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Article

Weighted E-Spaces and Epistemic Information Operators

Mark Burgin

Department of Mathematics, University of California, 405 Hilgard Ave. Los Angeles, Los Angeles, CA 90095, USA; E-Mail: markburg@cs.ucla.edu; Tel.: +1-310 251 4897

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Abstract: Information is usually related to knowledge. Here, we present a broader picture in which information is associated with epistemic structures, which form cognitive infological systems as basic recipients and creators of cognitive information. Infological systems are modeled by epistemic spaces, while operators in these spaces are mathematical models of information. Information that acts on epistemic structures is called cognitive information, while information that acts on knowledge structures is called epistemic information. The latter brings new and updates existing knowledge, being of primary importance to people. In this paper, both types of information are studied as operators in epistemic spaces based on the general theory of information. As a synthetic approach, which reveals the essence of information, organizing and encompassing all main directions in information theory, the general theory of information provides efficient means for such a study. Different types of information dynamics representation use tools from various mathematical disciplines, such as the theory of categories, functional analysis, mathematical logic and algebra. In this paper, we base our exploration of information and knowledge dynamics on functional analysis further developing the mathematical stratum of the general theory of information.

Keywords: information; knowledge; epistemic structure; epistemic space; weighted epistemic space; epistemic information operator; vector bundle; continuity; boundedness

1. Introduction

Mathematical models play the pivotal role in science. In information theory mathematical models have been mostly used for efficient and constructive representation and study of information measures. The most popular mathematical information measures are Shannon's entropy, Kolmogorov complexity and Fisher information (measure). At the same time, the most advanced approach to modeling information and its measures is developed in the general theory of information, which systematizes and encompasses all main directions in information theory [1].

In this paper, we further develop the mathematical stratum of the general theory of information. According to the main principles of the general theory of information, information causes changes in infological systems. Thus, it is natural to model information by epistemic information operators that change mathematical representation of infological systems. Here, we are concerned with the cognitive information, which acts on cognitive infological systems, which are modeled by epistemic information spaces. They comprise symbolic representations of epistemic structures, which are basic units in cognition. Examples of epistemic structures employed by cognitive processes are concepts, notions, statements, ideas, images, opinions, texts, beliefs, knowledge, values, measures, problems, schemas, procedures, tasks, goals, *etc.* Epistemic structures form epistemic spaces.

Here, we construct and utilize weighted epistemic spaces, which represent, not only epistemic structures, but also their characteristics. The mathematical structure used for representing weighted epistemic spaces in the formal context is called a generalized vector bundle [2]. Informally, it consists of epistemic elements connected by relations—the base of the vector bundle-and a vector space attached to each of these elements. Note that as an epistemic space is a set of epistemic structures or their representations with relations and operations, it encompasses many other mathematical structures, such as lattices, groups or partially ordered sets.

Epistemic information operators are transformations and mappings of epistemic spaces. A special case of epistemic spaces—knowledge spaces—and knowledge information operators were studied in [1,3,4]. Here, we study knowledge spaces and information operators in a more general context of epistemic structures, epistemic spaces, and epistemic information operators. In addition, we extend epistemic spaces to weighted epistemic spaces, which include weighted knowledge spaces, and study properties of information operators in these spaces. To do this, we use structures and methods from functional analysis.

Note that there are information operators that are not epistemic. Examples of such operators are emotional and instructional information operators [1]. They are studied elsewhere as here we explore only epistemic information operators.

This paper is organized in the following way. In Section 2, we define epistemic structures and study their properties. In Section 3, we introduce epistemic spaces and study their properties. In Section 4, we show how epistemic structures acquire weights turning epistemic spaces into weighted epistemic spaces. Mathematical models of epistemic and cognitive information in the form of epistemic information operators acting in epistemic and weighted epistemic spaces are studied in Section 5. In Conclusion, some open problems related to epistemic and cognitive information are given.

2. Epistemic Structures in the Context of Information

Information is closely related to transformations [5]. Cognitive information is associated with transformations of epistemic structures [1]. Knowledge is a kind of epistemic structures. Thus, it is natural to treat knowledge in the context of epistemic structures.

Definition 1. An *epistemic structure* is a basic structure of cognition.

It is possible to find different definitions of *structure* in general and a unifying approach to this concept in [6].

Note that although according to the conventional understanding, a domain usually comprises several objects one object can be also treated as the domain of an epistemic structure. In essence, a domain is any part of reality, while reality as a whole includes all three types—physical reality, mental reality and structural reality [6]. Actually it is possible to consider any object, e.g., a system or a process, as a domain.

The essence of epistemic structures is represented by the following diagrams in Figures 1 and 2:

Figure 1. A reflection epistemic triad (direct epistemic unit *U*).

$$\text{Epistemic Structure } ES \xleftarrow[\text{reflection}]{\text{representation}} \text{Domain } D$$

Figure 2. A substantiation epistemic triad (inverse epistemic unit *V*).

$$\text{Epistemic Structure } ES \xrightarrow[\text{substantiation}]{\text{interpretation}} \text{Domain } D$$

Both pivotal cognitive and behavioral concepts—beliefs and knowledge—are basic epistemic structures. At the same time, there are many other epistemic structures, such as concepts, notions, statements, questions, ideas, images, algorithms, tasks, procedures, problems, values, measures, opinions, and goals. Note that rational behavior of people is essentially based on their beliefs and knowledge.

Definition 1 is essentially informal. A formalized definition of epistemic structures is recursive as it is constructed based on recursion.

Epistemic triads (Figures 1 and 2) describe and represent extended epistemic units, each of which reflects a definite domain or an aspect of definite domain.

Definition 2. (a) A reflection epistemic triad (cf. Figure 1) describes a *direct extended epistemic unit*, while a substantiation epistemic triad (cf. Figure 2) describes an *inverse extended epistemic unit*.
(b) The epistemic structure *ES* of an epistemic triad *U* (or *V*) is called the *cognitive part* of the epistemic unit *U* (unit *V*) or simply, an *epistemic unit*.
(c) The domain *D* of the epistemic triad *U* (or *V*) is called the *substantial part* of the extended epistemic unit *U* (unit *V*).

Usually epistemic structures are represented by systems of symbols, which are called symbolic epistemic structures. *Symbolic knowledge units* are examples of symbolic epistemic units. For instance, the sentence "Ten is larger than five" is a symbolic knowledge unit, as well as a symbolic epistemic unit. The logical expressions, such as $\varphi \vee \psi$, $\varphi \wedge \psi$, and $\varphi \rightarrow \psi$, are also symbolic knowledge units, as well as symbolic epistemic units.

In general, epistemic structures are components of extended epistemic units.

In what follows, we consider only symbolic epistemic units and symbolic knowledge units. That is why for simplicity, symbolic epistemic units are called *epistemic units* and symbolic knowledge units are called *knowledge units*. In this, we follow the longstanding tradition where knowledge means only symbolic representation of knowledge.

It is natural to separate the domain of all epistemic structures into two classes (types)—static and dynamic epistemic structures.

Definition 3. *Static epistemic structures* describe domains that are not changing.

Definition 4. *Dynamic epistemic structures* either describe domains that are changing or have changes as their domain.

For instance, if a system is described as a collection of elements and relations between these elements, this description is a static epistemic structure. We obtain a dynamic epistemic structure when the processes going on in the system are also included in the description. An algorithm or a procedure is another example of a dynamic epistemic structure, the domain of which is a process or a system of processes.

We also separate dynamic epistemic structures into two classes (types)—functional epistemic structures and process epistemic structures.

Definition 5. *Functional epistemic structures* represent changes as transitions from the initial state to the final state.

For instance, operations, relations and goals are functional epistemic structures.

Definition 6. *Process epistemic structures* represent changes as processes.

For instance, algorithms, flowcharts, scenarios, inferences and stories are process epistemic structures. In these examples, algorithms are compressed epistemic structures, inferences and stories are expanded epistemic structures, while flowcharts and scenarios can be either compressed epistemic structures or expanded epistemic structures.

Epistemic item is a more general concept than *epistemic unit* because an epistemic item can be a part of an epistemic unit or consists of several epistemic units. In particular, *knowledge item* is a more general concept than *knowledge unit* because a knowledge item can be a part of a knowledge unit or consists of several knowledge units. For instance, taking a knowledge base, it is possible to consider all knowledge from this knowledge base as one knowledge item. However, usually this knowledge consists of many knowledge units. All knowledge from a textbook is a knowledge item.

Epistemic structures form the base for the traditional interpretation of information as the essence that gives or changes knowledge. According to the general theory of information, symbolic epistemic items (structures) constitute cognitive infological systems [1].

This understanding is formalized in the general theory of information based on the basic ontological principle O2c [1].

Ontological Principle O2c (**the** ***Cognitive Transformation Principle***). *Cognitive information* for a system R is a (potential) capacity to cause changes in the cognitive infological system CIF(R) of the system R.

In this context, a *cognitive infological system* CIF(R) contains, acquires, stores and processes various epistemic structures, such as knowledge, data, ideas, beliefs, images, algorithms, tasks, procedures, problems, schemas, scenarios, values, measures, opinions, goals, ideals, fantasies, abstractions, *etc.* Cognitive infological systems are very important, especially, for intelligent systems. Indeed, the majority of researchers believe that information in general is intrinsically connected to cognition, while cognitive information is one of the three basic types of anthropic information studied in [3]. Moreover, some researchers believe that people's knowledge about physical reality is the result of information they obtain from external sources [7–10]. Understanding that physicists study physical systems not directly but only through information they get from these systems has created a school of thought about the role of information processing in physical processes and its influence on physical theories. According to one of the outstanding physicists of the 20th century, John Archibald Wheeler (1911–2008), it means that every physical quantity derives its ultimate significance from information. He called this idea "It from Bit" where It stands for things, while Bit impersonates information as the most popular information unit [10]. For Wheeler and his followers, space-time itself must be understood and described in terms of a more fundamental pregeometry without dimensions and classical causality. These features of the physical world only appear as emergent properties in the ideal modeling the physical reality based on information about complex interactions of very simple basic elements, such as subatomic particles.

As the cognitive infological system contains knowledge of the system it belongs to, cognitive information is the main source of knowledge changes. When a system (it may be a person, a group of people, a community, society as whole or an intelligent agent) receives cognitive information, this system may convert it to knowledge or miss this information. For instance, a teacher is not giving knowledge to his students but only provides cognitive information and students themselves have to convert this information into knowledge [11].

It is important to discern epistemic structures and their representations, as well as knowledge and its representations. Epistemic structures are usually represented by symbolic systems in general and symbols, in particular. As a rule, one epistemic structure (knowledge unit) has several representations. For instance, knowledge that a person A is seven feet tall can be represented by: the statement/sentence/proposition "A is seven feet tall"; the statement/sentence/proposition "the height of A is seven feet"; the equality $H(A) = 7$ ft where $H(X)$ is the property *height* of a person; the truth of the predicate $H(A, 7)$ where $H(X, h)$ is the predicate with X being a person and h being a number of feet; and the element (A; 7) of a relational database.

At the same time, the same symbolic system can represent different epistemic structures. For instance, a statement can represent knowledge, a belief or an idea.

A symbolic representation of an epistemic structure (cognitive epistemic unit) is called a *symbolic epistemic structure* (*symbolic epistemic unit*). Thus, it is possible to discern structural cognitive infological systems, which consist of epistemic structures, and symbolic infological systems, which consist of symbolic representations of epistemic structures.

In a similar way, domains are usually represented by their models. This brings us to the *extended direct epistemic unit* characterized by Diagram (1) and *extended inverse epistemic unit* described by Diagram (2).

$$\begin{tikzcd}
\textbf{epistemic structure} \arrow[d, "\text{representing}"'] & \textbf{domain} \arrow[l, "\text{reflecting}"'] \arrow[d, "\text{modeling}"] \\
\textbf{representation} & \textbf{model} \arrow[l, "\text{reflecting}"]
\end{tikzcd}$$

(1)

$$\begin{tikzcd}
\textbf{epistemic structure} \arrow[r, "\text{referring}"] & \textbf{domain} \\
\textbf{representation} \arrow[u, "\text{interpreting}"] \arrow[r, "\text{attributing}"'] & \textbf{model} \arrow[u, "\text{substantiating}"']
\end{tikzcd}$$

(2)

Very often people make no distinction between epistemic structures and their representations. For instance, people assume that the statement "The Earth rotates around the Sun" is knowledge. However, it is only some representation of knowledge about the Earth and the Sun. It is possible to represent the same knowledge by the statement "The Earth moves about the Sun" or by the statement "The Earth travels about the Sun". This shows that one epistemic structure can have and usually has several symbolic, e.g., linguistic, representations.

3. Epistemic Spaces

Cognitive systems, e.g., intelligent agents or cognitive actors, store and employ a variety of epistemic structures in different forms and shapes. The memory of a cognitive system contains epistemic structures, which are organized in the system called an epistemic space.

Informally, an epistemic space is a set of epistemic structures or their representations with relations and operations. Epistemic structures (cognitive epistemic units) form *abstract epistemic spaces*, while their representations (symbolic epistemic units) create *symbolic epistemic spaces*. However, in what follows, we consider only symbolic epistemic spaces.

In modeling epistemic systems in general and knowledge systems in particular and studying information processes in these systems, we consider two basic structures—sets and multisets. Sets of epistemic structures enhanced by relations between these structures become *epistemic spaces*, while multisets of epistemic structures enhanced by relations between these structures become *epistemic multispaces*. In essence, using the classical mathematical background it is possible to consider only sets, which make the model simpler serving as the first approximation to real epistemic/knowledge systems and information processes. However, many real cognitive systems contain several copies of the same element. For instance, the same element of knowledge can be stored in different parts of a computer memory or of the brain. This makes utilization of multisets necessary. We remind that a *multiset* is a collection that is like a set but can include identical or indistinguishable elements. For instance, $M = \{a, a, b, b, b\}$ is a multiset that contains two elements a and three elements b. It is usually assumed that in a multiset, elements are indistinguishable if and only if they have the same type.

If a is an element from a multiset M, then the number of copies of an element a is called the *multiplicity* of a in M and is denoted by $m_M(a)$. In the considered example, $m_M(a) = 2$ and $m_M(b) = 3$.

A multiset M is a *multisubset* of a multiset N if $m_M(a) \leq m_N(a)$ for all elements a from M.

However, when for building epistemic spaces, we use not only sets but structures, *i.e.*, sets with relations, we are able to include multisets in this schema because multisets can be treated as sets with the indistinguishability relation. That is why we use only sets and structures forming epistemic spaces from epistemic structures.

To give an exact definition of an abstract epistemic space, we consider a set $\boldsymbol{W}_{\mathbf{es}}$ of epistemic structures, while for an exact definition of an symbolic epistemic space, we consider a set W_{ses} of symbolic epistemic units, e.g., of symbolic knowledge units or knowledge items, taking it as the *state base* of an epistemic space. For instance, it is possible to regard a set W_K of symbolic parts of elementary knowledge units as a set W_{es}. This allows us obtaining an efficient formalization of the concept of a knowledge space. The set W_L of propositions and/or predicates in a logical language L gives an example of $W_{\text{ses.}}$ Propositions and predicates are symbolic knowledge units in the logical approach to information theory developed in works of Bar-Hillel and Carnap [12], Hintikka [13,14] and some other authors. Shreider [15] interpreted symbolic knowledge units as texts in a thesaurus. Many researchers employ mental schemas as cognitive/symbolic knowledge units in the brain (cf., for example, [16–18]) One more possibility for W_{ses} is the set, or more exactly, a multiset, W_{Sit} of logical representations of situations possible in a world U. Situations themselves form the substantial parts of knowledge units, while their logical descriptions, e.g., in the form of propositions, are the cognitive/symbolic parts of knowledge units. These logical descriptions are also called situations playing the role of knowledge items or knowledge units (cf., for example, [19]).

Some readers may be confused by variability of epistemic structures. However, mathematics provides means to decrease non-uniformity in the diversity of epistemic structures. Let us assume that we study such diversity as a collection B of epistemic structures by means of epistemic spaces. One way to reduce complexity is to use homogeneous approximations and, at first, to study only uniform epistemic spaces, which model uniform collections of epistemic structures. For instance, a knowledge space is a uniform collection of epistemic structures or algorithmic space is a uniform collection of epistemic structures.

On the second step, it is possible to introduce and study epistemic spaces with a low degree of uniformity (A formal definition of degrees of uniformity based on measures of uniformity of general systems is given in [20,21]). For instance, the degree of uniformity of a knowledge space is lower than the degree of uniformity of a space of knowledge, beliefs and ideas. At the same time, logical knowledge space has higher degree of uniformity than the predicate knowledge space

Another way to deal with extremely non-uniform systems is unification. Here, we describe unification of collections of epistemic structures. To do this, we observe that it is possible to represent each epistemic structure by a concept. For instance, taking a knowledge item "Now it is 5 p.m.", we express the content of this statement by the equality relation between two temporal points (intervals) 5 p.m. and "now". This relation is a concept. The meaning of this concept is expressed by (consists of) its relations to other concepts, such as "time", "equality", "identity", "point", *etc.* In such a way, unification converts any collection of epistemic structures into a uniform system, which consists only of concepts. It is possible to model this system by a semantic network or by a more advanced epistemic space.

In the theory of epistemic spaces, often it is possible not to distinguish sets W_{es} and W_{ses} because they have many common properties and only these properties are important in many theoretical constructions. When this is the case, we denote W_{es} or W_{ses} by the same letter W without causing confusion.

Some sets W can reflect (represent) more objects (large domains), while others reflect (represent) less objects (small domains).

Definition 7. We call the set W *universal* for a collection **CIF** of cognitive infological systems, e.g., for systems of knowledge, when the following axiom is true.

UCIF1 (the Internal Representation Axiom). For any infological system R from **CIF**, any state of R is a subset of the set W.

For instance, we can take a group G of intelligent agents and define the collection **CIF** as the set of their knowledge systems playing the role of cognitive infological systems from **CIF**. Then the Internal Representation Axiom states that any possible state K_{Ai} of the knowledge system K_A of an agent A from G is a subset of the set W. In this case, it is possible to interpret W as the base of all knowledge that agents are able to have about their environment.

Another aspect of universality of the set W is expressed by the possibility to describe all possible (existing) worlds or/and situations utilizing epistemic elements (symbolic epistemic elements), e.g., knowledge (symbolic knowledge units), only from W. For instance, when W is the set W_L of propositions and/or predicates from some logical language L, then universality implies that it is possible to build all descriptions of all possible worlds by combining elements from W_L. This possibility is reflected in the following concept.

Let us consider a domain, D. This domain may be a part of the real world, the set of all (possible) situations in a part of the real world, the set of all possible (existing) worlds in the sense of logical semantics or the set of all possible (existing) states of the environment in some area.

Definition 8. We call the set W *universal* for the domain D when the following axiom is true.

UCEF2 (the External Representation Axiom). For any environment (situation, world or state) R of the domain D, there is a subset W_R of the set W that contains all epistemic structures that reflect R.

In particular, it means that if W_{es} consists only of knowledge structures, then for any environment (situation, world or state) R of the domain D, there is a subset W_R of the set W_{es} that contains all (accessible or representable) knowledge about R.

Taking axioms UCIF1 and UCEF2 as the foundation, we develop a theory of cognitive systems (cognitive agents) called the *theory of E-spaces*. At first, we define free epistemic spaces, taking W_{es} for abstract epistemic spaces and W_{ses} for symbolic epistemic spaces.

Usually an epistemic space E is a dynamic system, which is permanently changing its content. This content at a given moment of time is called the state of the epistemic space E.

Definition 9. (a) An *epistemic space*, also called an *E-space*, V with the base W is a subset of the set W.
(b) Appropriate subsets of V are called *states of the epistemic space V*.
(c) An epistemic space V with a collection $\mathrm{St}V \subseteq 2^V$ is called an *epistemic system*.

For instance, we can take the set *Id* of all ideas about which Professor Angstrem can think as an epistemic space. However, at any moment of time, he can think only about one or two ideas. Thus, modeling his thinking, only one-element and two-element subsets of *Id* will be appropriate as states of this epistemic space. These states will represent the projection of the Professor Angstrem's mentality on the space of ideas.

Note that that any state of an epistemic space is itself an epistemic space.

It is natural to consider epistemic spaces that have elements of the same type. For instance, free epistemic spaces that consist of symbolic knowledge units are called *knowledge spaces* or *Mizzaro spaces*. They are studied in [1,4].

Definition 10. (a) A *type of structures* is a system of conditions (axioms) that all these structures, *i.e.,* sets with relations, satisfy.
(b) An epistemic space *V* is called *uniform* if all its elements have the same type.
(c) An epistemic system (*V*, St*V*) in which all states are uniform is called *uniform*.

For instance, the epistemic space of all logical propositions is uniform. Algorithms are knowledge items explaining how to solve a problem or how a system is functioning. Finite automata form a uniform class of algorithms, which is a uniform epistemic space. However, a class of algorithms that consists of finite automata and Turing machines is an epistemic space that is not uniform.

In general, epistemic multispaces spaces have different relations between and operations with their elements and states. Thus, let us assume that *W* is not simply a set or a multiset but is a structure, *i.e.*, a set with relations and operations [6].

Definition 11. (a) A *structured epistemic space*, also called a *structured E-space*, *V* with the base *W* is a substructure of the structure *W*.
(b) A substructure of V is called a state of the structured epistemic space (multispace) V.

For instance, stratified M-spaces [3] give an example of structured epistemic spaces.

Let us consider other examples of epistemic spaces. Usually these epistemic spaces are structured.

Example 1. A propositional epistemic space *EPS*, in which propositions are epistemic elements, is a knowledge space as propositions constitute one of the basic forms of knowledge representation. There are many relations in this space, *i.e.*, it is a structured epistemic space. One of the main relations is implication denoted by the symbol →, where $p \rightarrow q$ means that whenever the proposition *p* is true, the proposition *q* is also true. Other important relations in the propositional epistemic space *EPS*:

The *deducibility relation* means "a proposition *p* is deduced from a proposition *q*", for example, the proposition *q* is deducible from the proposition q & p. This is the most popular logical relation between propositions from a propositional language. The deducibility relation is usually denoted by the symbol ⊢, e.g., $A \vdash \varphi$.

The *generality relation* means that one proposition is more general than another one, for example, the proposition "stars give light" is more general than the proposition "stars from our galaxy give light".

Another important logical relation between propositions or statements is "a proposition (statement) p directly implies a proposition (statement) q", *i.e.*, there is an inference rule, e.g., *modus ponens* or *modus tollens*, such that p is the argument and q is the conclusion.

One more important logical relation between propositions or statements is "a proposition (statement) p *entails* a proposition (statement) q (in a theory T)", meaning that in every model (of T) where p is true, q is also true. The *entailment relation* is usually denoted by the symbol $\vDash$, e.g., $p \vDash q$.

There are also operations in the propositional epistemic space *EPS*, for example, classical logical operations—conjunction, disjunction, and negation.

It is also possible to induce topology in the propositional epistemic space *EPS*.

Example 2. We can build a *conceptual epistemic space* based on formal concept analysis [22]. A conceptual epistemic space consists of formal contexts, concepts, concept intents, and concept extents, which are defined as rigorous mathematical structures.

A *formal context* **C** is a triad (named set) **C** = (G, I, M), where

- G is a set of objects,
- M is a set of attributes,
- I is a relation between G and M, in which the relation $(g, m) \in I$ means, the object g has the attribute m, *i.e.*, I is the connection between objects and attributes.

Thus, we can see that formal contexts are named sets and it is possible to apply to them different named set operations [2]. There are also various relations between formal contexts that come from the named set theory. For instance, a formal context **C** = (G, I, M) is a *subcontext* of a formal context **D** = (H, J, L) if $G \subseteq H$, $M \subseteq L$ and I is the restriction of J on G and M, *i.e.*, $I = J|_{(G, M)}$.

It is also possible to build an epistemic space from informal concepts. Taking concepts used in some society as symbolic epistemic elements, we obtain an epistemic space with different relations, *i.e.*, a structured epistemic space. For instance, it is natural to use the relation "to be a subconcept", e.g., the concept *dog* is a subconcept of the concept *animal*. Another relation between informal concepts is the *foundational relation*, which shows when one concept is used in a definition of another concept.

Semantic networks are examples of structured epistemic spaces built from informal concepts. *Semantic network* or *semantic net* is a knowledge representation formalism (graphic notation) that describes objects and their relationships in the form of a network consisting of labelled (named) nodes and (usually directed) links in the form of arcs or arrows. The nodes represent objects or concepts by their names, while the links represent relations between nodes also by their names.

Note that in a general case, epistemic spaces are graphs. So, conceptual epistemic spaces of Ganter and Wille [22] are special cases of general epistemic spaces.

Example 3. The epistemic scenarios together constitute an *epistemic space* in the sense of Chalmers [23]. The most natural way of scenario interpretations, at least initially, are possible worlds. More exactly, an epistemic scenario describes possible (in some sense) ways a world might be. Defining this, Chalmers uses the notion of possibility that is different from the notion of possibility usually associated with possible worlds. Here, possibility is a sort of epistemic possibility, whereas possible worlds are usually understood to be associated with a sort of "metaphysical" possibility.

Another way to describe epistemic scenarios is to identify them with equivalence classes of thoughts or with maximal classes of thoughts with equivalence defined as mutual implication. In this context, it is possible to assume that thoughts are composed of concepts.

Having defined a space of scenarios for each subject at a time, it is possible to form a common space of scenarios for all subjects. To do this, Chalmers uses the principle of *translation* between the maximal thoughts (or the complexes) of one subject, and those of another. This principle is based on the basic notion of translatability represented by the *translation relation*, which is an equivalence relation on the set of thoughts. As a result, we obtain a structured epistemic space (cf. Definition 11).

Example 4. A *knowledge information space M* is an important kind of epistemic information spaces. Knowledge information spaces are studied in [3].

Example 5. It is possible to represent a logical variety or a prevariety $\boldsymbol{M}$ [24] as a structured epistemic space V. In it, the components of $\boldsymbol{M}$ are treated as states of the space V, while the mappings f_i: $A_i \rightarrow L$ and g_i : $T_i \rightarrow L$ ($i \in I$), which form connections between components of the variety (prevariety), constitute relations between elements in this space. Thus, $\boldsymbol{M}$ is a structured epistemic space.

Example 6. Taking words from a natural language, such as English, Spanish, or German, as symbolic epistemic elements, it is natural to treat two words as linked when they express similar concepts. In such a way, Motter, *et al.* [25] built topology of the conceptual network of a language. This network gives one more example of structured epistemic spaces.

Given a set X with a binary relation R, it is possible to introduce a metric in this space [26].

An *R-path* in X between elements x and y is a sequence $p(x, y) = (x_1, x_2, x_3, \dots, x_n)$ of elements from X such that all pairs (x_{i-1}, x_i) belong to R, $x_1 = x$ and $x_n = x$. The number n is called the length of the path $p(x, y)$, *i.e.*, $l(p(x, y)) = n$. Then we define the distance by the following rule:

$$\mathrm{d_R}(x,y) = \begin{cases} \min\{l(p(x,y)); p(x,y) \text{ is path in } X \text{ between elements } x \text{ and } y\} & \text{if, at least, one path exists} \\ \infty & \text{if there are no paths in } X \text{ between elements } x \text{ and } y \\ 0 & \text{when } x = y \end{cases}$$

Lemma 1. The distance $\mathrm{d}_R(x, y)$ defines metric in the space X.

Indeed, by definition, $\mathrm{d}_R(x, y) = 0$ if and only if $x = y$. The relation d_R is symmetric, *i.e.*, $\mathrm{d}_R(x, y) = \mathrm{d}_R(y, x)$. In addition, if $(x_1, x_2, x_3, \dots, x_n)$ is path in X between elements x and z and $(y_1, y_2, y_3, \dots, y_m)$ is path in X between elements y and y, then $(x_1, x_2, x_3, \dots, x_n, y_1, y_2, y_3, \dots, y_m)$ is path in X between elements x and y. Thus, $\mathrm{d}_R(x, y) \leq \mathrm{d}_R(x, z) + \mathrm{d}_R(z, y)$.

We will call this metric by the name the *relational metric* in the set X.

Note that in the relational metric, X is a discrete topological space.

Lemma 1 shows that W_{es} and W_{ses} are metric spaces. Structures in the spaces W_{es} and W_{ses} are inherited by epistemic spaces and their states. Thus, as any subspace of a metric space is a metric space [26], an epistemic space E and all its states of are metric spaces.

4. Weighted Epistemic Spaces

Epistemic structures and epistemic units have many properties. Basic characteristics of epistemic structures (epistemic units) in general and of knowledge in particular are called *dimensions*. Usually dimensions are gradual compound properties or attributes.

It is possible to discern the following dimensions of epistemic structures (of knowledge):

(1) The *relevance dimension* reflects relevance of an epistemic structure, e.g., of knowledge unit, to its domain.
(2) The *confidence* or *certainty dimension* reflects the relevance estimation of an epistemic structure, e.g., of knowledge unit, to its domain.
(3) The *justification dimension* reflects justification of the relevance estimation of an epistemic structure, e.g., of knowledge unit, to its domain.
(4) The *complexity dimension* reflects complexity of an epistemic structure, e.g., of knowledge unit, in utilization.
(5) The *significance dimension* reflects significance of an epistemic structure, e.g., of knowledge unit.
(6) The *efficiency dimension* reflects the role of an epistemic structure, e.g., of knowledge unit, in achieving some goals.
(7) The *reliability dimension* reflects reliability of an epistemic structure, e.g., of knowledge unit.
(8) The *abstractness dimension* reflects the level of abstraction of an epistemic structure, e.g., of knowledge unit.
(9) The *generality dimension* reflects degree of generalization achieved by an epistemic structure, e.g., of knowledge unit.
(10) The *meaning dimension* reflects meaning of an epistemic structure, e.g., of knowledge unit.

The first three dimensions are the *separation dimensions* as these traits are often used to separate knowledge from other epistemic structures, e.g., from beliefs.

The next six dimensions are the *feature dimensions*.

The tenth dimension is the *integration dimension* as all other dimensions are projected into it.

Each dimension is a composite attribute comprising several basic epistemic attributes. For instance, the efficiency dimension of an epistemic structure e includes the efficiency of e for reaching some goal, e.g., for reaching the Mars, the efficiency of e for understanding e, the efficiency of e for understanding people, the efficiency of e for building some object A and the efficiency of e for obtaining knowledge about some object D.

Another example of efficiency is given by such an epistemic structure *int* as knowledge of mathematical integration and its efficiency for a student C. In this case, efficiency of *int* for getting a high grade in the class is rather high, while efficiency of *int* for getting from home to the college is rather low (usually it is zero).

Complexity comprises such properties as compression, understandability and hardness.

Taking the justification dimension, we see that there are different ways and strategies of epistemic structure (knowledge) justification. It is possible to treat each kind of justification as an attribute component of the justification dimension. At the same time, there are three basic approaches to justification, which are similar to the approaches to knowledge acquisition: by practice/experience, by reasoning/thinking and by authority/opinion.

Justification by practice/experience means that an epistemic structure (knowledge) is justified if it works well in practice, e.g., it allows better achieving some goals, and our experience gives evidence for this.

Justification by reasoning/thinking is performed in the mentality of the justifier and is explicit justification. However, the brain has three basic components-the *System of Rational Intelligence* (also called *System of Reasoning*), the *System of Emotions* (*Affective States*) and the *System of Will and Instinct* [1]. Consequently, there are two other kinds of mental justification—by emotions and by instructions/assertions.

Justification by authority/opinion means that an epistemic structure (knowledge) is justified if there is the corresponding opinion, which is usually held as an authoritative one. Note that it may be an opinion of an individual, of a social group taken from some source, such as a book, magazine, or the Internet.

In addition to attributes that constitute dimensions, there are other properties/attributes of epistemic structures and epistemic units. For instance, an important epistemic attribute is *novelty* with respect to the infological system of an intelligent agent (cognitive system). This attribute is a (fuzzy) function of another attribute that shows the time of attribution of epistemic structure to the given infological system.

Epistemic structures (knowledge units) described in two previous sections are pure. Properties/ attributes of epistemic units (structures) and characteristics of objects they reflect (represent) induce weights of these epistemic units. For instance, taking such an epistemic unit as a statement P, we can consider its properties (attributes): (1) time when this statement P was made; (2) person(s) who made this statement; (3) people who supported this statement; (4) time needed to prove validity (truthfulness) of P; (5) truth value of P; and so on. The value of the first attribute is the first weight w_1 of P. It is a numerical value. The value of the second attribute is the second weight w_2 of P. It is a nominal value, *i.e.*, it is a name or names of people who made statement P, or a functional value, *i.e.*, it is the indicator function of the names of people who made statement P. The value of the third attribute is the third weight w_3 of P. It is similar to the second weight. The value of the fourth attribute is the fourth weight w_4 of P. It is numerical and similar to the first weight. The value of the fifth attribute is the fifth weight w_5 of P and it takes two values from the two-element set $\{true, false\}$ if we utilize the classical logic. If we evaluate P by means of fuzzy logic, the fifth weight w_5 takes values in the interval [0, 1].

An algorithm is knowledge of how to solve a problem or a class of problems. Thus, taking an algorithm A as a symbolic structure, we have such properties as (cf., [27]): (1) the length l_A (weight w_1); (2) its time complexity T_A (weight w_2); (3) its space complexity S_A (weight w_3). The values of the first weight are positive numbers, while the values of the second and third weights are functions (cf., [27]).

Dimensions, which are basic complex properties, also add weights to knowledge units (epistemic structures). For instance, the knowledge unit A represented by the sentence "Now it is ten o'clock in the morning" is the symbolic part of pure knowledge. However, it can be true or false depending on current time. This estimate defines the weight the knowledge unit in the relevance dimension. Namely, if the estimate "true" is represented by 1 and the estimate "false" is represented by 0, then the weight of A is 1 when it is really ten o'clock in the morning and the weight of A is 0 when this is wrong.

As a result, dimensions and other properties/attributes bring us from *pure epistemic structures* (knowledge units) to *weighted epistemic structures* (*weighted knowledge units*). To determine weights, we fix a vector of attributes (A_1, …, A_k). Then we change a pure epistemic structure (pure knowledge unit) e, to the weighted epistemic structure (weighted knowledge unit) $B = (e; w_1, \ldots, w_k)$, where w_i is the weight of e with respect to the attributes A_i (the dimension i). The value of the weight w_i of the epistemic structure e with respect to the attribute A_i reflects to what extent e has the attribute A_i. When the attributes A_i is an abstract property in the sense of [2], then w_i is the value of this property for the epistemic structure e.

It is possible to consider the system of weights $w_1, \ldots, w_k$ of a weighted epistemic structure (knowledge unit) $B = (e; w_1, \ldots, w_k)$ as the state of e.

It is interesting to know that weights can describe arbitrary structures in epistemic spaces. In particular, weights in an epistemic space can turn it into an epistemic multispace by inducing an indistinguishability relation between elements of this space. Indeed, in a multiset (cf. Section 3), elements are indistinguishable if and only if they have the same type. Thus, we can introduce the weight w_t the values of which are types of elements (epistemic structures) from an epistemic space. Making elements that have the same value of the weight w_t indistinguishable, we obtain an epistemic multispace.

An epistemic space E is called *weighted* if all its elements are weighted epistemic structures. By construction, there is a natural projection π of a weighted epistemic space E onto a pure epistemic space E_0 where $\pi(e; w_1, \ldots, w_k) = e$ for any pure epistemic structure e.

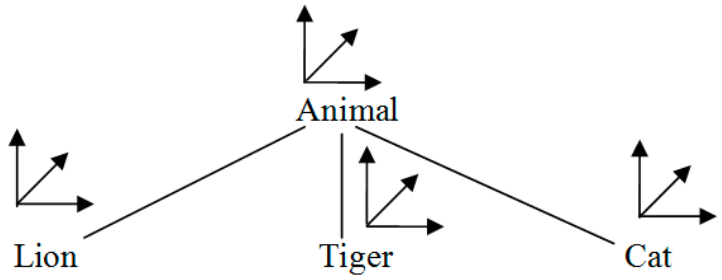
Example 7. Gardenfors offers his theory of conceptual representations as a bridge between the symbolic and connectionist approaches [28,29]. Symbolic representation is particularly weak at modeling concept learning, which is paramount for understanding many cognitive phenomena. Concept learning is closely tied to the notion of similarity, which is also poorly served by the symbolic approach. Gardenfors's theory of conceptual spaces presents a framework for representing information on the conceptual level. A conceptual space is built up from geometrical structures based on a number of quality dimensions. The main applications of the theory are on the constructive side of cognitive science: as a constructive model the theory can be applied to the development of artificial systems capable of solving cognitive tasks. Gardenfors also shows how conceptual spaces can serve as an explanatory framework for a number of empirical theories, in particular those concerning concept formation, induction, and semantics. His aim is to present a coherent research program that can be used as a basis for more detailed investigations.

Example 8. Osgood, Suci, and Tannenbaum [30] use semantic spaces for building their theory of meaning and its measurement. A *semantic space* is a set of concepts with their meaning. The *meaning of a concept to an individual subject* is defined as the set of the factor scores based on the data from this individual. The *meaning of a concept in the culture* is defined as the set of the averaged factor scores [30]. This shows that a semantic space is a special kind of weighted epistemic spaces. Although very often the factor scores are integers, it is possible to conjecture that using real numbers as the factor scores allows better evaluation of meaning turning the meaning of a concept into a real vector.

The mathematical structure used for representing weighted epistemic spaces in the formal context is called a generalized vector bundle [2]. Informally, it consists of epistemic elements connected by relations and a vector space attached to each of these elements. Below we give explicit examples of such spaces.

Example 9. We can also build a weighted conceptual epistemic space *WECS* in which concepts are epistemic elements. Below we give a sample of weighted conceptual epistemic spaces, which consists of four concepts, to each of which a three-dimensional weight space is attached. For instance, it is possible to construct each of these weight spaces using such properties as the level of abstraction, fuzziness and connectedness, *i.e.*, the number of other concepts to which this concept is connected.

Figure 3. A simple graphical example of a weighted conceptual epistemic space *WECS*.



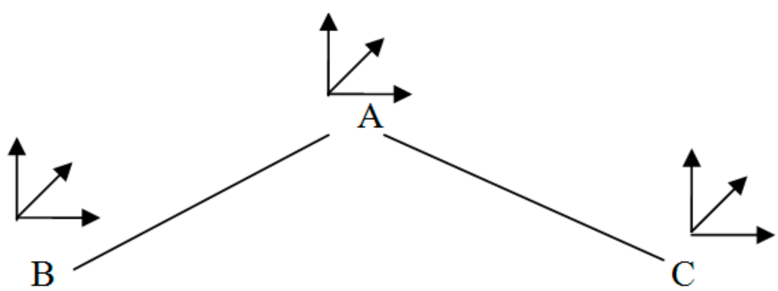
Example 10. A weighted propositional epistemic space *WEPS*, in which propositions are epistemic elements, is a knowledge space as propositions constitute one of the basic forms of knowledge representation. There are many relations in this space, *i.e.*, it is a structured epistemic space. One of the main relations is implication denoted by the symbol →, where r, $p \to q$ means that whenever the propositions p and r are true, the proposition q is also true.

Below we give a sample of weighted conceptual epistemic spaces, which consists of three propositions:

A: ABCD is a square.
B: ABCD is a rectangle.
C: ABCD is a rhombus.

To each of these propositions, a three-dimensional weight space is attached. For instance, it is possible to construct each of these weight spaces using such properties as the level of abstraction, fuzziness and connectedness, *i.e.*, the number of other concepts to which this concept is connected.

Figure 4. A simple graphical example of a weighted conceptual epistemic space *WECS*.



In what follows, we assume that all weighted epistemic elements of the form $(a; w_1, \ldots, w_k)$ with the fixed a is a real vector space—the space of weights, which is denoted by L_a, or even a topological vector space [31]. Note that in general, not all weights are numbers. For instance, there are functional weights. However, it is possible to immerse any domain of weight values into an appropriate vector space and assume that the whole weight space is the vector space equal to the Cartesian product of weight spaces of individual weights w.

In addition, it is possible to assume that all vector spaces L_a of weights have the same dimension. If it is not so, when dimensions of all L_a are bounded, we can take the space L_{a_0} of weights with the maximal dimension as the common space for all weights denoting it by L_e. In the case when dimensions of all L_a are unbounded, we come to the necessity to use an infinite-dimensional vector space as the common space of all weights L_e. That is why, exploring the general situation, we acknowledge that the common vector spaces L_e of weights is not necessarily finite dimensional.

In this context, the space W_{esw} of weighted epistemic structures from W_{es} has the structure of a vector bundle with the base W_{es}, *i.e.*, $\boldsymbol{W}_{\text{esw}} = (W_{\text{esw}}, \pi_{\text{es}}, W_{\text{es}})$ where $\pi_{\text{es}} : W_{\text{esw}} \to W_{\text{es}}$ is a projection, while the space $\boldsymbol{W}_{\text{sesw}}$ of weighted symbolic epistemic structures from W_{ses} has the structure of a vector bundle with the base W_{ses}, *i.e.*, $\boldsymbol{W}_{\text{sesw}} = (W_{\text{sesw}}, \pi_{\text{ses}}, W_{\text{ses}})$ where $\pi_{\text{ses}} : W_{\text{sesw}} \to W_{\text{ses}}$ is a projection.

We remind [32] that a vector bundle $\mathbf{E}$ is a triad (named set) $\mathbf{E} = (E, p, B)$ where the topological space E is called the *total space* or simply, *space* of the vector bundle $\mathbf{E}$; the topological space B is called the *base space* or simply, *base* of the vector bundle $\mathbf{E}$**;** and p is the topological projection of E onto B such that there is a vector space F is called the *fiber* of the vector bundle $\mathbf{E}$, for all points b from B, $p^{-1}(b) = F_b \simeq F$ and every point in the base space has a neighborhood U for which the space $p^{-1}(U)$ is homeomorphic to the direct product $U \times F$. In the case of the epistemic spaces W_{esw} and W_{sesw}, F is a vector space isomorphic the common vector space L_e of weights.

Consequently, taking a weighted epistemic space $E \subseteq W_{\text{esw}}$, we obtain the vector bundle $\boldsymbol{E} = (E, p_E, E_e)$ in which p_E is the restriction of π_{ses} on E and $E_e = \pi_{\text{ses}}(E)$.

In general, we have the set W_{w} of weighted epistemic structures and the vector bundle $\boldsymbol{E} = (E, p_E, E_e)$ where $E \subseteq W_{\text{w}}$ and $E_e \subseteq W$.

Assuming that E_e and the fiber F of the vector bundle $\boldsymbol{E}$ are metric spaces with distances (metrics) d and d_{v}, correspondingly, we are able to define a distance $\mathbf{d}$ between elements $(e; w_1, \ldots, w_n)$ and $(l; u_1, \ldots, u_m)$ from the space E in the following way

$$\mathbf{d}((e; w_1, \cdots, w_{\mathrm{n}}), (l; u_1, \cdots, u_{\mathrm{m}})) = \mathrm{d}_{\mathrm{v}}((w_1, \cdots, w_{\mathrm{n}}), (u_1, \cdots, u_{\mathrm{m}})) + \mathrm{d}(e, l) \tag{1}$$

when $n = m$;

$$\mathbf{d}((e; w_1, \cdots, w_{\mathrm{n}}), (l; u_1, \cdots, u_{\mathrm{m}})) = \mathrm{d}_{\mathrm{v}}((w_1, \cdots, w_{\mathrm{n}}), (u_1, \cdots, u_{\mathrm{m}}, 0, \cdots, 0)) + \mathrm{d}(e, l) \tag{2}$$

when $n > m$; and

$$\mathbf{d}((e; w_1, \cdots, w_{\mathrm{n}}), (l; u_1, \cdots, u_{\mathrm{m}})) = \mathrm{d}_{\mathrm{v}}((w_1, \cdots, w_{\mathrm{n}}, 0, \cdots, 0), (u_1, \cdots, u_{\mathrm{m}})) + \mathrm{d}(e, l) \tag{3}$$

when $n < m$.

In finite-dimensional vector spaces, we can take the Euclidean metric as d_{v}, defining the distance $\mathrm{d}_{\mathrm{v}}((x_1, \ldots, x_n), (y_1, \ldots, y_n))$.

However, as we discussed before, it is natural to assume that the fiber F is an infinite-dimensional vector space. In this case, we simply postulate existence of a metric in it. Usually, metrics in vector spaces are defined by norms [33,34]. Note that in this case, we use only formula (1) because all fibers F_a have the same dimension.

Proposition 1. The distance $\mathrm{d}((e; w_1, \ldots, w_n), (l; u_1, \ldots, u_m))$ defines a metric in the space W_{w}.

Proof. By definition, $\mathbf{d}((e; w_1,\ldots, w_n), (e; w_1,\ldots, w_n)) = 0$. When $\mathbf{d}((e; w_1,\ldots, w_n), (l; u_1,\ldots, u_m)) = 0$, then $\mathrm{d}(e, l) = 0$ and, thus, $e = l$ because d is a metric in W. Besides, $\mathrm{d_v}((w_1, \ldots, w_n), (u_1, \ldots, u_m)) = 0$ and thus, $(w_1,\ldots, w_n) = (u_1,\ldots, u_m)$ because $\mathrm{d_v}$ is a metric in a vector space. Consequently, $\mathrm{d}((e; w_1,\ldots, w_n), (l; u_1,\ldots, u_m)) = 0$ if and only if $(e; w_1,\ldots, w_n) = (l; u_1,\ldots, u_m)$.

The function **d** is symmetric because the function d is symmetric in W, while the function $\mathrm{d_v}$ is symmetric in a vector space.

In addition, let us take arbitrary weighted (symbolic) epistemic structures $(e; w_1,\ldots, w_n)$, $(l; u_1,\ldots, u_m)$ and $(h; v_1,\ldots, v_p)$ from W and denote $\mathrm{d}(e, l) = a$, $\mathrm{d}(l, h) = b$, $\mathrm{d}(e, h) = c$, $\mathrm{d_v}((w_1,\ldots, w_n), (u_1,\ldots, u_m)) = d$, $\mathrm{d_v}((u_1,\ldots, u_m), (v_1,\ldots, v_p)) = k$ and $\mathrm{d_v}((w_1,\ldots, w_n), (v_1,\ldots, v_p)) = r$. Then we have:

$c \leq a + b$ because d is a metric in W

$r \leq d + k$ because $\mathrm{d_v}$ is a metric in a vector space.

Consequently,

$\mathbf{d}((e; w_1,\ldots, w_n), (h; v_1,\ldots, v_p)) \leq \mathbf{d}((e; w_1,\ldots, w_n), (l; u_1,\ldots, u_m)) + \mathbf{d}((l; u_1,\ldots, u_m), (h; v_1,\ldots, v_p))$

i.e., the third axiom of metric spaces is true. □

Corollary 1. $\mathbf{d}((e; w_1,\ldots, w_n),(h; v_1,\ldots, v_n)) \geq \mathrm{d}(e, h)$ and $\mathbf{d}((e; w_1,\ldots, w_n), (h; v_1,\ldots, v_n)) \geq \mathrm{d_v}((w_1,\ldots, w_n), (v_1,\ldots, v_n))$

Corollary 2. $\mathbf{d}((e; w_1,\ldots, w_n), (h; v_1,\ldots, v_n)) < k$, then $\mathrm{d}(e, h) < k$.

There are other ways to define metrics in the spaces W_{esw} and W_{sesw} based on metrics in the base and fiber of the corresponding vector bundle. For instance, it is possible to use the following formulas:

$$\mathbf{d}((e; w_1, \cdots, w_{\mathrm{n}}), (l; u_1, \cdots, u_{\mathrm{m}})) = \sqrt{\mathrm{d_v}((e; w_1, \cdots, w_{\mathrm{n}}), (l; u_1, \cdots, u_{\mathrm{m}}))^2 + \mathrm{d}(e, l)^2} \quad (4)$$

when $n = m$;

$$\mathbf{d}((e; w_1, \cdots, w_{\mathrm{n}}), (l; u_1, \cdots, u_{\mathrm{m}})) = \sqrt{\mathrm{d_v}((w_1, \cdots, w_{\mathrm{n}}), (u_1, \cdots, u_{\mathrm{m}}, 0, \cdots, 0))^2 + \mathrm{d}(e, l)^2} \quad (5)$$

when $n > m$; and

$$\mathbf{d}((e; w_1, \cdots, w_{\mathrm{n}}), (l; u_1, \cdots, u_{\mathrm{m}})) = \sqrt{\mathrm{d_v}((w_1, \cdots, w_{\mathrm{n}}, 0, \cdots, 0), (u_1, \cdots, u_{\mathrm{m}}))^2 + \mathrm{d}(e, l)^2} \quad (6)$$

when $n < m$.

Structures in the spaces W_{es}, W_{ses}, W_{esw} and W_{sesw} are inherited by epistemic spaces and their states. In particular, a weighted epistemic space E and each its state is a vector bundle $\boldsymbol{E} = (E, p_E, E_e)$ with the metric **d**.

We remind that a set X in a metric space E with a metric d is called *bounded* if there is a number k such that for any points a and b from X, $\mathrm{d}(a, b) < k$.

To study bounded sets in metric spaces that are spaces of vector bundles, we need additional concepts.

Example 11. Osgood, Suci and Tannenbaum [30] define *distance* in semantic spaces (cf. Example 8) by the formula from the m-dimensional Euclidean spaces:

$$\mathbf{d}(e, l) = \sqrt{\sum_{j=1}^{m} \mathrm{d}_{elj}^2}$$

In this formula, m is the number of factors and d_{elj} is the difference between the coordinates of the elements e and l with respect to the same factor (dimension) j. In the most refined models, the number m is equal to 3 [30].

Let us consider a vector bundle $\boldsymbol{E} = (E, p_E, E_e)$ with the fiber F.

Definition 12. A set $X \subseteq E$ is called *rectangular* in E if $X = \{(b, u) \mid b \in X_e, u \in F$ and for any $a \in X_e$ and $v \in F$ $((a, v) \in X \Rightarrow (b, v) \in X)\}$.

Example 12. Let us consider a trivial vector bundle $\boldsymbol{H} = (H = \{a, b, c\} \times \boldsymbol{R}, p_h, H_e = \{a, b, c\})$. Then the set $X = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$ is rectangular in H, while the set $Z = \{(a, 1), (a, 3), (b, 1), (b, 5), (c, 1), (c, 3)\}$ is not rectangular in H.

Definition 13. If $X \subseteq E$, then the minimal rectangular in E set $\mathrm{R}(X)$ that contains X is called the *rectangular closure* of X in E.

Lemma 2. The rectangular closure of a set in E always exists and is unique.

Lemma 3. The operation of taking the rectangular closure of a set in E is a closure operation in the sense of ( ) on sets in metric spaces.

In particular, the operation of taking the rectangular closure of a set in E is idempotent, *i.e.*, $\mathrm{R}(\mathrm{R}(X)) = \mathrm{R}(X)$ for any $X \subseteq E$.

Lemma 4. A set X in E is rectangular if and only if $X = \mathrm{R}(X)$.

Definition 13. If $X \subseteq E$, then the *fiber projection* $\sigma(X)$ of X is defined as follows

$$\sigma(X) = \{ u \mid \exists b \in X_e ((b, u) \in X \}$$

For instance, taking sets X and Z from Example 8, we see that:

$$\sigma(X) = \{1, 2\}$$

and

$$\sigma(Z) = \{1, 3, 5\}$$

Definitions imply the following result.

Lemma 5. A set X in E is rectangular if and only if $X = X_e \times \sigma(X)$.

Some properties of sets and their rectangular closures are the same.

Proposition 2. A subset X of the space E of the vector bundle $\boldsymbol{E} = (E, p_E, E_e)$ is bounded if and only if its rectangular closure is bounded.

Proof. *Sufficiency*. By definition, any subset of a bounded set is bounded.

Necessity. Let us assume that X is bounded. It means that there is a positive number k such that $\mathbf{d}(x, z) < k$ for any two points $x = (a, u)$ and $z = (b, v)$ from X where $a, b \in X_e$.

Let us take two points p and q from R(X). Then $p = (c, w)$ and $q = (d, y)$ with $c, d \in X_{e.}$ By the definition of the rectangular closure R(X), there are points x and z from X such $x = (a, w)$ and $z = (b, y)$ with $a, b \in X_{e.}$ By the properties of metric,

$$\mathbf{d}(p, q) \leq \mathbf{d}(p, x) + \mathbf{d}(x, z) + \mathbf{d}(z, q)$$

By initial conditions, $\mathbf{d}(x, z) < k$. At the same time, by the definition of the metric $\mathbf{d}$ and Corollary 2, we have:

$$\mathbf{d}(p, x) = \mathbf{d}((c, w), (a, w)) = \mathbf{d}(c, a) < k$$

and

$$\mathbf{d}(z, q) = \mathbf{d}((b, y), (d, y)) = \mathbf{d}(b, d) < k$$

Consequently,

$$\mathbf{d}(p, q) < 3k$$

Proposition is proved because p and q are arbitrary points from R(X). □

Reducing the problem of boundedness to rectangular sets, now we find conditions of boundedness for rectangular sets.

Proposition 3. A rectangular subset X of the space E of the vector bundle $\boldsymbol{E} = (E, p_E, E_e)$ is bounded if and only if the projection $X_e = p_E(X)$ of X and the fiber projection $\sigma(X)$ of X are uniformly bounded.

Proof. *Necessity*. Let us assume that the projection $X_e = p_E(X)$ of X is unbounded. It means that for any positive number k, there are two points a and b in X_e such that $\mathbf{d}(a, b) > k$. As X_e is the projection of X, there are two points x and z in X such that $a = p_E(x)$ and $b = p_E(z)$. By the definition of the metric in the space E, $\mathbf{d}(x, z) \geq \mathbf{d}(a, b) > k$. Consequently, X is also unbounded.

Now, let us suppose that the fiber projection $\sigma(X)$ of X is not uniformly bounded. It means that for any positive number k, there are two points u and v in $\sigma(X)$ such that $d_V(u, v) > k$. As $\sigma(X)$ is a projection of X, there are points $x = (a, u)$ and $z = (b, v)$ from the space X. By Corollary 2, $\mathbf{d}(x, z) \geq k$ as by choice of the points u and v, $d_V(u, v) > k$. Thus, the space X is not bounded.

Then by the Law of Contraposition, if the space X is bounded, then the projection $X_e = p_E(X)$ of X and the fiber projection $\sigma(X)$ of X are bounded.

Sufficiency. Let us suppose that the projection $X_e = p_E(X)$ of X and the fiber projection $\sigma(X)$ of X are bounded. It means that there is a positive number k, such that for any two points a and b in X_e, we have $d(a, b) < k$ and there is a positive number h, such that for any two points u and v from $\sigma(X)$, we have $\mathbf{d}(x, z) < h$

Let us take two points x and z from X. Then $x = (a, u)$ and $z = (b, v)$ where $a, b \in X_e$, while u and v belong to the fiber F of the bundle $\mathbf{E}$. As X is a rectangular set, points (a, u) and (b, v) belong to X and by definition,

$$\mathbf{d}(x, y) = \mathbf{d}(a, b) + \mathbf{d}(u, v) < k + h$$

Consequently, the set X is bounded. □

Propositions 2 and 3 imply the following result.

Corollary 3. A subset X of the space E of the vector bundle $\boldsymbol{E} = (E, p_E, E_e)$ is bounded if and only if the projection $X_e = p_E(X)$ of X and the fiber projection $\sigma(X)$ of X are uniformly bounded.

Note that here we study weighted epistemic structures with real number weights and weighted epistemic space in which weights form real vector spaces. However, using the same technique, it is possible to obtain similar results for weighted epistemic structures with complex number weights or vector weights and for weighted epistemic space in which weights form complex vector spaces.

5. Information Operators and Dynamics in Epistemic Spaces

Epistemic information operators act in (weighted) epistemic spaces, transforming these spaces and describing dynamics of infological systems and epistemic spaces, which model infological systems.

Let us consider two (weighted) epistemic spaces E and H.

Definition 14. (a) A (partial) mapping A: $E \rightarrow H$ is called an *epistemic information operator*.
(b) If both epistemic spaces E and H have the same structure and an information operator A: $E \rightarrow H$ preserves this structure, then A is called a *structured information operator* or an *information homomorphism*.
(c) When $E = H$, the operator A is called an *inner epistemic information operator*.

An inner epistemic information operator changes elements or states or elements of (weighted) epistemic spaces and multispaces.

There are three basic types of inner epistemic information operators: content, bond and weight operators.

A content epistemic information operator acts on symbolic epistemic items.

For instance, all information operators studied in [1,3,4] are content epistemic information operators.

There are three key content inner epistemic information operators:

An *addition*/*deletion content operator* AD (DL) adds a symbolic epistemic item (knowledge item) to the state of the (weighted) epistemic space.

A *transformation*/*substitution content operator* TR (ST) transforms or substitutes a symbolic epistemic item in the state of the (weighted) epistemic space.

A *substantiation content operator* switches on or off an existing symbolic epistemic item in the state of the (weighted) epistemic space.

Example 13. When an intelligent agent learns, it usually adds knowledge items to its knowledge base changing in such a way the state of this base. The knowledge base is naturally represented by an appropriate epistemic space. Thus, growth of knowledge in the process of learning is modeled by application of adding content information operators.

Definition 15. (a) A *replica* of an epistemic (knowledge) item is another knowledge item equivalent to the initial one.
(b) A *replication epistemic information operator REPL* makes a replica of an epistemic (knowledge) item and adds it to the current epistemic (knowledge) state.

Note that copies of an epistemic (knowledge) item always are its replicas but a replica of an epistemic (knowledge) item is not always its copy.

Example 14. Let us consider logical knowledge representation in which knowledge items are propositions. Then according to laws of logic there are equivalent propositions. For instance, taking the proposition (1) "*B* implies *A*", we have equivalent propositions (2) "*A* follows from *B*", (3) "If *B*, then *A*", and (4) "*A* is a consequence of *B*". All of them are replicas of one another although they are not copies.

Addition of symbolic epistemic items can be performed by five operations:

- by generation of a new item inside the current state of the (weighted) epistemic space;
- by generation of a new item outside (the current state of) the epistemic space and its transition into the current state of the (weighted) epistemic space;
- by transition of an existing item from the epistemic space into the current state of the (weighted) epistemic space;
- by replication of an item from the current state of the (weighted) epistemic space;
- by replication of an item outside (the current state of) the epistemic space and transition of this replica into the current state of the (weighted) epistemic space.

Consequently, substitution of symbolic epistemic items can be performed by five operations because substitution is the sequential composition of elimination and addition.

In the case of stratified epistemic spaces, there is one more type of key content epistemic operators, namely, a moving operators MV, which moves epistemic items from one strata to another [3].

A *transformation epistemic information operator TR* takes a group of epistemic (knowledge) items (may be, one item) from the current epistemic (knowledge) state and transforms it into another group of epistemic (knowledge) items (may be, into one item).

A *generation epistemic information operator TR* takes a group of epistemic (knowledge) items (may be, one item) from the current epistemic (knowledge) state and generates another group of epistemic (knowledge) items (may be, one item).

The difference between transformation and generation is that in generation, the initial group of epistemic items is preserved, while in transformation, it is not preserved.

A *bond epistemic information operator* acts on connections (bonds or relations) between symbolic epistemic items.

Such operators as interpretation and reinterpretation of information/knowledge items [2] are bond epistemic information operators.

There are three key bond epistemic information operators:

(1) An *addition/deletion bond operator* adds or deletes a connection (bond or relation) between symbolic epistemic items in the current epistemic (knowledge) state of the (weighted) epistemic space.

(2) A *substitution bond operator* changes a connection (bond or relation) between symbolic epistemic items in the current epistemic (knowledge) state of the (weighted) epistemic space to another connection (bond or relation).

(3) A *substantiation bond operator* switches on or off an existing connection (bond or relation) between symbolic epistemic items in the current epistemic (knowledge) state of the (weighted)

epistemic space.

A *weight epistemic information operator* acts on weights of symbolic epistemic items.

There are three basic weight epistemic information operators:

(1) An *addition/deletion weight operator* adds or deletes a weight to symbolic epistemic items in the weighted epistemic space.
For instance, epistemic items had one weight *constructible*, which indicates whether the structure is constructible or not. Then the weight *complexity* was added by the adding operator. The new weight reflects complexity of the structure construction.

(2) A *transformation/substitution weight operator* substitutes one weight of symbolic epistemic items in the weighted epistemic space by another weight.
For instance, epistemic items had the weight *justification*, which is then substituted by the weight *provability*. In a different situation, a transformation operator changes the weight *complexity* for the weight *hardship*. Such situation happens in software engineering [35].

(3) A *value changing weight operator* changes weights of symbolic epistemic items in the weighted epistemic space.

For instance, let us assume the value of the weight *complexity* for texts was estimated based on recursive algorithms, such as Turing machines. Later the complexity estimate was obtained by means of super- recursive algorithms, such as inductive Turing machines. As it is proved that super-recursive algorithms decrease complexity [27], the value-changing operator has to be applied to give correct complexity of the texts.

There also mixed epistemic information operators. A *mixed epistemic information operator* acts on symbolic epistemic items in an epistemic state, their weights and their connections (bonds or relations).

For instance, a mixed epistemic information operator can act on knowledge items in a knowledge state, their weights and their connections (bonds or relations).

Operators of logical inference, such as rules of deduction, are mixed epistemic information operators act because they add new knowledge items in the form of propositions or/and predicates and establish relations of provability/deducibility between propositions or/and predicates.

Subspaces of knowledge spaces represent subsystems of knowledge systems. For instance, in large knowledge systems, such as a scientific theory, it is possible to separate the subsystem of denotational knowledge and the subsystem of operational knowledge.

It looks like it might be sufficient to consider only finite or at least, locally finite agents. However, if knowledge is represented by logical statements and it is assumed (as it is done, for example, in the theory of semantic information developed by Bar-Hillel and Carnap [12]) that any knowledge system contains all logical consequences of all its elements, then an agent with such knowledge system is infinite. In information algebras, portions of information are represented by close subsets of sentences from a logical language *L* [36].

However, in conventional logics closed with respect to such information operators as deduction, sets are infinite because any sentence p implies $p \vee q$ for any sentence q from L, which is, as a rule, infinite (cf., for example, [37]). Thus, in the context of classical logic and information algebras any portion of information has infinitely many representations. Consequently, such a portion generates a system with the infinite number of knowledge items.

Let us consider epistemic information operators from a weighted epistemic space E into a weighted epistemic space H.

Definition 16. An epistemic information operator A: $E \to H$ is called:
(a) *stationary* if for any epistemic structures e and l, the equality $A(e; w_1, \ldots, w_k) = (l; v_1, \ldots, v_h)$ implies the equality $A(e; u_1, \ldots, u_k) = (l; q_1, \ldots, q_h)$ for any $(e; u_1, \ldots, u_k)$.
(b) *permanent* if for any weighted epistemic structure $(e; w_1,\ldots, w_k)$, we have $A(e; w_1,\ldots, w_k) = (e; v_1,\ldots, v_k)$.
(c) *semipermanent* if for any epistemic structure e and any number k, there is a number h such that for any system of weights $(w_1, \ldots, w_k)$ of e, we have $A(e; w_1,\ldots, w_k) = (e; v_1,\ldots, v_h)$.

Definitions imply the following result.

Lemma 6. Any permanent epistemic information operator A is semipermanent, while any semipermanent epistemic information operator B is stationary.

Lemma 7. (a) Any weight epistemic information operator is semipermanent.
(b) Operators of adding weights and of deleting weights are semipermanent but not permanent.
(c) Operators of substituting weights and of changing values of weights are permanent.

Stationary epistemic information operator are related to morphisms of epistemic vector bundles.

We remind [32] that a morphism of a vector bundle $\mathbf{E} = (E, p, B)$ into a vector bundle $\mathbf{H} = (H, r, D)$ is a pair of continuous mappings f: $E \to H$ and g: $B \to D$ such that the following Diagram (3) is commutative.

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"] & H \arrow[d, "r"'] \\ B \arrow[r, "g"'] & D \end{tikzcd}$$

(3)

Note that morphisms of a vector bundles are mappings (morphisms) of named sets, which are studied in the theory of named sets [2]. Thus, weighted epistemic spaces form a category with vector bundle morphisms as its morphisms. It makes possible application of results for categorical information modeling [38,39] to epistemic information operators in weighted epistemic spaces.

Let us consider two weighted epistemic spaces E and H.

Proposition 4. An epistemic information operator A: $E \to H$ is stationary if and only if it induces a morphism of the vector bundle $\boldsymbol{E} = (E, p_E, E_e)$ into the vector bundle $\boldsymbol{H} = (H, p_H, H_e)$.

Proof. *Necessity*. Let us consider a stationary epistemic information operator A: $E \to H$. Then by Definition 16, each fiber F_a of the vector bundle $\mathbf{E} = (E, p_E, E_e)$ is mapped by A into a single fiber G_b of the vector bundle $\boldsymbol{H} = (H, p_H, H_e)$. That is why, we can build the mapping C: $E_e \to H_e$ defining $C(a) = b$. By construction, we have $C(p_E(x)) = p_H(A(x))$ for all elements x from E. This gives us the commutative Diagram (4).

$$\begin{tikzcd} E \arrow[r, "A"] \arrow[d, "p_E"] & H \arrow[d, "p_H"'] \\ E_E \arrow[r, "C"'] & H_E \end{tikzcd} \quad (4)$$

It means that the pair (A, C) is a morphism of the vector bundle $\boldsymbol{E} = (E, p_E, E_e)$ into the vector bundle $\boldsymbol{H} = (H, p_H, H_e)$.

Sufficiency. If the pair (A, C) is a morphism of the vector bundle $\boldsymbol{E} = (E, p_E, E_e)$ into the vector bundle $\boldsymbol{H} = (H, p_H, H_e)$, *i.e.*, the Diagram (4) is commutative. Consequently, each fiber F_a of the vector bundle $\mathbf{E} = (E, p_E, E_e)$ is mapped by A into a single fiber G_b of the vector bundle $\boldsymbol{H} = (H, p_H, H_e)$, *i.e.*, A is a stationary epistemic information operator. □

Definition 17. A stationary epistemic information operator A: $E \rightarrow H$ is called:

(a) *uniform* if for any real number a and any weighted epistemic structures $(e; w_1, \ldots, w_k)$ and $(l; v_1, \ldots, v_h)$, the equality $A(e; w_1, \ldots, w_k) = (l; v_1, \ldots, v_h)$ implies the equality $A(e; au_1, \ldots, au_k) = (l; av_1, \ldots, av_h)$.

(b) *additive* if $A(e; w_1 + u_1, \ldots, w_k + u_k) = A(e; w_1, \ldots, w_k) + A(e; u_1, \ldots, u_k)$.

(c) *linear* if it is uniform and additive.

Example 15. Let us consider a weighted epistemic space E, in which there are n epistemic structures $e_1, e_2, e_3, \ldots, e_n$, the distance between any two of them is 1 and each of them has one weight w the range of which is the real line $\boldsymbol{R}$. Taking a real number t, we define the epistemic information operator A by the following rule:

$$A(e_k, w) = (e_k, tw)$$

By definition, this operator is linear and thus, uniform and additive.

At the same time, the epistemic information operator B with $B(e_k, w) = (e_k, k)$ is neither uniform nor additive nor linear.

To study linear epistemic information operators, we need some topological constructions.

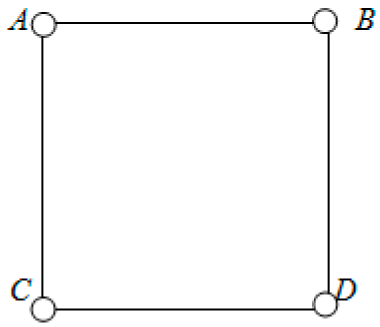
In [40], the concept of path Q-connectedness is introduced and studied. To find relations between linearity and boundedness of information operators, we need to further develop this concept for metric spaces. Path Q-connectedness is important for epistemic spaces because in many cases, epistemic spaces have the structure of a graph or network of epistemic items connected by various relations. For instance, Motter, *et al*. [25] build in such a way the conceptual network of a language.

Let C be a subspace of a metric space U with the distance function $\mathbf{d}$.

Definition 18. The space C is called *path* (q, r)*-connected* in U if for any two points a and b in C, there exists a sequence $a_1, a_2, a_3, \ldots, a_n$ of points in C such that $\mathbf{d}(a, a_1) \leq r$, $\mathbf{d}(a_n, b) \leq r$, $\mathbf{d}(a_i, a_{i+1}) \leq r$ for all $i = 1, 2, 3, \ldots, n - 1$ and $[\mathbf{d}(a, a_1) + \mathbf{d}(a_1, a_2) + \mathbf{d}(a_2, a_3) + \ldots + \mathbf{d}(a_{n-1}, a_n) + \mathbf{d}(a_n, b)] < q \cdot \mathbf{d}(a, b)$.

Example 16. Taking a square in which the length of the side is equal and which is situated in an Euclidean plane (cf. Figure 3) and the space of its vertices $C = \{A, B, C, D\}$, we see that it is path (2, 1)-connected but it is not path (1, 1)-connected and not path (2, ½)-connected. Indeed, $\mathbf{d}(A, C) = \surd 2$, the lengths of both paths (A, B, C) and (A, D, C) is equal to 2 with $\mathbf{d}(A, B) = \mathbf{d}(B, C) = \mathbf{d}(A, D) = \mathbf{d}(D, C) = 1$. Consequently, $\mathbf{d}(A, B) + \mathbf{d}(B, C) < 2\mathbf{d}(A, C)$ but $\mathbf{d}(A, B) + \mathbf{d}(B, C) > 1 \cdot \mathbf{d}(A, C)$. Thus, the space C is path (2, 1)-connected but it is not path (1, 1)-connected. In addition, it is not path (q, p)-connected when $p < 1$ because there no paths between A and C such that the distance between two consecutive points is less than 1.

Figure 5. A topological space that is path (2, ½)-connected but it is not path (1, 1)-connected.



However, not all sets in metric spaces are path (q, r)-connected.

Example 17. The parabola $y = x^2$ as the space C, we see that it is not path (q, r)-connected for any numbers q and r. Indeed, taking points $u = (x, x^2)$ and $w = (-x, x^2)$, we see that the distance between these points $\mathrm{d}(u, w) = 2x$. Now let us suppose that this parabola C is path (q, r)-connected for some numbers q and r. It means that there is a sequence $a_1, a_2, a_3, \ldots, a_n$ of points in C such that $\mathrm{d}(u, a_1) \leq r$, $\mathbf{d}(a_n, w) \leq r$, $\mathbf{d}(a_i, a_{i+1}) \leq r$ for all $i = 1, 2, 3, \ldots, n - 1$. As all points $a_1, a_2, a_3, \ldots, a_n$ belong to C, the sum $[\mathbf{d}(u, a_1) + \mathbf{d}(a_1, a_2) + \mathbf{d}(a_2, a_3) + \ldots + \mathbf{d}(a_{n-1}, a_n) + \mathbf{d}(a_n, w)]$ is larger than $x^2 - (½\, r)^2$. At the same time, we have $[\mathbf{d}(u, a_1) + \mathbf{d}(a_1, a_2) + \mathbf{d}(a_2, a_3) + \ldots + \mathbf{d}(a_{n-1}, a_n) + \mathbf{d}(a_n, w)] < q \cdot \mathbf{d}(u, w)$. Thus, $x^2 - (½\, r)^2 < 2qx$ because $\mathrm{d}(u, w) = 2x$. Transforming this inequality, we obtain

$$x^2 - 2qx < (½\, r)^2$$

$$x(x^2 - 2q) < (½\, r)^2$$

For sufficiently big x, this inequality cannot be valid because r is a fixed number. Thus, our assumption is not true and the parabola $y = x^2$ is not path (q, r)-connected for any numbers q and r.

Here we are mostly interested in content epistemic information operators, which we simply call epistemic information operators. Using classical concepts of continuity and boundedness [26,41], as well as the concept of (p, q)-continuity from neoclassical analysis [42], we explicate important classes of epistemic information operators.

Let us consider epistemic information operators from a weighted epistemic space E with a metric $\mathbf{d}$ into a weighted epistemic space H with a metric $\mathbf{d}$. In this case, it is possible to define the diameter $\mathbf{d}$ of sets in E and in H. Namely, if $X \subseteq E$, then $\mathbf{d}(X) = \sup \{\mathbf{d}(x, z); x, z \in X\}$ when this supremum exists and undefined otherwise.

Definition 19. An epistemic information operator A: $E \to H$ is called:
(a) *bounded* if given a state X of E, for any number r, there is a number t such that the condition $\mathbf{d}(X) \leq r$, implies the condition $\mathbf{d}(A(X)) \leq t$.
(b) *uniformly bounded* if for any number r, there is a number t such that for any state X of E, the condition $\mathrm{d}(X) \leq r$, implies the condition $\mathbf{d}(A(X)) \leq t$.
(c) *continuous* if A is a continuous mapping.
(d) (p, q)-*continuous* if A is a (p, q)-continuous mapping.

Boundedness of an epistemic information operator A means that when the distances between epistemic staructures in a set X are bounded, then the distances between epistemic staructures in the image $A(X)$ of the set X are bounded.

(p, q)-continuity of an epistemic information operator A informally means that when the distances between epistemic staructures in a set X are not larger than p, then the distances between epistemic staructures in the image $A(X)$ of the set X are not larger than q.

Lemma 8. Any uniformly bounded epistemic information operator A: $E \to H$ is bounded.

In a discrete metric space, any point is an open and a closed set [26]. Thus, any epistemic information operator in an epistemic space E is continuous, open and closed.

Results from [42] give us the following property of epistemic information operators.

Lemma 9. An epistemic information operator A: $E \to H$ continuous if and only if it is (0, 0)-continuous.

In addition, we need some constructions from neoclassical analysis, such as fuzzy limits and fuzzy continuity [42].

Let $r \in \boldsymbol{R}^+$.

Definition 20. (a) An element a from E is called an *r-limit of a sequence l* (it is denoted by $a = r\text{-}\lim_{i\to\infty} a_i$ or $a = r\text{-}\lim l$) if for any $\varepsilon \in \boldsymbol{R}^{++}$ the inequality $\mathbf{d}(a, a_i) < r + \varepsilon$ is valid for almost all a_i, *i.e.*, there is such n that for any $i > n$, we have $\mathbf{d}(a, a_i) < r + \varepsilon$.
(b) A sequence l that has an r-limit is called *r-convergent* and it is said that l *r-converges* to its r-limit a.

Informally, a is an r-limit of a sequence l if for an arbitrarily small ε, the distance between a and all but a finite number of elements from l is smaller than $r + \varepsilon$. In other words, an element a is an r-limit of a sequence l if for any $\varepsilon \in \boldsymbol{R}^{++}$ almost all a_i belong to the interval $(a - r - \varepsilon, a + r + \varepsilon)$.

It is a natural generalization of the classical concept of a limit as the following result demonstrates.

Lemma 10 [42]. A point a a limit of a sequence l if and only if it is a 0-limit of the sequence l.

We also need fuzzy continuity.

Definition 22. (a) A partial function f: $\boldsymbol{R} \to \boldsymbol{R}$ is called (q, r)-*continuous at a point* $a \in \boldsymbol{R}$ if for any sequence $l = \{ a_i \in \boldsymbol{R};\ i = 1, 2, 3, \ldots\}$, for which a is an q-limit, the point $f(a)$ is an r-limit of the sequence $\{ f(a_i) \in \boldsymbol{R};\ i = 1, 2, 3, \ldots\}$.
(b) A function f: $\boldsymbol{R} \to \boldsymbol{R}$ is called (q, r)-*continuous in* (*inside*) set $X \subseteq \boldsymbol{R}$ if $f(x)$ (the restriction of $f(x)$ on X) is (q, r)-continuous at each point a from $X \cap \mathrm{Dom} f$.

Fuzzy continuity is a natural generalization of the classical concept of continuity as the following result demonstrates.

Lemma 11 [42]**.** A function $f(x)$ is continuous at a point $a \in \boldsymbol{R}$ if and only if it is (0,0)-continuous at the point a.

These results show that the concept of (q, r)-continuity is a natural extension of the concept of conventional continuity.

Lemma 12 [42]**.** If $t > r$, and $p < q$, then any (q, r)-continuous at a function $f(x)$ is also (p, t)-continuous at a.

Note that if $q < p$, then it is possible that a (q, r)-continuous at a function is not (p, r)-continuous at a. For instance, the function $f(x) = x$ is (0, 0)-continuous at the point 0, but for any $p > 0$, it is not $(p, 0)$-continuous at 0.

Let us consider two (weighted) epistemic spaces E and H, assuming that the space E is a path $(q, 1)$-connected metric space with the metric $\mathbf{d}$ and the space H is a metric space with the metric $\mathbf{d}$. Denoting metrics in different spaces by the same letter $\mathbf{d}$ follows the mathematical tradition and does not cause confusion.

Theorem 1. A epistemic information operator A: $E \to H$ is $(1, k)$-continuous for some positive number k if and only if it is uniformly bounded.

Proof. *Necessity*. Let us consider a $(1, k)$-continuous epistemic information operator A: $E \to H$ and two points a and b from a set X in E. By Definition 17, there exists a path (sequence of points) $l = \{a_1, a_2, a_3, \ldots, a_n\}$ in E such that $\mathbf{d}(a, a_1) \leq 1$, $\mathbf{d}(a_n, b) \leq 1$, $\mathrm{d}(a_i, a_{i+1}) \leq 1$ for all $i = 1, 2, 3, \ldots, n-1$ and $[\mathbf{d}(a, a_1) + \mathbf{d}(a_1, a_2) + \mathbf{d}(a_2, a_3) + \ldots + \mathbf{d}(a_{n-1}, a_n) + \mathrm{d}(a_n, b)] < q \cdot \mathbf{d}(a, b)$. If for some i, $\mathbf{d}(a_i, a_{i+1}) < ½$ and $\mathbf{d}(a_{i+1}, a_{i+2}) < ½$, then it is possible to eliminate the point a_{i+1} from the path because by properties of metric, $\mathbf{d}(a_i, a_{i+2}) \leq \mathbf{d}(a_{i+1}, a_{i+2}) + \mathbf{d}(a_i, a_{i+1}) < ½ + ½ = 1$ and the new path is not longer than the previous one. We can reduce the initial path in such a way and assume that we have an irreducible path $l = \{a_1, a_2, a_3, \ldots, a_n\}$ between a and b.

As A is a $(1, k)$-continuous epistemic information operator, $\mathbf{d}(A(a), A(a_1)) < k$, $\mathbf{d}(A(a_n), A(b)) < k$, $\mathbf{d}(A(a_i), A(a_{i+1})) < k$ for all $i = 1, 2, 3, \ldots, n-1$. Thus,

$$\mathbf{d}(A(a), A(b)) \leq \mathbf{d}(A(a), A(a_1)) + \mathbf{d}(A(a_1), A(a_2)) + \ldots + \mathbf{d}(A(a_{i-1}), A(a_n)) + \mathbf{d}(A(a_n), A(b)) < k(n + 1)$$

Now let us estimate the number n. Taking an irreducible path $l = \{a_1, a_2, a_3, \ldots, a_n\}$, we know that between any two pairs (a_i, a_{i+1}) and (a_{i+2}, a_{i+3}) with the distance less than ½, there is at least, one pair (a_{i+1}, a_{i+2}) with the distance larger than ½. Thus, the number of pairs (a_i, a_{i+1}) the distance between which larger than ½ is more than $(¼)n$. Consequently, the length of the path $l = \{a_1, a_2, a_3, \ldots, a_n\}$ is larger than $½ \cdot (¼)n = (⅛)n$, *i.e.*, $(⅛)n < [\mathbf{d}(a, a_1) + \mathbf{d}(a_1, a_2) + \mathbf{d}(a_2, a_3) + \ldots + \mathbf{d}(a_{n-1}, a_n) + \mathbf{d}(a_n, b)] < q \cdot \mathbf{d}(a, b) = q \cdot d$ where the distance $\mathbf{d}(a, b)$ is equal to d. Thus, $n < 8q \cdot d$.

Let us assume that X is a bounded set. It means that there is a positive number h such that for two points a and b from X, the distance $\mathbf{d}(a, b)$ is less than h. It is possible to assume that $h > 1$. Then $n < 8q \cdot h$ and $\mathbf{d}(A(a), A(b)) < k(n + 1) < k(8q \cdot h + 1) = t < (8qk + k) \cdot h$. It means that the operator A is uniformly bounded because numbers k and q are constants and t depends only on one variable h.

Necessity is proved.

Sufficiency. Let us consider a uniformly bounded epistemic information operator A: $E \to H$. Then (cf. Definition 18) for any number r, there is a number t such that for any state X of E, the condition $\mathbf{d}(X) \leq r$, implies the condition $\mathbf{d}(A(X)) \leq t$. In particular, for the number 1, there is a number k such that for any points a and b from E, the condition $\mathbf{d}(a, b) \leq 1$, implies the condition $\mathbf{d}(A(a), A(b)) \leq k$. It means that the operator A is $(1, k)$-continuous. □

Remark 1. The proof of Theorem 1 is sufficiently general. So, this result remains true for general metric spaces that satisfy the necessary conditions.

One of the basic results of functional analysis is the theorem stating that a linear operator in Banach space is continuous if and only if it is uniformly bounded [34,43]. It is demonstrated that (1, 0)-continuity is stronger than continuity in metric spaces [42]. Thus, it is possible to ask a question whether it would be possible to change the condition of $(1, k)$-continuity to the condition of continuity in Theorem 1. The following example shows that it is impossible as there are linear epistemic information operators that are continuous but not uniformly bounded.

Example 18. Let us consider a weighted epistemic space E, in which there is a countable number of epistemic structures $e_1, e_2, e_3, \ldots, e_n, \ldots$, the distance between any two of them is 1 and each of them has one weight w the range of which is the real line $\boldsymbol{R}$. We define the epistemic information operator A by the following rule:

$$A(e_n, w) = (e_n, nw)$$

By definitions, this operator is linear and continuous but it is not uniformly bounded. Indeed, $\mathbf{d}((e_1, 1), (e_n, 1)) \leq 2$, while $\mathbf{d}(A(e_1, 1)\, A(e_n, 1)) > n$ for any n.

Even more, there are linear epistemic information operators that are continuous but not bounded.

Example 19. Let us consider a weighted epistemic space E, in which there is a countable number of epistemic items $e_1, e_2, e_3, \ldots, e_n, \ldots$ such that $\mathbf{d}(e_n, e_{n+1}) = (1/2^n)$ and $\mathbf{d}(e_n, e_{n+k}) = \sum_{i=n}^{n+k-1} (1/2^i)$. Each of these epistemic items has one weight w the range of which is the real line $\boldsymbol{R}$. We define the epistemic information operator A by the following rule

$$A(e_n, w) = (e_n, nw)$$

By definitions, the set $U = \{(e_1, 1), (e_2, 1), (e_3, 1), \ldots, (e_n, 1), \ldots\}$ is bounded as $\mathbf{d}(U) = \sup \mathbf{d}(e_n, e_{n+k}) < 3$. The operator A is linear and continuous. However, it is not bounded because in the set $A(U)$, there are pairs of points with the arbitrary big distance between them, e.g., $\mathbf{d}(A(e_1, 1), A(e_n, 1)) > n$.

Although continuity is insufficient for boundedness of a linear epistemic information operator in a general case, there are situations when boundedness is still equivalent to continuity.

Let us consider two weighted epistemic spaces E and H such that both spaces E_e and H_e are metric spaces with the metric $\mathbf{d}$, the base epistemic space E_e is finite and all fibers F_a of the vector bundle $\mathbf{E} = (E, p, E_e)$ and fibers G_a of the vector bundle $\mathbf{H} = (H, p, H_e)$ are hyperseminormed vector spaces.

Theorem 2. A linear epistemic information operator A: $E \to H$ is continuous in each fiber F_a of the vector bundle $\mathbf{E} = (E, p, E_e)$ if and only if A is bounded.

Proof. *Necessity*. Let us consider an epistemic information operator A: $E \to H$ continuous in each fiber F_a of the vector bundle $\mathbf{E} = (E, p, E_e)$ and a bounded set X in E. Then each intersection $X_a = X \cap F_a$ is a bounded set because a subset of a bounded set is also bounded. As the operator A is continuous in the fiber F_a, the image $A(X_a)$ of X_a is bounded [33]. As the union of a finite number of bounded sets is bounded, $X = \bigcup_{a \in E_e} X_a$ is a bounded set.

Sufficiency. Let us consider a bounded epistemic information operator A: $E \to H$. Then it is bounded on any subset of E, in particular, on each fiber F_a of the vector bundle $\mathbf{E} = (E, p, E_e)$. As each fiber F_a of the vector bundle $\mathbf{E} = (E, p, E_e)$ is a hyperseminormed vector space, the results from [33] show that the epistemic information operator A is continuous in each fiber F_a of the vector bundle $\mathbf{E} = (E, p, E_e)$. □

In many applications, the base epistemic space E_e is finite. For instance, a popular type of the base epistemic space E_e is a semantic network (cf. Example 2) and all known semantic networks are finite. Thus, the conditions from Theorem 2 are almost always satisfied.

As normed vector spaces are an important special case of hyperseminormed vector spaces [33], the following result is implied by Theorem 2.

Corollary 4. If all fibers F_a and G_a are normed vector spaces, then a linear epistemic information operator A: $E \to H$ is continuous in each fiber F_a of the vector bundle $\mathbf{E} = (E, p, E_e)$ if and only if A is bounded.

For linear operators in vector spaces uniform boundedness coincides with boundedness [43,34]. This gives us the following result.

Corollary 5. If all fibers F_a and G_a are seminormed, e.g., normed, vector spaces, then a linear epistemic information operator A: $E \to H$ is continuous in each fiber F_a of the vector bundle $\mathbf{E} = (E, p, E_e)$ if and only if A is uniformly bounded.

When E_e consists of single element, Theorem 2 gives us the classical result of functional analysis.

Corollary 6 [34,43]. A linear operator in Banach space is continuous if and only if it is uniformly bounded.

6. Conclusions

Here, we studied weighted epistemic information operators in weighted epistemic spaces with weights forming real vector spaces. However, using the same technique, it is possible to obtain similar results for epistemic information operators in weighted epistemic space with weights forming complex vector spaces.

An interesting problem is to study properties of the sequential composition of epistemic information operators. Sequential composition allows one to build new operators from the given ones. Besides, the categorical approach to information studies is based on sequential composition [39]. It would be also appealing to introduce and study other operations in the space of epistemic information operators, for example, the ones that mathematically represent operations with information studied in [44].

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Conflicts of Interest

The author declares no conflict of interest.

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