information

MDPI

Article

On Information Orders on Metric Spaces

Oliver Olela Otafudu [1] and Oscar Valero [2,3,*]

[1] Department of Mathematics and Applied Mathematics, North-West University, Potchefstroom Campus, Potchefstroom 2520, South Africa; olivier.olelaotafudu@nwu.ac.za
[2] Departament de Ciències, Matemàtiques i Informàtica, Universitat de les Illes Balears, 07122 Palma de Mallorca, Spain
[3] Institut d' Investigació Sanitària Illes Balears (IdISBa), Hospital Universitari Son Espases, 07120 Palma de Mallorca, Spain
* Correspondence: o.valero@uib.es

Abstract: Information orders play a central role in the mathematical foundations of Computer Science. Concretely, they are a suitable tool to describe processes in which the information increases successively in each step of the computation. In order to provide numerical quantifications of the amount of information in the aforementioned processes, S.G. Matthews introduced the notions of partial metric and Scott-like topology. The success of partial metrics is given mainly by two facts. On the one hand, they can induce the so-called specialization partial order, which is able to encode the existing order structure in many examples of spaces that arise in a natural way in Computer Science. On the other hand, their associated topology is Scott-like when the partial metric space is complete and, thus, it is able to describe the aforementioned increasing information processes in such a way that the supremum of the sequence always exists and captures the amount of information, measured by the partial metric; it also contains no information other than that which may be derived from the members of the sequence. R. Heckmann showed that the method to induce the partial order associated with a partial metric could be retrieved as a particular case of a celebrated method for generating partial orders through metrics and non-negative real-valued functions. Motivated by this fact, we explore this general method from an information orders theory viewpoint. Specifically, we show that such a method captures the essence of information orders in such a way that the function under consideration is able to quantify the amount of information and, in addition, its measurement can be used to distinguish maximal elements. Moreover, we show that this method for endowing a metric space with a partial order can also be applied to partial metric spaces in order to generate new partial orders different from the specialization one. Furthermore, we show that given a complete metric space and an inf-continuous function, the partially ordered set induced by this general method enjoys rich properties. Concretely, we will show not only its order-completeness but the directed-completeness and, in addition, that the topology induced by the metric is Scott-like. Therefore, such a mathematical structure could be used for developing metric-based tools for modeling increasing information processes in Computer Science. As a particular case of our new results, we retrieve, for a complete partial metric space, the above-explained celebrated fact about the Scott-like character of the associated topology and, in addition, that the induced partial ordered set is directed-complete and not only order-complete.

Keywords: information order; increasing sequence; directed-completeness; metric; partial metric; completeness

Citation: Otafudu, O.O.; Valero, O. On Information Orders on Metric Spaces. *Information* **2021**, *12*, 427. https://doi.org/10.3390/info12100427

Academic Editor: Willy Susilo

Received: 7 September 2021
Accepted: 13 October 2021
Published: 18 October 2021

1. Introduction: Information Orders and Partial Metric Spaces

In 1970, D.S. Scott introduced the domain theory with the aim of developing a suitable mathematical foundation of computation (see [1]). This theory is based on the notion of a domain. A domain is a partially ordered structure for modeling computing processes, where the "information" about the final "stage" of the process is increased successively in each step of the process. The partial orders used for this aim are called information orders.

Let us recall the basics about order theory that we will need in our subsequent discussion (see, for instance, [2,3]).

A partially ordered set is a pair $(X, \preceq)$ where X is a non-empty set and $\preceq$ is a binary relation on X satisfying, for all $x, y, z \in X$, the following axioms:

(i) $x \preceq x$; (reflexivity)

(ii) $x \preceq y$ and $y \preceq x \Rightarrow x = y$; (antisymmetry)

(iii) $x \preceq y$ and $y \preceq z \Rightarrow x \preceq z$; (transitivity)

An element $x \in X$ is a maximal element of $Y \subseteq X$ if $x \preceq y$ implies $y = x$. A least element of Y is an element $z \in Y$, such that $z \preceq y$ for all $y \in Y$. An upper bound of Y is an element $x \in X$ such that $y \preceq x$ for all $y \in Y$. The least upper bound (or supremum) of Y is the least of the set of all its upper bounds, provided it exists.

A non-empty subset $Y \subseteq X$ is directed if for every pair $x, y \in Y$, there exists $z \in Y$, such that $x \preceq z$ and $y \preceq z$.

An ordered set $(X, \preceq)$, in which every directed subset has a supremum, is called a directed-complete partially ordered set (dcpo for short). Directed complete partially ordered sets are also called pre-cpos in [3].

A sequence $(x_n)_{n \in \mathbb{N}}$ in $(X, \preceq)$ is said to be increasing, provided that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$, where $\mathbb{N}$ stands for the positive integers.

In a partially ordered structure $(X, \preceq)$, endowed with an information order $\preceq$, the condition $x \preceq y$ is interpreted as all information contained in the datum x is also contained in the datum y. Thus, the condition $x \preceq y$ is understood as the amount of information.

The use of algorithms, which obtain successively refined "approximations" of a desired result in the spirit of Scott is very usual in Computer Science. When an approximation is obtained in some stage of the computation, it seems natural to consider a specific question: How well does the computation approximate the result? In order to determine how "good" an approximation is, the computer scientist models this process using information orders. A computation of an element of the model is considered as a "sequence" of increasing elements in such a way that each element of the sequence is greater than (or equal to) the preceding one, i.e., each stage of the computation gives more information about the result. Hence, the approximated object is regarded as the supremum of the sequence of approximations. For a more full treatment of the topic, we refer the reader to [3].

However, under this point of view it is not possible to measure the amount of information in each approximation. So the necessity of reconciling the order-theoretic approach with a topological one arises in a natural way. A recent detailed account of the theory from this point of view can be found in the recent monograph [4].

In order to get a framework useful to unify topological and order-theoretic ideas and, in addition, to provide numerical quantifications of the aforementioned amount of information, several works have been developed for reasoning about programs using "metric" ideas. Among these works, the most prominent references are the paper by M.B. Smyth [5] and the paper by S.G. Matthews [6].

In the framework introduced by Matthews, partial metrics play the role of the metric tools.

On account of [6], a partial metric on a non-empty set X is a function $p : X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$, the following axioms are satisfied:

(i) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$; ($T_0$-separation)

(ii) $p(x, x) \leq p(x, y)$; (small self-distances)

(iii) $p(x, y) = p(y, x)$; (symmetry)

(iv) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$; (triangularity)

Of course, $\mathbb{R}^+$ denotes the set of non-negative real numbers.

A partial metric space is a pair (X, p), such that X is a non-empty set and p is a partial metric on X.

Note that a metric space is a partial metric space (X, d) where d satisfies, in addition, the condition: (v) $d(x, x) = 0$ for all $x \in X$.

Each partial metric p on X generates a T_0 topology $\mathcal{T}(p)$ on X, which has as a base in the family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Observe that, contrarily to the metric case, the topology induced by a partial metric is only T_0 and not Hausdorff.

From this fact, it immediately follows that a sequence $(x_n)_{n\in\mathbb{N}}$ in a partial metric space (X, p) converges to $x \in X$ ($\lim_{n\to\infty} x_n = x$ for short) with respect to $\mathcal{T}(p)$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.

According to [6], a partial metric p, defined on a non-empty set X, induces a partial order $\preceq_p$ on X, so-called the specialization order, as follows: $x \preceq_p y \Leftrightarrow p(x, y) = p(x, x)$.

Notice that the specialization order matches up with the flat order when the partial metric is exactly a metric.

Of course, $\preceq_p$ can be understood as an information order in the sense that $x \preceq_p y$ can be interpreted as all information contained if x is also contained in the information content of y. The amount of information is given by the numerical measure $p(x, x)$. Indeed, if $x \preceq_p y$, then $p(y, y) \leq p(x, x)$. Observe that those elements with $p(x, x) = 0$ are maximal from an information point of view.

Obviously, in those cases where the information content about the final stage of the computational process is increased successively in each step of such a process, the interest is focused on the study of increasing sequences of the form $x_0 \preceq_p x_1 \preceq_p x_2$..., called chains of increasing information, where $\preceq_p$ is an information order and the supremum of the sequence captures the amount of information and, besides, such an amount is measured by a partial metric.

In order to guarantee that such a supremum contains no information other than that which may be derived from the members of the chain, the supremum must be the "limit" of the mentioned chain. Matthews showed that this last condition can be modeled using a Scott-like topology [6]. Indeed, a topology $\mathcal{T}$ on a partially ordered set $(X, \preceq)$ is a Scott-like topology with respect to the partial order if each increasing sequence in $(X, \preceq)$ has a least upper bound as a limit point of the sequence with respect to $\mathcal{T}$. Let us recall that a partial ordered set $(X, \preceq)$ is $\preceq$-complete provided that every increasing sequence has a least upper bound [7]. Matthews proved that when a non-empty set X is endowed with a partial metric p, the induced partially-ordered set $(X, \preceq_p)$ is $\preceq_p$-complete and, in addition, the associated topology $\mathcal{T}(p)$ is, in fact, a Scott-like topology with respect to the specialization order $\preceq_p$ when the partial metric space (X, p) is complete.

Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ in a partial metric space (X, p) is called a Cauchy sequence if there exists $\lim_{n,m\to\infty} p(x_n, x_m)$ and, in addition, a partial metric space (X, p) is said to be complete if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X converges, with respect to $\mathcal{T}(p)$, to any element $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$. Of course, the completeness coincides with the standard completeness when the partial metric is exactly a metric.

Since Matthews introduced the notion of partial metrics, many works have delved into the study of topological and order-theoretical properties of domains through partial metrics as, for instance, [8–18].

In the context of metric spaces, a partial order can be induced by a non-negative real valued function. In particular, on account of [19,20], the classical method is given as follows:

If (X, d) is a metric space, then any real function $\varphi : X \to \mathbb{R}^+$, induces a partial order $\preceq_{d,\varphi}$ on X given by $x \preceq_{d,\varphi} y \Leftrightarrow d(x, y) \leq \varphi(x) - \varphi(y)$.

In the remainder of the paper, our target is two-fold.

On the one hand, we show that the method for generating the partial order $\preceq_{d,\varphi}$, on a metric space (X, d), through a function $\varphi : X \to \mathbb{R}^+$, captures the essence of information orders in such a way that the function φ is able to quantify the amount of information contained in the elements of the partially ordered set and its measurement can be used to distinguish between the maximal elements. Moreover, we show that this method for endowing a metric space with a partial order can also be applied to partial metric spaces in such a way that new partial orders, different from the specialization one, can be induced and, in addition, Matthews' method can be retrieved as a particular case.

On the other hand, we show that, given a complete metric space (X, d), the partial ordered set $(X, \preceq_{d,\varphi})$ induced by a function $\varphi : X \to \mathbb{R}^+$ enjoys rich properties from an information order viewpoint. Concretely we will show not only its $\preceq_{d,\varphi}$-completeness but the directed-completeness and, in addition, that the topology $\mathcal{T}(d)$ is a Scott-like topology when the metric space (X, d) is complete and the function φ enjoys a distinguished property that we have called inf-continuity. Therefore, the mathematical structure $(X, \preceq_{d,\varphi})$ could be used for developing metric-based tools for modeling increasing information processes in Computer Science. As a particular case, our new results we retrieve, for a complete partial metric space (X, p), the Scott-like character of the topology $\mathcal{T}(p)$ and, in addition, that the partial ordered set $(X, \preceq_p)$ is a dcpo, and not only $\preceq_p$-complete when the partial metric space (X, p) is complete.

2. The New Induced Order: Metric versus Partial-Metric Spaces

We next show that the method for generating the partial order $\preceq_{d,\varphi}$ on a metric space (X, d), through a function $\varphi : X \to \mathbb{R}^+$ provides a partial order which can be understood as an information order and extends the Matthews method when partial metric spaces are under consideration.

In [21], R. Heckmann, given a partial metric (X, p), characterized the specialization order $\preceq_p$ in terms of an induced metric δ_p on X in the following manner:

Proposition 1. *Let (X, p) be a partial metric space. Then the following holds:*

(1) *The function $\delta_p : X \times X \to \mathbb{R}^+$ given by $\delta_p(x, y) = p(x, y) - \frac{p(x,x)+p(y,y)}{2}$ is a metric.*
(2) *For $x, y \in X$, $x \preceq_p y \Leftrightarrow \delta_p(x, y) \leq \varphi_p(x) - \varphi_p(y)$ with $\varphi_p(x) = \frac{p(x,x)}{2}$.*

In view of the preceding proposition, the specialization order is induced by means of the above exposed classical method, where the metric space and the function under consideration are exactly (X, δ_p), and φ_p, respectively. However, the aforesaid classical method helps us to induce new partial orders, different from the specialization one, in partial-metric spaces, as follows:

If (X, p) is a partial metric space and $\varphi : X \to \mathbb{R}^+$ is any function on X, then the binary relation $\preceq_{\delta_p,\varphi}$ given by:

$$x \preceq_{\delta_p,\varphi} y \Leftrightarrow \delta_p(x, y) \leq \varphi(x) - \varphi(y),$$

is a partial order on X.

The value of partial metrics is given, among others, by the fact that there are many examples of spaces which arise in a natural way in Computer Science, whose order structure can be expressed in terms of a partial metric (see [11–14,16,22]. Following [22], partial metrics that capture the partial order of a partially ordered set are called satisfactory. Three samples of this type of situations are given in Examples 1–3, below. These examples show that partial metrics are relevant in several fields of Computer Science.

Example 1 (Domain of words)**.** *Let Σ be a non-empty alphabet. Denote by Σ^∞ the set of all finite and infinite sequences ("words") over Σ. As usual, if $w \in \Sigma^\infty$, then we will denote by $\ell(w)$ the length of w. Thus, $\ell(v) \in [1, +\infty]$. We will write $w := w_1 w_2 ... w_n$ when $w \in \Sigma^\infty$ with $\ell(w) = n < \infty$. Moreover, we will write $w := w_1 w_2 ....$ when w is an infinite word.*

G. Kahn introduced a model of parallel computation in order to describe mathematically communicating computing processes by sending unending streams of information (infinite words) *between them (see [6,23]). Thus, such a model was based on the set Σ^∞ endowed with the Baire metric. In order to study the existence of a possible deadlock in the communication processes (see [24]), Matthews defined the Baire partial metric on the set Σ^∞. The Baire partial metric $p_B : \Sigma^\infty \times \Sigma^\infty \to \mathbb{R}^+$ is given by:*

$$p_B(v,w) = \begin{cases} 2^{-\ell(v,w)} & \text{if } \ell(v,w) < +\infty \\ 0 & \text{otherwise} \end{cases},$$

where $\ell(v,w) = sup\{n \in \mathbb{N} : v_i = w_i \text{ whenever } i \leq n\}$ when v and w have a nonempty common prefix, and $\ell(v,w) = 0$ otherwise. Notice that $\ell(v,w) = +\infty$ and that this situation occurs when $\ell(v) = \ell(w) = +\infty$ and $u = v$. Moreover, observe that $\ell(u,w) \leq \min\{\ell(u), \ell(w)\}$.

Typically the set Σ^∞ is ordered by $\sqsubseteq$ in the following way:

$$v \sqsubseteq w \Leftrightarrow v \text{ is a prefix of } w.$$

Obviously the prefix order $\sqsubseteq$ coincides with the specialization order $\preceq_{p_B}$ and, thus, with $\preceq_{\varphi_{p_B}, p_B}$, i.e., the partial order induced by the metric δ_{p_B} through φ_{p_B}, where:

$$\varphi_{p_B}(v) = \begin{cases} 2^{-(\ell(v)+1)} & \text{if } \ell(v) < +\infty \\ 0 & \text{otherwise} \end{cases}.$$

The partial metric space (Σ^∞, p_B) endowed with the prefix order is called the domain of words.

The infinite words can be viewed as elements with total information content, while finite words can be considered as elements with partial information content. Note that the partial metric p_B allows us to distinguish between them. Indeed, $\ell(w) = +\infty \Leftrightarrow p_B(w,w) = 0$. The elements with total information content are maximal elements. Moreover, notice that when we observe $(\Sigma^\infty, \sqsubseteq)$ as the partially ordered set $(\Sigma^\infty, \preceq_{\delta_{p_B}, \varphi_{p_B}})$, we can appreciate that the function φ_{p_B} can be used to distinguish the words with total information content (maximal elements) from those with partial information content because $\ell(v) = +\infty \Leftrightarrow \varphi_{p_B}(v) = 0$).

Example 2 (Flat domain). *Let S be a non-empty set and $\perp \notin S$. Consider $X = S \cup \{\perp\}$ and the partial order $\preceq$ on X given by:*

$$x \preceq y \Leftrightarrow x = \perp \text{ or } x = y \in S.$$

According to [6], X becomes a partial metric space when we endowed it with the flat partial metric $p_\perp$ defined by:

$p_\perp : X \times X \to \{0,1\}$ by

$$p_\perp(x,y) = 0 \Leftrightarrow x = y \in S.$$

Clearly the partial order $\preceq$ coincides with the specialization order $\preceq_{p_\perp}$. The computational intuition underlying the ordered space $(X, \preceq)$ is given by the fact that the set S is formed by elements with totally defined information content (all of them have the same information content) and the element $\perp$ which is undefined and, thus, its information content is partial. Observe that the flat partial metric $p_\perp$ captures the notion of maximality from the information viewpoint, since $x \in S \Leftrightarrow p_\perp(x,x) = 0$. Hence all elements in S are maximal.

Note that that such a structure can also be induced by the new general method taking the function $\varphi_\perp : X \to \mathbb{R}^+$ given by:

$$\varphi_\perp(x) = \begin{cases} 1 & \text{if } x = \perp \\ 0 & \text{otherwise} \end{cases}.$$

Of course the partial order $\preceq$ is induced by the flat partial metric through $\varphi_\perp$, i.e., $\preceq$ coincides with $\preceq_{\delta_{p_\perp}, \varphi_\perp}$. Again, the function $\varphi_\perp$ can be used to distinguish the elements with totally defined information content and the element $\perp$, since $x \in S \Leftrightarrow \varphi_\perp(x) = 0$.

Example 3 (Domain of complexity functions)**.** *In [25], S. Oltra, S. Romaguera and E.A. Sánchez-Pérez introduced the partial metric complexity space* $(\mathcal{C}, p_{\mathcal{C}})$ *given by:*

$$\mathcal{C} = \left\{ f : \mathbb{N} \to]0, \infty] : \sum_{n=1}^{\infty} 2^{-n} \frac{1}{f(n)} < \infty \right\},$$

and

$$p_{\mathcal{C}}(f, g) = \sum_{n=1}^{\infty} 2^{-n} \max\left\{ \frac{1}{f(n)}, \frac{1}{g(n)} \right\}.$$

It must be pointed out that the convention $\frac{1}{\infty} = 0$ *is adopted.*

According to [15,25] (see [26] for detailed applications), the partial metric complexity space is suitable to develop a topological foundation for asymptotic complexity analysis of algorithms. In fact, one can assign a function in $\mathcal{C}$ *to the running time of the computing of an algorithm* P *in such a way that* $f(n)$ *represents the time taken by* P *to solve the problem for which it has been implemented. When an algorithm process of an input of size* n *provides an undefined output value, then* $f(n) = \infty$*. Observe that the partial order* $\preceq_{p_{\mathcal{C}}}$ *allows us to discuss the asymptotic complexity behaviors of the running time for computing of the algorithms. That is,* $f \preceq_{p_{\mathcal{C}}} g \Leftrightarrow f(n) \leq g(n)$ *for all* $n \in \mathbb{N}$*. Thus, from an information point of view,* $f \preceq_{p_{\mathcal{C}}} g$ *can be interpreted as* f *is "at least as efficient" as* g *on all inputs. Therefore,* g *provides an asymptotic upper bound of* f *and, hence, of the running time of computing that it represents.*

In this context, the element with its totally defined information content is the complexity function $f_\infty \in \mathcal{C}$*, such that* $f_\infty(n) = \infty$ *for all* $n \in \mathbb{N}$*, since* $p_{\mathcal{C}}(f_\infty, f_\infty) = 0$*. Thus, the information content is conceived in a reverse sense. The smaller* $p_{\mathcal{C}}(f, f)$*, the less the information about the running time of complexity. Thus, those elements with partial information content, providing information about running time, belong to* $\mathcal{C} \setminus \{f_\infty\}$*. Here, the maximal element is* f_∞*, that is, the element with less information about running time.*

Observe that the partial order $\preceq_{p_{\mathcal{C}}}$ *is exactly* $\preceq_{\delta_{p_{\mathcal{C}}}, \varphi_{\mathcal{C}}}$*, i.e., induced by the metric* $\delta_{p_{\mathcal{C}}}$ *through* $\varphi_{\mathcal{C}}$ *where:*

$$\varphi_{\mathcal{C}}(v) = \begin{cases} 0 & \text{if } f = f_\infty \\ \sum_{n=1}^{\infty} 2^{-(n+1)} \frac{1}{f(n)} & \text{otherwise} \end{cases}.$$

Consequently, the function $\varphi_{\mathcal{C}}$ *can be used to distinguish the functions with total information content (maximal elements) from those with partial information content because* $\varphi_{\mathcal{C}}(f) = 0 \Leftrightarrow f = f_\infty$*.*

Example 4 gives an instance of partial metric space which is not satisfactory and, thus, it shows that partial metrics are not always able to encode the partial order given in a non-empty set. Moreover, the example shows that it can be turned into a satisfactory result using our new method.

Example 4. *Consider the partial metric space* $(\mathbb{R}^+, p_m)$*, where* $p_m(x, y) = \max\{x, y\}$ *for all* $x, y \in \mathbb{R}^+$*. The restriction of* p_m *to* $p_m|_{[0,1]}$ *is denoted again by* p_m*. It is known that the partial metric* p_m *is not satisfactory when* $\mathbb{R}^+$ *is endowed with the usual partial order* $\leq$*. Indeed* $\preceq_{p_m}$ *(or equivalently* $\preceq_{\delta_{p_m}, \varphi_{p_m}}$*) does not coincide with* $\leq$*, since* $x \preceq_{p_m} y \Leftrightarrow y \leq x$*.*

Nevertheless we show that choosing a suitable function $\varphi : [0, 1] \to \mathbb{R}^+$*, the usual order* $\leq$ *can be induced by the partial metric* p_m *through* φ*, i.e.,* $\preceq_{\delta_{p_m}, \varphi}$ *coincides with* $\leq$*. Indeed, take* $\varphi(x) = -\frac{x}{2} + \frac{1}{2}$*. It is a simple matter to see that, given* $x, y \in \mathbb{R}^+$*,*

$$x \leq_{\delta_{p_m}, \varphi} y \Leftrightarrow x \vee y \leq y.$$

Indeed, $x \leq_{\delta_{p_m},\varphi} y \Leftrightarrow \delta_{p_m}(x,y) \leq \varphi(x) - \varphi(y)$. *Hence,* $x \leq_{\delta_{p_m},\varphi} y \Leftrightarrow$

$$\max\{x,y\} - \frac{x}{2} - \frac{y}{2} \leq -\frac{x}{2} + \frac{1}{2} + \frac{y}{2} - \frac{1}{2}.$$

Thus we obtain:

$$\max\{x,y\} \leq y.$$

Whence:

$$x \leq_{\delta_{p_m},\varphi} y \Leftrightarrow x \leq y.$$

If we interprete the fact that $x \leq y$ *as the number* y *has more information than* x*, then the function* φ *captures the amount of information. Since smaller* $\varphi(x)$ *values have more information content in* x*, in such a way that the element with total information content* 1 *(the maximal element) satisfies* $\varphi(1) = 0$*. Moreover, the function* φ *distinguishes between the maximal element and others because* $\varphi(x) = 0 \Leftrightarrow x = 1$*.*

Observe that the partial ordering method due to Heckmann, and introduced in Proposition 1, cannot turn the partial metric p_m *into a satisfactory result by means of the function* φ*.*

The above examples suggest that the function φ can be used as a feature to quantify the amount of information contained in the elements of the partially ordered set in such a way that the value $\varphi(x)$ allows us to distinguish maximal elements because x is maximally totally (defined), provided that $\varphi(x) = 0$.

In the following result, we formally prove that such a hypothesis is true. To this end, let us recall that, given two partially ordered sets, $(X, \preceq_1)$ and $(Y, \preceq_2)$, a mapping $\varphi : X \to Y$ is said to be decreasing provided that $\varphi(y) \preceq_2 \varphi(x)$, whenever $x \preceq_1 y$.

First of all, we stress that, given a metric space (X, d), a function $\varphi : X \to \mathbb{R}^+$ is decreasing with respect to $\preceq_{d,\varphi}$. Therefore, the smaller values $\varphi(x)$ match up with the more information content in x. Hence, $x \leq_{d,\varphi} y$ can be understood, as the element y has at least as much information content as the element x.

Proposition 2. *Let* (X, d) *be a metric space and let* $\varphi : X \to \mathbb{R}^+$ *be a function. Then the following assertions hold:*

(1) *For all* $z \in X$*,* $\varphi(z) = \min_{x \leq_{d,\varphi} z} \varphi(x)$*.*

(2) *Elements* $z \in X$ *with* $\varphi(z) = 0$ *are maximal.*

(3) *If* $\varphi(x) = \varphi(y)$ *and* $x \preceq_{d,\varphi} y$*, then* $x = y$*.*

Proof. (1). Let $z \in X$. Then $\varphi(z) \leq \varphi(x)$ for all $x \in X$, such that $x \preceq_{d,\varphi} z$. Whence we deduce that $\varphi(z) \leq \inf_{x \leq_{d,\varphi} z} \varphi(x)$. Suppose that $\varphi(z) < \inf_{x \leq_{d,\varphi} z} \varphi(x)$. Then, $\varphi(z) < \inf_{x \leq_{d,\varphi} z} \varphi(x) \leq \varphi(z)$, which is a contradiction. Since $z \preceq_{d,\varphi} z$, we have that $\min_{x \leq_{d,\varphi} z} \varphi(x) = \inf_{x \leq_{d,\varphi} z} \varphi(x)$.

(2). Let $z \in X$, such that $\varphi(z) = 0$. Assume that there exists $y \in X$ with $z \leq_{d,\varphi} y$. Then:

$$d(z,y) \leq \varphi(z) - \varphi(y) \leq 0.$$

It follows that $d(z,y) = 0$ and so $z = y$, which implies that z is a maximal element in $(X, \preceq_{d,\varphi})$.

(3). Let $x, y \in X$ such that $\varphi(x) = \varphi(y)$ and $x \leq_{d,\varphi} y$. Then we have:

$$d(x,y) \leq \varphi(x) - \varphi(y) = 0.$$

Therefore $x = y$. $\square$

3. Increasing Information Sequences

Taking into account the exposed facts, and that it has been suggested that the partial ordered set $(X, \preceq_{d,\varphi})$, induced from a metric space (X,d) and a non-negative real valued function φ on X, is able to quantify the amount of information contained in a sequence of elements by means of the function φ in Section 2. Next, we show the directed-completeness of $(X, \preceq_{d,\varphi})$, when the metric space (X,d) is complete and the function φ is inf-continuous. Moreover, under the same hypothesis, we prove that the topology $\mathcal{T}(d)$ is a Scott-like topology. Furthermore, we retrieve from our results the Scott-like character of the topology $\mathcal{T}(p)$ and that the partial ordered set $(X, \preceq_p)$ is a dcpo, and not only $\preceq_p$-complete, when (X,p) is a complete partial metric space.

3.1. Increasing Information Sequences and Direct-Completeness

Let us introduce the following notion that will play a central role in our subsequent discussion.

Definition 1. *Let (X,d) be a metric space. A function $\varphi : X \to \mathbb{R}^+$ will be said to be* inf-*continuous at $x \in X$ provided that $\lim_{n\to\infty} \varphi(x_n) = \varphi(x)$, whenever $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence in $(X, \preceq_{d,\varphi})$, such that $\lim_{n\to\infty} x_n = x$, with respect to $\mathcal{T}(d)$. A function φ that is* inf-*continuous at x for all $x \in X$ will be called simply* inf-*continuous.*

Let us recall that a function $\varphi : X \to \mathbb{R}^+$ is (lower semi) continuous provided that it is (lower semi) continuous from $(X, \mathcal{T}(d))$ into $(\mathbb{R}^+, \mathcal{T}(d_E))$, where d_E stands for the Euclidean metric. Lower semicontinuity and continuity of the function φ plays a crucial role in the study of the structure of $(X, \preceq_{d,\varphi})$ and its applications to metric fixed point theory (see [19,20]). Accordingly, it seems natural to wonder whether there exists any relationship between (lower semi) continuity and inf-continuity.

Clearly, every continuous function is always inf-continuous. However, the next example shows that the converse is not true in general. In fact, it gives an instance of a function which is inf-continuous and not lower semicontinuous and, thus, is not continous.

Example 5. *Consider the metric space $(\mathbb{R}^+, d_E)$. Define the function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(0) = 1$ and $\varphi(x) = 0$ for all $x \neq 0$. Observe that the unique increasing sequences in $(\mathbb{R}^+, \preceq_{d_E,\varphi})$ are those that are constants. It follows that φ is* inf-*continuos. Nevertheless, it is not lower semicontinuous. Indeed, take the sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. It is clear that $\lim_{n\to\infty} x_n = 0$ but $1 = \varphi(0) > \varphi(x_n) = 0$ for all $n \in \mathbb{N}$. So φ is not lower semicontinuos and, hence, it is not continuous.*

The next example gives an instance of a function that is lower semicontinuous and not inf-continuous (see [19]).

Example 6. *Consider the metric space $([0,1], d_E)$ and the function $\varphi : [0,1] \to \mathbb{R}^+$ defined by $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi(x) = x + 2$ for all $x \in]0,1[$. A straightforward computation shows that φ is lower semicontinuous. Clearly, it is not* inf-*continuous, since the sequence $(x_n)_{n\in\mathbb{N}}$, given by $x_n = \frac{1}{n+1}$ for all $n \in \mathbb{N}$, is increasing in $([0,1], \preceq_{d_E,\varphi})$ and $\lim_{n\to\infty} x_n = 0$ with respect to $\mathcal{T}(d_E)$. However, $\lim_{n\to\infty} \varphi(x_n) = 2 \neq \varphi(0) = 0$.*

The next results are useful to describe the interconnection between the order-theoretic and topological properties that will be key to prove the directed-compleness of $(X, \preceq_{d,\varphi})$.

Proposition 3. *Let (X,d) be a complete metric space and let $\varphi : X \to \mathbb{R}^+$ be an* inf-*continuous function. If $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence in $(X, \preceq_{d,\varphi})$, then the following assertions are true.*

(1) There exist $x^\star \in X$ such that $\lim_{n\to\infty} x_n = x^\star$ with respect to $\mathcal{T}(d)$.
(2) If $\lim_{n\to\infty} x_n = x^\star$, then $x^\star$ is the least upper bound of $(x_n)_{n\in\mathbb{N}}$ in $(X, \preceq_{d,\varphi})$.

Proof. (1). It is clear that $(\varphi(x_n))_{n\in\mathbb{N}}$ is a decreasing sequence in $\mathbb{R}^+$ and, thus, it is convergent. Thus, the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy in (X,d) because $(\varphi(x_n))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}^+$. Consequently, there exists $x^\star \in X$ such that $\lim_{n\to\infty} x_n = x^\star$ with respect to $\mathcal{T}(d)$.

(2). From $\lim_{n\to\infty} x_n = x^\star$ and from the fact that φ is inf-continuous we obtain that $\lim_{n\to\infty} \varphi(x_n) = \varphi(x^\star)$. Fix $n \in \mathbb{N}$. Then, given $\varepsilon > 0$, there exists $m \in \mathbb{N}$ with $m > n$, such that $d(x_m, x^\star) < \varepsilon$ and $\varphi(x^\star) < \varphi(x_m) + \varepsilon$. Since $x_n \leq_{\varphi,d} x_m$ we deduce that:

$$\begin{aligned} d(x_n, x^\star) &\leq d(x_n, x_m) + d(x_m, x^\star) < \varphi(x_n) - \varphi(x_m) + \varepsilon \\ &< \varphi(x_n) - \varphi(x^\star) + 2\varepsilon. \end{aligned}$$

Consequently, $d(x_n, x^\star) \leq \varphi(x_n) - \varphi(x^\star)$ and, hence, we have that $x_n \preceq_{d,\varphi} x^\star$ for all $n \in \mathbb{N}$. It follows that $x^\star$ is an upper bound of $(x_n)_{n\in\mathbb{N}}$.

It remains to prove that $x^\star$ is the least upper bound of $(x_n)_{n\in\mathbb{N}}$. With this aim, assume that $z \in X$ is an arbitrary upper bound of $(x_n)_{n\in\mathbb{N}}$. Then $x_n \leq_{\varphi,d} z$ for all $n \in \mathbb{N}$. Whence

$$\begin{aligned} d(x^\star, z) &\leq d(x^\star, x_m) + d(x_m, z) < \varepsilon + \varphi(x_m) - \varphi(z) \\ &< 2\varepsilon + \varphi(x^\star) - \varphi(z), \end{aligned}$$

since $d(x^\star, x_m) < \varepsilon$ and $\varphi(x_m) < \varphi(x^\star) + \varepsilon$ eventually. Therefore $d(x^\star, z) \leq \varphi(x^\star) - \varphi(z)$, and thus $x^\star \preceq_{d,\varphi} z$. This completes the proof. □

In the next result, for an element x of a partially ordered set $(X, \preceq)$, we will denote its upper set by $\uparrow_{\preceq} x$, that is, $\uparrow_{\preceq} x = \{y \in X : x \preceq y\}$.

Proposition 4. *Let (X,d) be a complete metric space and let $\varphi : X \to \mathbb{R}^+$ be an* inf-continuous *function. If $x_0 \in X$, then there exists an increasing sequence $(x_n)_{n\in\mathbb{N}}$ in $(X, \preceq_{d,\varphi})$ with least upper bound $x^\star$ such that $x^* \in \uparrow_{\preceq_{d,\varphi}} x_0$ and x^* is maximal.*

Proof. Let $x_0 \in X$. Next, we construct a sequence $(x_n)_{n\in\mathbb{N}}$ inductively as follows:

We choose $x_1 \in \uparrow_{\preceq_{d,\varphi}} x_0$, such that $\varphi(x_1) < 1 + \inf_{y \in \uparrow_{\preceq_{d,\varphi}} x_0} \varphi(y)$ and, for all $n \in \mathbb{N}$ with $n > 1$, we choose x_n with $x_n \in \uparrow_{\preceq_{d,\varphi}} x_{n-1}$, such that $\varphi(x_n) < \frac{1}{n} + \inf_{y \in \uparrow_{\preceq_{d,\varphi}} x_{n-1}} \varphi(y)$. We continue in this fashion, obtaining an increasing sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in \uparrow_{\leq_{d,\varphi}} x_{n-1}$ for all $n \in \mathbb{N}$.

Proposition 3 warranties the existence of $x^\star \in X$ such that $x^\star$ is the least upper bound of $(x_n)_{n\in\mathbb{N}}$ in $(X, \preceq_{d,\varphi})$ and, thus, that $x^* \in \uparrow_{\preceq_{d,\varphi}} x_0$.

It remains to prove that x^* is a maximal element. By Proposition 3 we have that $\lim_{n\to\infty} x_n = x^\star$ with respect to $\mathcal{T}(d)$. The inf-continuity of φ gives that $\lim_{n\to\infty} \varphi(x_n) = \varphi(x^\star)$. Now, assume that there exists $z \in X$ such that $x^\star \preceq_{d,\varphi} z$. Then, we have that $\varphi(z) \leq \varphi(x^\star)$. Moreover, $x_n \preceq_{d,\varphi} z$ for all $n \in \mathbb{N}$. Thus, $z \in \uparrow_{\preceq_{d,\varphi}} x_n$ for all $n \in \mathbb{N}$. Whence, we have that $\varphi(x_n) < \frac{1}{n} + inf_{y \in \uparrow_{\preceq_{d,\varphi}} x_{n-1}} \varphi(y) \leq \frac{1}{n} + \varphi(z)$ for all $n \in \mathbb{N}$. Whence, we deduce that $\varphi(x^\star) = lim_{n\to\infty} \varphi(x_n) \leq lim_{n\to\infty} \frac{1}{n} + \varphi(z) = \varphi(z)$. So $\varphi(x^\star) \leq \varphi(z) \leq \varphi(x^\star)$. It follows that, $\varphi(x^\star) = \varphi(z)$. By Proposition 2, we conclude that $x^\star = z$. This gives that $x^\star$ is maximal. □

It must be stressed that the counterparts of Propositions 3 and 4 have been obtained requiring lower semicontinuity for the function φ in [19,20], respectively. Observe that when inf-continuity is assumed, the proofs are different and are not based on the Cantor's Intersection Theorem.

The next result guarantees the directed-completeness of $(X, \preceq_{d,\varphi})$.

Theorem 1. *Let (X,d) be a complete metric space and let $\varphi : X \to \mathbb{R}^+$ be an* inf-continuous *function. Then $(X, \preceq_{d,\varphi})$ is a dcpo.*

Proof. Let D be a directed subset of X and let $x_0 \in D$. According to Proposition 4, we have guaranteed the existence of an increasing sequence $(x_n)_{n\in\mathbb{N}}$ in $(X, \preceq_{d,\varphi})$ with least upper bound $x^\star$ such that $x^* \in \uparrow_{\preceq_{d,\varphi}} x_0$ and $x^\star$ is maximal.

Let $y \in D$. Then, by directedness of D, there exists $z_0 \in D$, such that $x_0 \preceq_{d,\varphi} z_0$ and $y \preceq_{d,\varphi} z_0$. Now, let z_1 be an upper bound of z_0 and x_1 in D. In this way, we obtain an increasing sequence $(z_n)_{n\in\mathbb{N}} \subset D$ such that $z_n \in \uparrow_{\preceq_{d,\varphi}} x_n$ for all $n \in \mathbb{N}$. By Proposition 3 we have that there exists $z^\star \in X$ which is the least upper bound of $(z_n)_{n\in\mathbb{N}}$ and $z^\star \in \uparrow_{\preceq_{d,\varphi}} z_0$. Therefore, $y \preceq_{d,\varphi} z^\star$.

Since $x_n \preceq_{d,\varphi} z^\star$ for all $n \in \mathbb{N}$ and $x^\star$ is the least upper bound of $(x_n)_{n\in\mathbb{N}}$ we deduce that $x^\star \preceq_{d,\varphi} z^\star$. The maximality of $x^\star$ gives that $x^\star = z^\star$.

The above is true for all $y \in D$. Then we follow that x^* is an upper bound of D.

Next, we show that $x^\star$ is the unique upper bound of D, and thus the least upper bound of D. Otherwise, if we assume that there exists $d^\star \in X$ which is an upper bound of D then, $d^\star \in \uparrow_{\preceq_{d,\varphi}} x_n$ for all $n \in \mathbb{N}$. Thus, $d^\star$ is an upper bound of $(x_n)_{n\in\mathbb{N}}$. Since $x^\star$ is the least upper bound of $(x_n)_{n\in\mathbb{N}}$ we have that $x^\star \preceq_{d,\varphi} d^\star$. Since $x^\star$ is a maximal element, we conclude that $x^\star = d^\star$, which is our claim. □

The following example shows that there are metric spaces (X, d) such that $(X, \preceq_{d,\varphi})$ is a dcpo and, however, (X, d) is not complete. Therefore, the converse of the preceding result is not true.

Example 7. *Consider the metric space (X, d_E), where $X = \{\frac{1}{n} : n \in \mathbb{N}\}$ and d_E is the Euclidean metric on X. Define the function $\varphi : X \to \mathbb{R}^+$ by $\varphi(x) = 0$ for all $x \in X$. Clearly $(X, \preceq_{d_E,\varphi})$ is a dcpo (the unique directed sets are those formed by a singleton). Nevertheless, (X, d_E) is not complete because the sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n = \frac{1}{n}$ for all $n \in \mathbb{N}$ is Cauchy but does not converge with respect to $\mathcal{T}(d_E)$.*

Notice that Theorem 1 guarantees that the least upper bound of the dcpo $(X, \preceq_{d,\varphi})$ is always a maximal element.

From Theorem 1 we obtain the next result for partial metric spaces.

Corollary 1. *Let (X, p) be a complete partial metric space. Then $(X, \preceq_p)$ is a dcpo.*

Proof. According to [25], a partial metric space (X, p) is complete if and only if the associated metric space (X, δ_p) is complete. Moreover, Proposition 1 gives that $x \preceq_p y \Leftrightarrow x \preceq_{\delta_p,\varphi_p} y$, where the function $\varphi_p : X \to \mathbb{R}^+$ is given by $\varphi_p(x) = \frac{p(x,x)}{2}$ for all $x \in X$. The function φ_p is inf-continuous because it is the continuous from of $(X, \mathcal{T}(\delta_p))$ into $(\mathbb{R}^+, d_E)$. Applying Theorem 1, we have that $(X, \preceq_{\delta_p,\varphi_p})$ is a dcpo. Therefore, $(X, \preceq_p)$ is a dcpo. □

3.2. Increasing Information Sequences and Scott-like Topology

Next, we focus our efforts on discussing if the topology $\mathcal{T}(d)$ is a Scott-like topology with respect to the partial order $\preceq_{d,\varphi}$. With this aim we get the following results:

Proposition 5. *Let (X, d) be a complete metric space and let $\varphi : X \to \mathbb{R}^+$ be an* inf*-continuous function. If $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence in $(X, \preceq_{d,\varphi})$ with least upper bound $x^\star \in X$, then $\inf_{n\in\mathbb{N}} \varphi(x_n) = \varphi(x^\star)$.*

Proof. Clearly $(\varphi(x_n))_{n\in\mathbb{N}}$ is a decreasing sequence in $(\mathbb{R}^+, \leq)$ bounded below. Therefore, $\lim_{n\to\infty} \varphi(x_n) = \inf_{n\in\mathbb{N}} \varphi(x_n)$. By Proposition 3, there exists $y^\star \in X$ such that $\lim_{n\to\infty} x_n = y^\star$ with respect to $\mathcal{T}(d)$ and $y^\star \in X$ is the least upper bound of $(x_n)_{n\in\mathbb{N}}$ in $(X, \preceq_{d,\varphi})$. Whence we obtain that $x^\star = y^\star$. The inf-continuity of φ provides that $\lim_{n\to\infty} \varphi(x_n) = \varphi(x^\star)$. Thus, $\inf_{n\in\mathbb{N}} \varphi(x_n) = \varphi(x^\star)$. □

The next equivalence is crucial in our subsequent discussion.

Theorem 2. *Let (X,d) be a complete metric space and let $\varphi : X \to \mathbb{R}^+$ be an* inf*-continuous function. If $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence in $(X, \preceq_{d,\varphi})$, then the following statements are equivalent:*

(1) $x^\star \in X$ is the least upper bound of $(x_n)_{n\in\mathbb{N}}$.
(2) $x^\star \in X$ is an upper bound of $(x_n)_{n\in\mathbb{N}}$ and $\inf_{n\in\mathbb{N}} \varphi(x_n) = \varphi(x^\star)$.
(3) $(x_n)_{n\in\mathbb{N}}$ converges to $x^\star$ with respect to $\mathcal{T}(d)$.

Proof. $(1) \Rightarrow (2)$. Assume that $x^\star$ is the least upper bound of $(x_n)_{n\in\mathbb{N}}$, then $x^\star$ is an upper bound of $(x_n)_{n\in\mathbb{N}}$. By Proposition 5 we have that $\inf_{n\in\mathbb{N}} \varphi(x_n) = \varphi(x^\star)$.

$(2) \Rightarrow (3)$. Since $x^\star$ is an upper bound of $(x_n)_{n\in\mathbb{N}}$ we have that $d(x_n, x^\star) \leq \varphi(x_n) - \varphi(x^\star)$ for all $n \in \mathbb{N}$. Consequently, we deduce that $\lim_{n\to\infty} d(x_n, x^\star) = 0$ because the sequence $(\varphi(x_n))_{n\in\mathbb{N}}$ is decreasing and, in addition, $\lim_{n\to\infty} \varphi(x_n) = \inf_{n\in\mathbb{N}} \varphi(x_n) = \varphi(x^\star)$.

$(3) \Rightarrow (1)$. From Proposition 3, we deduce that $x^\star$ is the least upper bound of $(x_n)_{n\in\mathbb{N}}$. □

As a consequence of Theorem 2, we assert that the least upper bound of every chain of increasing information (increasing sequence) captures the amount of such information, and it does not contain more information that can be derived from the members of the chain, as happens in the case of the specialization order induced by a partial metric (see [6]).

In view of Theorems 1 and 2, we are able to show that $\mathcal{T}(d)$ is a Scott-like topology with respect to the partial order $\leq_{d,\varphi}$.

Theorem 3. *Let (X,d) be a complete metric space and let $\varphi : X \to \mathbb{R}^+$ be an* inf*-continuous function. Then the following assertions hold:*

(1) $(X, \preceq_{d,\varphi})$ is $\preceq_{d,\varphi}$-complete.
(2) Every increasing sequence $(x_n)_{n\in\mathbb{N}}$ in $(X, \preceq_{d,\varphi})$ converges with respect to $\mathcal{T}(d)$ to its least upper bound which is a maximal element in $(X, \preceq_{d,\varphi})$.

Proof. (1). Let $(x_n)_{n\in\mathbb{N}}$ be an increasing sequence in $(X, \preceq_{d,\varphi})$. Theorem 1 provides that there exists $x^\star \in X$ such that $x^\star$ is the supremum of $(x_n)_{n\in\mathbb{N}}$. Thus, $(X, \preceq_{d,\varphi})$ is $\preceq_{d,\varphi}$-complete.

(2). Assume that $(x_n)_{n\in\mathbb{N}}$ is increasing. From (1) we have that $x^\star$ is its least upper bound. Theorem 1 provides that $x^\star$ is maximal. By Theorem 2 we deduce that $\lim_{n\to\infty} x_n = x^\star$ with respect to $\mathcal{T}(d)$. □

It must be pointed out that Theorem 3 does not require continuity for the function φ (compare [19]).

Example 7 shows that there are metric spaces (X,d) such that $(X, \preceq_{d,\varphi})$ is $\preceq_{d,\varphi}$-complete but (X,d) is not complete. Therefore, the converse of Theorem 3 does not hold in general.

From Theorem 3 we can retrieve the $\preceq_p$-completeness of $(X, \preceq_p)$ and the fact that $\mathcal{T}(p)$ is a Scott-like topology when (X,p) is a complete partial metric space. Indeed, with this aim let us introduce the following result:

Proposition 6. *Let (X,p) be a partial metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. If $(x_n)_{n\in\mathbb{N}}$ is convergent with respect to $\mathcal{T}(\delta_p)$ then it is so with respect to $\mathcal{T}(p)$.*

Proof. Let $x \in X$ such that $\lim_{n\to\infty} x_n = x$ with respect to $\mathcal{T}(\delta_p)$. Then,

$$p(x, x_n) - \frac{p(x,x)}{2} - \frac{p(x_n, x_n)}{2} = \delta_p(x, x_n) < \varepsilon,$$

eventually. Thus, $2p(x, x_n) - p(x,x) - p(x_n, x_n) < \varepsilon$ and $p(x, x_n) < \varepsilon + p(x,x)$, eventually. Consequently, $(x_n)_{n\in\mathbb{N}}$ converges to x with respect to $\mathcal{T}(p)$. □

As a consequence of Proposition 1, Theorem 3, and Proposition 6 we retrieve the celebrated result below.

Corollary 2. *Let (X, p) be a complete partial metric space. Then the following assertions hold:*

(1) *$(X, \preceq_p)$ is $\preceq_p$-complete.*
(2) *Every increasing sequence $(x_n)_{n\in\mathbb{N}}$ in $(X, \preceq_p)$ converges with respect to $\mathcal{T}(p)$ to its least upper bound, which is a maximal element of $(X, \preceq_p)$.*

4. Conclusions

Information orders are suitable to describe processes in which the information increases successively in each step of the computation processes. S.G. Matthews introduced the notions of partial metric and Scott-like topology as a part of a mathematical method for providing numerical quantifications of the amount of information in the aforementioned processes. Later on, R. Heckmann showed that the method due to Matthews could be retrieved as a particular case of a celebrated method for generating partial orders through metrics and non-negative real-valued functions. In this paper, we have explored this general method from an information orders theory viewpoint. Specifically, we have shown that such a method captures the essence of information orders in such a way that the function under consideration is able to quantify the amount of information and, in addition, its measurement can be used to distinguish maximal elements. Moreover, we have shown that this method for endowing a metric space with a partial order can also be applied to partial metric spaces in order to generate new partial orders different from the specialization one. Furthermore, we have shown that, given a complete metric space and an inf-continuous function (a special continuous function that we have introduced), the partial ordered set induced by this general method enjoys rich properties. Concretely, we have shown not only its order-completeness but the directed-completeness and, in addition, that the topology induced by the metric is Scott-like. Therefore, such a mathematical structure could be useful for developing metric-based tools for modeling increasing information processes in Computer Science. As a particular case of our new results we have retrieved, for a complete partial metric space, the Scott-like character of the associated topology and, in addition, that the induced partial ordered set is a directed-complete and not only order-complete.

Author Contributions: O.O.O. and O.V. contributed to the development of this manuscript. Conceptualization, O.O.O. and O.V.; Writing, review and editing, O.O.O. and O.V.; Funding acquisition, O.V.; Project administration, O.V. All authors have read and agreed to the published version of the manuscript.

Funding: This work acknowledges financial support from Proyecto PGC2018-095709-B-C21 financiado por MCIN/AEI/10.13039/501100011033 y FEDER "Una manera de hacer Europa". This work is also partially supported by Programa Operatiu FEDER 2014-2020 de les Illes Balears, by project PROCOE/4/2017 (Direcció General d'Innovació i Recerca, Govern de les Illes Balears) and by projects ROBINS and BUGWRIGHT2. These two latest projects have received funding from the European Union's Horizon 2020 research and innovation programme under grant agreements No 779776 and No 871260, respectively. This publication reflects only the authors views and the European Union is not liable for any use that may be made of the information contained therein.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Scott, D.S. Outline of a mathematical theory of computation. In Proceedings of the IEEE 4th Annual Princeton Conference on Information Sciences and Systems, Princeton, NJ, USA, 26–27 March 1970; pp. 169–176.
2. Pitts, A.M.; Abramsky, S.; Gabbay, D.M.; Maibaum, T.S.E. Domain theory. In *Handbook of Logic in Computer Science*; Oxford University Press: New York, NY, USA, 1994; Volume III, pp. 1–168.
3. Davey, B.A.; Priestley, H.A. *Introduction to Lattices and Order*; Cambridge University Press: Cambridge, UK, 1990.
4. Goubault-Larecq, J. *Non-Hausdorff Topology and Domain Theory*; Cambridge University Press: Cambridge, UK, 2013.
5. Smyth, M.B. Quasi-uniformities: Reconciling domains with metric spaces. *Lect. Notes Comput. Sci.* **1987**, *298*, 236–253.
6. Matthews, S.G. Partial metric topology. *Ann. N. Y. Acad. Sci.* **1994**, *728*, 183–197. [CrossRef]
7. Baranga, A. The contraction principle as a particular case of Kleene's fixed point theorem. *Discret. Math.* **1991**, *98*, 75–79. [CrossRef]
8. Bukatin, M.A.; Scott, J.S. Towards computing distances between programs via Scott domains. *Lect. Notes Comput. Sci.* **1997**, *1234*, 33–43.
9. Bukatin, M.A.; Shorina, S.Y. Partial metrics and co-continuous valuations. *Lect. Notes Comput. Sci.* **1998**, *1378*, 125–139.
10. O'Neill, S.J. Partial metrics, valuations and domain theory. *Ann. N. Y. Acad. Sci.* **1996**, *806*, 304–315. [CrossRef]
11. Romaguera, S.; Schellekens, M. Weightable qusi-metric semigroups and semilattices. *Electron. Notes Theor. Comput. Sci.* **2001**, *40*, 347–358. [CrossRef]
12. Romaguera, S.; Schellekens, M. Partial metric monoids and semivaluation spaces. *Topol. Its Appl.* **2005**, *153*, 948–962. [CrossRef]
13. Schellekens, M. A characterization of partial metrizability: Domains are quantifiable. *Theor. Comput. Sci.* **2003**, *305*, 409–432. [CrossRef]
14. Schellekens, M. The correpondence between partial metrics and semivaluations. *Theor. Comput. Sci.* **2004**, *315*, 135–149. [CrossRef]
15. Schellekens, M. The Smyth completion: A common foundation for denotational semantics and complexity analysis. *Electron. Notes Theor. Comput. Sci.* **1995**, *1*, 535–556. [CrossRef]
16. Waszkierwicz, P. Quantitative continuous domains. *Appl. Categ. Struct.* **2003**, *11*, 41–67. [CrossRef]
17. Waszkierwicz, P. Distance and measurement in domain theory. *Electron. Notes Theor. Comput. Sci.* **2001**, *45*, 448–462 [CrossRef]
18. Waszkierwicz, P. Partial metrizability of continuous posets. *Math. Struct. Comput. Sci.* **2006**, *16*, 359–372. [CrossRef]
19. Jachymski, J. Order-theoretic aspects of metric fixed point theory. In *Handbook of Metric Fixed Point Theory*; Springer: Dordrech, The Netherlands, 2001; pp. 613–641.
20. Dugundi, J.; Granas, A. *Fixed Point Theory*; Springer: New York, NY, USA, 2003.
21. Heckmann, R. Approximation of metric spaces by partial metric spaces. *Appl. Categ. Struct.* **1999**, *7*, 71–83. [CrossRef]
22. O'Neill, S.J. *Two Topologies Are Better than One*; Research Report CS-RR-293; Department of Computer Science, University of Warwick: Coventry, UK, March 1995.
23. Kahn, G. The semantics of a simple language for parallel processing. In *Proceedings of the IFIC Congress 74*; Elsevrier: Amsterdam, The Netherlands, 1974; pp. 471–475.
24. Matthews, S.G. An extensional treatment of lazy data flow deadlok. *Theor. Comput. Sci.* **1995**, *151*, 195–205. [CrossRef]
25. Oltra, S.; Romaguera, S.; Sánchez-Pérez, E.A. Bicompleting weightable quasi-metric spaces and partial metric spaces. *Rend. Del Circ. Mat. Palermo* **2002**, *51*, 151–162. [CrossRef]
26. Alghamdi, M.A.; Shahzad, N.; Valero, O. Fixed point theorems in generalized metrics spaces with applications to computer science. *Fixed Point Theory Appl.* **2013**, *2013*, 118. [CrossRef]