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Improving Convergence Analysis of the Newton–Kurchatov Method under Weak Conditions

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Abstract: The technique of using the restricted convergence region is applied to study a semilocal convergence of the Newton–Kurchatov method. The analysis is provided under weak conditions for the derivatives and the first order divided differences. Consequently, weaker sufficient convergence criteria and more accurate error estimates are retrieved. A special case of weak conditions is also considered.

Keywords: Newton–Kurchatov method; ω-type conditions; ε-type conditions; method with decomposition of operator; semilocal convergence; divided differences; restricted convergence region

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1. Introduction

Nonlinear equations, in particular systems of nonlinear algebraic or transcendental equations, arise often when numerical methods are used for solving applied problems. A popular method for solving such equations is Newton's [1–3]. However, it requires differentiability of the nonlinear function. This is not a requirement for difference methods [1–4]. They can be applied to equations with a nondifferentiable function [5]. For some problems, the nonlinear function can be represented as the sum of the differentiable and nondifferentiable parts. In this case, methods with operator decomposition are often used [1,2,6–9]. Numerical examples show that the convergence is faster than difference methods and the Newton-type method [10–12].

Let us consider an equation

$$H(x) \equiv F(x) + G(x) = 0, \tag{1}$$

where F and $G : D \subset X \to Y$. Here F is a differentiable operator, G is a continuous operator, D is an open convex set, X and Y are Banach spaces.

We use Newton–Kurchatov method [8,13–16] for solving Equation (1) numerically

$$x_{n+1} = x_n - A_n^{-1} H(x_n), \quad n \geq 0, \tag{2}$$

where $A_n = F'(x_n) + G(2x_n - x_{n-1}; x_{n-1})$. It is a combination of Newton and Kurchatov methods [3,4,17]. Their formulas for solving equation $F(x) = 0$ are of the form

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n), \quad n \geq 0$$

and

$$x_{n+1} = x_n - F(2x_n - x_{n-1}; x_{n-1})^{-1} F(x_n), \quad n \geq 0,$$

respectively; $G(\cdot;\cdot)$ and $F(\cdot;\cdot)$ denote a first-order divided difference. Let $\tilde{x} \in D$. Denote

$$B(\tilde{x}, R) = \{x \in X : \|x - \tilde{x}\| < R\},$$

and

$$\overline{B(\tilde{x}, R)} = \{x \in X : \|x - \tilde{x}\| \leq R\}.$$

Our semilocal convergence is based on some generalized conditions. Suppose that for each $x, y \in D$:

$$\|A_0^{-1}(F'(x) - F'(x_0))\| \leq \omega_1^0(\|x - x_0\|), \tag{3}$$

$$\|A_0^{-1}(G(x; y) - G(2x_0 - x_{-1}; x_{-1}))\| \leq \omega_2^0(\|x - (2x_0 - x_{-1})\|, \|y - x_{-1}\|), \tag{4}$$

where $\omega_1^0 : \mathbb{R}_+ \to \mathbb{R}_+$ and $\omega_2^0 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing functions. Let $a > 0$. Suppose that equation

$$\omega_1^0(u) + \omega_2^0(3u + a, u + a) = 1 \tag{5}$$

has at least one positive solution. Denote by R_0 the smallest such solution. Set $D_0 = D \cap B(x_0, 3R_0)$ and suppose that for each $x, y, u, v \in D_0$ with $2y - x \in D_0$

$$\|A_0^{-1}(F'(x) - F'(y))\| \leq \omega_1(\|x - y\|), \tag{6}$$

$$\|A_0^{-1}(G(2y - x; x) - G(u; v))\| \leq \omega_2(\|2y - x - u\|, \|x - v\|), \tag{7}$$

where $\omega_1 : \mathbb{R}_+ \to \mathbb{R}_+$ and $\omega_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are nondecreasing functions. Moreover, $\omega_1(tr) \leq h(t)\omega_1(r)$, $t \in [0, 1]$, $r \in [0, R]$, $h : [0, 1] \to \mathbb{R}$ [11].

In this article, we also consider ε-type conditions for $x, y \in D$

$$\|A_0^{-1}(F'(x) - F'(x_0))\| \leq \varepsilon_1^0, \tag{8}$$

$$\|A_0^{-1}(G(x; y) - G(2x_0 - x_{-1}; x_{-1}))\| \leq \varepsilon_2^0, \tag{9}$$

for $\bar{x}, \bar{y}, u, v \in D_0$ with $2\bar{y} - \bar{x} \in D_0$

$$\|A_0^{-1}(F'(x) - F'(y))\| \leq \varepsilon_1, \tag{10}$$

$$\|A_0^{-1}(G(2\bar{y} - \bar{x}; \bar{x}) - G(u; v))\| \leq \varepsilon_2, \tag{11}$$

where ε_1^0, ε_1, ε_2^0 and ε_2 are positive constants, for some $D_0 \subseteq D$.

2. Semilocal Analysis

Set $\Phi = \int_0^1 h(t)dt$.

Theorem 1. *Let F and G be nonlinear operators with the specified properties. Assume that:*

- *linear operator A_0, where x_{-1} and $x_0 \in D$, is invertible;*
- $$\|A_0^{-1}(F(x_0) + G(x_0))\| \leq \eta, \ \|x_0 - x_{-1}\| \leq \alpha; \tag{12}$$
- *Equations (3) and (4) hold on D, Equations (6) and (7) hold on D_0;*
- *equation*

$$u\left(1 - \frac{m}{1 - \omega_1^0(u) - \omega_2^0(3u + \alpha, u + \alpha)}\right) - \eta = 0, \tag{13}$$

where $m = \Phi\omega_1(\eta) + \max\{\omega_2(\eta+\alpha,\alpha), \omega_2(2\eta,\eta)\}$, *has at least one positive solution greater than* η *and* α. *Denote by* R *the smallest such solution;*

- $\omega_1^0(R) + \omega_2^0(3R+\alpha, R+\alpha) < 1$, $M = \dfrac{m}{1-\omega_1^0(R)-\omega_2^0(3R+\alpha,R+\alpha)} < 1$, $B(x_0, 3R) \subset D$.

Then, the sequence $\{x_n\}_{n\geq 0}$, *generated by Equation* (2), *is well-defined, remains in* $B(x_0,R)$ *and converges to a unique solution* $x^* \in \overline{B(x_0,R)}$ *of Equation* (1), *and* $R < R_0$.

Proof. The proof of Theorem 1 is carried out by mathematical induction and is similar to the one in [8] but there are some differences. By Equations (2) and (12), for $n = 0$ we have

$$\|x_1 - x_0\| \leq \|A_0^{-1}(F(x_0)+G(x_0))\| \leq \eta < R$$

and $x_1 \in B(x_0,R)$. Using the conditions in Equations (3) and (4), we get

$$\begin{gathered}\|I - A_0^{-1}A_1\| = \|A_0^{-1}(A_0 - A_1)\| \\ \leq \omega_1^0(\|x_1-x_0\|) + \omega_2^0(\|2x_0 - x_{-1} - 2x_1 + x_0\|, \|x_{-1}-x_0\|) \\ \leq \omega_1^0(\eta) + \omega_2^0(2\eta+\alpha,\alpha) \leq \omega_1^0(R) + \omega_2^0(2R+\alpha,\alpha) \\ \\ \leq \omega_1^0(R) + \omega_2^0(3R+\alpha, R+\alpha) < 1.\end{gathered}$$

According to the Banach lemma on inverse operators [1] $A_1^{-1}A_0$ exists and

$$\|A_1^{-1}A_0\| \leq \frac{1}{1-\omega_1^0(R)-\omega_2^0(3R+\alpha,R+\alpha)}.$$

Then, we have

$$\begin{gathered}A_0^{-1}(F(x_1)+G(x_1)) \\ = A_0^{-1}(F(x_1)-F(x_0)-F'(x_0)(x_1-x_0)) \\ +A_0^{-1}(G(x_1)-G(x_0)-G(2x_0-x_{-1};x_{-1})(x_1-x_0)) \\ = \int_0^1 A_0^{-1}(F'(x_0+t(x_1-x_0))-F'(x_0))\,dt(x_1-x_0) \\ +A_0^{-1}(G(x_1;x_0)-G(2x_0-x_{-1};x_{-1}))(x_1-x_0).\end{gathered}$$

Hence, by the conditions in Equations (6) and (7), we obtain

$$\begin{gathered}\|x_2-x_1\| = \|A_1^{-1}(F(x_1)+G(x_1))\| \leq \|A_1^{-1}A_0\|\|A_0^{-1}(F(x_1)+G(x_1))\| \\ \leq \frac{\Phi\omega_1(\|x_1-x_0\|)+\omega_2(\|2x_0-x_{-1}-x_1\|,\|x_{-1}-x_0\|)}{1-\omega_1(R)-\omega_2(3R+\alpha,R+\alpha)}\|x_1-x_0\| \\ \leq \frac{\Phi\omega_1(\eta)+\omega_2(\eta+\alpha,\alpha)}{1-\omega_1^0(R)-\omega_2^0(3R+\alpha,R+\alpha)}\|x_1-x_0\| \leq M\|x_1-x_0\| < \eta.\end{gathered}$$

On the other hand,

$$\begin{gathered}\|x_2-x_0\| \leq \|x_2-x_1\| + \|x_1-x_0\| \\ \leq (M+1)\|x_1-x_0\| \leq (M+1)\eta = \frac{1-M^2}{1-M}\eta < \frac{1}{1-M}\eta = R.\end{gathered}$$

Therefore, $x_2 \in B(x_0,R)$. Suppose that for $k = 1,\ldots,n-1$ holds:

- $A_k^{-1}A_0$ exists and $\|A_k^{-1}A_0\| \le \dfrac{1}{1-\omega_1^0(R)-\omega_2^0(3R+\alpha,R+\alpha)}$;
- $\|x_{k+1}-x_k\| \le M\|x_k-x_{k-1}\| \le M^k\|x_1-x_0\| < \eta$;
- $x_{k+1} \in B(x_0,R)$.

Then, using the conditions in Equations (3) and (4), for $k=n$ we have

$$\|I-A_0^{-1}A_n\| = \|A_0^{-1}(A_0-A_n)\|$$
$$\le \omega_1^0(\|x_0-x_n\|)+\omega_2^0(\|2x_0-x_{-1}-2x_n+x_{n-1}\|,\|x_{-1}-x_{n-1}\|)$$
$$\le \omega_1^0(R)+\omega_2^0(3R+\alpha,R+\alpha)<1.$$

According to the Banach lemma on inverse operators [1] $A_n^{-1}A_0$ exists and

$$\|A_n^{-1}A_0\| \le \frac{1}{1-\omega_1^0(R)-\omega_2^0(3R+\alpha,R+\alpha)}.$$

By equality

$$A_0^{-1}(F(x_n)+G(x_n)) = A_0^{-1}(F(x_n)-F(x_{n-1})-F'(x_{n-1})(x_n-x_{n-1}))$$
$$+A_0^{-1}(G(x_n)-G(x_{n-1})-G(2x_{n-1}-x_{n-2};x_{n-2})(x_n-x_{n-1}))$$
$$=\int_0^1 A_0^{-1}(F'(x_{n-1}+t(x_n-x_{n-1}))-F'(x_{n-1}))\,dt(x_n-x_{n-1})$$
$$+A_0^{-1}(G(x_n;x_{n-1})-G(2x_{n-1}-x_{n-2};x_{n-2}))(x_n-x_{n-1})$$

and the conditions in Equations (6) and (7), we have

$$\|x_{n+1}-x_n\| = \|A_n^{-1}(F(x_n)+G(x_n))\| \le \|A_n^{-1}A_0\|\|A_0^{-1}(F(x_n)+G(x_n))\|$$
$$\le \frac{1}{1-\omega_1^0(R)-\omega_2^0(3R+\alpha,R+\alpha)}\Big(\Phi\omega_1(\|x_n-x_{n-1}\|)$$
$$+\omega_2(\|2x_{n-1}-x_{n-2}-x_n\|,\|x_{n-1}-x_{n-2}\|)\Big)\|x_n-x_{n-1}\|$$
$$\le \frac{\Phi\omega_1(\eta)+\omega_2(2\eta,\eta)}{1-\omega_1^0(R)-\omega_2^0(3R+\alpha,R+\alpha)}\|x_n-x_{n-1}\|$$
$$\le M\|x_n-x_{n-1}\| \le M^n\|x_1-x_0\| < \eta.$$

Next, we show that $x_{n+1} \in B(x_0,R)$. Indeed,

$$\|x_{n+1}-x_0\| \le \|x_{n+1}-x_n\|+\|x_n-x_{n-1}\|+\ldots+\|x_1-x_0\|$$
$$\le (M^n+M^{n-1}+\ldots+M+1)\|x_1-x_0\| \le \frac{1-M^{n+1}}{1-M}\eta < \frac{1}{1-M}\eta = R,$$

and $x_{n+1} \in B(x_0,R)$. Moreover, we show that sequence $\{x_n\}_{n\ge0}$ is fundamental. Indeed, for $p \ge 1$

$$\|x_{n+p}-x_n\| \le \|x_{n+p}-x_{n+p-1}\|+\|x_{n+p-1}-x_{n+p-2}\|+\ldots+\|x_{n+1}-x_n\|$$
$$\le (M^{p-1}+M^{p-2}+\ldots+1)\|x_{n+1}-x_n\|$$
$$=\frac{1-M^p}{1-M}\|x_{n+1}-x_n\| \le \frac{1-M^p}{1-M}M^n\eta < \frac{M^n}{1-M}\eta.$$

Therefore, $\{x_n\}_{n\ge0}$ is a fundamental sequence and converges to $x^* \in \overline{B(x_0,R)}$. Furthermore, we prove that x^* is a unique solution of Equation (1). Since

$$\|A_0^{-1}H(x_n)\| \leq (\Phi\omega_1(\eta) + \omega_2(2\eta, \eta))\|x_n - x_{n-1}\|$$

and $\|x_n - x_{n-1}\| \to 0$ for $n \to \infty$, so $H(x^*) = 0$. Finally, suppose there exists $y^* \in \overline{B(x_0, R)}$, $y^* \neq x^*$ such that $H(y^*) = 0$. Denote

$$A = \int_0^1 F'(x^* + t(y^* - x^*))\, dt + G(y^*; x^*).$$

Using the conditions in Equations (3) and (4), we get

$$\begin{aligned}
\|A_0^{-1}(A_0 - A)\| &\leq \int_0^1 \omega_1^0(\|x_0 - x^* - t(y^* - x^*)\|)\, dt \\
&+\omega_2^0(\|2x_0 - x_{-1} - y^*\|, \|x_{-1} - x^*\|) \\
&\leq \int_0^1 \omega_1^0((1-t)\|x_0 - x^*\| + t\|x_0 - y^*\|)\, dt \\
&+\omega_2^0(\|x_0 - x_{-1}\| + \|x_0 - y^*\|, \|x_{-1} - x_0\| + \|x_0 - x^*\|) \\
&\leq \omega_1^0(R) + \omega_2^0(R + \alpha, R + \alpha) < 1.
\end{aligned}$$

According to the Banach lemma on inverse operators A is invertible and in view of

$$A(y^* - x^*) = H(y^*) - H(x^*)$$

it follows $y^* = x^*$. □

Theorem 2. *Let F and G be nonlinear operators with the specified properties. Assume that*

- *linear operator A_0, where x_{-1} and $x_0 \in D$, is invertible;*
- *Equations (8)–(11) hold;*
- *numbers $\eta > 0$, γ and $R > 0$ such that*

$$\|A_0^{-1}(F(x_0) + G(x_0))\| \leq \eta, \quad \|x_0 - x_{-1}\| < R, \quad 0 < \varepsilon_1^0 + \varepsilon_2^0 < 1,$$
$$\gamma = \frac{\varepsilon_1 + \varepsilon_2}{1 - (\varepsilon_1^0 + \varepsilon_2^0)} < 1, \quad \frac{\eta}{1-\gamma} < R, \quad B(x_0, 3R) \subset D.$$

Then, the sequence $\{x_n\}_{n\geq 0}$, generated by Equation (2), is well-defined, remains in $B(x_0, R)$ and converges to a unique solution $x^* \in \overline{B(x_0, R)}$ of Equation (1). Moreover, the following inequality holds for each $n \geq 0$

$$\|x_n - x^*\| \leq \frac{\gamma^n}{1-\gamma}\eta. \tag{14}$$

Notice that one possibility is ε_1^0 and ε_2^0 in practice may be functions of $\|x - x_0\|$. That is

$$\varepsilon_1^0 : [0, \infty) \to [0, \infty) \text{ and } \varepsilon_2^0 : [0, \infty) \to [0, \infty)$$

to be nondecreasing and continuous functions.

Suppose that equation

$$\varepsilon_1^0(u) + \varepsilon_2^0(u) = 1$$

has a minimal positive solution ρ. Then, the set D_0 can be defined as $D_0 = D \cap B(x_0, \rho)$. Moreover, in this case we have

$$\|A_0^{-1}(A_0 - A_n)\| \leq \varepsilon_1^0 + \varepsilon_2^0 < 1,$$

so $\|A_n^{-1}A_0\| \leq \dfrac{1}{1-(\varepsilon_1^0+\varepsilon_2^0)}$.

However, D_0 can also be defined in other ways too depending on the construction of F and G. Let

$$\omega_1^0(\|x-y\|) = 2\ell_0\|x-y\|,\ \omega_2^0(\|x-u\|, \|y-v\|) = p_0(\|x-u\|+\|y-v\|),$$
$$\omega_1(\|x-y\|) = 2\ell\|x-y\|,\ \omega_2(\|x-u\|, \|y-v\|) = p(\|x-u\|+\|y-v\|).$$

Next, we obtain from Theorem 1 the convergence analysis of the method in Equation (2) under the Lipschitz conditions.

Corollary 1. *Let F and G be nonlinear operators with the specified properties. Assume that:*

- *linear operator A_0, where x_{-1} and $x_0 \in D$, is invertible;*
- *numbers $\eta > 0$ and $\alpha > 0$ such that Equation (12) is satisfied;*
- *the Lipschitz conditions are fulfilled for each $x, y \in D$*

$$\|A_0^{-1}(F'(x_0) - F'(y))\| \leq 2\ell_0\|x-y\|,$$
$$\|A_0^{-1}(G(x;y) - G(2x_0 - x_{-1}; x_{-1}))\| \leq p_0(\|x-(2x_0-x_{-1})\| + \|y-x_{-1}\|),$$

and for each $x, y, u, v \in D_0$

$$\|A_0^{-1}(F'(x) - F'(y))\| \leq 2\ell\|x-y\|,$$
$$\|A_0^{-1}(G(x;y) - G(u;v))\| \leq p(\|x-u\| + \|y-v\|);$$

- *equation*

$$s\left(1 - \frac{m}{1-2\ell_0 s - p_0(4s+2\alpha)}\right) - \eta = 0,$$

where $m = \ell\eta + \max\{p(\eta+2\alpha), 3p\eta\}$ has at least one positive solution greater than η and α. Denote by R the smallest such solution;

- $2\ell_0 R + p_0(4R+2\alpha) < 1$, $M = \dfrac{m}{1-2\ell_0 R - p_0(4R+2\alpha)} < 1$, $B(x_0, 3R) \subset D$.

Then, the sequence $\{x_n\}_{n\geq 0}$, generated by Equation (2), is well-defined, remains in $B(x_0, R)$ and converges to a unique solution $x^* \in \overline{B(x_0, R)}$ of Equation (1).

Remark 1. *The corresponding Equations (6) and (7) to conditions in [8] are given for each $x, y, u, v \in D$ by*

$$\|A_0^{-1}(F'(x) - F'(y))\| \leq \omega_1^1(\|x-y\|), \tag{15}$$
$$\|A_0^{-1}(G(x;y) - G(u;v))\| \leq \omega_2^1(\|x-u\|, \|y-v\|). \tag{16}$$

Notice that $\omega_1 = \omega_1(\omega_1^0)$ and $\omega_2 = \omega_2(\omega_2^0)$, i.e. they are functions of ω_1^0 and ω_2^0, and since $D_0 \subseteq D$

$$\omega_1^0(t) \leq \omega_1^1(t), \tag{17}$$
$$\omega_2^0(t) \leq \omega_2^1(t), \tag{18}$$
$$\omega_1(t) \leq \omega_1^1(t), \tag{19}$$
$$\omega_2(t) \leq \omega_2^1(t), \tag{20}$$
$$m \leq m^1, \tag{21}$$
$$\varepsilon_1^0 \leq \varepsilon_1^1,\ \varepsilon_2^0 \leq \varepsilon_2^1,\ \varepsilon_1 \leq \varepsilon_1^1,\ \varepsilon_2 \leq \varepsilon_2^1, \tag{22}$$
$$l_0 \leq l^1, \tag{23}$$
$$\text{and}\ p_0 \leq p^1, \tag{24}$$

where $m^1 = \Phi\omega_1^1(\eta) + \max\{\omega_2^1(\eta+\alpha,\alpha), \omega_2^1(2\eta,\eta)\}$, $M^1 = \dfrac{m^1}{1-\omega_1^1(R^1)-\omega_2^1(3R^1+\alpha, R^1+\alpha)}$.

It's easy to see that if $R^1 \le R$ *then* $M^1 \le M$, *and if* $R^1 \ge R$ *then* $M^1 \ge M$.

If ω_1^1, ω_2^1 *are constant functions*

$$\omega_1^1(\|x-y\|) = 2\ell^1\|x-y\|,\ \omega_2^1(\|x-u\|, \|y-v\|) = p^1(\|x-u\| + \|y-v\|).$$

It follows from the above that we obtain the following improvements:

1. Weaker sufficient convergence criteria, since

$$\begin{aligned}&\omega_1^0(\|x_0 - x_n\|) + \omega_2^0(\|2x_0 - x_{-1} - 2x_n + x_{n-1}\|, \|x_{-1} - x_{n-1}\|)\\ &\le \omega_1^1(\|x_0 - x_n\|) + \omega_2^1(\|2x_0 - x_{-1} - 2x_n + x_{n-1}\|, \|x_{-1} - x_{n-1}\|)\end{aligned}$$

and

$$\begin{aligned}&\Phi\omega_1(\|x_n - x_{n-1}\|) + \omega_2(\|2x_{n-1} - x_{n-2} - x_n\|, \|x_{n-1} - x_{n-2}\|)\\ &\le \Phi\omega_1^1(\|x_n - x_{n-1}\|) + \omega_2^1(\|2x_{n-1} - x_{n-2} - x_n\|, \|x_{n-1} - x_{n-2}\|)\\ &\Rightarrow \quad M_n \le M_n^1 \quad \Rightarrow \quad M_n\|x_n - x_{n-1}\| \le M_n^1\|x_n - x_{n-1}\|,\end{aligned}$$

where

$$\begin{aligned}M_n &= \frac{\Phi\omega_1(\|x_n - x_{n-1}\|) + \omega_2(\|2x_{n-1} - x_{n-2} - x_n\|, \|x_{n-1} - x_{n-2}\|)}{1 - \omega_1^0(\|x_0 - x_n\|) - \omega_2^0(\|2x_0 - x_{-1} - 2x_n + x_{n-1}\|, \|x_{-1} - x_{n-1}\|)} \le M,\\ M_n^1 &= \frac{\Phi\omega_1^1(\|x_n - x_{n-1}\|) + \omega_2^1(\|2x_{n-1} - x_{n-2} - x_n\|, \|x_{n-1} - x_{n-2}\|)}{1 - \omega_1^1(\|x_0 - x_n\|) - \omega_2^1(\|2x_0 - x_{-1} - 2x_n + x_{n-1}\|, \|x_{-1} - x_{n-1}\|)} \le M^1,\end{aligned}$$

but not necessarily vice versa unless, if

$$\omega_1^1 = \omega_1^0 \text{ and } \omega_2^1 = \omega_2^0.$$

2. Fewer iterates to obtain a desired error accuracy on $\|x_{n+1} - x_n\|$.

3. Better information on the location of the solution x^*.

Notice that (ω_1^0, ω_1), (ω_2^0, ω_2) are special cases of the old functions ω_1^1, ω_2^1, respectively. So no additional information or computational effort are required to obtain these improvements. This technique of using the restricted convergence region can be used to extend the applicability of other iterative methods along the same lines. Finally, Lipschitz functions and parameters can become at least as small, if D_0 is replaced by $D_1 := D \cap B(x_1, R_0 - b)$, $b = \max\{\eta, \alpha\}$. Notice that $D_1 \subseteq D_0$. The results then can be adjusted in this setting. Numerical examples where Equations (17)–(24) hold as strict inequalities can be found in [1,6].

3. Numerical Results

In this Section, we test the old and the new convergence criteria.

Let $X = Y = \mathbb{R}$. In this case $\|x\| = |x|$ for $x \in X$ or $x \in Y$, $D = (a, b)$, $D_0 = (a_0, b_0)$. Let us define function $F + G : \mathbb{R} \to \mathbb{R}$, where

$$F(x) = e^{x-0.5} + x^3 - 1.3,\ \ G(x) = 0.2x|x^2 - 2|.$$

The exact solution of $F(x) + G(x) = 0$ is $x_* = 0.5$. Let $D = (0, 1.4)$. Then, we have

$$F'(x) = e^{x-0.5} + 3x^2,$$

$$G(x,y) = \frac{0.2x(2-x^2) - 0.2y(2-y^2)}{x-y} = 0.2(2 - x^2 - xy - y^2).$$

$$A_0 = e^{x_0 - 0.5} + 3x_0^2 + 0.2(2 - x_{-1}^2 - x_{-1}x_0 - x_0^2),$$

and

$$|A_0^{-1}(F'(x) - F'(y))| \le \frac{e^{b-0.5} + 3|x+y|}{|A_0|}|x-y|,$$

$$|A_0^{-1}(G(x,y) - G(u,v))| \le \frac{0.2}{|A_0|}\Big(|u+x+y||u-x| + |v+y+u||v-y|\Big).$$

In view of this, we can write

$$\omega_1^0(|x-x_0|) = \frac{e^{b-0.5} + 3|1.4 + x_0|}{|A_0|}|x-x_0|,$$

$$\omega_2^0(|x-(2x_0-x_{-1})|, |y-x_{-1}|) = \frac{0.2}{|A_0|}\Big(|2x_0 - x_{-1} + 2b||x-(2x_0-x_{-1})| + |2x_0+b||y-x_{-1}|\Big),$$

$$\omega_1(|x-y|) = \frac{e^{b_0-0.5} + 6b_0}{|A_0|}|x-y|, \quad \omega_2(|2y-x-u||, |x-v|) = \frac{0.6b_0}{|A_0|}(|2y-x-u| + |x-v|),$$

$$\omega_1^1(|x-y|) = \frac{e^{b-0.5} + 6b}{|A_0|}|x-y|, \quad \omega_2^1(|x-u|, |y-v|) = \frac{0.6b}{|A_0|}\Big(|x-u| + |y-v|\Big).$$

Let $x_0 = 0.55$, $x_{-1} = 0.551$. Then, we get $\alpha = 0.001$, $\eta \approx 0.0479$, $R_0 \approx 0.2011$, $m \approx 0.1431$, $m^1 \approx 0.1750$, $D_0 = D \cap B(x_0, 3R_0) = (0,\ 1.153)$. To get the radius of convergence we solve Equation (13) and similar one from [8]. Every such equation has two positive solutions. The smallest solutions satisfy conditions of appropriate theorems. So, we get $R \approx 0.0602$, $R^1 \approx 0.0714$, $M \approx 0.2043$, $M^1 \approx 0.3283$, $B(x_0, 3R) = (0.3693, 0.7307) \subset D$ and $B(x_0, 3R^1) = (0.3359, 0.7641) \subset D$. The error estimates are given in Table 1. For error $|x_{n+1} - x_n|$, $n \ge 1$, holds

$$|x_{n+1} - x_n| \le M_n|x_n - x_{n-1}| \le M|x_n - x_{n-1}|$$

and

$$|x_{n+1} - x_n| \le M_n^1|x_n - x_{n-1}| \le M^1|x_n - x_{n-1}|.$$

Table 1. Results for $\varepsilon = 10^{-15}$.

n	$\lvert x_{n+1} - x_n\rvert$	$M_n\lvert x_n - x_{n-1}\rvert$	$M\lvert x_n - x_{n-1}\rvert$	$M_n^1\lvert x_n - x_{n-1}\rvert$	$M^1\lvert x_n - x_{n-1}\rvert$
1	2.0602×10^{-3}	6.8518×10^{-3}	9.7951×10^{-3}	9.1409×10^{-3}	1.5738×10^{-2}
2	3.1428×10^{-6}	8.9551×10^{-5}	4.2097×10^{-4}	1.1958×10^{-4}	6.7636×10^{-4}
3	7.0617×10^{-12}	5.2824×10^{-9}	6.4218×10^{-7}	7.0457×10^{-9}	1.0318×10^{-6}
4	0	1.8032×10^{-17}	1.4429×10^{-12}	2.4050×10^{-17}	2.3184×10^{-12}

Let $x_0 = 0.53$, $x_{-1} = 0.6$. Then, we get $\alpha = 0.07$, $\eta \approx 0.0293$, $R_0 \approx 0.1892$, $m \approx 0.1114$, $m^1 \approx 0.1431$, $D_0 = D \cap B(x_0, 3R_0) = (0,\ 1.0975)$. In this case only the largest solutions satisfy conditions of appropriate theorems. So, we get $R \approx 0.1624$, $R^1 \approx 0.1109$, and $M \approx 0.8199$, $M^1 \approx 0.7362$. Moreover, $B(x_0, 3R) = (0.3676, 0.6924) \subset D$ and $B(x_0, 3R^1) = (0.4191, 0.6409) \subset D$. The error estimates are given in Table 2.

So, more accurate error estimates are retrieved because $M_n|x_n - x_{n-1}| \le M_n^1|x_n - x_{n-1}|$. Although $M|x_n - x_{n-1}| \le M^1|x_n - x_{n-1}|$ in the first case and $M|x_n - x_{n-1}| \ge M^1|x_n - x_{n-1}|$ in the second case.

Table 2. Results for $\varepsilon = 10^{-15}$.

n	$\lvert x_{n+1} - x_n \rvert$	$M_n \lvert x_n - x_{n-1} \rvert$	$M \lvert x_n - x_{n-1} \rvert$	$M_n^1 \lvert x_n - x_{n-1} \rvert$	$M^1 \lvert x_n - x_{n-1} \rvert$
1	7.3779×10^{-4}	3.1227×10^{-3}	2.3990×10^{-2}	4.2444×10^{-3}	2.1541×10^{-2}
2	3.9991×10^{-7}	1.6564×10^{-5}	6.0489×10^{-4}	2.2443×10^{-5}	5.4314×10^{-4}
3	1.1419×10^{-13}	2.1230×10^{-10}	3.2787×10^{-7}	2.8737×10^{-10}	2.9440×10^{-7}
4	0	3.2808×10^{-20}	9.3616×10^{-14}	4.4410×10^{-20}	8.4060×10^{-14}

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