Neutrosophic $\mathcal{N}$-Structures Applied to $BCK/BCI$-Algebras

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Abstract: Neutrosophic $\mathcal{N}$-structures with applications in $BCK/BCI$-algebras is discussed. The notions of a neutrosophic $\mathcal{N}$-subalgebra and a (closed) neutrosophic $\mathcal{N}$-ideal in a $BCK/BCI$-algebra are introduced, and several related properties are investigated. Characterizations of a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal are considered, and relations between a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal are stated. Conditions for a neutrosophic $\mathcal{N}$-ideal to be a closed neutrosophic $\mathcal{N}$-ideal are provided.

Keywords: neutrosophic $\mathcal{N}$-structure; neutrosophic $\mathcal{N}$-subalgebra; (closed) neutrosophic $\mathcal{N}$-ideal

MSC: 06F35, 03G25, 03B52

1. Introduction

$BCK$-algebras entered into mathematics in 1966 through the work of Imai and Iséki [1], and they have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. Such algebras generalize Boolean rings as well as Boolean $D$-posets ($MV$-algebras). Additionally, Iséki introduced the notion of a $BCI$-algebra, which is a generalization of a $BCK$-algebra (see [2]).

A (crisp) set $A$ in a universe $X$ can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set $A$ and the value 0 for elements excluded from the set $A$. So far, most of the generalizations of the crisp set have been conducted on the unit interval $[0, 1]$, and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply a mathematical tool. To attain such an object, Jun et al. [3] introduced a new function, called a negative-valued function, and constructed $\mathcal{N}$-structures. Zadeh [4] introduced the degree of membership/truth ($t$) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [5] introduced the degree of nonmembership/falsehood ($f$) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/neutrality ($i$) as an independent component in 1995 (published in 1998) and defined the neutrosophic set on three components:

$$(t, i, f) = \text{(truth, indeterminacy, falsehood)}$$
For more details, refer to the following site:

http://fs.gallup.unm.edu/FlorentinSmarandache.htm

In this paper, we discuss a neutrosophic \( N \)-structure with an application to BCK/BCI-algebras. We introduce the notions of a neutrosophic \( N \)-subalgebra and a (closed) neutrosophic \( N \)-ideal in a BCK/BCI-algebra, and investigate related properties. We consider characterizations of a neutrosophic \( N \)-subalgebra and a neutrosophic \( N \)-ideal. We discuss relations between a neutrosophic \( N \)-subalgebra and a neutrosophic \( N \)-ideal. We provide conditions for a neutrosophic \( N \)-ideal to be a closed neutrosophic \( N \)-ideal.

2. Preliminaries

We let \( K(\tau) \) be the class of all algebras with type \( \tau = (2,0) \). A BCI-algebra refers to a system \( X := (X,*,\theta) \in K(\tau) \) in which the following axioms hold:

(I) \( ((x*y)*(x*z))*(z+y) = \theta, \)

(II) \( (x*(x*y))*y = \theta, \)

(III) \( x*x = \theta, \)

(IV) \( x*y = y*x = \theta \Rightarrow x = y, \)

for all \( x,y,z \in X \). If a BCI-algebra \( X \) satisfies \( \theta*x = \theta \) for all \( x \in X \), then we say that \( X \) is a BCK-algebra. We can define a partial ordering \( \leq \) by

\[
(\forall x,y \in X) (x \leq y \Rightarrow x*y = \theta)
\]

In a BCK/BCI-algebra \( X \), the following hold:

\[
(\forall x \in X) (x*\theta = x) \quad (1)
\]

\[
(\forall x,y,z \in X) ((x*y)*z = (x*z)*y) \quad (2)
\]

A non-empty subset \( S \) of a BCK/BCI-algebra \( X \) is called a subalgebra of \( X \) if \( x*y \in S \) for all \( x,y \in S \).

A subset \( I \) of a BCK/BCI-algebra \( X \) is called an ideal of \( X \) if it satisfies the following:

(I1) \( 0 \in I, \)

(I2) \( (\forall x,y \in X)(x*y \in I, y \in I \Rightarrow x \in I), \)

We refer the reader to the books [6,7] for further information regarding BCK/BCI-algebras.

For any family \( \{a_i \mid i \in \Lambda\} \) of real numbers, we define

\[
\bigvee \{a_i \mid i \in \Lambda\} := \begin{cases} \max \{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite} \\ \sup \{a_i \mid i \in \Lambda\} & \text{otherwise} \end{cases}
\]

\[
\bigwedge \{a_i \mid i \in \Lambda\} := \begin{cases} \min \{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite} \\ \inf \{a_i \mid i \in \Lambda\} & \text{otherwise} \end{cases}
\]

We denote by \( \mathcal{F}(X,[-1,0]) \) the collection of functions from a set \( X \) to \([-1,0]\). We say that an element of \( \mathcal{F}(X,[-1,0]) \) is a negative-valued function from \( X \) to \([-1,0]\) (briefly, \( N \)-function on \( X \)). An \( N \)-structure refers to an ordered pair \( (X,f) \) of \( X \) and an \( N \)-function \( f \) on \( X \) (see [3]). In what follows, we let \( X \) denote the nonempty universe of discourse unless otherwise specified.

A neutrosophic \( N \)-structure over \( X \) (see [8]) is defined to be the structure:

\[
X_N := \left\{ \left( x, \frac{x}{(i(x),j(x),k(x))} \right) \mid x \in X \right\}
\]

(3)
where $T_N$, $I_N$ and $F_N$ are $N$-functions on $X$, which are called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function, respectively, on $X$.

We note that every neutrosophic $N$-structure $X_N$ over $X$ satisfies the condition:

$$\forall x \in X \ (-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0)$$

3. Application in BCK/BCI-Algebras

In this section, we take a $BCK$/$BCI$-algebra $X$ as the universe of discourse unless otherwise specified.

**Definition 1.** A neutrosophic $N$-structure $X_N$ over $X$ is called a neutrosophic $N$-subalgebra of $X$ if the following condition is valid:

$$\forall x, y \in X \left( T_N(x \ast y) \leq \bigvee \{T_N(x), T_N(y)\} \right) \left( I_N(x \ast y) \geq \bigwedge \{I_N(x), I_N(y)\} \right) \left( F_N(x \ast y) \leq \bigvee \{F_N(x), F_N(y)\} \right)$$

(4)

**Example 1.** Consider a BCK-algebra $X = \{\theta, a, b, c\}$ with the following Cayley table.

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The neutrosophic $N$-structure

$$X_N = \left\{ (\theta, a, b, c) \in (-0.7, -0.2, -0.6), (-0.5, -0.3, -0.4), (-0.5, -0.3, -0.4), (-0.3, -0.8, -0.5) \right\}$$

over $X$ is a neutrosophic $N$-subalgebra of $X$.

Let $X_N$ be a neutrosophic $N$-structure over $X$ and let $\alpha, \beta, \gamma \in [-1,0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Consider the following sets:

$$T_\alpha^N := \{ x \in X \mid T_N(x) \leq \alpha \}$$

$$I_\beta^N := \{ x \in X \mid I_N(x) \geq \beta \}$$

$$F_\gamma^N := \{ x \in X \mid F_N(x) \leq \gamma \}$$

The set

$$X_N(\alpha, \beta, \gamma) := \{ x \in X \mid T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma \}$$

is called the $(\alpha, \beta, \gamma)$-level set of $X_N$. Note that

$$X_N(\alpha, \beta, \gamma) = T_\alpha^N \cap I_\beta^N \cap F_\gamma^N$$

**Theorem 1.** Let $X_N$ be a neutrosophic $N$-structure over $X$ and let $\alpha, \beta, \gamma \in [-1,0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $X_N$ is a neutrosophic $N$-subalgebra of $X$, then the nonempty $(\alpha, \beta, \gamma)$-level set of $X_N$ is a subalgebra of $X$. 

Proof. Let \( a, b, \gamma \in [-1, 0] \) be such that \(-3 \leq a + b + \gamma \leq 0\) and \( X_N(a, \beta, \gamma) \neq \emptyset \). If \( x, y \in X_N(a, \beta, \gamma) \), then \( T_N(x) \leq a, I_N(x) \geq \beta, F_N(x) \leq \gamma \), \( T_N(y) \leq a, I_N(y) \geq \beta \) and \( F_N(y) \leq \gamma \). It follows from Equation (4) that

\[
T_N(x * y) \leq \bigvee \{ T_N(x), T_N(y) \} \leq a, \\
I_N(x * y) \geq \bigwedge \{ I_N(x), I_N(y) \} \geq \beta, \text{ and} \\
F_N(x * y) \leq \bigvee \{ F_N(x), F_N(y) \} \leq \gamma.
\]

Hence, \( x * y \in X_N(a, \beta, \gamma) \), and therefore \( X_N(a, \beta, \gamma) \) is a subalgebra of \( X \). \( \Box \)

**Theorem 2.** Let \( X_N \) be a neutrosophic \( N \)-structure over \( X \) and assume that \( T_N^a, I_N^b \) and \( F_N^c \) are subalgebras of \( X \) for all \( a, \beta, \gamma \in [-1, 0] \) with \(-3 \leq a + \beta + \gamma \leq 0\). Then \( X_N \) is a neutrosophic \( N \)-subalgebra of \( X \).

Proof. Assume that there exist \( a, b \in X \) such that \( T_N(a * b) > \bigvee \{ T_N(a), T_N(b) \} \). Then \( T_N(a * b) \geq \bigvee \{ T_N(a), T_N(b) \} \) for some \( t_\alpha \in [-1, 0] \). Hence, \( a, b \in T_N^a \) but \( a * b \notin T_N^a \), which is a contradiction. Thus

\[
T_N(x * y) \leq \bigvee \{ T_N(x), T_N(y) \}
\]

for all \( x, y \in X \). If \( I_N(a * b) < \bigwedge \{ I_N(a), I_N(b) \} \) for some \( a, b \in X \), then

\[
I_N(a * b) < t_\beta \leq \bigwedge \{ I_N(a), I_N(b) \}
\]

where \( t_\beta := \frac{1}{2} \{ I_N(a * b) + \bigwedge \{ I_N(a), I_N(b) \} \} \). Thus \( a, b \in I_N^b \) and \( a * b \notin I_N^b \), which is a contradiction. Therefore

\[
I_N(x * y) \leq \bigwedge \{ I_N(x), I_N(y) \}
\]

for all \( x, y \in X \). Now, suppose that there exist \( a, b \in X \) and \( t_\gamma \in [-1, 0] \) such that

\[
F_N(a * b) > t_\gamma \geq \bigvee \{ F_N(a), F_N(b) \}
\]

Then \( a, b \in F_N^c \) and \( a * b \notin F_N^c \), which is a contradiction. Hence

\[
F_N(x * y) \leq \bigvee \{ F_N(x), F_N(y) \}
\]

for all \( x, y \in X \). Therefore \( X_N \) is a neutrosophic \( N \)-subalgebra of \( X \). \( \Box \)

Because \([-1, 0]\) is a completely distributive lattice with respect to the usual ordering, we have the following theorem.

**Theorem 3.** If \( \{ X_N : i \in \mathbb{N} \} \) is a family of neutrosophic \( N \)-subalgebras of \( X \), then \( \{ \{ X_N : i \in \mathbb{N} \}, \subseteq \} \) forms a complete distributive lattice.

**Proposition 1.** If a neutrosophic \( N \)-structure \( X_N \) over \( X \) is a neutrosophic \( N \)-subalgebra of \( X \), then \( T_N(\theta) \leq T_N(x) \), \( I_N(\theta) \geq I_N(x) \) and \( F_N(\theta) \leq F_N(x) \) for all \( x \in X \).

Proof. Straightforward. \( \Box \)

**Theorem 4.** Let \( X_N \) be a neutrosophic \( N \)-subalgebra of \( X \). If there exists a sequence \( \{ a_n \} \) in \( X \) such that \( \lim_{n \to \infty} T_N(a_n) = -1 \), \( \lim_{n \to \infty} I_N(a_n) = 0 \) and \( \lim_{n \to \infty} F_N(a_n) = -1 \), then \( T_N(\theta) = -1 \), \( I_N(\theta) = 0 \) and \( F_N(\theta) = -1 \).

Proof. By Proposition 1, we have \( T_N(\theta) \leq T_N(x) \), \( I_N(\theta) \geq I_N(x) \) and \( F_N(\theta) \leq F_N(x) \) for all \( x \in X \). Hence \( T_N(\theta) \leq T_N(a_n) \), \( I_N(\theta) \leq I_N(a_n) \) and \( F_N(\theta) \leq F_N(a_n) \) for every positive integer \( n \). It follows that
Using Equations (1) and (5), we have for all $x$

Hence $T_N(\theta) = -1$, $I_N(\theta) = 0$ and $F_N(\theta) = -1$. □

**Proposition 2.** If every neutrosophic $\mathcal{N}$-subalgebra $X_N$ of $X$ satisfies:

$$T_N(x \ast y) \leq T_N(y), \quad I_N(x \ast y) \geq I_N(y), \quad F_N(x \ast y) \leq F_N(y)$$

(5)

for all $x, y \in X$, then $X_N$ is constant.

**Proof.** Using Equations (1) and (5), we have $T_N(x) = T_N(x \ast \theta) \leq T_N(\theta)$, $I_N(x) = I_N(x \ast \theta) \geq I_N(\theta)$ and $F_N(x) = F_N(x \ast \theta) \leq F_N(\theta)$ for all $x \in X$. It follows from Proposition 1 that $T_N(x) = T_N(\theta)$, $I_N(x) = I_N(\theta)$ and $F_N(x) = F_N(\theta)$ for all $x \in X$. Therefore $X_N$ is constant. □

**Definition 2.** A neutrosophic $\mathcal{N}$-structure $X_N$ over $X$ is called a neutrosophic $\mathcal{N}$-ideal of $X$ if the following assertion is valid:

$$\left( \forall x, y \in X \right) \quad \begin{cases}
T_N(\theta) \leq T_N(x) \leq \bigvee \{T_N(x \ast y), T_N(y)\} \\
I_N(\theta) \geq I_N(x) \geq \bigwedge \{I_N(x \ast y), I_N(y)\} \\
F_N(\theta) \leq F_N(x) \leq \bigvee \{F_N(x \ast y), F_N(y)\}
\end{cases}$$

(6)

**Example 2.** The neutrosophic $\mathcal{N}$-structure $X_N$ over $X$ in Example 1 is a neutrosophic $\mathcal{N}$-ideal of $X$.

**Example 3.** Consider a BCI-algebra $X := Y \times \mathbb{Z}$ where $(Y, \ast, \theta)$ is a BCI-algebra and $(\mathbb{Z}, +, 0)$ is the adjoint BCI-algebra of the additive group $(\mathbb{Z}, +, 0)$ of integers (see [6]). Let $X_N$ be a neutrosophic $\mathcal{N}$-structure over $X$ given by

$$X_N = \left\{ \frac{x}{(a, \beta, \gamma)} \mid x \in Y \times (\mathbb{N} \cup \{0\}) \right\} \cup \left\{ \frac{x}{(0, \beta, \gamma)} \mid x \notin Y \times (\mathbb{N} \cup \{0\}) \right\}$$

where $a, \gamma \in [-1, 0)$ and $\beta \in (-1, 0]$. Then $X_N$ is a neutrosophic $\mathcal{N}$-ideal of $X$.

**Proposition 3.** Every neutrosophic $\mathcal{N}$-ideal $X_N$ of $X$ satisfies the following assertions:

$$(x, y \in X) \left( x \preceq y \Rightarrow T_N(x) \leq T_N(y), \quad I_N(x) \geq I_N(y), \quad F_N(x) \leq F_N(y) \right)$$

(7)

**Proof.** Let $x, y \in X$ be such that $x \preceq y$. Then $x \ast y = \theta$, and so

$$T_N(x) \leq \bigvee \{T_N(x \ast y), T_N(y)\} = \bigvee \{T_N(\theta), T_N(y)\} = T_N(y)$$

$$I_N(x) \geq \bigwedge \{I_N(x \ast y), I_N(y)\} = \bigwedge \{I_N(\theta), I_N(y)\} = I_N(y)$$

$$F_N(x) \leq \bigvee \{F_N(x \ast y), F_N(y)\} = \bigvee \{F_N(\theta), F_N(y)\} = F_N(y)$$

This completes the proof. □

**Proposition 4.** Let $X_N$ be a neutrosophic $\mathcal{N}$-ideal of $X$. Then

1. $T_N(x \ast y) \leq T_N((x \ast y) \ast y) \iff T_N((x \ast z) \ast (y \ast z)) \leq T_N((x \ast y) \ast z)$
2. $I_N(x \ast y) \geq I_N((x \ast y) \ast y) \iff I_N((x \ast z) \ast (y \ast z)) \geq I_N((x \ast y) \ast z)$
3. $F_N(x \ast y) \leq F_N((x \ast y) \ast y) \iff F_N((x \ast z) \ast (y \ast z)) \leq F_N((x \ast y) \ast z)$

for all $x, y, z \in X$. 

Theorem 5. Note that

\[(x \ast (y \ast z)) \ast z \preceq (x \ast y) \ast z\]  \hspace{1cm} (8)

for all \(x, y, z \in X\). Assume that \(T_N(x \ast y) \leq T_N((x \ast y) \ast y)\), \(I_N(x \ast y) \geq I_N((x \ast y) \ast y)\) and \(F_N(x \ast y) \leq F_N((x \ast y) \ast y)\) for all \(x, y \in X\). It follows from Equation (2) and Proposition 3 that

\[T_N((x \ast z) \ast (y \ast z)) = T_N((x \ast (y \ast z)) \ast z)\]
\[\leq T_N(((x \ast (y \ast z)) \ast z) \ast z)\]
\[\leq T_N((x \ast y) \ast z)\]

\[I_N((x \ast z) \ast (y \ast z)) = I_N((x \ast (y \ast z)) \ast z)\]
\[\geq I_N(((x \ast (y \ast z)) \ast z) \ast z)\]
\[\geq I_N((x \ast y) \ast z)\]

and

\[F_N((x \ast z) \ast (y \ast z)) = F_N((x \ast (y \ast z)) \ast z)\]
\[\leq F_N(((x \ast (y \ast z)) \ast z) \ast z)\]
\[\leq F_N((x \ast y) \ast z)\]

for all \(x, y \in X\).

Conversely, suppose

\[T_N((x \ast z) \ast (y \ast z)) \leq T_N((x \ast y) \ast z)\]
\[I_N((x \ast z) \ast (y \ast z)) \geq I_N((x \ast y) \ast z)\]
\[F_N((x \ast z) \ast (y \ast z)) \leq F_N((x \ast y) \ast z)\]  \hspace{1cm} (9)

for all \(x, y, z \in X\). If we substitute \(z \ast y\) in Equation (9), then

\[T_N(x \ast z) = T_N((x \ast z) \ast \theta) = T_N((x \ast z) \ast (z \ast z)) \leq T_N((x \ast z) \ast z)\]
\[I_N(x \ast z) = I_N((x \ast z) \ast \theta) = I_N((x \ast z) \ast (z \ast z)) \geq I_N((x \ast z) \ast z)\]
\[F_N(x \ast z) = F_N((x \ast z) \ast \theta) = F_N((x \ast z) \ast (z \ast z)) \leq F_N((x \ast z) \ast z)\]

for all \(x, z \in X\) by using (III) and Equation (1) \(\Box\).

Theorem 5. Let \(X_N\) be a neutrosophic \(N\)-structure over \(X\) and let \(\alpha, \beta, \gamma \in [-1, 0]\) be such that \(-3 \leq \alpha + \beta + \gamma \leq 0\). If \(X_N\) is a neutrosophic \(N\)-ideal of \(X\), then the nonempty \((\alpha, \beta, \gamma)\)-level set of \(X_N\) is an ideal of \(X\).

Proof. Assume that \(X_N(\alpha, \beta, \gamma) \neq \emptyset\) for \(\alpha, \beta, \gamma \in [-1, 0]\) with \(-3 \leq \alpha + \beta + \gamma \leq 0\). Clearly, \(\theta \in X_N(\alpha, \beta, \gamma)\). Let \(x, y \in X\) be such that \(x \ast y \in X_N(\alpha, \beta, \gamma)\) and \(y \in X_N(\alpha, \beta, \gamma)\). Then \(T_N(x \ast y) \leq \alpha\), \(I_N(x \ast y) \geq \beta\), \(F_N(x \ast y) \leq \gamma\), \(T_N(y) \leq \alpha\), \(I_N(y) \geq \beta\) and \(F_N(y) \leq \gamma\). It follows from Equation (6) that

\[T_N(x) \leq \max\{T_N(x \ast y), T_N(y)\} \leq \alpha\]
\[I_N(x) \geq \min\{I_N(x \ast y), I_N(y)\} \geq \beta\]
\[F_N(x) \leq \min\{F_N(x \ast y), F_N(y)\} \leq \gamma\]

so that \(x \in X_N(\alpha, \beta, \gamma)\). Therefore \(X_N(\alpha, \beta, \gamma)\) is an ideal of \(X\). \(\Box\)
Theorem 7. Let \( X_N \) be a neutrosophic \( N \)-structure over \( X \) and assume that \( T_N^\alpha, I_N^\beta, \) and \( F_N^\gamma \) are ideals of \( X \) for all \( \alpha, \beta, \gamma \in [-1,0] \) with \(-3 \leq \alpha + \beta + \gamma \leq 0\). Then \( X_N \) is a neutrosophic \( N \)-ideal of \( X \).

Proof. If there exist \( a, b, c \in X \) such that \( T_N(\theta) > T_N(a), I_N(\theta) < I_N(b) \) and \( F_N(\theta) > F_N(c) \), respectively, then \( T_N(\theta) > a_i \geq T_N(a), I_N(\theta) < b_i \leq I_N(b) \) and \( F_N(\theta) > c_f \geq F_N(c) \) for some \( a_i, c_f \in [-1,0) \) and \( b_i \in (-1,0] \). Then \( \theta \notin T_N^\alpha, \theta \notin I_N^\beta, \) and \( \theta \notin F_N^\gamma \). This is a contradiction. Hence, \( T_N(\theta) \leq T_N(x), I_N(\theta) \geq I_N(x) \) and \( F_N(\theta) \leq F_N(x) \) for all \( x \in X \). Assume that there exist \( a_i, b_i, a_j, b_j \in X \) such that \( T_N(a_i) > \bigvee \{ T_N(a_i \ast b_i), T_N(b_i) \}, I_N(a_i) < \bigwedge \{ I_N(a_i \ast b_i), I_N(b_i) \} \) and \( F_N(a_f) > \bigvee \{ F_N(a_f \ast b_f), F_N(b_f) \} \). Then there exist \( s_i, s_f \in [-1,0) \) and \( s_i \in (-1,0] \) such that

\[
T_N(a_i) > s_i \geq \bigvee \{ T_N(a_i \ast b_i), T_N(b_i) \}
\]

\[
I_N(a_i) < s_i \leq \bigwedge \{ I_N(a_i \ast b_i), I_N(b_i) \}
\]

\[
F_N(a_f) > s_f \geq \bigvee \{ F_N(a_f \ast b_f), F_N(b_f) \}
\]

It follows that \( a_i \ast b_i \in T_N^s, b_i \in T_N^s, a_i \ast b_i \in I_N^s, b_i \in I_N^s, a_f \ast b_f \in F_N^s, \) and \( b_f \in F_N^s \). However, \( a_i \notin T_N^s, a_i \notin I_N^s \) and \( a_f \notin F_N^s \). This is a contradiction, and so

\[
T_N(x) \leq \bigvee \{ T_N(x \ast y), T_N(y) \}
\]

\[
I_N(x) \geq \bigwedge \{ I_N(x \ast y), I_N(y) \}
\]

\[
F_N(x) \leq \bigvee \{ F_N(x \ast y), F_N(y) \}
\]

for all \( x, y \in X \). Therefore \( X_N \) is a neutrosophic \( N \)-ideal of \( X \). \( \square \)

Proposition 5. For any neutrosophic \( N \)-ideal \( X_N \) of \( X \), we have

\[ (\forall x, y, z \in X) \left( x \ast y \leq z \Rightarrow \begin{cases} T_N(x) \leq \bigvee \{ T_N(y), T_N(z) \} \\ I_N(x) \geq \bigwedge \{ I_N(y), I_N(z) \} \\ F_N(x) \leq \bigvee \{ F_N(y), F_N(z) \} \end{cases} \right) \]  

(10)

Proof. Let \( x, y, z \in X \) be such that \( x \ast y \leq z \). Then \( (x \ast y) \ast z = \theta \), and so

\[
T_N(x \ast y) \leq \bigvee \{ T_N((x \ast y) \ast z), T_N(z) \} = \bigvee \{ T_N(\theta), T_N(z) \} = T_N(z)
\]

\[
I_N(x \ast y) \geq \bigwedge \{ I_N((x \ast y) \ast z), I_N(z) \} = \bigwedge \{ I_N(\theta), I_N(z) \} = I_N(z)
\]

\[
F_N(x \ast y) \leq \bigvee \{ F_N((x \ast y) \ast z), F_N(z) \} = \bigvee \{ F_N(\theta), F_N(z) \} = F_N(z)
\]

It follows that

\[
T_N(x) \leq \bigvee \{ T_N(x \ast y), T_N(y) \} \leq \bigvee \{ T_N(y), T_N(z) \}
\]

\[
I_N(x) \geq \bigwedge \{ I_N(x \ast y), I_N(y) \} \geq \bigwedge \{ I_N(y), I_N(z) \}
\]

\[
F_N(x) \leq \bigvee \{ F_N(x \ast y), F_N(y) \} \leq \bigvee \{ F_N(y), F_N(z) \}
\]

This completes the proof. \( \square \)

Theorem 7. In a BCK-algebra, every neutrosophic \( N \)-ideal is a neutrosophic \( N \)-subalgebra.

Proof. Let \( X_N \) be a neutrosophic \( N \)-ideal of a BCK-algebra \( X \). For any \( x, y \in X \), we have
$T_N(x \ast y) \leq \bigvee \{T_N((x \ast y) \ast x), T_N(x)\} = \bigvee \{T_N((x \ast x) \ast y), T_N(x)\}$
$= \bigvee \{T_N(\theta \ast y), T_N(x)\} = \bigvee \{T_N(\theta), T_N(x)\}$
$\leq \bigvee \{T_N(x), T_N(y)\}$
$I_N(x \ast y) \geq \bigwedge \{I_N((x \ast y) \ast x), I_N(x)\} = \bigwedge \{I_N((x \ast x) \ast y), I_N(x)\}$
$= \bigwedge \{I_N(\theta \ast y), I_N(x)\} = \bigwedge \{I_N(\theta), I_N(x)\}$
$\geq \bigwedge \{I_N(y), I_N(x)\}$

and

$F_N(x \ast y) \leq \bigvee \{F_N((x \ast y) \ast x), F_N(x)\} = \bigvee \{F_N((x \ast x) \ast y), F_N(x)\}$
$= \bigvee \{F_N(\theta \ast y), F_N(x)\} = \bigvee \{F_N(\theta), F_N(x)\}$
$\leq \bigvee \{F_N(x), F_N(y)\}$

Hence $X_N$ is a neutrosophic $N'$-subalgebra of a BCK-algebra $X$. □

The converse of Theorem 7 may not be true in general, as seen in the following example.

**Example 4.** Consider a BCK-algebra $X = \{\theta, 1, 2, 3, 4\}$ with the following Cayley table.

<table>
<thead>
<tr>
<th></th>
<th>$\theta$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>$\theta$</td>
<td>$\theta$</td>
<td>$\theta$</td>
<td>$\theta$</td>
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<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>$\theta$</td>
<td>1</td>
<td>$\theta$</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>$\theta$</td>
<td>$\theta$</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>$\theta$</td>
</tr>
</tbody>
</table>

Let $X_N$ be a neutrosophic $N'$-structure over $X$, which is given as follows:

$X_N = \left\{ \begin{array}{c} \theta \\ \frac{1}{2} \end{array}, \begin{array}{c} \frac{2}{3} \\ \frac{4}{7} \end{array}, \begin{array}{c} \frac{-1}{2} \\ \frac{-3}{4} \end{array}, \begin{array}{c} \frac{-4}{5} \\ \frac{-5}{6} \end{array}, \begin{array}{c} \frac{-6}{7} \\ \frac{-7}{8} \end{array} \right\}$

Then $X_N$ is a neutrosophic $N'$-subalgebra of $X$, but it is not a neutrosophic $N'$-ideal of $X$ as $T_N(2) = -0.2 > -0.7 = \bigvee \{T_N(2 \ast 3), T_N(3)\}$, $I_N(4) = -0.8 < -0.4 = \bigwedge \{I_N(4 \ast 3), I_N(3)\}$, or $F_N(4) = -0.3 > -0.7 = \bigvee \{F_N(4 \ast 3), F_N(3)\}$.

Theorem 7 is not valid in a BCI-algebra; that is, if $X$ is a BCI-algebra, then there is a neutrosophic $N'$-ideal that is not a neutrosophic $N'$-subalgebra, as seen in the following example.

**Example 5.** Consider the neutrosophic $N'$-ideal $X_N$ of $X$ in Example 3. If we take $x := (\theta, 0)$ and $y := (\theta, 1)$ in $Y \times (\mathbb{N} \cup \{0\})$, then $x \ast y = (\theta, 0) \ast (\theta, 1) = (\theta, -1) \notin Y \times (\mathbb{N} \cup \{0\})$. Hence

$T_N(x \ast y) = 0 > \alpha = \bigvee \{T_N(x), T_N(y)\}$
$I_N(x \ast y) = \beta < 0 = \bigwedge \{I_N(x), I_N(y)\}$ or
$F_N(x \ast y) = 0 > \gamma = \bigvee \{F_N(x), F_N(y)\}$

Therefore $X_N$ is not a neutrosophic $N'$-subalgebra of $X$. 
Theorem 8. Let \( \omega_i, \omega_i, \omega_f \in X \) be any elements of X. If \( X_N \) is a neutrosophic \( N \)-ideal of X, then \( X_N^{\omega_i}, X_N^{\omega_i} \) and \( X_N^{\omega_f} \) are ideals of X.

Proof. Clearly, \( \theta \in X_N^{\omega_i}, \theta \in X_N^{\omega_i} \) and \( \theta \in X_N^{\omega_f} \). Let \( x, y \in X \) be such that \( x \neq y \in X_N^{\omega_i} \cap X_N^{\omega_i} \cap X_N^{\omega_f} \) and \( y \in X_N^{\omega_i} \cap X_N^{\omega_i} \cap X_N^{\omega_f} \). Then

\[
T_N(x \ast y) \leq T_N(\omega_i), \quad T_N(y) \leq T_N(\omega_i)
\]

\[
I_N(x \ast y) \geq I_N(\omega_i), \quad I_N(y) \geq I_N(\omega_i)
\]

\[
F_N(x \ast y) \leq F_N(\omega_f), \quad F_N(y) \leq F_N(\omega_f)
\]

It follows from Equation (6) that

\[
T_N(x) \leq \bigvee \{T_N(x \ast y), T_N(y)\} \leq T_N(\omega_i)
\]

\[
I_N(x) \geq \bigwedge \{I_N(x \ast y), I_N(y)\} \geq I_N(\omega_i)
\]

\[
F_N(x) \leq \bigvee \{F_N(x \ast y), F_N(y)\} \leq F_N(\omega_f)
\]

Hence \( x \in X_N^{\omega_i} \cap X_N^{\omega_i} \cap X_N^{\omega_f} \), and therefore \( X_N^{\omega_i}, X_N^{\omega_i} \) and \( X_N^{\omega_f} \) are ideals of X. \( \square \)

Theorem 9. Let \( \omega_i, \omega_i, \omega_f \in X \) and let \( X_N \) be a neutrosophic \( N \)-structure over X. Then

(1) If \( X_N^{\omega_i}, X_N^{\omega_i} \) and \( X_N^{\omega_f} \) are ideals of X, then the following assertion is valid:

\[
(\forall x, y, z \in X) \begin{cases} T_N(x) \geq \bigvee \{T_N(y \ast z), T_N(z)\} \Rightarrow T_N(x) \geq T_N(y) \\ I_N(x) \geq \bigwedge \{I_N(y \ast z), I_N(z)\} \Rightarrow I_N(x) \leq I_N(y) \\ F_N(x) \geq \bigvee \{F_N(y \ast z), F_N(z)\} \Rightarrow F_N(x) \geq F_N(y) \end{cases} \tag{11}
\]

(2) If \( X_N \) satisfies Equation (11) and

\[
(\forall x \in X) (T_N(\theta) \leq T_N(x), I_N(\theta) \geq I_N(x), F_N(\theta) \leq F_N(x)) \tag{12}
\]

then \( X_N^{\omega_i}, X_N^{\omega_i} \) and \( X_N^{\omega_f} \) are ideals of X for all \( \omega_i \in \text{Im}(T_N), \omega_i \in \text{Im}(I_N) \) and \( \omega_f \in \text{Im}(F_N) \).

Proof. (1) Assume that \( X_N^{\omega_i}, X_N^{\omega_i} \) and \( X_N^{\omega_f} \) are ideals of X for \( \omega_i, \omega_i, \omega_f \in X \). Let \( x, y, z \in X \) be such that \( T_N(x) \geq \bigvee \{T_N(y \ast z), T_N(z)\}, I_N(x) \leq \bigwedge \{I_N(y \ast z), I_N(z)\} \) and \( F_N(x) \geq \bigvee \{F_N(y \ast z), F_N(z)\} \). Then \( y \ast z \in X_N^{\omega_i} \cap X_N^{\omega_i} \cap X_N^{\omega_f} \) and \( z \in X_N^{\omega_i} \cap X_N^{\omega_i} \cap X_N^{\omega_f} \), where \( \omega_i = \omega_i = \omega_f = x \). It follows from (12) that \( y \in X_N^{\omega_i} \cap X_N^{\omega_i} \cap X_N^{\omega_f} \) for \( \omega_i = \omega_i = \omega_f = x \). Hence \( T_N(y) \leq T_N(\omega_i) = T_N(x), I_N(y) \geq I_N(\omega_i) = I_N(x) \) and \( F_N(y) \leq F_N(\omega_f) = F_N(x) \).
which implies that

\[ \text{Theorem 10.} \]

\[ \text{Let } X \text{ be a BCI-algebra, for any } x, y \in X \text{ and } \omega \in \text{Im}(T_N) \text{ and } \omega' \in \text{Im}(F_N) \text{ and suppose that } X \text{ satisfies Equations (11) and (12). Clearly, } \theta \in X_{N}^{\omega} \cap X_{N}^{\omega'} \cap X_{N}^{\omega'} \text{ by Equation (12). Let } x, y \in X \text{ be such that } x \ast y \in X_{N}^{\omega} \cap X_{N}^{\omega'} \cap X_{N}^{\omega'} \text{ and } y \in X_{N}^{\omega} \cap X_{N}^{\omega'} \cap X_{N}^{\omega'} \text{. Then} \]

\[ T_N(x \ast y) \leq T_N(\omega), \quad T_N(y) \leq T_N(\omega') \]

\[ I_N(x \ast y) \geq I_N(\omega), \quad I_N(y) \geq I_N(\omega') \]

\[ F_N(x \ast y) \leq F_N(\omega), \quad F_N(y) \leq F_N(\omega') \]

which implies that \( \{T_N(x \ast y), T_N(y)\} \leq T_N(\omega), \{I_N(x \ast y), I_N(y)\} \geq I_N(\omega), \text{ and } \{F_N(x \ast y), F_N(y)\} \leq F_N(\omega). \) It follows from \( \text{Equation (11)} \) that \( T_N(\omega) \geq T_N(x), I_N(\omega) \leq I_N(x) \text{ and } F_N(\omega) \geq F_N(x). \) Thus, \( x \in X_{N}^{\omega} \cap X_{N}^{\omega'} \cap X_{N}^{\omega'} \text{, and therefore } X_{N}^{\omega}, X_{N}^{\omega'}, \text{ and } X_{N}^{\omega'} \text{ are ideals of } X. \]

\[ \text{Definition 3.} \]

A neutrosophic \( \mathcal{N} \)-ideal \( X \) of \( X \) is said to be closed if it is a neutrosophic \( \mathcal{N} \)-subalgebra of \( X \).

\[ \text{Example 6.} \]

Consider a BCI-algebra \( X = \{\theta, 1, a, b, c\} \) with the following Cayley table.

<table>
<thead>
<tr>
<th></th>
<th>( \theta )</th>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta )</td>
<td>( \theta )</td>
<td>( \theta )</td>
<td>( a )</td>
<td>( b )</td>
<td>( c )</td>
</tr>
<tr>
<td>1</td>
<td>( 1 )</td>
<td>( \theta )</td>
<td>( a )</td>
<td>( b )</td>
<td>( c )</td>
</tr>
<tr>
<td>a</td>
<td>( a )</td>
<td>( \theta )</td>
<td>( c )</td>
<td>( b )</td>
<td>( a )</td>
</tr>
<tr>
<td>b</td>
<td>( b )</td>
<td>( \theta )</td>
<td>( c )</td>
<td>( b )</td>
<td>( \theta )</td>
</tr>
<tr>
<td>c</td>
<td>( c )</td>
<td>( b )</td>
<td>( a )</td>
<td>( \theta )</td>
<td>( \theta )</td>
</tr>
</tbody>
</table>

Let \( X \) be a neutrosophic \( \mathcal{N} \)-structure over \( X \) which is given as follows:

\[ X_N = \left\{ \begin{array}{c}
\frac{\theta}{(-0.9, -0.3, -0.8)} \cup \frac{a_i}{(-0.9, -0.4, -0.5)} \cup \frac{\theta}{(-0.6, -0.8, -0.5)}, \\
\frac{b}{(-0.2, -0.6, -0.3)} \cup \frac{c}{(-0.2, -0.8, -0.5)}
\end{array} \right\} \]

Then \( X \) is a closed neutrosophic \( \mathcal{N} \)-ideal of \( X \).

\[ \text{Theorem 10.} \]

Let \( X \) be a BCI-algebra. For any \( a_1, a_2, \gamma_1, \gamma_2 \in [-1, 0) \) and \( \beta_1, \beta_2 \in (-1, 0] \) with \( a_1 < a_2, \gamma_1 < \gamma_2 \text{ and } \beta_1 > \beta_2 \), let \( X_N := \left\{ \frac{X}{T_N, I_N, F_N} \right\} \) be a neutrosophic \( \mathcal{N} \)-structure over \( X \) given as follows:

\[ T_N : X \rightarrow [-1, 0], \quad x \mapsto \left\{ \begin{array}{ll}
a_1 & \text{if } x \in X_+ \\
a_2 & \text{otherwise}
\end{array} \right. \]

\[ I_N : X \rightarrow [-1, 0], \quad x \mapsto \left\{ \begin{array}{ll}
\beta_1 & \text{if } x \in X_+ \\
\beta_2 & \text{otherwise}
\end{array} \right. \]

\[ F_N : X \rightarrow [-1, 0], \quad x \mapsto \left\{ \begin{array}{ll}
\gamma_1 & \text{if } x \in X_+ \\
\gamma_2 & \text{otherwise}
\end{array} \right. \]

where \( X_+ = \{x \in X \mid \theta \preceq x\} \). Then \( X_N \) is a closed neutrosophic \( \mathcal{N} \)-ideal of \( X \).

\[ \text{Proof.} \] Because \( \theta \in X_+ \), we have \( T_N(\theta) = a_1 \leq T_N(x), I_N(\theta) = \beta_1 \geq I_N(x) \) and \( F_N(\theta) = \gamma_1 \leq F_N(x) \) for all \( x \in X \). Let \( x, y \in X \). If \( x \in X_+ \), then

\[ T_N(x) = a_1 \leq \bigvee \{T_N(x \ast y), T_N(y)\} \]

\[ I_N(x) = \beta_1 \geq \bigwedge \{I_N(x \ast y), I_N(y)\} \]

\[ F_N(x) = \gamma_1 \leq \bigvee \{F_N(x \ast y), F_N(y)\} \]
Suppose that \( x \not\in X_+ \). If \( x \ast y \in X_+ \) then \( y \not\in X_+ \), and if \( y \in X_+ \) then \( x \ast y \not\in X_+ \). In either case, we have

\[
T_N(x) = a_2 = \sqrt{\{T_N(x \ast y), T_N(y)\}} \\
I_N(x) = \beta_2 = \Lambda\{I_N(x \ast y), I_N(y)\} \\
F_N(x) = \gamma_2 = \sqrt{\{F_N(x \ast y), F_N(y)\}}
\]

For any \( x, y \in X \), if any one of \( x \) and \( y \) does not belong to \( X_+ \), then

\[
T_N(x \ast y) \leq a_2 = \sqrt{\{T_N(x), T_N(y)\}} \\
I_N(x \ast y) \geq \beta_2 = \Lambda\{I_N(x), I_N(y)\} \\
F_N(x \ast y) \leq \gamma_2 = \sqrt{\{F_N(x), F_N(y)\}}
\]

If \( x, y \in X_+ \), then \( x \ast y \in X_+ \). Hence

\[
T_N(x \ast y) = a_1 = \sqrt{\{T_N(x), T_N(y)\}} \\
I_N(x \ast y) = \beta_1 = \Lambda\{I_N(x), I_N(y)\} \\
F_N(x \ast y) = \gamma_1 = \sqrt{\{F_N(x), F_N(y)\}}
\]

Therefore \( X_N \) is a closed neutrosophic \( N \)-ideal of \( X \). \( \square \)

**Proposition 6.** Every closed neutrosophic \( N \)-ideal \( X_N \) of a BCI-algebra \( X \) satisfies the following condition:

\[
(\forall x \in X) \ (T_N(\theta \ast x) \leq T_N(x), \ I_N(\theta \ast x) \geq I_N(x), \ F_N(\theta \ast x) \leq F_N(x)) \tag{13}
\]

**Proof.** Straightforward. \( \square \)

We provide conditions for a neutrosophic \( N \)-ideal to be closed.

**Theorem 11.** Let \( X \) be a BCI-algebra. If \( X_N \) is a neutrosophic \( N \)-ideal of \( X \) that satisfies the condition of Equation (13), then \( X_N \) is a neutrosophic \( N \)-subalgebra and hence is a closed neutrosophic \( N \)-ideal of \( X \).

**Proof.** Note that \( (x \ast y) \ast x \leq \theta \ast y \) for all \( x, y \in X \). Using Equations (10) and (13), we have

\[
T_N(x \ast y) \leq \sqrt{\{T_N(x), T_N(\theta \ast y)\}} \leq \sqrt{\{T_N(x), T_N(y)\}} \\
I_N(x \ast y) \geq \Lambda\{I_N(x), I_N(\theta \ast y)\} \geq \Lambda\{I_N(x), I_N(y)\} \\
F_N(x \ast y) \leq \sqrt{\{F_N(x), F_N(\theta \ast y)\}} \leq \sqrt{\{F_N(x), F_N(y)\}}
\]

Hence \( X_N \) is a neutrosophic \( N \)-subalgebra and is therefore a closed neutrosophic \( N \)-ideal of \( X \). \( \square \)

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**References**