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Two-Dimensional Uniform and Non-Uniform Haar Wavelet Collocation Approach for a Class of Nonlinear PDEs

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Abstract: In this paper, we introduce a novel approach employing two-dimensional uniform and non-uniform Haar wavelet collocation methods to effectively solve the generalized Burgers–Huxley and Burgers–Fisher equations. The demonstrated method exhibits an impressive quartic convergence rate. Several test problems are presented to exemplify the accuracy and efficiency of this proposed approach. Our results exhibit exceptional accuracy even with a minimal number of spatial divisions. Additionally, we conduct a comparative analysis of our results with existing methods.

Keywords: Burgers–Huxley; Burgers–Fisher; Haar wavelet; uniform; non-uniform; nonlinear; PDEs; collocation; quartic convergence

1. Introduction

Nonlinear partial differential equations (PDEs) play a fundamental role in describing various phenomena in science and engineering. However, due to their inherent complexity, finding analytical solutions to these equations is often impractical or even impossible. As a result, researchers have developed numerous numerical techniques to approximate solutions for such PDEs. These methods encompass a wide range of approaches, including the homotopy analysis method [1], iterative differential quadrature method [2], variational iteration method [3], spectral collocation method [4–6], meshless method of lines [7], and polynomial differential quadrature method [8]. Among the PDEs that have received significant attention in the literature are the generalized Burgers–Huxley (B–H) equation and the Burgers–Fisher (B–F) equation.

In this article, we consider the following B–H equation in the generalized form with the initial condition (IC) and boundary conditions (BCs):

$$w_t + \alpha w^\delta w_x - w_{xx} = \beta w\left(1 - w^\delta\right)\left(w^\delta - \gamma\right), \; x \in [0,1], \; t \in [0,1], \tag{1}$$

$$\textbf{IC:} \quad w(x,0) = g(x) = \left(\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\delta c \gamma x)\right)^{\frac{1}{\delta}}, \tag{2}$$

$$\textbf{BCs:} \quad w(0,t) = \phi(t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left[-\delta c\gamma\left(\frac{\gamma\alpha}{1+\delta} - 2c(1+\delta-\gamma)\right)t\right]\right)^{\frac{1}{\delta}}, \tag{3}$$

$$w(1,t) = \xi(t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left[\delta c\gamma\left(1 - \left(\frac{\gamma\alpha}{1+\delta} - 2c(1+\delta-\gamma)\right)t\right)\right]\right)^{\frac{1}{\delta}}. \tag{4}$$

Equation (1) has the exact solution $\tilde{w}(x,t)$ and was derived by Wang et al. [9]. It is given as

$$\tilde{w}(x,t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left[\delta c\gamma\left(x - \left(\frac{\gamma\alpha}{1+\delta} - 2c(1+\delta-\gamma)\right)t\right)\right]\right)^{\frac{1}{\delta}},$$

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where $c = \frac{-\alpha+\sqrt{\alpha^2+4\beta(1+\delta)}}{4(1+\delta)}$ and $\alpha, \beta, \gamma, \delta$ are parameters such that $\beta \geq 0, \delta > 0$, and $\gamma \in (0,1)$. Here, the subscripts t and x represent differentiation with respect to time and space, respectively.

In this article, we consider another equation, known as the B–F equation, in the generalized form, subject to the initial and boundary conditions

$$w_t + \alpha w^\delta w_x - w_{xx} = \beta w\left(1 - w^\delta\right), \; x \in [0,1], \; t \in [0,1], \tag{5}$$

$$\textbf{IC:} \quad w(x,0) = g(x) = \left(\frac{1}{2} + \frac{1}{2}\tanh(\frac{-\delta\rho}{2}x)\right)^{\frac{1}{\delta}}, \tag{6}$$

$$\textbf{BCs:} \quad w(0,t) = \phi(t) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left[\frac{\delta\rho}{2}\left(\rho + \frac{\beta}{\rho}\right)t\right]\right)^{\frac{1}{\delta}}, \tag{7}$$

$$w(1,t) = \xi(t) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left[\frac{-\delta\rho}{2}\left(1 - \left(\rho + \frac{\beta}{\rho}\right)t\right)\right]\right)^{\frac{1}{\delta}}, \tag{8}$$

with the exact solution $\tilde{w}(x,t)$,

$$\tilde{w}(x,t) = \left(\frac{1}{2} + \frac{1}{2}\tanh\left[\frac{-\delta\rho}{2}\left(x - \left(\rho + \frac{\beta}{\rho}\right)t\right)\right]\right)^{\frac{1}{\delta}},$$

where $\rho = \frac{\alpha}{\delta+1}$.

Ismail et al. [10] were probably the first to provide a numerical solution for the B–H equation using the Adomian decomposition method (ADM). Subsequently, Hashim et al. [11] conducted a convergence analysis of the ADM for solving the B–H equation. Batiha et al. [12] employed the variational iteration method (VIM) to solve the generalized B–H equation. Other methods used to solve the B–H equation include the splitting method [13], the compact operator method [14], the homotopy perturbation method [15], and the cubic splines approximation method [16–18]. Additionally, Díaz [19] described a modified exponential finite-difference method for solving the B–H equation.

The B–H equation is an extension of the widely used Huxley and Burgers equations. VIM [20], the HPM technique [21], and the ADM scheme [22] have all been used to solve the generalized Huxley equation for the special case when $\alpha = 0$. For this case, the Equation (1) becomes the generalized Huxley equation. Additionally, Equation (1) reduces to the Huxley equation for $\alpha = 0$ and $\delta = 1$, which has applications in modeling wall motion in liquid crystals and nerve pulse propagation in nerve fibers [9].

Equation (5) is known as the generalized Burgers equation when $\beta = 0$. Ismail et al. [10] obtained the numerical solution to the B–F equation using the ADM technique. Moghimi and Hejazi [23] subsequently described the VIM method for solving generalized Burgers–Fisher and Burgers equations. The cubic B-splines collocation method was used by Mittal and Tripathi [16] to solve the generalized B–F problem.

Mickens [24–27] introduced concepts for solving numerous ordinary differential equations (ODEs) and PDEs, which have become well-known as exact finite difference (EFD) and non-standard finite difference (NSFD) schemes. Recently, various forms of Burgers equations [28–32] have been solved by NSFD.

The Haar wavelet is an effective and popular tool for addressing a wide range of practical issues. Its applications in diverse fields have been investigated by several researchers. Due to its distinctive qualities of orthogonality, compact support, and symmetry, Haar wavelets have grown in prominence. They are particularly good at solving differential equations because of these characteristics. The Haar wavelet approach has gained popularity in numerical techniques because of its ease of use and effectiveness, particularly when dealing with nonlinearities and singularities. For a comprehensive understanding of the theory and applications of Haar wavelets, several references can be consulted, including [33–52] and references therein. Jiwari [36] used the uniform Haar wavelet method along with the quasilinearization strategy to solve the Burgers equation numerically. The uniform Haar wavelet was suggested by Celik et al. [53] to study various applications of the generalized B–H equation. Shukla and Kumar [54] used a combination of the uniform Haar wavelet analysis and the Crank–Nicolson finite difference approach to numerically solve the B–H problem. Recently, Verma et al. [55] developed a numerical technique based on the uniform Haar wavelet and non-standard finite difference scheme for solving a class of extended Burgers equations. A higher-order non-uniform Haar wavelet approach was also suggested by Ratas et al. [56] for solving nonlinear PDEs. Recently, the two-dimensional Haar wavelet method (2DHWM) [57] was developed as an enhancement of the Haar wavelet method (HWM). The 2DHWM has demonstrated improved accuracy and convergence compared to the standard HWM [58]. For a comprehensive review of 2DHWM and its applications, one may refer to [51,59–61].

In this article, our attention is directed towards the exploration of the generalized B–H and B–F equations. To address these equations effectively, we introduce a pair of methodologies: the two-dimensional uniform Haar wavelet collocation method (UHWCM) and the two-dimensional non-uniform Haar wavelet collocation method (NUHWCM), both of which are seamlessly integrated with the Newton–Raphson technique. Through our investigation, we establish and substantiate the remarkable quartic convergence rate inherent in each of these methods. We obtain numerical solutions for the generalized B–H and B–F equations, present absolute errors, and compare the results with existing methods. Furthermore, we report the computational time (in seconds). The proposed methods and numerical results are novel, providing accurate solutions with a minimal number of spatial divisions. The outcomes outlined in this paper are novel and have not been previously documented in existing literature. To the best of our knowledge, the techniques introduced in this study have not been employed for addressing the generalized B–H and B–F equations in prior research. We employed Mathematica 11.3 for computing the numerical results.

This article is organized as follows: Section 2 introduces the uniform and non-uniform Haar wavelets. Section 3 details the proposed method combining Haar wavelets and the Newton–Raphson method. Section 4 presents numerical illustrations to validate the method. The findings are summarized in Section 5 of the article's conclusion.

2. Preliminaries

Here, we define uniform and non-uniform Haar wavelets in detail and discuss some of their key properties. This will enable a complete understanding of Haar wavelets and their characteristics.

2.1. Uniform Haar Wavelet

Lepik and Hein took into account the interval $[0,1]$ in their analysis in 2014 [62]. Additionally, they divided this interval into $2M$ sub-intervals with equal $\Delta t = \frac{1}{2M}$ step size. The Haar wavelet's mother wavelet function $\mathscr{H}_i(t)$ is defined as follows for $i > 1$:

$$\mathscr{H}_i(t) = \begin{cases} 1, & y_1(i) \le t < y_2(i), \\ -1, & y_2(i) \le t < y_3(i), \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

such that the values $y_1(i)$, $y_2(i)$, and $y_3(i)$ are given by

$$y_1(i) = 2k\left(\frac{M}{m}\right)\Delta t, \quad y_2(i) = (2k+1)\left(\frac{M}{m}\right)\Delta t, \quad y_3(i) = 2(k+1)\left(\frac{M}{m}\right)\Delta t, \tag{10}$$

where the parameters i, m, M, j, k, and J are defined as:

$$\begin{aligned}
&i = m + k + 1,\\
&m = 2^j,\ M = 2^J,\\
&j = 0, 1, \cdots, J,\\
&k = 0, 1, \cdots, m-1,\\
&J = \text{Maximum level of resolution.}
\end{aligned}$$

For $i = 1$, the Haar function $\mathscr{H}_1(t)$ is defined as:

$$\mathscr{H}_1(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

The integral of the Haar function, $\mathscr{P}_{\nu,i}(t)$, is defined as follows for $i > 1$:

$$\mathscr{P}_{\nu,i}(t) = \begin{cases} 0, & t < y_1(i), \\ \frac{1}{\nu!}[t - y_1(i)]^\nu, & y_1(i) \le t \le y_2(i), \\ \frac{1}{\nu!}\{[t - y_1(i)]^\nu - 2[t - y_2(i)]^\nu\}, & y_2(i) \le t \le y_3(i), \\ \frac{1}{\nu!}\{[t - y_1(i)]^\nu - 2[t - y_2(i)]^\nu + [t - y_3(i)]^\nu\}, & t > y_3(i). \end{cases} \tag{12}$$

For $i = 1$, we have $y_1 = 0, y_2 = y_3 = 1$, and the expression for $\mathscr{P}_{\nu,1}(t)$ becomes

$$\mathscr{P}_{\nu,1}(t) = \frac{t^\nu}{\nu!}. \tag{13}$$

Collocation Points: We define the collocation points as follows:

$$t_{cl} = 0.5(\tilde{t}_{cl-1} + \tilde{t}_{cl}), \quad cl = 1, \cdots, 2M, \tag{14}$$

where

$$\text{Grid Points } (\tilde{t}_{cl}) = cl\Delta t, \quad cl = 0, 1, \cdots, 2M.$$

For computation, we introduce the matrices $H, P_1, P_2, \cdots$, which are the order of $2M \times 2M$ and defined by the following rules:

$$H(i, cl) = \mathscr{H}_i(t_{cl}), \quad P_\nu(i, cl) = \mathscr{P}_{\nu,i}(t_{cl}), \quad \nu = 1, 2, \cdots.$$

2.2. Non-Uniform Haar Wavelet

In their study, Ratas et al. [56] partitioned the interval [0,1] into 2M sub-intervals of varying step sizes using the following method:

$$\omega^g(r) = \frac{q^r - 1}{q^{2M} - 1}, \ r = 0, 1, 2, \cdots, 2M, \ \omega^g(r+1) > \omega^g(r) \forall r, \omega^g(0) = 0, \ \omega^g(2M) = 1, \tag{15}$$

where q represents an arbitrary constant with $q < 1$. The Haar wavelet's mother wavelet function $\tilde{\mathscr{H}}_i(t)$ is defined as follows for $i > 1$:

$$\tilde{\mathscr{H}}_i(t) = \begin{cases} 1, & y_1(i) \le t < y_2(i), \\ -\mathscr{C}_i, & y_2(i) \le t < y_3(i), \\ 0, & \text{otherwise,} \end{cases} \tag{16}$$

such that the values of $y_1(i)$, $y_2(i)$, and $y_3(i)$ are defined as:

$$y_1(i) = \omega^g\left(2k\left(\frac{M}{m}\right)\right), \quad y_2(i) = \omega^g\left((2k+1)\left(\frac{M}{m}\right)\right), \quad y_3(i) = \omega^g\left(2(k+1)\left(\frac{M}{m}\right)\right),$$

where,

$$\begin{aligned}
&i = m + k + 1,\\
&m = 2^j,\ M = 2^J,\\
&j = 0, 1, \cdots, J,\\
&k = 0, 1, \cdots, m-1,\\
&J = \text{Maximum level of resolution},\\
&\mathscr{C}_i = \frac{y_2(i) - y_1(i)}{y_3(i) - y_2(i)}.
\end{aligned}$$

For $i = 1$, the Haar function $\tilde{\mathscr{H}}_1(t)$ is defined as:

$$\tilde{\mathscr{H}}_1(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & \text{otherwise}. \end{cases} \tag{17}$$

The integral of the Haar function, $\tilde{\mathscr{P}}_{\nu,i}(t)$, is defined as follows for $i > 1$:

$$\tilde{\mathscr{P}}_{\nu,i}(t) = \begin{cases} 0, & 0 \le t < y_1(i), \\ \frac{1}{\nu!}[t - y_1(i)]^\nu, & y_1(i) \le t < y_2(i), \\ \frac{1}{\nu!}\{[t - y_1(i)]^\nu - (1 + \mathscr{C}_i)[t - y_2(i)]^\nu\}, & y_2(i) \le t < y_3(i), \\ \frac{1}{\nu!}\{[t - y_1(i)]^\nu - (1 + \mathscr{C}_i)[t - y_2(i)]^\nu + \mathscr{C}_i[t - y_3(i)]^\nu\}, & y_3(i) \le t \le 1. \end{cases} \tag{18}$$

For $i = 1$, we have $y_1 = 0, y_2 = y_3 = 1$, and

$$\tilde{\mathscr{P}}_{\nu,1}(t) = \frac{t^\nu}{\nu!}. \tag{19}$$

Collocation Points: To determine the collocation points t_r, the following definition is introduced as follows:

$$t_r = 0.5(\omega^g(r) + \omega^g(r+1)), \quad r = 0, \cdots, 2M - 1. \tag{20}$$

3. Novel Approach and Convergence Analysis

In this section, we formulate the solution technique for the proposed problem by employing both the uniform Haar wavelet collocation method (UHWCM) and non-uniform Haar wavelet collocation method (NUHWCM). Furthermore, we verify the stability of these newly introduced techniques.

3.1. Derivation: UHWCM

For the sake of simplicity, let us assume that the interval $[0, 1]$ represents the computational domain for both x and t. We divide both intervals into $2M$ sub-intervals with equal step sizes and assume that the solution is expressed in the following form:

$$w_{xxt}(x, t) = \sum_{i=1}^{2M} \sum_{l=1}^{2M} a_{il} \mathscr{H}_i(x) \mathscr{H}_l(t), \tag{21}$$

where a_{il} are the wavelet coefficients. Now, integrating Equation (21) from 0 to t, we obtain

$$w_{xx}(x, t) = \sum_{i=1}^{2M} \sum_{l=1}^{2M} a_{il} \mathscr{H}_i(x) \mathscr{P}_{1,l}(t) + w_{xx}(x, 0), \tag{22}$$

where $\mathscr{P}_{\nu,l}(t)$ represents the integral of the Haar function of order ν. By applying the initial condition $w(x,0) = g(x)$ in Equation (22), we have:

$$w_{xx}(x,t) = \sum_{i=1}^{2M}\sum_{l=1}^{2M} a_{il}\mathscr{H}_i(x)\mathscr{P}_{1,l}(t) + g_{xx}(x). \tag{23}$$

Now, we perform integration on Equation (23) from 0 to x twice, and we obtain:

$$w_x(x,t) = \sum_{i=1}^{2M}\sum_{l=1}^{2M} a_{il}\mathscr{P}_{1,i}(x)\mathscr{P}_{1,l}(t) + g_x(x) - g_x(0) + w_x(0,t), \tag{24}$$

$$w(x,t) = \sum_{i=1}^{2M}\sum_{l=1}^{2M} a_{il}\mathscr{P}_{2,i}(x)\mathscr{P}_{1,l}(t) + g(x) - g(0) - xg_x(0) + xw_x(0,t) + w(0,t). \tag{25}$$

Applying boundary conditions $w(0,t) = \phi(t), w(1,t) = \xi(t)$ on the Equation (25), we obtain:

$$w_x(0,t) = \xi(t) - \phi(t) + g_x(0) + g(0) - g(1) - \sum_{i=1}^{2M}\sum_{l=1}^{2M} a_{il}\mathscr{P}_{2,i}(1)\mathscr{P}_{1,l}(t). \tag{26}$$

We substitute the obtained value from Equation (26) into Equations (24) and (25) and obtain:

$$w_x(x,t) = \sum_{i=1}^{2M}\sum_{l=1}^{2M} a_{il}[\mathscr{P}_{1,i}(x)\mathscr{P}_{1,l}(t) - \mathscr{P}_{2,i}(1)\mathscr{P}_{1,l}(t)] + g_x(x) + \xi(t) - \phi(t) + g(0) - g(1), \tag{27}$$

$$w(x,t) = \sum_{i=1}^{2M}\sum_{l=1}^{2M} a_{il}[\mathscr{P}_{2,i}(x)\mathscr{P}_{1,l}(t) - x\mathscr{P}_{2,i}(1)\mathscr{P}_{1,l}(t)] + g(x) - g(0) + x[\xi(t) - \phi(t) + g(0) - g(1)] + \phi(t). \tag{28}$$

Differentiating Equation (28) with respect to t, we obtain:

$$w_t(x,t) = \sum_{i=1}^{2M}\sum_{l=1}^{2M} a_{il}[\mathscr{P}_{2,i}(x)\mathscr{H}_l(t) - x\mathscr{P}_{2,i}(1)\mathscr{H}_l(t)] + x[\xi_t(t) - \phi_t(t)] + \phi_t(t). \tag{29}$$

Now, we discretize the obtained results using the method of collocation defined in Equation (14). After that, we substitute the values of $w_{xx}(x_{cl},t_{cl}), w_x(x_{cl},t_{cl}), w(x_{cl},t_{cl}), w_t(x_{cl},t_{cl})$ from Equations (21) and (27)–(29) into the proposed problem (1) or (5) and obtain the following system of nonlinear equations:

$$\begin{aligned}
&\theta_1(a_{1,1}, a_{1,2}, \cdots, a_{2M,2M}) = 0,\\
&\theta_2(a_{1,1}, a_{1,2}, \cdots, a_{2M,2M}) = 0,\\
&\theta_3(a_{1,1}, a_{1,2}, \cdots, a_{2M,2M}) = 0,\\
&\vdots\\
&\theta_{(2M)^2}(a_{1,1}, a_{1,2}, \cdots, a_{2M,2M}) = 0.
\end{aligned}$$

We solve the above system of nonlinear equations by the Newton–Raphson method and obtain the wavelet coefficients. We substitute obtained wavelet coefficients into Equation (28) and obtain the required numerical solution.

3.2. Convergence: UHWCM

Theorem 1. *Assuming that the function $w_{xxt}(x,t)$ satisfies the Lipschitz condition on the interval $[0,1] \times [0,1]$, denoted by the existence of a positive constant μ such that for any $(x_1,t), (x_2,t) \in$*

$[0,1] \times [0,1]$, *the following inequality holds:* $|w_{xxt}(x_1,t) - w_{xxt}(x_2,t)| \leq \mu|x_1 - x_2|$. *Under these conditions, we can derive an error bound for* $||E_J(x,t)||_2$ *as follows:*

$$||E_J(x,t)||_2 \leq \frac{\mu}{112}\left(\frac{1}{2^J}\right)^4,$$

where $E_J(x,t) = |w(x,t) - w_J(x,t)|$. *Here,* $w_J(x,t)$ *represents the approximate solution obtained using the UHWCM method. Furthermore, the UHWCM method exhibits convergence, meaning that* $E_J(x,t)$ *approaches zero as* J *tends to infinity. The convergence is of order* 4, *indicated by the fact that* $||E_J(x,t)||_2 = O\left(\frac{1}{2^J}\right)^4$.

Proof. In order to distinguish between an exact solution and an approximate solution, we can represent the approximate solution obtained through the utilization of the UHWCM method in the following manner:

$$w_J(x,t) = \sum_{i=1}^{2M}\sum_{l=1}^{2M} a_{il}[\mathscr{P}_{2,i}(x)\mathscr{P}_{1,l}(t) - x\mathscr{P}_{2,i}(1)\mathscr{P}_{1,l}(t)] \\ + g(x) - g(0) + x[\xi(t) - \phi(t) + g(0) - g(1)] + \phi(t).$$

Then, error at the J^{th} level of resolution is defined as:

$$\begin{aligned} |E_J(x,t)| &= |w(x,t) - w_J(x,t)| \\ &= \left|\sum_{i=1}^{2M}\sum_{l=1}^{2M} a_{il}[\mathscr{P}_{2,i}(x)\mathscr{P}_{1,l}(t) - x\mathscr{P}_{2,i}(1)\mathscr{P}_{1,l}(t)]\right|. \end{aligned}$$

Now, expanding the L^2 norm of the error function, we obtain:

$$\begin{aligned} ||E_J(x,t)||_2^2 &= \int_0^1\int_0^1 \left(\sum_{i=1}^{2M}\sum_{l=1}^{2M} a_{il}[\mathscr{P}_{2,i}(x)\mathscr{P}_{1,l}(t) - x\mathscr{P}_{2,i}(1)\mathscr{P}_{1,l}(t)]\right)^2 dxdt \\ &= \int_0^1\int_0^1 \left(\sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1}\sum_{u=J+1}^{\infty}\sum_{v=0}^{2^u-1} a_{2^j+k+1,2^u+v+1}[\mathscr{P}_{2,2^j+k+1}(x) - \right. \\ &\qquad \left. x\mathscr{P}_{2,2^j+k+1}(1)]\mathscr{P}_{1,2^u+v+1}(t)\right)^2 dxdt. \end{aligned}$$

After simplifying above expression, we obtain:

$$\begin{aligned} ||E_J(x,t)||_2^2 = &\sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1}\sum_{u=J+1}^{\infty}\sum_{v=0}^{2^u-1}\sum_{j'=J+1}^{\infty}\sum_{k'=0}^{2^{j'}-1}\sum_{u'=J+1}^{\infty}\sum_{v'=0}^{2^{u'}-1} a_{2^j+k+1,2^u+v+1}a_{2^{j'}+k'+1,2^{u'}+v'+1} \\ &\int_0^1\int_0^1 [\mathscr{P}_{2,2^j+k+1}(x) - x\mathscr{P}_{2,2^j+k+1}(1)][\mathscr{P}_{2,2^{j'}+k'+1}(x) - x\mathscr{P}_{2,2^{j'}+k'+1}(1)] \\ &\mathscr{P}_{1,2^u+v+1}(t)\mathscr{P}_{1,2^{u'}+v'+1}(t)dxdt. \end{aligned} \tag{30}$$

To evaluate a_{il}, we use the following expression:

$$a_{il} = \int_0^1\int_0^1 w_{xxt}(x,t)\mathscr{H}_i(x)\mathscr{H}_l(t)dxdt \quad = \langle\mathscr{H}_i(x), \langle w_{xxt}(x,t), \mathscr{H}_l(t)\rangle\rangle, \tag{31}$$

where $\langle .,. \rangle$ denotes the inner product. Using the definition of the uniform Haar wavelet (9), we have:

$$\langle w_{xxt}(x,t), \mathscr{H}_l(t)\rangle = \int_0^1 w_{xxt}(x,t)\mathscr{H}_l(t)dt \quad = \int_{y_1(l)}^{y_2(l)} w_{xxt}(x,t)dt - \int_{y_2(l)}^{y_3(l)} w_{xxt}(x,t)dt.$$

Applying the mean value theorem for integrals, there exist $\Lambda_1 \in [y_1(l), y_2(l)]$ and $\Lambda_2 \in [y_2(l), y_3(l)]$ such that:

$$\langle w_{xxt}(x,t), \mathscr{H}_l(t)\rangle = (y_2(l) - y_1(l))w_{xxt}(x,\Lambda_1) - (y_3(l) - y_2(l))w_{xxt}(x,\Lambda_2).$$

From Equation (10), we have $y_2(l) - y_1(l) = y_3(l) - y_2(l) = \frac{1}{2^{u+1}}$. Therefore, the above expression simplifies to:

$$\langle w_{xxt}(x,t), \mathscr{H}_l(t)\rangle = \frac{1}{2^{u+1}}[w_{xxt}(x,\Lambda_1) - w_{xxt}(x,\Lambda_2)]. \tag{32}$$

By substituting Equation (32) into Equation (31) and utilizing Equation (9), we can derive the following expression:

$$\begin{aligned} a_{il} &= \langle \mathscr{H}_i(x), \frac{1}{2^{u+1}}[w_{xxt}(x,\Lambda_1) - w_{xxt}(x,\Lambda_2)]\rangle \\ &= \frac{1}{2^{u+1}}\left[\int_{y_1(i)}^{y_2(i)}[w_{xxt}(x,\Lambda_1) - w_{xxt}(x,\Lambda_2)]dx - \int_{y_2(i)}^{y_3(i)}[w_{xxt}(x,\Lambda_1) - w_{xxt}(x,\Lambda_2)]dx\right]. \end{aligned}$$

Once again, applying the mean value theorem for integrals, there exist η_1 and η_2 within the interval $[y_1(i), y_2(i)]$, as well as η_3 and η_4 within the interval $[y_2(i), y_3(i)]$, such that:

$$\begin{aligned} a_{il} =& \frac{1}{2^{u+1}}[(y_2(i) - y_1(i))[w_{xxt}(\eta_1,\Lambda_1) - w_{xxt}(\eta_2,\Lambda_1)] - \\ & (y_3(i) - y_2(i))[w_{xxt}(\eta_3,\Lambda_1) - w_{xxt}(\eta_4,\Lambda_1)]]. \end{aligned}$$

Using Equation (10), we know that $y_2(i) - y_1(i) = y_3(i) - y_2(i) = \frac{1}{2^{j+1}}$. Simplifying the expression above, we obtain:

$$a_{il} = \frac{1}{2^{u+j+2}}[w_{xxt}(\eta_1,\Lambda_1) - w_{xxt}(\eta_3,\Lambda_1) + w_{xxt}(\eta_4,\Lambda_1) - w_{xxt}(\eta_2,\Lambda_1)]. \tag{33}$$

Since $w_{xxt}(x,t)$ is Lipschitz in $[0,1] \times [0,1]$:

$$\begin{aligned} |a_{il}| &= \frac{1}{2^{u+j+2}}|w_{xxt}(\eta_1,\Lambda_1) - w_{xxt}(\eta_3,\Lambda_1) + w_{xxt}(\eta_4,\Lambda_1) - w_{xxt}(\eta_2,\Lambda_1)| \\ &\leq \frac{1}{2^{u+j+2}}(|w_{xxt}(\eta_3,\Lambda_1) - w_{xxt}(\eta_1,\Lambda_1)| + |w_{xxt}(\eta_4,\Lambda_1) - w_{xxt}(\eta_2,\Lambda_1)|) \\ &\leq \frac{1}{2^{u+j+2}}(\mu_1|\eta_3 - \eta_1| + \mu_2|\eta_4 - \eta_2|) \\ &\leq \frac{1}{2^{u+j+2}}(\mu_1|y_3(i) - y_1(i)| + \mu_2|y_3(i) - y_1(i)|) \\ &\leq \frac{\mu}{2^{u+2j+2}}; \ \text{ where } \mu = \max(\mu_1, \mu_2). \end{aligned}$$

Thus, we have obtained the bound for coefficients a_{il},

$$|a_{il}| \leq \frac{\mu}{2^{u+2j+2}}. \tag{34}$$

Since $\mathscr{P}_{2,i}(x) = 0$ for all x in the interval $[0, y_1(i)]$, we can observe that $\mathscr{P}_{2,i}(x)$ monotonically increases in the interval $[y_1(i), y_2(i)]$. Therefore, $\mathscr{P}_{2,i}(x)$ reaches its maximum value at $x = y_2(i)$. By the definition of $\mathscr{P}_{2,i}(x)$, we have:

$$\begin{aligned}\mathscr{P}_{2,i}(x) &\leq \frac{(y_2(i) - y_1(i))^2}{2} \\ &= \frac{1}{2}\left(\frac{1}{2^{j+1}}\right)^2; \quad x \in [y_1(i), y_2(i)].\end{aligned}$$

In the interval $x \in [y_2(i), y_3(i)]$, $\mathscr{P}_{2,i}(x)$ is monotonically increasing if $x \leq y_3(i)$. We can obtain this condition by using the definition of $\mathscr{P}_{2,i}(x)$ and $\frac{d\mathscr{P}_{2,i}(x)}{dx} \geq 0$. Therefore, $\mathscr{P}_{2,i}(x)$ reaches its upper bound at $x = y_3(i)$, which is given as:

$$\mathscr{P}_{2,i}(x) \leq \left(\frac{1}{2^{j+1}}\right)^2; \quad x \in [y_2(i), y_3(i)].$$

For $x \in [y_3(i), 1]$, we can conclude that:

$$\mathscr{P}_{2,i}(x) \leq \left(\frac{1}{2^{j+1}}\right)^2.$$

Hence, the upper bound of $\mathscr{P}_{2,i}(x)$ in the interval $[0, 1]$ is given by:

$$\mathscr{P}_{2,i}(x) \leq \left(\frac{1}{2^{j+1}}\right)^2. \tag{35}$$

Similarly, we can establish an upper bound for $\mathscr{P}_{1,l}(t)$ in the interval $[0, 1]$ as follows:

$$\mathscr{P}_{1,l}(t) \leq \frac{1}{2^{u+1}}. \tag{36}$$

Now, inserting Equation (34) into (30), we obtain:

$$\begin{aligned}||E_J(x,t)||_2^2 \leq &\mu^2 \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \sum_{u=J+1}^{\infty} \sum_{v=0}^{2^u-1} \sum_{j'=J+1}^{\infty} \sum_{k'=0}^{2^{j'}-1} \sum_{u'=J+1}^{\infty} \sum_{v'=0}^{2^{u'}-1} \frac{1}{2^{u+2j+u'+2j'+4}} \\ &\left(\int_0^1 \mathscr{P}_{1,2^u+v+1}(t)\mathscr{P}_{1,2^{u'}+v'+1}(t)dt\right) \\ &\left(\int_0^1 [\mathscr{P}_{2,2^j+k+1}(x) - x\mathscr{P}_{2,2^j+k+1}(1)][\mathscr{P}_{2,2^{j'}+k'+1}(x) - x\mathscr{P}_{2,2^{j'}+k'+1}(1)]dx\right).\end{aligned} \tag{37}$$

By Equation (36), we have:

$$\mathscr{P}_{1,2^u+v+1}(t) \leq \frac{1}{2^{u+1}}, \;\; \mathscr{P}_{1,2^{u'}+v'+1}(t) \leq \frac{1}{2^{u'+1}}, \;\; \forall t \in [0,1]. \tag{38}$$

Now, inserting Equation (38) into (37), we obtain:

$$\begin{aligned}||E_J(x,t)||_2^2 \leq &\mu^2 \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \sum_{u=J+1}^{\infty} \sum_{v=0}^{2^u-1} \sum_{j'=J+1}^{\infty} \sum_{k'=0}^{2^{j'}-1} \sum_{u'=J+1}^{\infty} \sum_{v'=0}^{2^{u'}-1} \left(\frac{1}{2^{2u+2j+2u'+2j'+6}}\right) \\ &\left(\int_0^1 [\mathscr{P}_{2,2^j+k+1}(x) - x\mathscr{P}_{2,2^j+k+1}(1)][\mathscr{P}_{2,2^{j'}+k'+1}(x) - x\mathscr{P}_{2,2^{j'}+k'+1}(1)]dx\right).\end{aligned} \tag{39}$$

Also, by using Equation (35), we deduce the following inequalities:

$$\left|\mathscr{P}_{2,2^j+k+1}(x) - x\mathscr{P}_{2,2^j+k+1}(1)\right| \leq \left|\mathscr{P}_{2,2^j+k+1}(x)\right| + |x|\left|\mathscr{P}_{2,2^j+k+1}(1)\right| \leq 2\left(\frac{1}{2^{j+1}}\right)^2, \quad (40)$$

$$\left|\mathscr{P}_{2,2^{j'}+k'+1}(x) - x\mathscr{P}_{2,2^{j'}+k'+1}(1)\right| \leq \left|\mathscr{P}_{2,2^{j'}+k'+1}(x)\right| + |x|\left|\mathscr{P}_{2,2^{j'}+k'+1}(1)\right| \leq 2\left(\frac{1}{2^{j'+1}}\right)^2. \quad (41)$$

Inserting inequalities (40) and (41) into (39), we obtain:

$$\begin{aligned}||E_J(x,t)||_2^2 &\leq \mu^2 \sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1}\sum_{u=J+1}^{\infty}\sum_{v=0}^{2^u-1}\sum_{j'=J+1}^{\infty}\sum_{k'=0}^{2^{j'}-1}\sum_{u'=J+1}^{\infty}\sum_{v'=0}^{2^{u'}-1}\left(\frac{1}{2^{2u+4j+2u'+4j'+8}}\right)\\ &= \mu^2 \sum_{j=J+1}^{\infty}\sum_{u=J+1}^{\infty}\sum_{j'=J+1}^{\infty}\sum_{u'=J+1}^{\infty}\left(\frac{2^j 2^{j'} 2^u 2^{u'}}{2^{2u+4j+2u'+4j'+8}}\right)\\ &= \frac{\mu^2}{2^8}\sum_{j=J+1}^{\infty}\sum_{u=J+1}^{\infty}\sum_{j'=J+1}^{\infty}\sum_{u'=J+1}^{\infty}\left(\frac{1}{2^u}\right)\left(\frac{1}{2^{u'}}\right)\left(\frac{1}{2^{3j}}\right)\left(\frac{1}{2^{3j'}}\right)\\ &= \frac{\mu^2}{2^8}\left(\frac{1}{49}\frac{1}{2^{8J}}\right).\end{aligned}$$

Thus,

$$||E_J(x,t)||_2 \leq \frac{\mu}{112}\left(\frac{1}{2^J}\right)^4.$$

Therefore, $\lim_{J\to\infty} E_J(x,t) = 0$. Moreover, the convergence is of order 4, that is,

$$||E_J(x,t)||_2 = O\left(\frac{1}{2^J}\right)^4.$$

This completes the proof. □

3.3. Derivation: NUHWCM

Here, we consider the computational domain for x and t as $[0,1]$ each. To discretize the intervals for x and t, we partition them into $2M$ sub-intervals of varying step sizes. This discretization approach is achieved by employing Equation (15). Moreover, we consider the solution to be sought in the following form:

$$w_{xxt}(x,t) = \sum_{i=1}^{2M}\sum_{l=1}^{2M} a_{il}\tilde{\mathscr{H}}_i(x)\tilde{\mathscr{H}}_l(t), \quad (42)$$

where a_{il} are the wavelet coefficients. Now, integrating Equation (42) multiple times similar to the method described in Section 3.1 and applying the initial and boundary conditions, we arrive at the following equations:

$$w_x(x,t) = \sum_{i=1}^{2M}\sum_{l=1}^{2M} a_{il}[\tilde{\mathscr{P}}_{1,i}(x)\tilde{\mathscr{P}}_{1,l}(t) - \tilde{\mathscr{P}}_{2,i}(1)\tilde{\mathscr{P}}_{1,l}(t)] + g_x(x) + \xi(t) - \phi(t) + g(0) - g(1), \quad (43)$$

$$w(x,t) = \sum_{i=1}^{2M}\sum_{l=1}^{2M} a_{il}[\tilde{\mathscr{P}}_{2,i}(x)\tilde{\mathscr{P}}_{1,l}(t) - x\tilde{\mathscr{P}}_{2,i}(1)\tilde{\mathscr{P}}_{1,l}(t)] \\ + g(x) - g(0) + x[\xi(t) - \phi(t) + g(0) - g(1)] + \phi(t). \tag{44}$$

Differentiating Equation (44) with respect to t, we obtain:

$$w_t(x,t) = \sum_{i=1}^{2M}\sum_{l=1}^{2M} a_{il}[\tilde{\mathscr{P}}_{2,i}(x)\mathscr{H}_l(t) - x\tilde{\mathscr{P}}_{2,i}(1)\mathscr{H}_l(t)] + x[\xi_t(t) - \phi_t(t)] + \phi_t(t). \tag{45}$$

Now, we discretize the obtained results by the use of the method of collocation defined in Equation (20). After that, we substitute the value of $w_{xx}(x_r,t_r)$, $w_x(x_r,t_r)$, $w(x_r,t_r)$, $w_t(x_r,t_r)$ from Equations (42)–(45) into the proposed problem (1) or (5) and obtain the following system of nonlinear equations:

$$\begin{aligned} &\zeta_1(a_{1,1}, a_{1,2}, \cdots, a_{2M,2M}) = 0,\\ &\zeta_2(a_{1,1}, a_{1,2}, \cdots, a_{2M,2M}) = 0,\\ &\zeta_3(a_{1,1}, a_{1,2}, \cdots, a_{2M,2M}) = 0,\\ &\vdots\\ &\zeta_{(2M)^2}(a_{1,1}, a_{1,2}, \cdots, a_{2M,2M}) = 0. \end{aligned}$$

We solve the above system of nonlinear equations by the Newton–Raphson method and obtain the wavelet coefficients. We substitute obtained wavelet coefficients into Equation (44) and obtain the required numerical solution.

3.4. Convergence: NUHWCM

Theorem 2. *Assuming that the function $w_{xxt}(x,t)$ satisfies the Lipschitz condition on the interval $[0,1]\times[0,1]$, denoted by the existence of a positive constant μ such that for any $(x_1,t),(x_2,t)\in[0,1]\times[0,1]$, the following inequality holds: $|w_{xxt}(x_1,t) - w_{xxt}(x_2,t)| \le \mu|x_1 - x_2|$. Under these given conditions, particularly when q approaches 1, we can derive an error bound for $||E_J(x,t)||_2$ as follows:*

$$||E_J(x,t)||_2 \le \frac{\mu}{7}\left(\frac{1}{2^J}\right)^4,$$

where $E_J(x,t) = |w(x,t) - w_J(x,t)|$. Here, $w_J(x,t)$ represents the approximate solution obtained using the NUHWCM method. Furthermore, the NUHWCM method exhibits convergence, meaning that $E_J(x,t)$ approaches zero as J approaches infinity. The convergence is of order 4, indicated by the fact that $||E_J(x,t)||_2 = O\left(\frac{1}{2^J}\right)^4$.

Proof. In order to distinguish between an exact solution and an approximate solution, we can represent the approximate solution obtained through the utilization of the NUHWCM method in the following manner:

$$w_J(x,t) = \sum_{i=1}^{2M}\sum_{l=1}^{2M} a_{il}[\tilde{\mathscr{P}}_{2,i}(x)\tilde{\mathscr{P}}_{1,l}(t) - x\tilde{\mathscr{P}}_{2,i}(1)\tilde{\mathscr{P}}_{1,l}(t)] \\ + g(x) - g(0) + x[\xi(t) - \phi(t) + g(0) - g(1)] + \phi(t).$$

Then, the error at the J^{th} level of resolution is defined as:

$$\begin{aligned} |E_J(x,t)| &= |w(x,t) - w_J(x,t)| \\ &= \left|\sum_{i=1}^{2M}\sum_{l=1}^{2M} a_{il}[\tilde{\mathscr{P}}_{2,i}(x)\tilde{\mathscr{P}}_{1,l}(t) - x\tilde{\mathscr{P}}_{2,i}(1)\tilde{\mathscr{P}}_{1,l}(t)]\right|. \end{aligned}$$

Now, expanding the L^2 norm of the error function, we obtain:

$$||E_J(x,t)||_2^2 = \int_0^1 \int_0^1 \left(\sum_{i=1}^{2M} \sum_{l=1}^{2M} a_{il} [\tilde{\mathscr{P}}_{2,i}(x) \tilde{\mathscr{P}}_{1,l}(t) - x \tilde{\mathscr{P}}_{2,i}(1) \tilde{\mathscr{P}}_{1,l}(t)] \right)^2 dxdt$$
$$= \int_0^1 \int_0^1 \left(\sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \sum_{u=J+1}^{\infty} \sum_{v=0}^{2^u-1} a_{2^j+k+1, 2^u+v+1} [\tilde{\mathscr{P}}_{2,2^j+k+1}(x) - x \tilde{\mathscr{P}}_{2,2^j+k+1}(1)] \tilde{\mathscr{P}}_{1,2^u+v+1}(t) \right)^2 dxdt.$$

After simplifying above expression, we obtain:

$$||E_J(x,t)||_2^2 = \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \sum_{u=J+1}^{\infty} \sum_{v=0}^{2^u-1} \sum_{j'=J+1}^{\infty} \sum_{k'=0}^{2^{j'}-1} \sum_{u'=J+1}^{\infty} \sum_{v'=0}^{2^{u'}-1} a_{2^j+k+1, 2^u+v+1} a_{2^{j'}+k'+1, 2^{u'}+v'+1}$$
$$\int_0^1 \int_0^1 [\tilde{\mathscr{P}}_{2,2^j+k+1}(x) - x \tilde{\mathscr{P}}_{2,2^j+k+1}(1)] [\tilde{\mathscr{P}}_{2,2^{j'}+k'+1}(x) - x \tilde{\mathscr{P}}_{2,2^{j'}+k'+1}(1)]$$
$$\tilde{\mathscr{P}}_{1,2^u+v+1}(t) \tilde{\mathscr{P}}_{1,2^{u'}+v'+1}(t) dxdt. \quad (46)$$

To evaluate a_{il}, we use the following expression:

$$a_{il} = \int_0^1 \int_0^1 w_{xxt}(x,t) \tilde{\mathscr{H}}_i(x) \tilde{\mathscr{H}}_l(t) dxdt \quad = \langle \tilde{\mathscr{H}}_i(x), \langle w_{xxt}(x,t), \tilde{\mathscr{H}}_l(t) \rangle \rangle, \quad (47)$$

where $\langle .,. \rangle$ denotes the inner product. Using the definition of the non-uniform Haar wavelet (16), we have:

$$\langle w_{xxt}(x,t), \tilde{\mathscr{H}}_l(t) \rangle = \int_0^1 w_{xxt}(x,t) \tilde{\mathscr{H}}_l(t) dt \quad = \int_{y_1(l)}^{y_2(l)} w_{xxt}(x,t) dt - \mathscr{C}_l \int_{y_2(l)}^{y_3(l)} w_{xxt}(x,t) dt.$$

Applying the mean value theorem for integrals, there exist $\Lambda_1 \in [y_1(l), y_2(l)]$ and $\Lambda_2 \in [y_2(l), y_3(l)]$ such that:

$$\langle w_{xxt}(x,t), \tilde{\mathscr{H}}_l(t) \rangle = (y_2(l) - y_1(l)) w_{xxt}(x, \Lambda_1) - (y_3(l) - y_2(l)) \mathscr{C}_l w_{xxt}(x, \Lambda_2).$$

By definition, we have $\mathscr{C}_l = \frac{y_2(l) - y_1(l)}{y_3(l) - y_2(l)}$. Therefore, the above expression simplifies to:

$$\langle w_{xxt}(x,t), \tilde{\mathscr{H}}_l(t) \rangle = (y_2(l) - y_1(l)) [w_{xxt}(x, \Lambda_1) - w_{xxt}(x, \Lambda_2)]. \quad (48)$$

By substituting Equation (48) into Equation (47) and utilizing Equation (16), we can derive the following expression:

$$a_{il} = \langle \tilde{\mathscr{H}}_i(x), (y_2(l) - y_1(l)) [w_{xxt}(x, \Lambda_1) - w_{xxt}(x, \Lambda_2) \rangle$$
$$= (y_2(l) - y_1(l)) \left[\int_{y_1(i)}^{y_2(i)} [w_{xxt}(x, \Lambda_1) - w_{xxt}(x, \Lambda_2)] dx - \mathscr{C}_i \int_{y_2(i)}^{y_3(i)} [w_{xxt}(x, \Lambda_1) - w_{xxt}(x, \Lambda_2)] dx \right].$$

Again, by applying the mean value theorem for integrals, there exist η_1 and η_2 within the interval $[y_1(i), y_2(i)]$, as well as η_3 and η_4 within the interval $[y_2(i), y_3(i)]$, such that:

$$a_{il} = (y_2(l) - y_1(l)) [(y_2(i) - y_1(i)) [w_{xxt}(\eta_1, \Lambda_1) - w_{xxt}(\eta_2, \Lambda_2)] - (y_3(i) - y_2(i)) \mathscr{C}_i [w_{xxt}(\eta_3, \Lambda_1) - w_{xxt}(\eta_4, \Lambda_2)]].$$

From definition, we have $\mathscr{C}_i = \frac{y_2(i)-y_1(i)}{y_3(i)-y_2(i)}$. Therefore, the above expression simplifies to:

$$\begin{aligned} a_{il} =& (y_2(l) - y_1(l))(y_2(i) - y_1(i))[w_{xxt}(\eta_1, \Lambda_1) - w_{xxt}(\eta_3, \Lambda_1) + \\ & w_{xxt}(\eta_4, \Lambda_2) - w_{xxt}(\eta_2, \Lambda_2)]. \end{aligned} \tag{49}$$

Since $w_{xxt}(x,t)$ is Lipschitz in $[0,1] \times [0,1]$:

$$\begin{aligned} |a_{il}| =& |y_2(l) - y_1(l)||y_2(i) - y_1(i)||w_{xxt}(\eta_1, \Lambda_1) - w_{xxt}(\eta_3, \Lambda_1) + \\ & w_{xxt}(\eta_4, \Lambda_1) - w_{xxt}(\eta_2, \Lambda_1)| \\ \leq& |y_2(l) - y_1(l)||y_2(i) - y_1(i)|(|w_{xxt}(\eta_3, \Lambda_1) - w_{xxt}(\eta_1, \Lambda_1)| + \\ & |w_{xxt}(\eta_4, \Lambda_1) - w_{xxt}(\eta_2, \Lambda_1)|) \\ \leq& |y_2(l) - y_1(l)||y_2(i) - y_1(i)|(\mu_1|\eta_3 - \eta_1| + \mu_2|\eta_4 - \eta_2|) \\ \leq& |y_2(l) - y_1(l)||y_2(i) - y_1(i)|(\mu_1|y_3(i) - y_1(i)| + \mu_2|y_3(i) - y_1(i)|) \\ \leq& \mu|y_2(l) - y_1(l)||y_2(i) - y_1(i)||y_3(i) - y_1(i)|; \text{ where } \mu = \max(\mu_1, \mu_2). \end{aligned}$$

Also, we have the relation $y_2 - y_1 \leq y_3 - y_1$. Thus, we obtain the following bound for coefficients a_{il} by using the values of y_1, y_2, and y_3 from the definition:

$$|a_{il}| \leq \frac{\mu q^{(2^{(J-j+2)}k + 2^{(J-u+1)}v)}(q^{2^{(J-j+1)}} - 1)^2(q^{2^{(J-u+1)}} - 1)}{(q^{2^{(J+1)}} - 1)^3}. \tag{50}$$

Now, we evaluate the bounds for $\tilde{\mathscr{P}}_{1,l}(t)$ and $\tilde{\mathscr{P}}_{2,i}(x)$. To start with $\tilde{\mathscr{P}}_{1,l}(t)$, we use definition (18):

$$\tilde{\mathscr{P}}_{1,l}(t) = \begin{cases} t - y_1(l), & y_1(l) \leq t < y_2(l), \\ (y_2(l) - y_1(l)) - \mathscr{C}_l(t - y_2(l)), & y_2(l) \leq t < y_3(l), \\ 0, & \text{otherwise}. \end{cases}$$

From the above equation, $\tilde{\mathscr{P}}_{1,l}(t) = 0 \; \forall \; t \in [0, y_1(l)) \cup [y_3(l), 1]$. $\tilde{\mathscr{P}}_{1,l}(t)$ is monotonically increasing in $y_1(l) \leq t \leq y_2(l)$. Thus, $\tilde{\mathscr{P}}_{1,l}(t)$ achieves its upper bound at $t = y_2(l)$. Therefore, $\tilde{\mathscr{P}}_{1,l}(t) \leq (y_2(l) - y_1(l)) \leq (y_3(l) - y_1(l))$. Also, when $y_2(l) \leq t \leq y_3(l)$, $\tilde{\mathscr{P}}_{1,l}(t)$ is monotonically decreasing, so the maximum value will be obtained at $t = y_2(l)$. Thus, $\tilde{\mathscr{P}}_{1,l}(t) \leq (y_2(l) - y_1(l)) \leq (y_3(l) - y_1(l))$. Thus, we have obtained the upper bound $\tilde{\mathscr{P}}_{1,l}(t)$ in the interval $[0,1]$:

$$\tilde{\mathscr{P}}_{1,l}(t) \leq \frac{q^{2^{(J-u+1)}v}(q^{2^{(J-u+1)}} - 1)}{q^{2^{(J+1)}} - 1}. \tag{51}$$

Similarly, we can obtain the bound for $\tilde{\mathscr{P}}_{2,i}(x)$. By definition (18):

$$\tilde{\mathscr{P}}_{2,i}(x) = \begin{cases} 0, & 0 \leq x < y_1(i), \\ \frac{1}{2!}[x - y_1(i)]^2, & y_1(i) \leq x < y_2(i), \\ \frac{1}{2!}\{x - y_1(i)]^2 - (1 + \mathscr{C}_i)[x - y_2(i)]^2\}, & y_2(i) \leq x < y_3(i), \\ \frac{1}{2!}\{[x - y_1(i)]^2 - (1 + \mathscr{C}_i)[x - y_2(i)]^2 + \mathscr{C}_i[x - y_3(i)]^2\}, & y_3(i) \leq x \leq 1. \end{cases}$$

Now, the derivative of the above function is given as:

$$\frac{\tilde{\mathscr{P}}_{2,i}(x)}{dx} = \begin{cases} x - y_1(i), & y_1(i) \leq x < y_2(i), \\ \mathscr{C}_i(y_3(i) - x), & y_2(i) \leq x < y_3(i), \\ 0, & \text{otherwise}. \end{cases}$$

From the above equation, we can say that $\tilde{\mathscr{P}}_{2,i}(x)$ is monotonically increasing in the interval $y_1(i) \leq x < y_2(i)$. Therefore, the upper bound can be achieved at $x = y_2(i)$. Thus, we have $\tilde{\mathscr{P}}_{2,i}(x) \leq \frac{[y_2(i)-y_1(i)]^2}{2} \leq \frac{[y_3(i)-y_1(i)]^2}{2}$. In $y_2(i) \leq x < y_3(i)$, function $\tilde{\mathscr{P}}_{2,i}(x)$ is monotonically increasing if $x \leq y_3(i)$. This condition can be obtained by using $\frac{\tilde{\mathscr{P}}_{2,i}(x)}{dx} \geq 0$ in $y_2(i) \leq x < y_3(i)$. Hence, the maximum value of $\tilde{\mathscr{P}}_{2,i}(x)$ can be obtained by substituting $x = y_3(i)$ in the definition of $\tilde{\mathscr{P}}_{2,i}(x)$ as follows:

$$\begin{aligned}
\tilde{\mathscr{P}}_{2,i}(x) &\leq \frac{1}{2!}\Big\{ y_3(i) - y_1(i)]^2 - (1+\mathscr{C}_i)[y_3(i) - y_2(i)]^2 \Big\} \\
&= \frac{(y_3(i) - y_1(i))(y_2(i) - y_1(i))}{2} \\
&\leq \frac{[y_3(i) - y_1(i)]^2}{2}; \;\; y_2(i) \leq x < y_3(i).
\end{aligned}$$

In $y_3(i) \leq x < 1$, $\tilde{\mathscr{P}}_{2,i}(x)$ is maximum when $x = 1$. Therefore, the upper bound will be:

$$\begin{aligned}
\tilde{\mathscr{P}}_{2,i}(x) &\leq \frac{1}{2!}\Big\{ [1 - y_1(i)]^2 - (1+\mathscr{C}_i)[1 - y_2(i)]^2 + \mathscr{C}_i[1 - y_3(i)]^2 \Big\} \\
&= \frac{(y_3(i) - y_1(i))(y_2(i) - y_1(i))}{2} \\
&\leq \frac{[y_3(i) - y_1(i)]^2}{2}; \;\; y_3(i) \leq x < 1.
\end{aligned}$$

Thus, we have obtained the upper bound $\tilde{\mathscr{P}}_{2,i}(x)$ in the interval $[0,1]$:

$$\tilde{\mathscr{P}}_{2,i}(x) \leq \frac{q^{2(J-j+2)}k(q^{2(J-j+1)} - 1)^2}{2(q^{2(J+1)} - 1)^2}. \tag{52}$$

Now, inserting Equation (50) into (46), we obtain:

$$\begin{aligned}
||E_J(x,t)||_2^2 \leq & \mu^2 \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \sum_{u=J+1}^{\infty} \sum_{v=0}^{2^u-1} \sum_{j'=J+1}^{\infty} \sum_{k'=0}^{2^{j'}-1} \sum_{u'=J+1}^{\infty} \sum_{v'=0}^{2^{u'}-1} q^{(2^{(J-j+2)}k + 2^{(J-u+1)}v)} \\
& q^{(2^{(J-j'+2)}k' + 2^{(J-u'+1)}v')} \left(\frac{(q^{2^{(J-j+1)}} - 1)^2 (q^{2^{(J-u+1)}} - 1)}{(q^{2^{(J+1)}} - 1)^3} \right) \\
& \left(\frac{(q^{2^{(J-j'+1)}} - 1)^2 (q^{2^{(J-u'+1)}} - 1)}{(q^{2^{(J+1)}} - 1)^3} \right) \left(\int_0^1 \tilde{\mathscr{P}}_{1,2^u+v+1}(t) \tilde{\mathscr{P}}_{1,2^{u'}+v'+1}(t) dt \right) \\
& \left(\int_0^1 [\tilde{\mathscr{P}}_{2,2^j+k+1}(x) - x\tilde{\mathscr{P}}_{2,2^j+k+1}(1)][\tilde{\mathscr{P}}_{2,2^{j'}+k'+1}(x) - x\tilde{\mathscr{P}}_{2,2^{j'}+k'+1}(1)] dx \right).
\end{aligned} \tag{53}$$

By Equation (51), we have:

$$\tilde{\mathscr{P}}_{1,l}(t) \leq \frac{q^{2^{(J-u+1)}v}(q^{2^{(J-u+1)}} - 1)}{q^{2^{(J+1)}} - 1}, \;\; \tilde{\mathscr{P}}_{1,l}(t) \leq \frac{q^{2^{(J-u'+1)}v'}(q^{2^{(J-u'+1)}} - 1)}{q^{2^{(J+1)}} - 1}, \forall t \in [0,1]. \tag{54}$$

Now, inserting Equation (54) into (53), we obtain:

$$
\begin{aligned}
||E_J(x,t)||_2^2 \leq &\mu^2 \sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1}\sum_{u=J+1}^{\infty}\sum_{v=0}^{2^u-1}\sum_{j'=J+1}^{\infty}\sum_{k'=0}^{2^{j'}-1}\sum_{u'=J+1}^{\infty}\sum_{v'=0}^{2^{u'}-1} q^{(2^{(J-j+2)}k+2^{(J-u+2)}v)} \\
&q^{(2^{(J-j'+2)}k'+2^{(J-u'+2)}v')}\left(\frac{(q^{2^{(J-j+1)}}-1)^2(q^{2^{(J-u+1)}}-1)^2}{(q^{2^{(J+1)}}-1)^4}\right) \\
&\left(\frac{(q^{2^{(J-j'+1)}}-1)^2(q^{2^{(J-u'+1)}}-1)^2}{(q^{2^{(J+1)}}-1)^4}\right) \\
&\left(\int_0^1[\tilde{\mathscr{P}}_{2,2^j+k+1}(x)-x\tilde{\mathscr{P}}_{2,2^j+k+1}(1)][\tilde{\mathscr{P}}_{2,2^{j'}+k'+1}(x)-x\tilde{\mathscr{P}}_{2,2^{j'}+k'+1}(1)]dx\right).
\end{aligned} \tag{55}
$$

Also, by using Equation (52), we deduce the following inequalities:

$$\left|\tilde{\mathscr{P}}_{2,2^j+k+1}(x)-x\tilde{\mathscr{P}}_{2,2^j+k+1}(1)\right| \leq \frac{q^{2^{(J-j+2)}k}(q^{2^{(J-j+1)}}-1)^2}{(q^{2^{(J+1)}}-1)^2}, \tag{56}$$

$$\left|\tilde{\mathscr{P}}_{2,2^{j'}+k'+1}(x)-x\tilde{\mathscr{P}}_{2,2^{j'}+k'+1}(1)\right| \leq \frac{q^{2^{(J-j'+2)}k'}(q^{2^{(J-j'+1)}}-1)^2}{(q^{2^{(J+1)}}-1)^2}. \tag{57}$$

Inserting inequalities (56) and (57) into (55), we obtain:

$$
\begin{aligned}
||E_J(x,t)||_2^2 \leq &\mu^2 \sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1}\sum_{u=J+1}^{\infty}\sum_{v=0}^{2^u-1}\sum_{j'=J+1}^{\infty}\sum_{k'=0}^{2^{j'}-1}\sum_{u'=J+1}^{\infty}\sum_{v'=0}^{2^{u'}-1} q^{(2^{(J-j+3)}k+2^{(J-u+2)}v)} \\
&q^{(2^{(J-j'+3)}k'+2^{(J-u'+2)}v')}\left(\frac{(q^{2^{(J-j+1)}}-1)^4(q^{2^{(J-u+1)}}-1)^2}{(q^{2^{(J+1)}}-1)^6}\right) \\
&\left(\frac{(q^{2^{(J-j'+1)}}-1)^4(q^{2^{(J-u'+1)}}-1)^2}{(q^{2^{(J+1)}}-1)^6}\right).
\end{aligned}
$$

Since $0<q<1$, as q approaches 1, the above expression simplifies as:

$$
\begin{aligned}
||E_J(x,t)||_2^2 &\leq \mu^2 \sum_{j=J+1}^{\infty}\sum_{k=0}^{2^j-1}\sum_{u=J+1}^{\infty}\sum_{v=0}^{2^u-1}\sum_{j'=J+1}^{\infty}\sum_{k'=0}^{2^{j'}-1}\sum_{u'=J+1}^{\infty}\sum_{v'=0}^{2^{u'}-1}\left(\frac{1}{2^{4j}}\right)\left(\frac{1}{2^{4j'}}\right)\left(\frac{1}{2^{2u}}\right)\left(\frac{1}{2^{2u'}}\right) \\
&= \mu^2 \sum_{j=J+1}^{\infty}\sum_{u=J+1}^{\infty}\sum_{j'=J+1}^{\infty}\sum_{u'=J+1}^{\infty} 2^j 2^{j'} 2^u 2^{u'}\left(\frac{1}{2^{4j}}\right)\left(\frac{1}{2^{4j'}}\right)\left(\frac{1}{2^{2u}}\right)\left(\frac{1}{2^{2u'}}\right) \\
&= \mu^2 \sum_{j=J+1}^{\infty}\sum_{u=J+1}^{\infty}\sum_{j'=J+1}^{\infty}\sum_{u'=J+1}^{\infty}\left(\frac{1}{2^{u}}\right)\left(\frac{1}{2^{u'}}\right)\left(\frac{1}{2^{3j}}\right)\left(\frac{1}{2^{3j'}}\right) \\
&= \mu^2\left(\frac{1}{49}\frac{1}{2^{8J}}\right).
\end{aligned}
$$

Thus, we obtain:

$$||E_J(x,t)||_2 \leq \frac{\mu}{7}\left(\frac{1}{2^J}\right)^4.$$

Therefore, $\lim_{J\to\infty} E_J(x,t)=0$. Moreover, the convergence is of order 4, that is,

$$||E_J(x,t)||_2 = O\left(\frac{1}{2^J}\right)^4.$$

This completes the proof. □

Remark 1. *Lepik et al. [62] have considered linear PDEs and developed the solution technique by using the two-dimensional uniform Haar wavelet method, and they obtained the linear system of equations. To evaluate the wavelet coefficients, they proposed a transformation technique to reduce the system from fourth-order to second-order matrices. After performing the transformation, they again converted the coefficients into the original form to obtain the required numerical solution.*

In this article, we consider nonlinear PDEs. We use the Newton–Raphson method to solve the system of nonlinear equations after applying the two-dimensional uniform and non-uniform Haar wavelet collocation method. We do not require any transformation technique to solve the nonlinear system of equations, which reduces the computational time. The proposed method takes very few iterations to obtain the required numerical solution. CPU time is also much less.

4. Numerical Illustration

In this section, we explore specific cases of the proposed problems by considering different parameter values. To obtain the numerical solutions, we apply the methods described in Section 3. In all the examples, we set $q = 0.99$ and use an initial guess of $[1, 1, \ldots, 1]$ for the computation of numerical solutions. To show the accuracy of the proposed method, we present the absolute error, which is defined as follows:

$$\text{Absolute error} = |w(x,t) - \tilde{w}(x,t)|,$$

where $\tilde{w}(x,t)$ represents the exact solution.

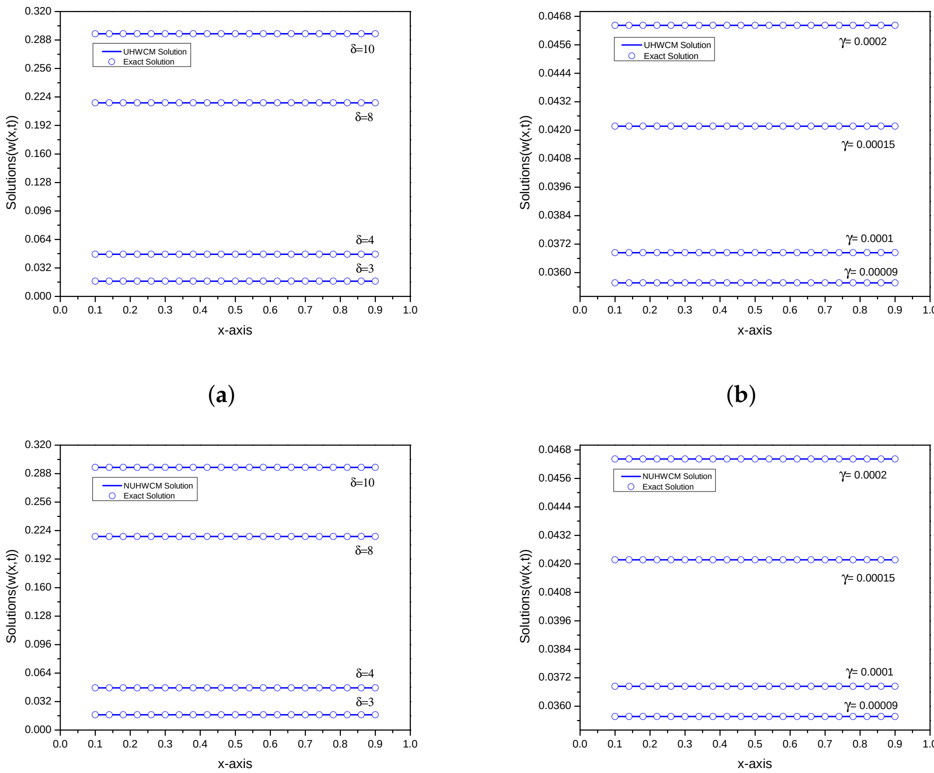
Example 1. *Consider the generalized Burgers–Huxley Equation* (1) *with the corresponding initial condition* (2)*, and boundary conditions* (3) *and* (4) *for the following parameter values: $\alpha = 1$, $\beta = 1$, $\delta = 1$, and $\gamma = 0.001$. We employ both the UHWCM and the NUHWCM approaches and record the absolute errors in Table 1. We compare our results with the other existing methods in the table. It can be observed through the analysis of the table that our method provides highly accurate results even with very few spatial divisions. When we make alterations to the parameters δ and γ, the solutions remain almost the same (see Figure 1). However, as δ and γ increase, we obtain very accurate results. Furthermore, with respect to the parameters α and β, we have omitted the presentation of solution variations, as they consistently yield the same result across different parameter settings. It is interesting to note that variations in the initial guess have no effect on the final solution. This shows the robustness and stability of the proposed method.*

Example 2. *Let us consider Equation* (1) *with initial condition* (2) *and boundary conditions* (3) *and* (4)*. When $\alpha = 0$, this becomes the generalized Huxley equation. For our computations, we consider the other parameter values $\beta = 1$, $\delta = 1$, and $\gamma = 0.001$. The absolute errors achieved using the UHWCM and NUHWCM approaches, as well as those obtained using other methods, are presented in Table 2. When compared to existing methods, our method clearly offers findings that are significantly more accurate, even with very few spatial divisions, as shown in the table. When varying the parameters δ and γ, the solutions demonstrate minimal variation, as illustrated in Figure 2. Notably, as we increase δ and γ, we consistently obtain highly accurate results. Furthermore, we have omitted the presentation of solution variations as ours consistently yields the same result across different parameter settings of β. It is significant to observe that the final solution holds true despite variations in the initial guess. It shows that our proposed method is stable and accurate.*

Example 3. *Consider the generalized Burgers–Fisher Equation* (5) *and its corresponding initial and boundary conditions* (6)*,* (7)*, and* (8)*, respectively. We select three specific parameter values for our calculations: $\alpha = 0.001$, $\beta = 0.001$, and $\delta = 1$. After that, we examine the outcomes of several existing techniques with the absolute errors obtained from our proposed techniques, UHWCM and NUHWCM. This comparative study is shown in Table 3. The table makes it evident that, especially when only a few spatial divisions are used, our method yields the most accurate results. When we vary the parameters α, β, and δ, the solutions exhibit minimal variation while maintaining a high level of accuracy, as illustrated in Figure 3. It is noteworthy that the final solution consistency holds* *true even when the initial guess is changed. This highlights the stability and robustness that are built into our method.*

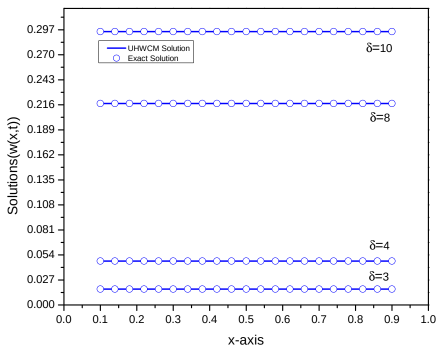
Table 1. Comparison of absolute errors in $w(x,t)$ for $\alpha = 1, \beta = 1, \delta = 1$, and $\gamma = 0.001$.

x	t	Proposed Method				HWCFD [54]	VIM [12]	ADM [11]	ADM [10]
		UHWCM		NUHWCM					
		$J = 1$	$J = 2$	$J = 1$	$J = 2$				
0.1	0.05	4.02996×10^{-9}	5.94152×10^{-9}	3.99135×10^{-9}	5.8416×10^{-9}	$5.09065400001 \times 10^{-9}$	1.87405×10^{-8}	1.87406×10^{-8}	1.93715×10^{-7}
0.1	0.1	8.05993×10^{-9}	1.1883×10^{-8}	7.9827×10^{-9}	1.16832×10^{-8}	$3.94874100008 \times 10^{-9}$	3.74813×10^{-8}	3.74812×10^{-8}	3.87434×10^{-7}
0.1	1	1.59406×10^{-8}	1.65469×10^{-8}	1.59536×10^{-8}	1.65483×10^{-8}	$1.66056969999 \times 10^{-8}$	3.74812×10^{-7}	3.74812×10^{-7}	3.87501×10^{-6}
0.5	0.05	1.02564×10^{-8}	1.38108×10^{-8}	1.01733×10^{-8}	1.36407×10^{-8}	$9.99043449999 \times 10^{-8}$	1.87405×10^{-8}	1.87406×10^{-8}	1.9373×10^{-7}
0.5	0.1	2.05129×10^{-8}	2.76215×10^{-8}	2.03466×10^{-8}	2.72815×10^{-8}	$1.06041257999 \times 10^{-7}$	1.37481×10^{-8}	3.74812×10^{-8}	3.87464×10^{-7}
0.5	1	4.75675×10^{-8}	4.68696×10^{-8}	4.75675×10^{-8}	4.68694×10^{-8}	$2.16505682999 \times 10^{-7}$	3.74813×10^{-7}	3.74812×10^{-7}	3.87531×10^{-6}
0.9	0.05	4.03039×10^{-9}	5.94204×10^{-9}	3.99424×10^{-9}	5.83573×10^{-9}	$2.04899342000 \times 10^{-7}$	1.87405×10^{-8}	1.87406×10^{-8}	1.93745×10^{-7}
0.9	0.1	8.06078×10^{-9}	1.18841×10^{-8}	7.98848×10^{-9}	1.16715×10^{-8}	$2.16031254999 \times 10^{-7}$	3.74813×10^{-8}	3.74812×10^{-8}	3.87494×10^{-7}
0.9	1	1.5943×10^{-8}	1.65492×10^{-8}	1.59305×10^{-8}	1.65496×10^{-8}	$4.16405665000 \times 10^{-7}$	$3.74813 \times 10^{-7} \times 10^{-7}$	3.74812×10^{-7}	3.87561×10^{-6}
CPU Time (In Sec.)		6.265	24.202	11.765	62.828				

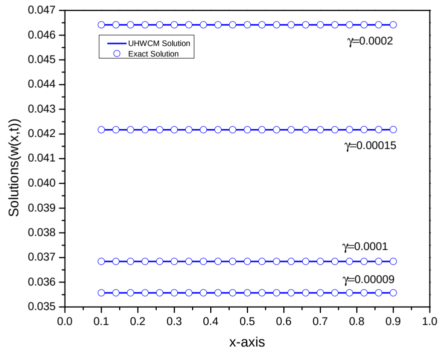


(**a**) (**b**) (**c**) (**d**)

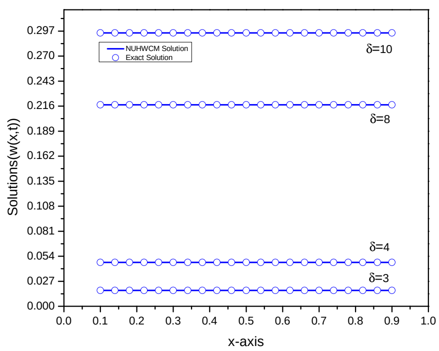
Figure 1. Graphs of exact, UHWCM, and NUHWCM solutions for various values of δ and γ. (**a**) Exact and UHWCM solutions with fixed parameters: $\alpha = 1, \beta = 1, \gamma = 0.00001, t = 0.8$. (**b**) Exact and UHWCM solutions with fixed parameters: $\alpha = 4, \beta = 1, \delta = 3, t = 0.1$. (**c**) Exact and NUHWCM solutions with fixed parameters: $\alpha = 1, \beta = 1, \gamma = 0.00001, t = 0.8$. (**d**) Exact and NUHWCM solutions with fixed parameters: $\alpha = 4, \beta = 1, \delta = 3, t = 0.1$.



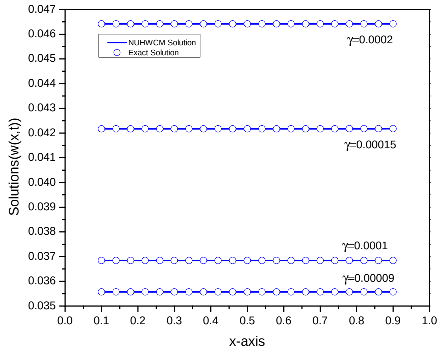
(**a**)



(**b**)

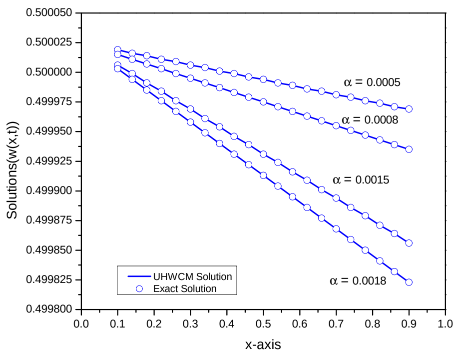


(**c**)

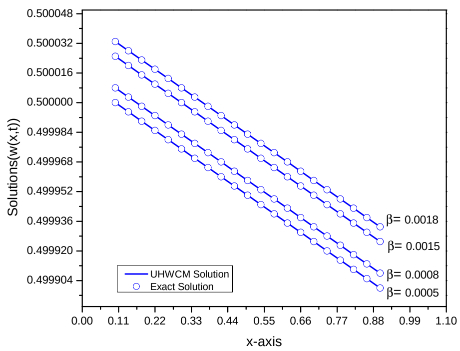


(**d**)

Figure 2. Graph of exact, UHWCM, and NUHWCM solutions for various values of δ and γ. (**a**) Exact and UHWCM solutions with fixed parameters: $\beta = 1, \gamma = 0.00001, t = 0.8$. (**b**) Exact and UHWCM solutions with fixed parameters: $\beta = 1, \delta = 3, t = 0.1$. (**c**) Exact and NUHWCM solutions with fixed parameters: $\beta = 1, \gamma = 0.00001, t = 0.8$. (**d**) Exact and NUHWCM solutions with fixed parameters: $\beta = 1, \delta = 3, t = 0.1$.

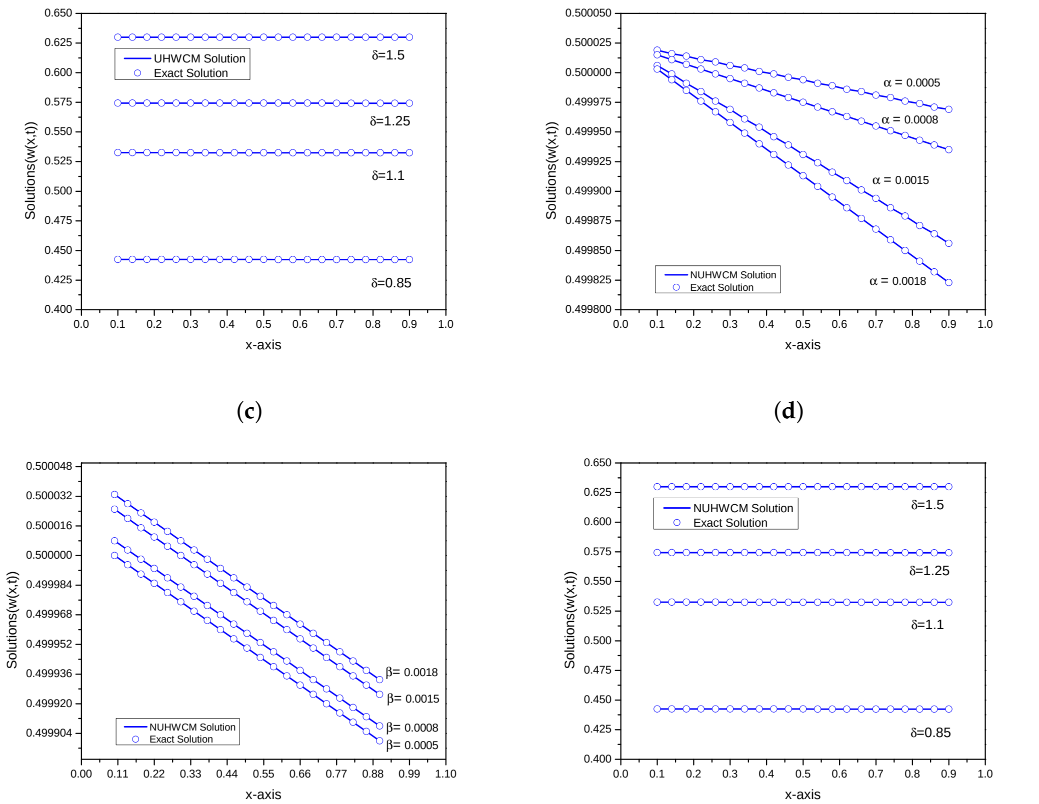


(**a**)



(**b**)

Figure 3. *Cont*.



(**c**) (**d**)

(**e**) (**f**)

Figure 3. Graph of exact, UHWCM, and NUHWCM solutions for various values of α, β, and δ. (**a**) Exact and UHWCM solutions with fixed parameters: $\delta = 1, \beta = 0.001, t = 0.1$. (**b**) Exact and UHWCM solutions with fixed parameters: $\delta = 1, \alpha = 0.001, t = 0.1$. (**c**) Exact and UHWCM solutions with fixed parameters: $\beta = 0.001, \alpha = 0.001, t = 0.1$. (**d**) Exact and NUHWCM solutions with fixed parameters: $\delta = 1, \beta = 0.001, t = 0.1$. (**e**) Exact and NUHWCM solutions with fixed parameters: $\delta = 1, \alpha = 0.001, t = 0.1$. (**f**) Exact and NUHWCM solutions with fixed parameters: $\beta = 0.001, \alpha = 0.001, t = 0.1$.

Table 2. Comparison of absolute errors in $w(x,t)$ for $\alpha = 0$, $\beta = 1$, $\delta = 1$, and $\gamma = 0.001$.

x	t	Proposed Method				HWCFD [54]	ADM [22]	ADM [10]
		UHWCM		NUHWCM				
		$J=1$	$J=2$	$J=1$	$J=2$			
0.1	0.05	5.37357×10^{-9}	7.92237×10^{-9}	5.32208×10^{-9}	7.78914×10^{-9}	$6.40665700005 \times 10^{-9}$	2.49875×10^{-8}	1.87465×10^{-7}
0.1	0.1	1.07471×10^{-8}	1.58447×10^{-8}	1.06442×10^{-8}	1.55783×10^{-8}	$8.69100700000 \times 10^{-9}$	4.99750×10^{-8}	3.74934×10^{-7}
0.1	1	2.12557×10^{-8}	2.20641×10^{-8}	2.12731×10^{-8}	2.20659×10^{-8}	$4.98093070000 \times 10^{-8}$	$4.9PO9750 \times 10^{-7}$	3.75002×10^{-6}
0.5	0.05	1.36753×10^{-8}	1.84144×10^{-8}	1.35644×10^{-8}	1.81877×10^{-8}	$1.57823008000 \times 10^{-7}$	2.49875×10^{-8}	1.87486×10^{-7}
0.5	0.1	2.73505×10^{-8}	3.68287×10^{-8}	2.71288×10^{-8}	3.63753×10^{-8}	$1.70102357000 \times 10^{-7}$	4.99750×10^{-8}	3.74977×10^{-7}
0.5	1	6.34233×10^{-8}	6.24928×10^{-8}	6.34234×10^{-8}	6.24925×10^{-8}	$3.91130589000 \times 10^{-7}$	4.99750×10^{-7}	3.75044×10^{-6}
0.9	0.05	5.37357×10^{-9}	7.92237×10^{-9}	5.32537×10^{-9}	7.78062×10^{-9}	$3.09239355999 \times 10^{-7}$	2.49875×10^{-8}	1.87508×10^{-7}
0.9	0.1	1.07471×10^{-8}	1.58447×10^{-8}	1.06507×10^{-8}	1.55612×10^{-8}	$3.31513703000 \times 10^{-7}$	4.99750×10^{-8}	3.75019×10^{-7}
0.9	1	2.12557×10^{-8}	2.20641×10^{-8}	2.12391×10^{-8}	2.20646×10^{-8}	$7.32451854999 \times 10^{-7}$	4.99750×10^{-7}	3.75086×10^{-6}
CPU Time (In Sec.)		3.14	20.453	12.548	58.845			

Example 4. *Let us consider Equation* (5) *with initial condition* (6) *and boundary conditions* (7) *and* (8)*. When $\beta = 0$, this becomes the generalized Burgers equation. For our computations, we have opted for the parameter values $\alpha = 1$, $\beta = 0$, and $\delta = 1$. The absolute errors for the Burgers*

equation are computed using both the UHWCM and NUHWCM methods and compared with the existing methods. The results are presented in Table 4. Upon a thorough review of the table, it becomes evident that our method yields the most accurate results, especially when employing a minimal number of spatial divisions. Upon varying the parameters α and δ, the solutions show minimal variation and are very accurate, as illustrated in Figure 4. Worth noting is that the final solution remains consistent even when we alter the initial guess, showcasing the stability and robustness of our method.

Table 3. Comparison of absolute errors in $w(x,t)$ for $\alpha = 0.001$, $\beta = 0.001$, and $\delta = 1$.

x	t	Proposed Method				NSFD [31]	CFDM [63]	RPA [64]	ADM [10]
		UHWCM		NUHWCM					
		$J=1$	$J=2$	$J=1$	$J=2$				
0.1	0.005	1.11022×10^{-16}	1.11022×10^{-16}	1.11022×10^{-16}	1.11022×10^{-16}	2.50063×10^{-8}	4.38×10^{-7}	9.75×10^{-6}	9.68763×10^{-6}
0.1	0.001	5.55112×10^{-17}	5.55112×10^{-17}	5.55112×10^{-17}	5.55112×10^{-17}	2.50063×10^{-8}	1.01×10^{-7}	1.75×10^{-6}	1.93753×10^{-6}
0.1	0.01	0	0	0	0	2.50064×10^{-8}	7.53×10^{-7}	1.90×10^{-5}	1.93752×10^{-5}
0.5	0.005	5.55112×10^{-17}	5.55112×10^{-17}	5.55112×10^{-17}	5.55112×10^{-17}	2.50063×10^{-8}	5.21×10^{-7}	9.75×10^{-6}	9.68691×10^{-6}
0.5	0.001	5.55112×10^{-17}	5.55112×10^{-17}	5.55112×10^{-17}	5.55112×10^{-17}	2.50063×10^{-8}	1.01×10^{-7}	1.75×10^{-6}	1.93738×10^{-6}
0.5	0.01	0	0	0	0	2.50065×10^{-8}	1.04×10^{-7}	1.90×10^{-5}	1.93738×10^{-5}
0.9	0.005	0	0	0	0	2.50063×10^{-8}	4.38×10^{-7}	9.75×10^{-6}	9.68619×10^{-6}
0.9	0.001	1.11022×10^{-16}	1.11022×10^{-16}	1.11022×10^{-16}	1.11022×10^{-16}	2.50063×10^{-8}	1.01×10^{-7}	1.75×10^{-6}	1.93724×10^{-6}
0.9	0.01	5.55112×10^{-17}	5.55112×10^{-17}	5.55112×10^{-17}	5.55112×10^{-17}	2.50064×10^{-8}	7.53×10^{-7}	1.90×10^{-5}	1.93724×10^{-5}
CPU Time (In Sec.)		3.939	13.924	7.328	27.407				

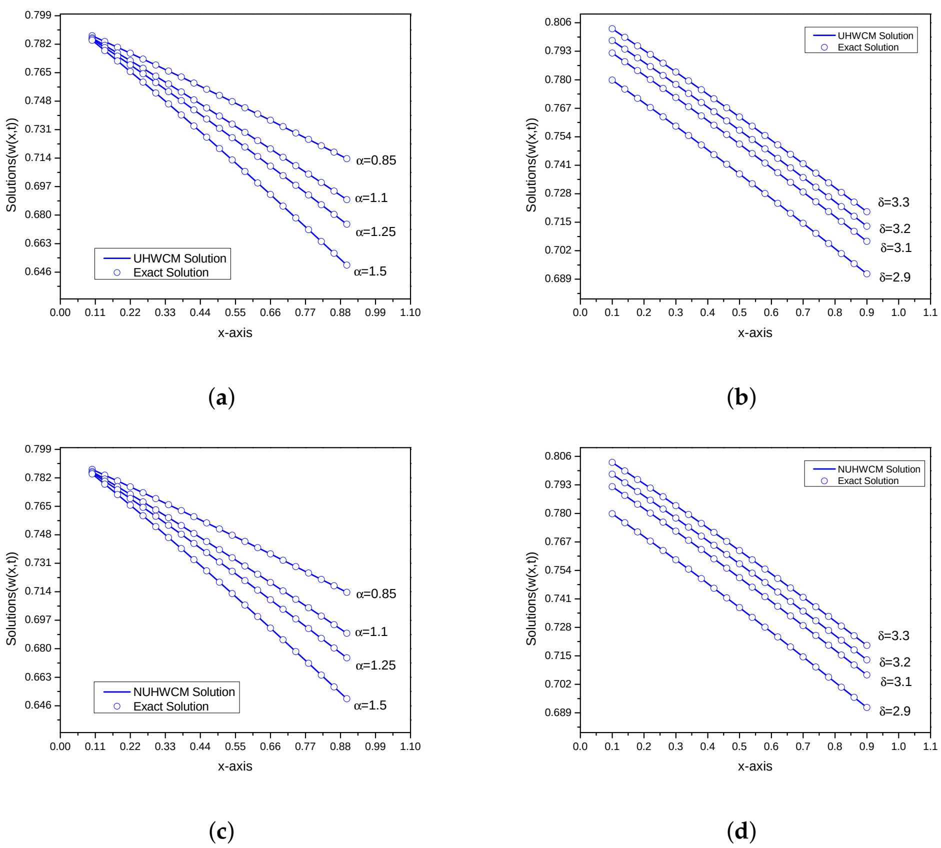


Figure 4. Graph of exact, UHWCM, and NUHWCM solutions for various values of α and δ. (**a**) Exact and UHWCM solutions with fixed parameters: $\delta = 3, t = 0.1$. (**b**) Exact and UHWCM solutions with fixed parameters: $\alpha = 1, t = 0.1$. (**c**) Exact and NUHWCM solutions with fixed parameters: $\delta = 3, t = 0.1$. (**d**) Exact and NUHWCM solutions with fixed parameters: $\alpha = 1, t = 0.1$.

Table 4. Comparison of absolute errors in $w(x,t)$ for $\alpha = 1$, $\beta = 0$, and $\delta = 3$.

x	t	Proposed Method				NSFD [31]	RPA [64]
		UHWCM		NUHWCM			
		$J = 1$	$J = 2$	$J = 1$	$J = 2$		
0.1	0.0005	8.94873×10^{-10}	5.48376×10^{-10}	9.06105×10^{-10}	5.7152×10^{-10}	2.90×10^{-7}	0.000444925
0.1	0.0001	1.7942×10^{-10}	1.10121×10^{-10}	1.81667×10^{-10}	1.1475×10^{-10}	7.94×10^{-8}	0.000446088
0.1	0.001	1.78417×10^{-9}	1.10121×10^{-10}	1.80664×10^{-9}	1.13747×10^{-9}	5.39×10^{-7}	0.000444222
0.5	0.0005	2.51474×10^{-9}	1.27795×10^{-9}	2.58877×10^{-9}	1.33266×10^{-9}	3.27×10^{-8}	0.001854465
0.5	0.0001	5.03903×10^{-10}	2.56544×10^{-10}	5.18708×10^{-10}	2.67486×10^{-10}	2.85×10^{-8}	0.001860448
0.5	0.001	5.01757×10^{-9}	2.54398×10^{-9}	5.16562×10^{-9}	2.6534×10^{-9}	3.82×10^{-8}	0.001847737
0.9	0.0005	6.75557×10^{-10}	3.46276×10^{-10}	6.89016×10^{-10}	3.58104×10^{-10}	2.18×10^{-7}	0.00092005
0.9	0.0001	1.35346×10^{-10}	6.94895×10^{-11}	1.38038×10^{-10}	7.18552×10^{-11}	2.14×10^{-8}	0.000931582
0.9	0.001	1.34818×10^{-9}	6.89623×10^{-10}	1.3751×10^{-9}	7.13279×10^{-10}	4.49×10^{-7}	0.000904635
CPU Time (In Sec.)		3.766	8.984	9.234	26.906		

5. Conclusions

In our article, we introduce a novel approach, combining the two-dimensional uniform and non-uniform Haar wavelet collocation method with the Newton–Raphson method. This approach effectively solves various cases of the generalized Burgers–Huxley and Burgers–Fisher equations. We establish quartic convergence for each method and demonstrate its high accuracy by comparing with existing techniques through absolute error analysis. Our method stands out for its accuracy and efficiency, requiring minimal spatial divisions and boasting notably low CPU times. Overall, the combined approach proves stable, accurate, and efficient, offering a promising solution for these equations compared to existing methods.

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