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Structure Fault Tolerance of Bubble-Sort Star Graphs

Lantao You [1,2,3], Jianfeng Jiang [1] and Yuejuan Han [3,*]

[1] School of Information Engineering, Suzhou Industrial Park Institute of Services Outsourcing, Suzhou 215123, China
[2] Provincial Key Laboratory for Computer Information Processing Technology, Soochow University, Suzhou 215006, China
[3] School of Computer Science and Technology, Soochow University, Suzhou 215006, China
[*] Correspondence: hyj@suda.edu.cn

Abstract: As two significant performance indicators, structure connectivity and substructure connectivity have been widely studied, and they are used to judge a network's fault tolerance properties from the perspective of the structure becoming faulty. An n-dimensional bubble-sort star graph BS_n is a popular interconnection network with many good properties. We find the upper bounds of $\kappa(BS_n; K_{1,3})$ and $\kappa^s(BS_n; K_{1,3})$ in this paper. Furthermore, we establish $\kappa(BS_n; H)$ and $\kappa^s(BS_n; H)$ of BS_n, where $H \in \{K_1, K_{1,1}, K_{1,2}\}$.

Keywords: connectivity; bubble-sort star graph; fault tolerance; interconnection network; structure connectivity

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1. Introduction

With the development of parallel and distributed computer systems, the number of processors in an interconnection network is increasing at a great rate. The topology of a high-performance computer can be indicated by an undirected graph G, represented by $G(V(G), E(G))$, where we use $V(G)$ to represent the processor set and $E(G)$ to represent the link set.

As a significant performance indicator, connectivity is widely studied, and it is used to judge a network's fault tolerance properties [1]. In addition, some other connectivities with restrictions have been proposed, such as conditional connectivity [2], g-extra connectivity [3], h-restricted connectivity [4,5], and R_g-connectivity[6]. Most works have only focused on the impact on the network when individual nodes fail. In an actual network environment, the vertices connected to a fault vertex are more prone to fail, which means that some network structures or substructures may fail. Based on this thought, Lin et al. [7] considered the impact on the network from the perspective of structure failure and proposed two connectivities, which are called structure and substructure connectivity. These two connectivities can be used to evaluate a network's fault tolerance properties. A network has good structure fault tolerance properties if its (sub)structure connectivity is high.

We use $F = \{H_1, H_2, \ldots, H_t\}$ to express one of the subgraph sets in G. Here, each $H_i \in F (1 \leq i \leq t)$ denotes a connected subgraph of G. F is called a subgraph cut of graph G if removing $V(F)$ from G disconnects G or makes G trivial. If each $H_i \in F (1 \leq i \leq t)$ is isomorphic to H(or a connected subgraph of H), where H denotes a connected subgraph of G, we say that F is an H-structure-cut (or H-substructure-cut). The minimum cardinality of all H-structure-cuts (or H-substructure-cuts) of G is defined as the H-structure-connectivity (or H-substructure-connectivity) of G, which is denoted by $\kappa(G; H)$ (or $\kappa^s(G; H)$). With the definitions above, we have $\kappa(G; H) \geq \kappa^s(G; H)$.

Paths, cycle, and stars are three common structures that exist in all networks. Recently, most of the research on structure connectivity was based on these three structures. For example, star/cycle structure fault tolerance in a hypercube [7], k-ary n-cube [8], balanced hypercube [9], and twisted hypercube [10] was studied. Star/cycle/path structure fault tolerance in a folded hypercube [11] and alternating group graph [12,13] was investigated. Cycle/path structure fault tolerance in a bubble-sort star graph [14], bubble-sort graph [15], and wheel network [16] was studied.

The bubble-sort graph B_n and star graph S_n, which were introduced by Akers and Krishnamurthy [17], are two alternatives to the hypercube. These two graphs have many attractive features, except for the embeddability of S_n and the diameter of B_n. To improve the performance of these two graphs, Chou et al. [18] proposed the bubble-sort star graph BS_n, which is a combination of B_n and S_n. It was proven that BS_n had a better embeddability than that of S_n and a smaller diameter than that of B_n. Hence, BS_n has the advantages of both B_n and S_n.

In [14], Zhang et al. gave $\kappa(BS_n;H)$ and $\kappa^s(BS_n;H)$ for BS_n, where H is a path or a cycle. In this paper, we determine the star structure fault tolerance in BS_n. We present the upper bounds for $\kappa(BS_n;K_{1,3})$ and $\kappa^s(BS_n;K_{1,3})$. Furthermore, we establish $\kappa(BS_n;H)$ and $\kappa^s(BS_n;H)$ of BS_n, where $H \in \{K_1, K_{1,1}, K_{1,2}\}$. We will get the following results for BS_n with $n \geq 4$:

$$\kappa(BS_n;H) = \begin{cases} 2n-3 & \text{if } H = K_1 \\ 2n-3 & \text{if } H = K_{1,1} \\ n-1 & \text{if } H = K_{1,2}, \end{cases}$$

$$\kappa^s(BS_n;H) = \begin{cases} 2n-3 & \text{if } H = K_1 \\ 2n-3 & \text{if } H = K_{1,1} \\ n-1 & \text{if } H = K_{1,2} \end{cases}$$

and

$$\kappa^s(BS_n;K_{1,3}) \leq \kappa(BS_n;K_{1,3}) \leq \begin{cases} \frac{3(n-1)}{4} & \text{if } (n-1)\%4 = 0 \\ \frac{3(n-2)}{4}+1 & \text{if } (n-2)\%4 = 0 \\ \frac{3(n-3)}{4}+1 & \text{if } (n-3)\%4 = 0 \\ \frac{3(n-4)}{4}+2 & \text{if } (n-4)\%4 = 0. \end{cases}$$

The structure of this paper is organized as follows. First, the preliminaries will be given in Section 2. Then, we determine the results of the structure connectivity $\kappa(BS_n;H)$ and substructure connectivity $\kappa^s(BS_n;H)$ in Section 3. Finally, we give a summary of the paper in Section 4.

2. Preliminaries

Two vertices μ, ν are adjacent if $(\mu,\nu) \in E(G)$. Let $N_G(\mu)$ denote a vertex set in which each element is adjacent to μ. Suppose that S is a vertex set of G. We can define the neighborhood of S as $N_G(S) = (\bigcup_{x\in S} N(x)) - S$ (or $N(S)$ for short).

Let $[n] = \{1,2,\ldots,n\}$. BS_n has $n!$ vertices, each of which is labeled with a permutation $\mu = \mu_1\mu_2\ldots\mu_n$ on $[n]$, where $\mu_i \neq \mu_j$ and $i \neq j$. For example, 1234 and 1243 are vertex labels when $n = 4$. We use vertex labels to represent the nodes in this paper. Let μ be any vertex of BS_n. We define an operator on μ, which is denoted by μ_i^j, where $1 \leq i \neq j \leq n$, such that the ith bit and jth bit of μ are exchanged. If $\mu = 12345$, then $\mu_1^2 = 21345$ and $\mu_3^4 = 12435$. The neighbor of μ_i^j can be denoted by $(\mu_i^j)_k^l$, where $1 \leq k \neq l \leq n$. For example, $(\mu_1^2)_3^4 = 21435$ and $(\mu_1^2)_1^3 = 31245$. We give the following definition of BS_n.

Definition 1 (see [18])**.** *There exist $n!$ vertices in BS_n, each of which is labeled with a permutation on $[n]$. Any two vertices μ and ν of BS_n are adjacent if and only if $\nu = \mu_1^i$ for $2 \leq i \leq n$ or $\nu = \mu_i^{i+1}$ for $2 \leq i \leq n-1$.*

The graphs BS_3 and BS_4 are depicted in Figure 1. We can see from the definition that BS_n is a $(2n-3)$-regular bipartite graph. For any vertex μ in BS_n, $N(\mu) = \{\mu_1^j | 2 \leq j \leq$

$n\} \bigcup \{\mu_j^{j+1} | 2 \leq j \leq n-1\}$. In addition, BS_n is a Cayley graph with vertex symmetry. BS_n is composed of n subgraphs $BS_n^1, BS_n^2, \ldots, BS_n^n$, where each $BS_n^i (1 \leq i \leq n)$ is isomorphic to BS_{n-1}.

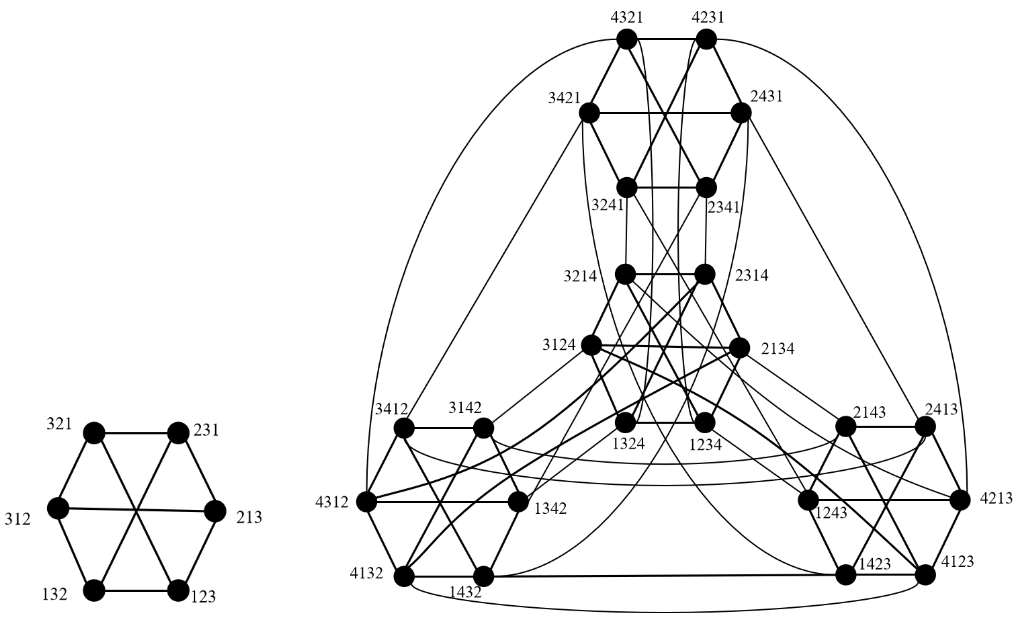

Figure 1. The graphs (**a**) BS_3 and (**b**) BS_4.

3. Structure Fault Tolerance of BS_n

We first prove $\kappa(BS_n; H)$ and $\kappa^s(BS_n; H)$ for any integer $n \geq 4$, where $H \in \{K_1, K_{1,1}, K_{1,2}\}$. Then, we study the upper bounds for $\kappa(BS_n; K_{1,3})$ and $\kappa^s(BS_n; K_{1,3})$.

Lemma 1 (see [19]). *For $n \geq 3$, $\kappa(BS_n) = \lambda(BS_n) = 2n - 3$.*

By Lemma 1, we can easily get the results as follows.

Theorem 1. $\kappa(BS_n; K_1) = 2n - 3$ and $\kappa^s(BS_n; K_1) = 2n - 3$ for any integer $n \geq 4$.

Lemma 2 (see [19]). *For $n \geq 2$, $\kappa\lambda(BS_n) = 2n - 3$.*

Lemma 3. *Let μ be any vertex in BS_n and let $\nu = \mu_2^3$ for $n \geq 4$. Then, $(\mu_1^i, \nu_1^i) \in E(BS_n)$ for $2 \leq i \leq n$ and $(\mu_i^{i+1}, \nu_i^{i+1}) \in E(BS_n)$ for $4 \leq i \leq n-1$.*

Proof. Let $\mu = \mu_1\mu_2\mu_3 \ldots \mu_i\mu_{i+1} \ldots \mu_n$. Then, $\nu = \mu_1\mu_3\mu_2 \ldots \mu_i\mu_{i+1} \ldots \mu_n$. We have $\mu_1^i = \mu_i\mu_2\mu_3 \ldots \mu_1\mu_{i+1} \ldots \mu_n$ and $\nu_1^i = \mu_i\mu_3\mu_2 \ldots \mu_1\mu_{i+1} \ldots \mu_n$ for $4 \leq i \leq n$. Since $(\mu_1^i)_2^3 = \nu_1^i$, then $(\mu_1^i, \nu_1^i) \in E(BS_n)$ for $4 \leq i \leq n$. Again, we have $\mu_1^2 = \mu_2\mu_1\mu_3 \ldots \mu_i\mu_{i+1} \ldots \mu_n$, $\mu_1^3 = \mu_3\mu_2\mu_1 \ldots \mu_i\mu_{i+1} \ldots \mu_n$, $\nu_1^2 = \mu_3\mu_1\mu_2 \ldots \mu_i\mu_{i+1} \ldots \mu_n$, and $\nu_1^3 = \mu_2\mu_3\mu_1 \ldots \mu_i\mu_{i+1} \ldots \mu_n$. Then, $(\mu_1^2)_1^3 = \nu_1^2$ and $(\mu_1^3)_1^2 = \nu_1^3$. Hence, $(\mu_1^i, \nu_1^i) \in E(BS_n)$ for $2 \leq i \leq n$. Similarly, $\mu_i^{i+1} = \mu_1\mu_2\mu_3 \ldots \mu_{i+1}\mu_i \ldots \mu_n$ and $\nu_i^{i+1} = \mu_1\mu_3\mu_2 \ldots \mu_{i+1}\mu_i \ldots \mu_n$ for $4 \leq i \leq n-1$. Since $(\mu_i^{i+1})_2^3 = \nu_i^{i+1}$, then $(\mu_i^{i+1}, \nu_i^{i+1}) \in E(BS_n)$ for $4 \leq i \leq n-1$. □

Lemma 4. *For $n \geq 4$, $\kappa(BS_n; K_{1,1}) \leq 2n - 3$ and $\kappa^s(BS_n; K_{1,1}) \leq 2n - 3$.*

Proof. Let $\mu = 12 \ldots n$ and $\nu = \mu_2^3$. We set $H = \{\{\mu_1^i, \nu_1^i\} | 2 \leq i \leq n\} \cup \{\{\mu_i^{i+1}, \nu_i^{i+1}\} | 4 \leq i \leq n-1\} \cup \{\{\mu_3^4, (\mu_3^4)_1^2\}, \{\nu_3^4, (\nu_3^4)_1^2\}\}$. Then, $|H| = n - 1 + n - 4 + 2 = 2n - 3$. Obviously, $N(\{\mu, \nu\}) \subset V(H)$. Then, $BS_n - V(H)$ is disconnected and $\{\mu, \nu\}$ is a component of

$BS_n - V(H)$. By Lemma 3, each $H_i \in H$ is isomorphic to $K_{1,1}$, and we get $\kappa(BS_n; K_{1,1}) \leq 2n-3$ and $\kappa^s(BS_n; K_{1,1}) \leq 2n-3$. See Figure 2 for an illustration. □

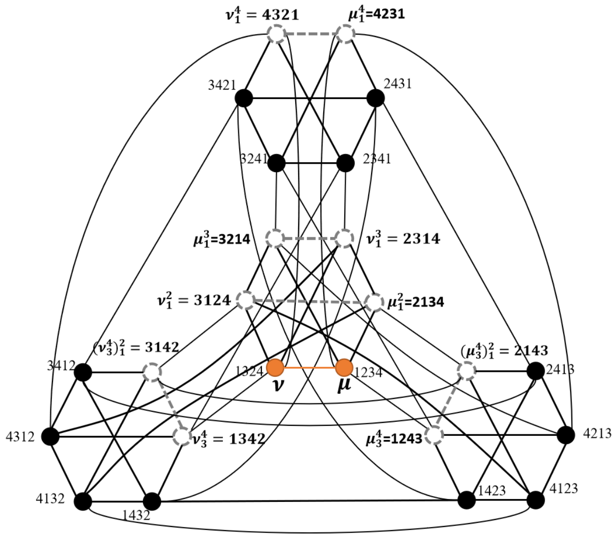

Figure 2. An example of $\kappa(BS_n; K_{1,1}) \leq 2n-3$ and $\kappa^s(BS_n; K_{1,1}) \leq 2n-3$, where $n=4$.

Lemma 5. *For $n \geq 4$, $\kappa^s(BS_n; K_{1,1}) \geq 2n-3$ and $\kappa(BS_n; K_{1,1}) \geq 2n-3$.*

Proof. Let $H = \{T_i | 1 \leq i \leq 2n-4\}$ be a subgraph set of BS_n, where each T_i is isomorphic to a connected subgraph of $K_{1,1}$. To prove $\kappa^s(BS_n; K_{1,1}) \geq 2n-3$, we need to show that $BS_n - V(H)$ is connected. Let $H_e = H \cap E(BS_n)$ and $H_v = H - H_e$. We consider the following three cases.

Case 1. $|H_v| = |H|$.

In this case, $|V(H)| \leq 2n-4$. Since $\kappa(BS_n) = 2n-3$, $BS_n - V(H)$ is connected.

Case 2. $|H_e| = |H|$.

In this case, $|E(H)| \leq 2n-4$. Since $\lambda(BS_n) = 2n-3$, $BS_n - V(H)$ is connected.

Case 3. $0 < |H_v| < |H|$ and $0 < |H_e| < |H|$.

In this case, $|H_e| + |H_v| = |H| \leq 2n-4$. By Lemma 2, $BS_n - V(H)$ is connected.

Hence, $\kappa^s(BS_n; K_{1,1}) \geq 2n-3$ and $\kappa(BS_n; K_{1,1}) \geq 2n-3$. □

According to Lemmas 4 and 5, we have Theorem 2.

Theorem 2. $\kappa(BS_n; K_{1,1}) = 2n-3$ and $\kappa^s(BS_n; K_{1,1}) = 2n-3$ for any integer $n \geq 4$.

Lemma 6. *For $n \geq 4$, $\kappa(BS_n; K_{1,2}) \leq n-1$ and $\kappa^s(BS_n; K_{1,2}) \leq n-1$.*

Proof. Let $\mu = 12\ldots n$ and $H = \{\{\mu_1^i, (\mu_1^i)_1^{i+1}, \mu_i^{i+1}\} | 2 \leq i \leq n-1\} \cup \{\mu_1^n, (\mu_1^n)_1^2, (\mu_1^n)_2^3\}$. Then, $|H| = n-2+1 = n-1$. Since $N(\mu) \subset V(H)$, $BS_n - V(H)$ is disconnected, and one component of $BS_n - V(H)$ is $\{\mu\}$. Since $(\mu_1^i, (\mu_1^i)_1^{i+1}), ((\mu_1^i)_1^{i+1}, \mu_i^{i+1}) \in E(BS_n)$, each element in H is isomorphic to $K_{1,2}$, where $(\mu_1^i)_1^{i+1}$ is the center vertex. Then, we get $\kappa(BS_n; K_{1,2}) \leq n-1$ and $\kappa^s(BS_n; K_{1,2}) \leq n-1$. See Figure 3 for an illustration. □

Lemma 7. *Let μ be any vertex in BS_n, let $M \cong K_{1,2}$ be any connected subgraph in BS_n, and let $c(M \cap N(\mu))$ be the maximum number of neighbors of μ that can be contained in M. Then, $c(M \cap N(\mu)) \leq 2$.*

Proof. Let $V(M) = \{\mu_i | 0 \leq i \leq 2\}$, where μ_1 is the center vertex of M. Suppose that $c(M \cap N(\mu))$=3. Then, we have two C_3: $\{\mu_0, \mu_1, \mu\}$ and $\{\mu_2, \mu_1, \mu\}$. Since BS_n is bipartite, there is no C_3 in BS_n, and we get a contradiction. Hence, $c(M \cap N(\mu)) \leq 2$. □

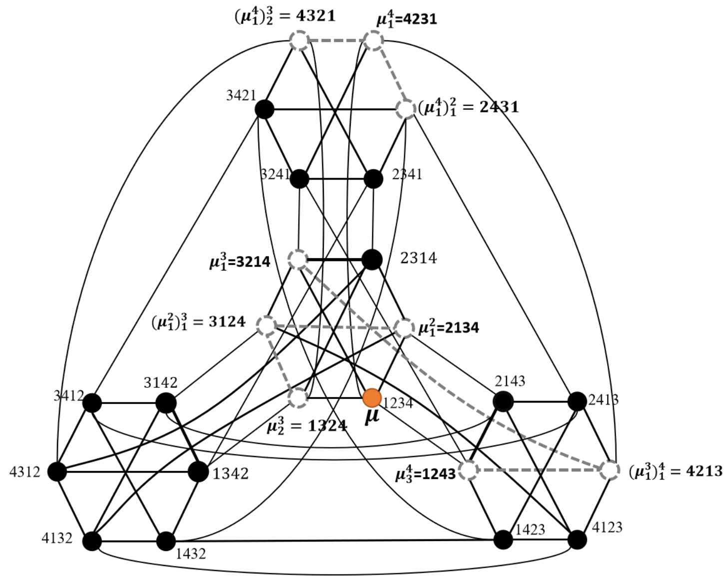

Figure 3. An example of $\kappa(BS_n; K_{1,2}) \leq n-1$ and $\kappa^s(BS_n; K_{1,2}) \leq n-1$ where $n = 4$.

Lemma 8. *Let $H = \{H_1, H_2, \ldots, H_t\}$ be a subgraph set of BS_n, where each $H_i \in H$ is isomorphic to a connected subgraph of $K_{1,2}$. If $t \leq n-2$, then $BS_n - V(H)$ is connected.*

Proof. Suppose that μ is any vertex in BS_n. Let $s = \sum_{i=1}^{t} c(H_i, N(\mu))$. By Lemma 7, $s \leq 2(n-2) = 2n-4 < 2n-3 = \kappa(BS_n)$. Then, by Lemma 1, $BS_n - V(H)$ is connected. □

According to Lemma 8, we have Lemma 9.

Lemma 9. *For $n \geq 4$, $\kappa(BS_n; K_{1,2}) \geq n-1$ and $\kappa^s(BS_n; K_{1,2}) \geq n-1$.*

According to Lemmas 6 and 9, we have Theorem 3.

Theorem 3. $\kappa(BS_n; K_{1,2}) = n-1$ and $\kappa^s(BS_n; K_{1,2}) = n-1$ for any integer $n \geq 4$.

Lemma 10. *Let μ be any vertex in BS_n, and let μ_1^i, μ_1^{i+1} and μ_i^{i+1} have a common neighbor in BS_n for $2 \leq i \leq n-1$ and $n \geq 4$.*

Proof. Let $\mu = x_1 \ldots x_i x_{i+1} \ldots x_n$. Then, $\mu_1^i = x_i \ldots x_1 x_{i+1} \ldots x_n$, $\mu_1^{i+1} = x_{i+1} \ldots x_i x_1 \ldots x_n$, and $\mu_i^{i+1} = x_1 \ldots x_{i+1} x_i \ldots x_n$. Let $\nu = (\mu_1^i)_1^{i+1} = x_{i+1} \ldots x_1 x_i \ldots x_n$. Then, $(\mu_1^i, \nu) \in E(BS_n)$. Since $(\mu_1^{i+1})_i^{i+1} = x_{i+1} \ldots x_1 x_i \ldots x_n = \mu$ and $(\mu_i^{i+1})_1^i = x_{i+1} \ldots x_1 x_i \ldots x_n = \mu$, $(\mu_1^{i+1}, \nu), (\mu_i^{i+1}, \nu) \in E(BS_n)$. Hence, ν is the common neighbor of μ_1^i,μ_1^{i+1} and μ_i^{i+1}. □

Lemma 11. *Let μ be any vertex in BS_n, and let μ_i^{i+1} and μ_{i+2}^{i+3} have a common neighbor in BS_n for i with $2 \leq i \leq n-3$ and $n \geq 4$.*

Proof. Let $\mu = x_1 \ldots x_i x_{i+1} x_{i+2} x_{i+3} \ldots x_n$. Then, $\mu_i^{i+1} = x_1 \ldots x_{i+1} x_i x_{i+2} x_{i+3} \ldots x_n$, $\mu_{i+2}^{i+3} = x_1 \ldots x_i x_{i+1} x_{i+3} x_{i+2} \ldots x_n$. Let $\nu = (\mu_i^{i+1})_{i+2}^{i+3} = x_1 \ldots x_{i+1} x_i x_{i+3} x_{i+2} \ldots x_n$. Then, $(\mu_i^{i+1}, \nu) \in E(BS_n)$. Since $(\mu_{i+2}^{i+3})_i^{i+1} = x_1 \ldots x_{i+1} x_i x_{i+3} x_{i+2} \ldots x_n = \nu$, then $(\mu_{i+2}^{i+3}, \nu) \in E(BS_n)$. Hence, ν is the common neighbor of μ_i^{i+1} and μ_{i+2}^{i+3}. □

Lemma 12. *For $n \geq 4$,*

$$\kappa^s(BS_n;K_{1,3}) \leq \kappa(BS_n;K_{1,3}) \leq \begin{cases} \frac{3(n-1)}{4} & if\ (n-1)\%4=0 \\ \frac{3(n-2)}{4}+1 & if\ (n-2)\%4=0 \\ \frac{3(n-3)}{4}+1 & if\ (n-3)\%4=0 \\ \frac{3(n-4)}{4}+2 & if\ (n-4)\%4=0. \end{cases}$$

Proof. Let μ be any vertex in BS_n. Each μ has n neighbors $N(\mu) = \{\mu_1^i|2 \leq i \leq n\} \bigcup \{\mu_i^{i+1}|2 \leq i \leq n-1\}$, so we can construct $K_{1,3}$ with $N(\mu)$. According to Lemma 10, we can construct $H_1 = \{\{\mu_1^{2i}, \mu_1^{2i+1}, \mu_{2i}^{2i+1}, (\mu_1^{2i})_1^{2i+1}\}|1 \leq i \leq \frac{n-1}{2}\}$, which has $\frac{n-1}{2}$ $K_{1,3}$ when n is odd. In addition, we can construct $H_1 = \{\{\mu_1^{2i}, \mu_1^{2i+1}, \mu_{2i}^{2i+1}, (\mu_1^{2i})_1^{2i+1}\}|1 \leq i \leq \frac{n-2}{2}\}$, which has $\frac{n-2}{2}$ $K_{1,3}$ when n is even. By Lemma 11, we can construct $K_{1,3}$ with μ_{4i-1}^{4i} and μ_{4i+1}^{4i+2} for $i \geq 1$. Then, if there are still vertices left, we can build $K_{1,3}$ with these vertices and their neighbors. We have the following four cases.

Case 1. $(n-1)\%4=0$.

We can construct $K_{1,3}$ as follows:
$H_1 = \{\{\mu_1^{2i}, \mu_1^{2i+1}, \mu_{2i}^{2i+1}, (\mu_1^{2i})_1^{2i+1}\}|1 \leq i \leq \frac{n-1}{2}\}$,
$H_2 = \{\{\mu_{4i-1}^{4i}, \mu_{4i+1}^{4i+2}, (\mu_{4i-1}^{4i})_{4i+1}^{4i+2}, ((\mu_{4i-1}^{4i})_{4i+1}^{4i+2})_1^2\}|1 \leq i \leq \frac{n-5}{4}\}$,
$H_3 = \{\{\mu_{n-2}^{n-1}, (\mu_{n-2}^{n-1})_1^2, (\mu_{n-2}^{n-1})_1^3, (\mu_{n-2}^{n-1})_1^4\}\}$.

Let $H = H_1 \cup H_2 \cup H_3$. Then, $|H| = \frac{n-1}{2} + \frac{n-5}{4} + 1 = \frac{3(n-1)}{4}$. Since $N(\mu) \subset V(H)$, $BS_n - V(H)$ is disconnected and μ is a component of $BS_n - V(H)$. For each H_i in H is isomorphic to $K_{1,3}$, we have $\kappa(BS_n;K_{1,3}) \leq \frac{3(n-1)}{4}$ and $\kappa^s(BS_n;K_{1,3}) \leq \frac{3(n-1)}{4}$.

Case 2. $(n-2)\%4=0$.

We can construct $K_{1,3}$ as follows:
$H_1 = \{\{\mu_1^{2i}, \mu_1^{2i+1}, \mu_{2i}^{2i+1}, (\mu_1^{2i})_1^{2i+1}\}|1 \leq i \leq \frac{n-2}{2}\}$,
$H_2 = \{\{\mu_{4i-1}^{4i}, \mu_{4i+1}^{4i+2}, (\mu_{4i-1}^{4i})_{4i+1}^{4i+2}, ((\mu_{4i-1}^{4i})_{4i+1}^{4i+2})_1^2\}|1 \leq i \leq \frac{n-2}{4}\}$,
$H_3 = \{\{\mu_1^n, (\mu_1^n)_1^2, (\mu_1^n)_1^3, (\mu_1^n)_1^4\}\}$.

Let $H = H_1 \cup H_2 \cup H_3$. Then, $|H| = \frac{n-2}{2} + \frac{n-2}{4} + 1 = \frac{3(n-2)}{4} + 1$. Since $N(\mu) \subset V(H)$, $BS_n - V(H)$ is disconnected and μ is a component of $BS_n - V(H)$. For each H_i in H is isomorphic to $K_{1,3}$, we have $\kappa(BS_n;K_{1,3}) \leq \frac{3(n-2)}{4} + 1$ and $\kappa^s(BS_n;K_{1,3}) \leq \frac{3(n-2)}{4} + 1$.

Case 3. $(n-3)\%4=0$.

We can construct $K_{1,3}$ as follows:
$H_1 = \{\{\mu_1^{2i}, \mu_1^{2i+1}, \mu_{2i}^{2i+1}, (\mu_1^{2i})_1^{2i+1}\}|1 \leq i \leq \frac{n-1}{2}\}$,
$H_2 = \{\{\mu_{4i-1}^{4i}, \mu_{4i+1}^{4i+2}, (\mu_{4i-1}^{4i})_{4i+1}^{4i+2}, ((\mu_{4i-1}^{4i})_{4i+1}^{4i+2})_1^2\}|1 \leq i \leq \frac{n-3}{4}\}$,

Let $H = H_1 \cup H_2$. Then, $|H| = \frac{n-1}{2} + \frac{n-3}{4} = \frac{3(n-3)}{4} + 1$. Since $N(\mu) \subset V(H)$, $BS_n - V(H)$ is disconnected and μ is a component of $BS_n - V(H)$. For each H_i in H is isomorphic to $K_{1,3}$, we have $\kappa(BS_n;K_{1,3}) \leq \frac{3(n-3)}{4} + 1$ and $\kappa^s(BS_n;K_{1,3}) \leq \frac{3(n-3)}{4} + 1$.

Case 4. $(n-4)\%4=0$.

We can construct $K_{1,3}$ as follows:
$H_1 = \{\{\mu_1^{2i}, \mu_1^{2i+1}, \mu_{2i}^{2i+1}, (\mu_1^{2i})_1^{2i+1}\}|1 \leq i \leq \frac{n-2}{2}\}$,
$H_2 = \{\{\mu_{4i-1}^{4i}, \mu_{4i+1}^{4i+2}, (\mu_{4i-1}^{4i})_{4i+1}^{4i+2}, ((\mu_{4i-1}^{4i})_{4i+1}^{4i+2})_1^2\}|1 \leq i \leq \frac{n-4}{4}\}$,
$H_3 = \{\{\mu_1^n, \mu_{n-1}^n, (\mu_{n-1}^n)_1^n, ((\mu_{n-1}^n)_1^n)_1^2\}\}$.

Let $H = H_1 \cup H_2 \cup H_3$. Then, $|H| = \frac{n-2}{2} + \frac{n-4}{4} + 1 = \frac{3(n-4)}{4} + 2$. Since $N(\mu) \subset V(H)$, $BS_n - V(H)$ is disconnected and μ is a component of $BS_n - V(H)$. For each H_i in H is isomorphic to $K_{1,3}$, we have $\kappa(BS_n;K_{1,3}) \leq \frac{3(n-4)}{4} + 2$ and $\kappa^s(BS_n;K_{1,3}) \leq \frac{3(n-4)}{4} + 2$. See Figure 4 for an illustration. □

According to the proof, we give an algorithm for calculating the upper bounds of the $K_{1,3}$-(sub)structure connectivity of BS_n (see Algorithm 1). We performed a simulation based on this algorithm to get the upper bounds of the $K_{1,3}$-(sub)structure connectivity when the dimension was $n \leq 9$. The results obtained from the algorithm are consistent with those of Lemma 12, please see Table 1 for reference.

Algorithm 1: Calculate the upper bounds of $K_{1,3}$-(sub)structure connectivity

Input: node μ, dimension n
Output: upper bounds of $K_{1,3}$-(sub)structure connectivity
switch $n\%4$ do
case 0:
    for $i \leftarrow 1$ **to** $\frac{n-2}{2}$ do
        construct $K_{1,3}$ with $\{\mu_1^{2i}, \mu_1^{2i+1}, \mu_{2i}^{2i+1}, (\mu_1^{2i})_1^{2i+1}\}$;
    for $i \leftarrow 1$ **to** $\frac{n-4}{4}$ do
        construct $K_{1,3}$ with $\{\mu_{4i-1}^{4i}, \mu_{4i+1}^{4i+2}, (\mu_{4i-1}^{4i})_{4i+1}^{4i+2}, ((\mu_{4i-1}^{4i})_{4i+1}^{4i+2})_1^2\}$;
    construct $K_{1,3}$ with $\{\mu_1^n, \mu_{n-1}^n, (\mu_{n-1}^n)_1^n, ((\mu_{n-1}^n)_1^n)_1^2\}\}$;
case 1:
    for $i \leftarrow 1$ **to** $\frac{n-1}{2}$ do
        construct $K_{1,3}$ with $\{\mu_1^{2i}, \mu_1^{2i+1}, \mu_{2i}^{2i+1}, (\mu_1^{2i})_1^{2i+1}\}$;
    for $i \leftarrow 1$ **to** $\frac{n-5}{4}$ do
        construct $K_{1,3}$ with $\{\mu_{4i-1}^{4i}, \mu_{4i+1}^{4i+2}, (\mu_{4i-1}^{4i})_{4i+1}^{4i+2}, ((\mu_{4i-1}^{4i})_{4i+1}^{4i+2})_1^2\}$;
    construct $K_{1,3}$ with $\{\mu_{n-2}^{n-1}, (\mu_{n-2}^{n-1})_1^2, (\mu_{n-2}^{n-1})_1^3, (\mu_{n-2}^{n-1})_1^4\}$;
case 2:
    for $i \leftarrow 1$ **to** $\frac{n-2}{2}$ do
        construct $K_{1,3}$ with $\{\mu_1^{2i}, \mu_1^{2i+1}, \mu_{2i}^{2i+1}, (\mu_1^{2i})_1^{2i+1}\}$;
    for $i \leftarrow 1$ **to** $\frac{n-2}{4}$ do
        construct $K_{1,3}$ with $\{\mu_{4i-1}^{4i}, \mu_{4i+1}^{4i+2}, (\mu_{4i-1}^{4i})_{4i+1}^{4i+2}, ((\mu_{4i-1}^{4i})_{4i+1}^{4i+2})_1^2\}$;
    construct $K_{1,3}$ with $\{\mu_1^n, (\mu_1^n)_1^2, (\mu_1^n)_1^3, (\mu_1^n)_1^4\}$;
case 3:
    for $i \leftarrow 1$ **to** $\frac{n-1}{2}$ do
        construct $K_{1,3}$ with $\{\mu_1^{2i}, \mu_1^{2i+1}, \mu_{2i}^{2i+1}, (\mu_1^{2i})_1^{2i+1}\}$;
    for $i \leftarrow 1$ **to** $\frac{n-3}{4}$ do
        construct $K_{1,3}$ with $\{\mu_{4i-1}^{4i}, \mu_{4i+1}^{4i+2}, (\mu_{4i-1}^{4i})_{4i+1}^{4i+2}, ((\mu_{4i-1}^{4i})_{4i+1}^{4i+2})_1^2\}$;
end

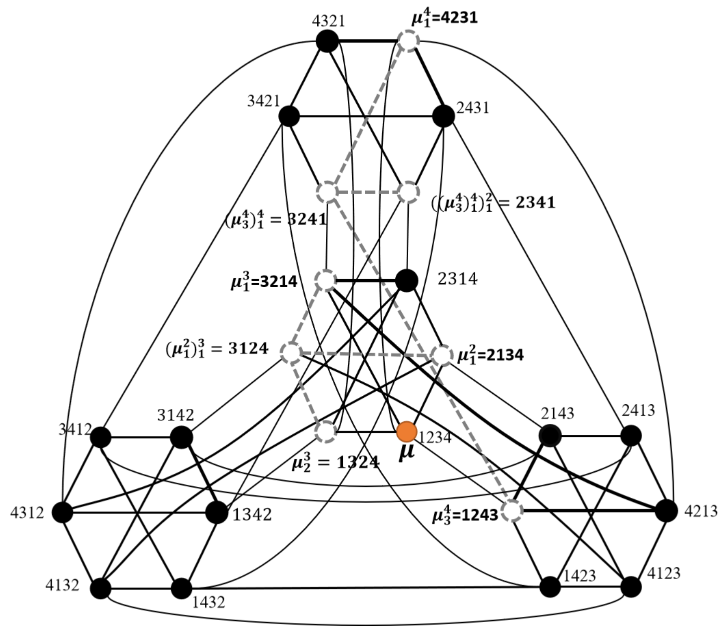

Figure 4. An example of $\kappa(BS_n; K_{1,3}) \leq \frac{3(n-4)}{4} + 2$ and $\kappa^s(BS_n; K_{1,3}) \leq \frac{3(n-4)}{4} + 2$ where $n = 4$.

Table 1. The upper bounds of $K_{1,3}$-(sub)structure connectivity.

	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
$\kappa(BS_n, K_{1,3}) \leq$	2	3	4	4	5	6
$\kappa^s(BS_n; K_{1,3}) \leq$	2	3	4	4	5	6

4. Conclusions

The connectivity of a network is a significant indicator for measuring that network's fault tolerance properties. In order to assess the impact of structure failure, structure connectivity and substructure connectivity are presented. In this paper, we find the upper bounds of $\kappa(BS_n; K_{1,3})$ and $\kappa^s(BS_n; K_{1,3})$. Furthermore, we establish $\kappa(BS_n; H)$ and $\kappa^s(BS_n; H)$ of BS_n, where $H \in \{K_1, K_{1,1}, K_{1,2}\}$.

A hypercube Q_n is an efficient symmetric network that has been used for commercial high-performance computers. The star graph S_n and bubble-sort graph B_n are two alternatives to the hypercube. BS_n, which is generated by merging B_n and S_n, has the advantages of both B_n and S_n. Here, we compare the H-(sub)structure connectivity of Q_n, B_n, S_n, and BS_n for $H \in \{K_1, K_{1,1}, K_{1,2}\}$. As shown in Table 2, BS_n has the highest K_1-(sub)structure connectivity and $K_{1,1}$-(sub)structure connectivity among these four networks. In addition, BS_n has the same $K_{1,2}$-(sub)structure connectivity as that of S_n, which is larger than those of Q_n and B_n. The comparison shows that BS_n is more stable than Q_n, S_n, and B_n, when structure faults occur.

Table 2. Comparison of $\{K_1, K_{1,1}, K_{1,2}\}$-(sub)structure connectivity.

	Dimension	K_1	$K_{1,1}$	$K_{1,2}$
Q_n	7	7	6	4
S_n	7	6	6	6
B_n	7	6	6	3
BS_n	7	11	11	6
Q_n	8	8	7	4
S_n	8	7	7	7
B_n	8	7	7	4
BS_n	8	13	13	7
Q_n	9	9	8	5
S_n	9	8	8	8
B_n	9	8	8	4
BS_n	9	15	15	8
Q_n	10	10	9	5
S_n	10	9	9	9
B_n	10	9	9	5
BS_n	10	17	17	9

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