differential operator $F_2$ is a quite "massive" operator (30,391 characters), which is (non-trivially) homomorphic to the order-two linear differential operator $L_2$ which is the telescoper of the rational function (103) and where the solutions of $G_2$,

$J_2$, and $K_2$ are two *algebraic functions*. The order-two linear differential operator $F_2$ is of the form

$$F_2 \; = \; D_x^2 \, + \frac{A_1(x)}{D_1(x)} \cdot D_x \, + \frac{A_0(x)}{D_0(x)}, \tag{A43}$$

where $A_1(x)$ and $A_0(x)$ are polynomials of degree 41 and 55, respectively, where $D_1(x)$ and $D_0(x)$ read

$$D_1(x) \; = \; \lambda(x) \cdot P_{14}(x) \cdot P_{20}(x), \qquad D_0(x) \; = \; x \cdot \lambda(x) \cdot P_{14}(x) \cdot P_{20}(x)^2, \tag{A44}$$

with:

$$\begin{aligned} \lambda(x) \; = \; & (219024 - 6916931\,x - 23604075\,x^2) \cdot (7 - 225\,x) \cdot (5 - 243\,x) \\ & \times (1 - 27\,x) \cdot (35 - x) \cdot (21 + x) \cdot x, \end{aligned} \tag{A45}$$

where $P_{14}(x)$ and $P_{20}(x)$ are polynomials of degrees 14 and 20, respectively. The order-two operator linear differential $G_2$ yielding algebraic solutions is also quite a "large" linear differential operator.

**Remark A8.** *The factorization of (A42) is far from being unique. As usual, we have a mix between the product and direct-sum of factors. With the order-ten linear differential operator being quite large, it is difficult to obtain the direct-sum factorization of $L_{10}$ in (A42). One finds, however, quite easily that $L_{10}$ has two simple rational function solutions*

$$\frac{1}{(x-35) \cdot (x+21)}, \qquad \frac{x}{(x-35) \cdot (x+21)}, \tag{A46}$$

*corresponding to two order-one operators $L_1 = D_x + 2\,(x-7)/(x+21)/(x-35)$ and $M_1 = D_x + 2\,(x-7)/(x+21)/(x-35) - 1/x$ and, thus, can be rightdivided by the LCLM of $L_1$ and $M_1$. More interestingly, the product $H_1 \cdot I_1 \cdot J_2 \cdot K_2$ in the decomposition (A42) of $L_{10}$ can be seen as the direct sum of $L_1$, $M_1$, and $K_2$ and two new (and simpler) order-two linear differential operators $\tilde{G}_2$ and $\tilde{J}_2$:*

$$G_2 \cdot H_1 \cdot I_1 \cdot J_2 \cdot K_2 \; = \; L_1 \oplus M_1 \oplus \tilde{G}_2 \oplus \tilde{J}_2 \oplus K_2. \tag{A47}$$

*In contrast, note that the product $F_2 \cdot G_2$ in the decomposition (A42) is* not *a direct-sum. It was easy to see that the log-derivatives of the solutions of the order-two operator $J_2$ were algebraic functions but harder to see that these solutions were actually algebraic. One now finds immediately that the solutions of $\tilde{J}_2$ are algebraic functions.*

**Remark A9.** *If one considers, instead of (A41), the rational function with the same denominator $\mathcal{D}$ but where the numerator is normalized to 1,*

$$\mathcal{R} \; = \; \frac{1}{\mathcal{D}}. \tag{A48}$$

*Its telescoper is an order-seven linear differential operator*

$$L_7 \; = \; F_2 \cdot G_1 \cdot G_2 \cdot H_2, \tag{A49}$$

*where the order-two linear differential operator $F_2$ is (non-trivially) homomorphic to the order-two linear differential operator $L_2$, which is the telescoper of the rational function (103) and where the order-two linear differential operators $G_2$ and $H_2$ have simple algebraic solutions.*

*Appendix B.4. Birational Symmetries from Collineations: Another Example*

Let us consider the following birational transformation associated with a collineation:

$$(x,\, y,\, z) \quad \longrightarrow \quad \Big(\frac{x}{1\,-\,x\,+2\,y},\ \ \frac{y}{1\,-\,x\,+2\,y},\ \ z\cdot\,(1\,-x\,+2\,y)^2\Big), \tag{A50}$$

which preserves the product $p \,=\, x\,y\,z$ and the origin $(x,\, y,\, z) \,=\, (0,\, 0,\, 0)$. Again, if one transforms the simple rational function (103) with the birational transformation (A50), one obtains the rational function of the form:

$$\mathcal{R} \;=\; \frac{1\,-\,x\,+2\,y}{\mathcal{D}}, \tag{A51}$$

again, the intersection of the algebraic surface $\mathcal{D} \,=\, 0$ with the algebraic surface $p \,=\, x\,y\,z$ is an elliptic curve, corresponding to eliminate $z \,=\, p/x/y$ in $\mathcal{D} \,=\, 0$. One immediately obtains the same Hauptmodul (A24) for this new elliptic curve. The telescoper of the rational function (A51) is an order-four linear differential operator

$$L_4 \;=\; F_2 \cdot G_2, \tag{A52}$$

where the order-two linear differential operator $F_2$ is (non-trivially) homomorphic to the order-two linear differential operator $L_2$, which is the telescoper of the rational function (103) and where the solutions of $G_2$ are two *algebraic functions* of series expansion:

$$\begin{aligned} s_0 \;&=\; 1\,+\frac{105}{4}\cdot x\,+\frac{12753}{16}\cdot x^2\,+\frac{876225}{32}\cdot x^3\,+\frac{251403765}{256}\cdot x^4\;+\;\cdots \\ s_1 \;&=\; x\,+\frac{105}{4}\cdot x^2\,+\frac{7385}{8}\cdot x^3\,+\frac{2111725}{64}\cdot x^4\,+\frac{155849463}{128}\cdot x^5\;+\;\cdots \end{aligned} \tag{A53}$$

The series $s \,=\, s_1$ is, for instance, the solution of the polynomial equation $P(s,x) \,=\, 0$, where $P(s,x)$ reads:

$$\begin{aligned} P(s,x) \;&=\; 2847312\cdot\, p(x)^3\cdot\, s^6 \;+158184\cdot\, p(x)^2\cdot\, s^4 \;+5040\cdot\, p(x)^2\cdot\, s^3 \\ &\quad +2197\cdot\, p(x)\cdot\, s^2 \;+140\cdot\, p(x)\cdot\, s \;+4\,x\cdot\,(243\,x+35), \end{aligned} \tag{A54}$$

with $p(x) \;=\; 243\,x^2\,+35\,x\,-1$. The series expansions of the algebraic solutions of $P(s,x) \,=\, 0$ read:

$$\begin{aligned} \mathcal{S}(u) \;&=\; u\;+\frac{448451640\,u^4-38438712\,u^3-20761650\,u^2+1377667\,u+221830}{17710}\cdot x \\ &\quad +3\cdot\frac{448451640\,u^4-38438712\,u^3-20761650\,u^2+1450531\,u+221830}{2024}\cdot x^2\;+\;\cdots \end{aligned}$$

where $u \,=\, 0,\, -1/6,\, 1/6,\, 5/26,\, -4/39,\, -7/78$. One finds that

$$\begin{aligned} &15\cdot\mathcal{S}\Big(\frac{1}{6}\Big)\,+8\cdot\mathcal{S}\Big(-\frac{1}{6}\Big)\,+13\cdot\mathcal{S}\Big(-\frac{7}{78}\Big) \;=\; 0, \\ &13\cdot\mathcal{S}\Big(\frac{1}{6}\Big)\,+8\cdot\mathcal{S}\Big(-\frac{4}{39}\Big)\,+15\cdot\mathcal{S}\Big(-\frac{7}{78}\Big) \;=\; 0, \\ &15825411\cdot\mathcal{S}\Big(\frac{1}{6}\Big)\;-1771\cdot\mathcal{S}\Big(\frac{5}{6}\Big)\;+29373604\cdot\mathcal{S}\Big(-\frac{7}{78}\Big) \;=\; 0, \end{aligned} \tag{A55}$$

and that the two solutions (A53) of $G_2$ read:

$$s_0 \;=\; \mathcal{S}(0), \qquad\quad s_1 \;=\; \frac{521}{32}\cdot\mathcal{S}\Big(\frac{1}{6}\Big)\;+\frac{611}{32}\cdot\mathcal{S}\Big(-\frac{7}{78}\Big). \tag{A56}$$

The homomorphism between $F_2$ and $L_2$ gives

$$F_2 \cdot X_1 \; = \; Y_1 \cdot L_2, \qquad \text{where:}$$
$$X_1 \; = \; \alpha(x) \cdot \Big( (3240\,x^2 \, + 6\,x \, + 1) \cdot D_x \; + 1080\,x \, - 6 \Big), \qquad \text{with:}$$
$$\alpha(x) \; = \; \frac{81}{10 \cdot \, (1 \, - 35\,x \, - 243\,x^2) \cdot \, (1 - 27\,x)}. \tag{A57}$$

Consequently, a solution $\mathcal{S}$ of the telescoper $L_4$ (but not of $G_2$ in (A52)) will be related to the hypergeometric solution ${}_2F_1([1/3, 2/3], [1], 27\,x)$ of the order-two linear differential operator $L_2$, as follows:

$$X_1 \Big( {}_2F_1 \Big( [\frac{1}{3}, \frac{2}{3}], [1], 27\,x \Big) \Big) \; = \; G_2 \cdot \mathcal{S}. \tag{A58}$$

The formal series solutions of the order-four linear differential operator (A52) are (of course) the two (algebraic) solutions (A53) of $G_2$, together with a solution with a $\ln(x)^1$, and a series $s_2$, analytic at $x \, = \, 0$:

$$s_2 \; = \; x^2 \; + \frac{93}{2} \cdot x^3 \, + \frac{31185}{16} \cdot x^4 \, + \frac{2488035}{32} \cdot x^5 \, + \frac{1953542437}{640} \cdot x^6 \; + \; \cdots \tag{A59}$$

Relation (A58) is actually satisfied with $\mathcal{S} \, = \, 5103 \cdot s_2$. Note that the series for (A58) is a series with *integer* coefficients:

$$\frac{1}{2} \cdot \frac{1}{5103} \cdot X_1 \Big( {}_2F_1 \Big( [\frac{1}{3}, \frac{2}{3}], [1], 27\,x \Big) \Big) \; = \; 1 \, + 87\,x \, + 5358\,x^2 \, + 282459\,x^3$$
$$+13662531\,x^4 \, + 626640714\,x^5 \, + 27758265651\,x^6 \, + 1200939383487\,x^7 \; + \; \cdots$$

**Remark A10.** *Note that the diagonal $\delta$ of the rational function (A51) reads:*

$$\delta \; = \; 1 \, + 4\,x \, + 108\,x^2 \, + 1960\,x^3 \, + 43240\,x^4 \, + 965664\,x^5 \, + 22377600\,x^6$$
$$+528712272\,x^7 \, + 12698698320\,x^8 \, + 308814134200\,x^9 \, + \; \cdots \tag{A60}$$

*We expect this diagonal to be a solution of the order-four telescoper (A52). This series is actually a linear combination of the three series $s_0$, $s_1$, and $s_2$, analytic at $x \, = \, 0$:*

$$\delta \; = \; s_0 \, - \frac{89}{4} \cdot s_1 \; - 105 \cdot s_2. \tag{A61}$$

*It is interesting to see how the three globally bounded series $s_0$, $s_1$, and $s_2$ conspire to give a series with integer coefficients, the diagonal (A61).*

**Remark A11.** *These results must be compared with the calculations for the rational function*

$$\mathcal{R} \; = \; \frac{1}{\mathcal{D}}, \tag{A62}$$

*where the denominator $\mathcal{D}$ is the same as the one in (A51). In this case, where the numerator has been normalized to 1, the diagonal is the same as the diagonal of $1/(1 - x - y - z)$, namely, ${}_2F_1([1/3, 2/3], [1], 27\,x)$, and the telescoper is the same telescoper as the one for $1/(1 - x - y - z)$.*

*Appendix B.5. Birational Symmetries from Collineations: Another Example*

Let us consider the following birational transformation associated with a collineation:

$$(x, \, y, \, z) \quad \longrightarrow$$
$$\Big( \frac{x \, + 3\,y}{1 \, - \, x \, + 2\,y}, \; \frac{y}{1 \, - \, x \, + 2\,y}, \; \frac{x\,z \cdot \, (1 \, - x \, + 2\,y)^2}{x \, + 3\,y} \Big), \tag{A63}$$

which preserves the product $p = x\,y\,z$, and the origin $(x, y, z) = (0, 0, 0)$. Again, if one transforms the simple rational function (103) with the birational transformation (A63), one obtains the rational function of the form:

$$\mathcal{R} \;=\; \frac{(1 - x + 2\,y) \cdot (x + 3\,y)}{\mathcal{D}}, \tag{A64}$$

again, the intersection of the algebraic surface $\mathcal{D} = 0$ with the algebraic surface $p = x\,y\,z$ is an elliptic curve, corresponding to eliminate $z = p/x/y$ in $\mathcal{D} = 0$. One immediately obtains the same Hauptmodul (A24) for this new elliptic curve. The telescoper of the rational function (A64) is an order-seven linear differential operator

$$L_7 \;=\; F_2 \cdot G_2 \cdot H_1 \cdot H_2, \tag{A65}$$

where the order-two linear differential operator $F_2$ is (non-trivially) homomorphic to the order-two linear differential operator $L_2$, which is the telescoper of the rational function (103); where the order-two linear differential operators $G_2$ and $H_2$ have algebraic solutions (one finds easily that the log-derivative of these solutions are algebraic functions); and where $H_1$ is an order-one linear differential operator. This homomorphism between $F_2$ and $L_2$ gives

$$F_2 \cdot X_1 \;=\; Y_1 \cdot L_2 \qquad \text{where:} \qquad X_1 \;=\; A(x) \cdot D_x + B(x), \tag{A66}$$

where $A(x)$ and $B(x)$ are rational functions. Consequently, a solution $\mathcal{S}$ of the telescoper $L_7$ (but not of the product $G_2 \cdot H_1 \cdot H_2$ in (A65)) will be related to the hypergeometric solution ${}_2F_1([1/3, 2/3], [1], 27\,x)$ of the order-two linear differential operator $L_2$, as follows:

$$X_1\Big({}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], 27\,x\Big)\Big) \;=\; G_2 \cdot H_1 \cdot H_2 \cdot \mathcal{S}. \tag{A67}$$

In that case, the solution of $\mathcal{S}$ of the telescoper $L_7$ reads

$$\mathcal{S} \;=\; x^4 \;+\; \frac{13316825310791}{231428221515} \cdot x^5 \;+\; \frac{30360140830595651}{11108554632720} \cdot x^6 \;+\; \cdots \tag{A68}$$

and the expansion of (A67) reads:

$$\begin{aligned} X_1\Big({}_2F_1\Big([\frac{1}{3}, \frac{2}{3}], [1], 27\,x\Big)\Big) \;&=\; \frac{1}{x} \;+\; \frac{85390121841387522079}{6298412854103317908} \\ &+\frac{906492811433323772155053002605}{771362364514926968178541 92} \cdot x \;+\; \cdots \end{aligned} \tag{A69}$$

**Remark A12.** *The factorization (A65) is far from being unique. Introducing the order-one linear differential operator* $L_1 = D_x + 4/(3 + 4\,x)$*, one has the following direct-sum decomposition:*

$$L_7 \;=\; L_1 \oplus L_6, \tag{A70}$$

$$G_2 \cdot H_1 \cdot H_2 \;=\; L_1 \oplus \tilde{G}_2 \oplus H_2, \tag{A71}$$

*where* $L_6$ *is an order-six linear differential operator and where the order-two linear differential operator operator* $\tilde{G}_2$ *is slightly simpler than* $G_2$*.*

**Remark A13.** *If one considers, instead of (A64), the rational function with the same denominator* $\mathcal{D}$ *but where the numerator is normalized to* $1$*,*

$$\mathcal{R} \;=\; \frac{1}{\mathcal{D}}. \tag{A72}$$

*its telescoper is an order-four linear differential operator*

$$L_4 \;=\; F_2 \cdot G_2. \tag{A73}$$

*The order-two linear differential operator* $F_2$ *is (non-trivially) homomorphic to the order-two linear differential operator* $L_2$*, which is the telescoper of the rational function (103), and the order-two linear differential operator* $G_2$ *has simple algebraic solutions.*

*Appendix B.6. Birational Symmetries from Collineations: Another Simpler Example*

Let us consider the following birational transformation associated with a collineation:

$$(x,\, y,\, z) \quad \longrightarrow \quad \Big( \frac{x\,+3\,y}{1\,-\,x\,+2\,y},\;\; \frac{1\,+y}{1\,-\,x\,+2\,y},\;\; \frac{x\,y\,z \cdot\,(1\,-x\,+2\,y)^2}{(x\,+3\,y)\cdot\,(1\,+y)} \Big), \tag{A74}$$

which preserves the product $p \;=\; x\,y\,z$. Again, if one transform the simple rational function (103) with the birational transformation (A74), one obtains the rational function of the form:

$$\mathcal{R} \;=\; \frac{(1\,-\,x\,+2\,y)\cdot\,(x\,+3\,y)\cdot\,(1\,+y)}{\mathcal{D}}, \tag{A75}$$

again, the intersection of the algebraic surface $\mathcal{D} \,=\, 0$ with the algebraic surface $p \,=\, x\,y\,z$ is an elliptic curve, corresponding to eliminate $z \,=\, p/x/y$ in $\mathcal{D} \,=\, 0$. One immediately obtains the same Hauptmodul (A24) for this new elliptic curve.

The telescoper of the rational function (A75) can now be calculated in only a few hours, and one obtains an order-nine linear differential operator of the form

$$L_9 \;=\; F_2 \cdot G_2 \cdot H_1 \cdot H_2 \cdot I_2, \tag{A76}$$

where the order-two linear differential operator $F_2$ is (non-trivially) homomorphic to the order-two linear differential operator $L_2$, which is the telescoper of the rational function (103); where the order-two linear differential operators $G_2$, $H_2$, and $I_2$ have algebraic solutions; and where $H_1$ is an order-one linear differential operator. This homomorphism between $F_2$ and $L_2$ gives

$$F_2 \cdot X_1 \;=\; Y_1 \cdot L_2 \qquad \text{where:} \qquad X_1 \;=\; A(x)\cdot D_x \,+B(x), \tag{A77}$$

where $A(x)$ and $B(x)$ are quite large rational functions. Consequently, a solution $\mathcal{S}$ of the telescoper $L_9$ (but not of the product $G_2 \cdot H_1 \cdot H_2 \cdot I_2$ in (A76)) will be related to the hypergeometric solution ${}_2F_1([1/3, 2/3], [1], 27\,x)$ of the order-two linear differential operator $L_2$, as follows:

$$X_1\Big({}_2F_1\Big([\frac{1}{3},\, \frac{2}{3}],\, [1],\, 27\,x\Big)\Big) \;=\; G_2 \cdot H_1 \cdot H_2 \cdot I_2 \cdot \mathcal{S}. \tag{A78}$$

If finding the emergence of the hypergeometric function ${}_2F_1([1/3, 2/3], [1], 27\,x)$ is easy to obtain from the (algebraic geometry) calculation of the Hauptmodul (A24), (see (129)), the telescoper of (A75), or equivalently, the solution $\mathcal{S}$ of that telescoper, requires one to find many linear differential operators, namely, the intertwinner $X_1$ and the right factors $G_2$, $H_1$, $H_2$, and $I_2$. In contrast with the birational transformations described in Section 6 (see (108), (111), and (112)), which simply preserve the diagonals of the rational functions, we have here, again, with the birational transformation (A74), *two birationally equivalent underlying elliptic curves* and a *much more convoluted "covariance"* requiring one to find many linear differential operators. The "elliptic curve skeleton" (the j-invariant or the Hauptmodul) is preserved, but the right factors dressing $G_2$, $H_1$, $H_2$, and $I_2$ and the intertwiner $X_1$ are quite involved.

**Remark A14.** *In fact, the order-nine operator (A76) is a direct sum. It can be written in the form*

$$L_9 \;=\; L_8 \oplus L_1, \tag{A79}$$

$$G_2 \cdot H_1 \cdot H_2 \cdot I_2 \;=\; L_1 \oplus \tilde{G}_2 \oplus \tilde{H}_2 \oplus I_2, \tag{A80}$$

*where the order-one operator reads:*

$$L_1 \;=\; D_x \,+ \frac{4}{3+4\,x}, \tag{A81}$$

*where* $L_8$ *is an order-eight operator and where the operators with a tilde are much simpler than the operators without a tilde.*

**Remark A15.** *Again if one considers, instead of (A75), the rational function with the same denominator* $\mathcal{D}$*, but where the numerator has been normalized to* 1*,*

$$\mathcal{R} \;=\; \frac{1}{\mathcal{D}}, \tag{A82}$$

*one finds an order-seven telescoper that factorizes as follows:*

$$L_7 \;=\; F_2 \cdot G_1 \cdot H_2 \cdot I_2, \tag{A83}$$

*where the order-two linear differential operator* $F_2$ *is (non-trivially) homomorphic to the order-two linear differential operator* $L_2$*, which is the telescoper of the rational function (103), and where the order-two linear differential operators* $H_2$ *and* $I_2$ *have algebraic solutions.*

**Remark A16.** *Again, the factorization (A83) is far from being unique. Introducing the order-one linear differential operator* $L_1 \,=\, D_x \,+1/x$*, one has the two following direct-sum decompositions*

$$L_7 \,=\, L_6 \oplus L_1, \tag{A84}$$

$$G_1 \cdot H_2 \cdot I_2 \;=\; L_1 \oplus \tilde{H}_2 \oplus I_2, \tag{A85}$$

*where the order-two linear differential operator* $\tilde{H}_2$ *is slightly simpler than* $H_2$*.*

**Remark A17.** *As far as an* algebraic geometry approach *of diagonals and telescopers is concerned (see [40]), we see that the concept of telescopers, which describes* all *the periods, can be more interesting than the concept of diagonals, which often yields to diagonals that can be almost trivial functions (being simple rational functions, or being simply equal to zero). The examples of Appendix B show that the differential algebra approach of creative telescoping cannot be totally replaced by an algebraic geometry approach [40]. The algebraic geometry approach very quickly provides some precious information on the telescoper (the Hauptmodul), but not the telescoper itself. In fact, one might consider the opposite point of view: creative telescoping could be seen as a tool to obtain effective algebraic geometry results.*

**Remark A18.** *The examples displayed in this appendix can be seen as an illustration of the "dialogue of the deaf" between mathematicians and physicists. Some mathematicians will point out the fact that the calculation of the Hauptmodul (A24) underlines the essence of the problem, namely, the existence of an underlying elliptic curve, and will see the explicit calculation of the telescoper, and all its periods, as a laborious and slightly useless piece of work. In particular, they will consider the "dressing" right-factors occurring in the decompositions (A34), (A42), ... as a totally and utterly spurious information, and they will also probably see the explicit expression of the large order-two operators* $F_2$ *as superfluous, retaining only the order-two linear differential operator* $L_2$*, prefering to ignore, or forget, the intertwiner* $X_1$ *in (A66) or (A77). Accordingly, they may consider the other solutions of the telescoper, namely, the "periods" (associated with non-vanishing cycles) that are not diagonals, as irrelevant. In contrast, for a physicist, obtaining all the periods, and the explicit expression of the telescoper, will be seen as essential. Recalling the* $\chi^{(n)}$ *components of* *the susceptibility of the Ising model; it is essential to obtain the explicit expression of the linear differential operators (telescopers) annihilating these* $\chi^{(n)}$*'s even if these (large) linear differential operators [26] are products (and direct sums) of a large set of factors. In the framework of integrable models, beyond diagonals, a physicist will always seek for a linear differential operator corresponding to an elliptic curve (resp. K3 surface, Calabi–Yau manifold, . . . ) even if it is "buried" as a left factor of a large telescoper, like the* $F_2$*'s in (A34) or (A42).*

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