

Concept Paper

Electromagnetic Wave Scattering by Small Impedance Particles of an Arbitrary Shape and Applications

Alexander G. Ramm

Mathematics Department, Kansas State University, Manhattan, KS 66506, USA;

E-Mail: ramm@math.ksu.edu

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Abstract: The proposal deals with electromagnetic (EM) wave scattering by one and many small impedance particles of an arbitrary shape. Analytic formula is derived for EM wave scattering by one small impedance particle of an arbitrary shape and an integral equation for the effective field in the medium where many such particles are embedded. These results are applied for creating a medium with a desired refraction coefficient. The proposed theory has no analogs in the literature. (Mathematical Subject Classification: 35J05, 35J25, 65N12, 78A25, 78A48.)

Keywords: Electromagnetic wave scattering; small impedance particles; many-body scattering problem; refraction coefficient; permeability; creating materials with a desired refraction coefficient

1. Statement of the Problem

Let

$$\nabla \times E = i\omega\mu H, \quad \nabla \times H = -i\omega\epsilon E \quad (1)$$

$$E = E_0 + v_E, \quad E_0 = \xi e^{ik\alpha \cdot x}, \quad \alpha \cdot \xi = 0 \quad (2)$$

$$\frac{\partial v_E}{\partial r} - ikv_E = o\left(\frac{1}{r}\right), \quad r = |x| \rightarrow \infty \quad (3)$$

Equation (1) hold in $D' := \mathbb{R}^3 \setminus D$, D is a bounded domain, $a := \frac{1}{2}\text{diam}D$, $ka \ll 1$, i.e., D is a small body, the boundary S of D is $C^{1,\gamma}$ -smooth, i.e., in the local coordinates the equation of S is $z = f(x, y)$ and f has first derivative, satisfying the Hölder condition with exponent $0 < \gamma \leq 1$. The impedance boundary condition on S is

$$[N, [E, N]] = \zeta[N, H] \quad (4)$$

where N is the unit normal to S pointing out of D , ζ is the impedance, $\text{Im}\zeta \geq 0$, $\zeta = \text{const}$. Vector ξ in Equation (2) is constant, $\alpha \in S^2$, S^2 is the unit sphere in \mathbb{R}^3 , condition $\alpha \cdot \xi = 0$ implies $\nabla \cdot E_0 = 0$. The $k = \omega\sqrt{\epsilon\mu}$ is the wave number, ω is the frequency, ϵ and μ are dielectric and magnetic constants.

The problem PI (Principal Investigator) proposes to investigate is the electromagnetic (EM) wave scattering problem by one and many small impedance bodies (particles) of an arbitrary shape.

If there are M small particles, then $D := \cup_{m=1}^M D_m$, $D' := \mathbb{R}^3 \setminus D$, $\zeta = \frac{\zeta_m}{a^\kappa}$ on S_m , the surface of D_m , $\zeta_m := h(x_m)$, $x_m \in D_m$ is an arbitrary point in D_m , $h(x)$, $\text{Im}h \geq 0$ is a given continuous function in \mathbb{R}^3 , and $\kappa \in [0, 1)$ is a parameter. The function $h(x)$ and the parameter κ can be chosen by the experimentalist as he wishes.

Equations (1)–(4) can be reduced to finding vector E , and if E is found then H is found from the equation $H = \frac{\nabla \times E}{i\omega\mu}$. The problem for E is:

$$\nabla \times \nabla \times E = k^2 E, \quad k^2 := \omega^2 \epsilon \mu \quad (5)$$

where we have assumed $\nabla \times \mu H = \mu \nabla \times H$

$$[N, [E, N]] = \frac{\zeta}{i\omega\mu} [N, \nabla \times E] \text{ on } S \quad (6)$$

and Equations (2) and (3) remain valid.

The *first basic result* PI proposes to obtain is an analytic, closed-form formula for the EM field scattered by the small impedance body D of an arbitrary shape.

The *second basic result* PI proposes to obtain is a numerical method for solving many-body EM wave scattering problem in the case of M small impedance bodies of an arbitrary shape, where M ranges from $O(10)$ to $O(10^{12})$. The usual approach, based on solving boundary integral equations, is not feasible when M is large. A new approach and new ideas are needed. These are described in Section 2.

The *third basic result* PI proposes to derive is an integral equation for the effective field in the medium in which many small impedance bodies (particles) are embedded. This allows PI to derive an analytic formula for the refraction coefficient of the above medium. PI proposes to give a method for creating materials with a desired refraction coefficient by embedding many small impedance particles in the given medium. The distribution of the small particles and their boundary impedances determine the desired refraction coefficient.

PI and his Ph.D students (research assistants) plan to obtain numerical implementation of the PI's theory, developed in this proposal. Since the number of small particles is very large, the numerical implementation requires solving very large linear algebraic systems. PI plans to use GMRES method ([18]) for solving such systems and PETSC libraries to do the computations in parallel ([1]).

2. Description of PI's Approach

Let us look for the solution Equations (1)–(4) of the form

$$E(x) = E_0(x) + \nabla \times \int_S g(x, t) \sigma(t) dt \quad (7)$$

where $\sigma(t)$ is some tangential to S field and dt is the element of the surface area. Let

$$Q := \int_S \sigma(t) dt \quad (8)$$

Then Equation (7) can be written as

$$E(x) = E_0(x) + [\nabla g(x, x_1), Q] + \nabla \times \int_S (g(x, t) - g(x, x_1))\sigma(t)dt \quad (9)$$

where $x_1 \in D$ is an arbitrary point and one has used the formula $\nabla \times g(x, x_1)Q = [\nabla g, Q]$, where $[A, B] = A \times B$ is the vector product, $A \cdot B = (A, B)$ is the scalar product. PI's main idea is to prove that if $ka \ll 1$ (small D) and $d := |x - x_1| \gg a$, then

$$\left| \nabla \times \int_S (g(x, t) - g(x, x_1))\sigma(t)dt \right| \ll |[\nabla g(x, x_1), Q]| \quad (10)$$

Therefore, the problem of finding a function $\sigma(t)$ from the boundary condition Equation (6), which amounts to solving a boundary integral equation (bie), the problem Equations (2), (3), (5), (6) is reduced to finding one pseudovector Q . Note that since E is a vector σ has to be a pseudo-vector field. PI derives an asymptotic analytic formula, as $a \rightarrow 0$, namely

$$Q = -\frac{\zeta|S|}{i\omega\mu}\tau\nabla \times E_0(x_1), \quad a \rightarrow 0 \quad (11)$$

where $|S|$ is the surface area of S and

$$\tau := (\tau_{jp}) := \delta_{jp} - \frac{1}{|S|} \int_S N_j(s)N_p(s)ds \quad (12)$$

$N_j(s)$ is the j -th component of the unit normal $N(s)$ at the point $s \in S$. Thus the solution to the EM wave scattering problem Equations (2), (3), (5), (6) is

$$E(x) = E_0(x) - \frac{\zeta|S|}{i\omega\mu}[\nabla_x g(x, x_1), \tau\nabla \times E(x_1)] \quad (13)$$

This formula is valid if $ka \ll 1$ and $|x - x_1| \gg a$. To derive formula Equation (13), PI plans to use boundary condition Equation (6) not for solving the result from Equation (6) equation for σ , but for finding asymptotic formula Equation (11).

If there are $M \gg 1$ small bodies, then PI looks for the solution to problem Equations (2), (3), (5), (6) with $D = \cup_{m=1}^M D_m$ of the form

$$E(x) = E_0(x) + \sum_{m=1}^M \nabla \times \int_{S_m} g(x, t)\sigma_m(t)dt \quad (14)$$

PI checks that the right-hand side of Equation (14) satisfies Equations (2), (3), (5) for any σ_m , so that problem Equations (2), (3), (5), (6) is solved if the boundary conditions Equation (6) is satisfied on every $S_m, 1 \leq m \leq M$. The field Equation (14) can be written as

$$E(x) = E_0(x) + \sum_{m=1}^M [\nabla g(x, x_m), Q_m] + \sum_{m=1}^M \nabla \times \int_{S_m} (g(x, t) - g(x, x_m))\sigma_m(t)dt \quad (15)$$

where $x_m \in D_m$ are arbitrary points. PI proves that

$$\left| \nabla \times \int_{S_m} (g(x, t) - g(x, x_m))\sigma_m(t)dt \right| \ll |[\nabla g(x, x_m), Q_m]| \quad (16)$$

Thus, the solution to many-body EM wave scattering problem is given by the formula

$$E(x) = E_0(x) - \sum_{m=1}^M \frac{\zeta_m |S_m|}{i\omega\mu} [\nabla_x g(x, x_m), \tau_m \nabla \times E_e(x_m)] \quad (17)$$

where $E_e(x_m)$ is the effective field acting on m -th particle. Let us assume that small particles D_m are of the same shape and size. Then $\tau_m = \tau$ and $|S_m| = c_S a^2$, where $c_S > 0$ is a constant. For example, the constant $c_S = 4\pi$ and $\tau = \frac{2}{3}\delta_{jp}$ if S_m are spheres. Let us assume that

$$\zeta_m = \frac{h(x_m)}{a^\kappa} \quad (18)$$

where $h(x)$ is a given continuous function, $\text{Re}h \geq 0$, in a bounded domain Ω in which the small bodies D_m are distributed, and $\kappa \in [0, 1)$ is a given parameter. The experimentalist can choose $h(x)$ and κ as he wants. Assume also that the number $N(\Delta)$ of small bodies (or points x_m) in any subdomain Δ is given by the formula

$$N(\Delta) = \frac{1}{a^{2-\kappa}} \int_{\Delta} N(x) dx (1 + o(1)) \quad a \rightarrow 0 \quad (19)$$

where $N(x) \geq 0$ is a continuous in Ω function that can be chosen by the experimentalist as he wants.

Let us define the effective field acting on j -th particle by the formula

$$E_e(x) = E_0(x) - \frac{c_S}{i\omega\mu} \sum_{m \neq j} a^{2-\kappa} h(x_m) [\nabla_x g(x, x_m), \tau \nabla \times E_e(x_m)] \quad (20)$$

To find vectors $\nabla \times E(x_m)$, PI applies the operator $\nabla \times$ to Equation (20) and then sets $x = x_j$ to get:

$$\nabla \times E(x_j) = \nabla \times E_0(x_j) - \frac{c_S a^{2-\kappa}}{i\omega\mu} \sum_{m \neq j} h(x_m) \nabla \times [\nabla g(x, x_m), \tau \nabla \times E(x_m)]|_{x=x_j} \quad (21)$$

This is a linear algebraic system for finding vectors $\nabla \times E(x_m)$ $1 \leq m \leq M$.

If M is very large, say $M \geq 10^5$, then one reduces the order of the system Equation (21) as follows. Consider a partition of the domain Ω in the union of $P \ll M$ small cubes Δ_p of side $b = b(a)$, $1 \leq p \leq P$. Assume that $b \gg d \gg a$, where $d = d(a)$ is the minimal distance between two neighboring particles. Let x_p be the center of the cube Δ_p , and denote $\nabla \times E(x_p) := A_p$, $\nabla \times E_0(x_p) = A_{0p}$. Then Equation (21) can be transformed as follows:

$$A_q = A_{0q} - \frac{c_S}{i\omega\mu} \sum_{p \neq q} \nabla \times [\nabla g(x, x_p), \tau A_p] h(x_p) \sum_{x_m \in \Delta_p} 1 \quad (22)$$

Using Equation (19) and denoting $|\Delta_p| := \text{volume } \Delta_p$, one gets:

$$a^{2-\kappa} \sum_{x_m \in \Delta_p} 1 = N(x_p) |\Delta_p|, \quad a \rightarrow 0 \quad (23)$$

Thus Equation (22) takes the form

$$A_q = A_{0q} - \frac{c_S}{i\omega\mu} \nabla \times \sum_{p \neq q} [\nabla g(x, x_p), \tau A_p] h(x_p) N_p |\Delta_p| \quad (24)$$

As $a \rightarrow 0$, Equation (24) tends to the limiting integral equation

$$A(x) = A_0(x) - \frac{c_S}{i\omega\mu} \nabla \times \nabla \times \int_{\Omega} g(x, y) \tau A(y) h(y) N(y) dy \quad (25)$$

The limiting form of Equation (20) is

$$E(x) = E_0(x) - \frac{c_S}{i\omega\mu} \nabla \times \int_{\Omega} g(x, y) \tau \nabla \times E(y) h(y) N(y) dy \quad (26)$$

To demonstrate possible application of Equation (26) to creating materials with a desired refraction coefficient, apply the operator $\nabla \times \nabla \times$ to Equation (26). The result is

$$\nabla \times \nabla \times E = k^2 E_0 - \nabla \times (\text{grad div} - \nabla^2) \int_{\Omega} g(x, y) \tau \nabla \times E h(y) N(y) dy \quad (27)$$

Since $\nabla \times \text{grad} = 0$ and $-\nabla^2 g = \delta(x - y) + k^2 g$, Equation (27) yields

$$\nabla \times \nabla \times E = k^2 E - \frac{c_S}{i\omega\mu} \nabla \times (\tau \nabla \times E(x) h(x) N(x)) \quad (28)$$

Assume for simplicity that $h(x)$ and $N(x)$ are constants and D_m are spheres, i.e., $\tau = \frac{2}{3}I$. Then Equation (28) yields

$$\nabla \times \nabla \times E = k^2 E - \frac{c_S}{i\omega\mu} \frac{2}{3} h N \nabla \times \nabla \times E \quad (29)$$

This means that the refraction coefficient of the new medium which is obtained by embedding of many small impedance balls into Ω , is equal to

$$n^2 = 1 + \frac{c_S}{i\omega\mu} \frac{2}{3} h N, \quad n_0^2 = \sqrt{\epsilon\mu} \quad (30)$$

Since h and N are at our disposal, $\text{Re}h \geq 0$, one can change the original refraction coefficient in a desired direction. For example, suppose one wants to have $\frac{n^2}{n_0^2} = b_1 + ib_2$. Denote $\frac{2}{3} \frac{c_S N}{\omega\mu} := C$, $h = h_1 + ih_2$. Then Equation (30) yields

$$b_1 = \frac{1 + Ch_2}{(1 + h_2 C)^2 + h_1^2 C^2}; \quad b_2 = \frac{h_1 C}{(1 + h_2 C)^2 + h_1^2 C^2}; \quad C := \frac{2c_S N}{3\omega\mu} \quad (31)$$

Formula Equation (28) gives also a recipe for creating materials with a desired magnetic permeability $\mu(x)$. To formulate this recipe apply the operator $\nabla \times$ to the first Equation (1) assuming $\mu = \mu(x)$. This yields

$$\nabla \times \nabla \times E = k^2 E + i\omega[\nabla\mu, H] = k^2 E + \left[\frac{\nabla\mu}{\mu}, \nabla \times E \right], \quad k^2 = \omega^2 \epsilon(x) \mu(x) \quad (32)$$

Compare Equations (28) and (32) assuming that D_m are balls of radius a , so that $\tau = \frac{2}{3}\delta_{jp}$, and denote $\mu = \mu_0$ in Equation (28), while $\mu(x)$ is the magnetic permeability of the new medium in which many small impedance particles are embedded.

Then Equation (28) can be rewritten as

$$\nabla \times \nabla \times E = \frac{k^2 E}{1 - i \frac{2c_S h(x) N(x)}{3\omega\mu_0}} + \frac{i2c_S}{3\omega\mu_0} \frac{[\nabla(h(x)N(x)), \nabla \times E]}{1 - i \frac{2c_S h(x) N(x)}{3\omega\mu_0}} \quad (33)$$

Compare Equation (33) with Equation (32) and remember that $H = \frac{\nabla \times E}{i\omega\mu_0}$. Then one obtains

$$\mu(x) = \frac{\mu_0}{1 - i\frac{2c_S h(x)N(x)}{3\omega\mu_0}}; \quad \frac{\nabla\mu}{\mu} = \frac{i2c_S \nabla(h(x)N(x))}{3\omega\mu_0(1 - i\frac{2c_S h(x)N(x)}{3\omega\mu_0})} \quad (34)$$

Thus, $\mu(x)$ can be changed by changing $h(x)$ and $N(x)$.

3. Comparison of the Proposed Research with the Known Results

EM wave scattering by small bodies is a classical field of theoretical and applied physics and mathematics. Rayleigh in 1871 [17] understood that the main term in the scattered field is given by the dipole radiation. He did not give formulas for the induced dipole moment for a body of an arbitrary shape. This was done about 100 years later by PI ([6–8]). In 1908 Mie [5] gave a solution of the EM wave scattering by one perfectly conducting sphere by separation of variables ([19]). In [2–4], these results are summarized. The origin of these results is the work [17]. There were no results which would give analytic formula, asymptotically exact as $a \rightarrow 0$, for the EM wave field scattered by one small impedance particle (body) of an arbitrary shape. Such a formula PI proposes to derive. There were no methods for solving many-body EM wave scattering problem when the number M of small impedance bodies is very large $M = O(10^5) - O(10^{10})$. Such a method PI proposes to develop. There were no methods for deriving an integral equation for the effective EM field in the medium in which many small impedance bodies are embedded. PI proposes to derive such an equation and give on this basis a method for creating materials with a desired refraction coefficient and/or desired magnetic permeability.

In Rayleigh theory of wave scattering by small bodies the order of the scattering amplitude is $O(a^3)$, where a is the characteristic size of the small body, $a \rightarrow 0$. In PI's theory of wave scattering by small impedance body the scattering amplitude is of the order $O(a^{2-\kappa})$, where $\kappa \in [0, 1)$. Thus, in PI's theory the order of the scattering amplitude is much larger than in Rayleigh's theory. This difference comes from the fact that in Rayleigh's theory the scattering amplitude is proportional to the volume of the small body, while in PI's theory it is proportional to $\zeta|S|$, where $|S| = O(a^2)$ is the surface area of the small body, and $\zeta = O(a^{-\kappa})$ is its boundary impedance. This difference may be used practically.

The ideas of this research were developed in [6–15]. In [16] numerical results, based on this proposal, are presented.

Remark. If Equation (7) is assumed for one small body D , where $v_E := \nabla \times \int_S g(x, t)\sigma(t)dt$ and $v_H := \frac{\nabla \times v_E}{i\omega\mu}$, then

$$\nabla \times v_E = i\omega\mu v_H \quad \text{in } D' \quad (35)$$

and one can derive the following equations:

$$\nabla \times v_H = -i\omega\epsilon v_E \quad \text{in } D' \quad (36)$$

$$\nabla \times \nabla \times v_E = k^2 v_E \quad \text{in } D' \quad (37)$$

$$\nabla \times \nabla \times v_H = k^2 v_H \quad \text{in } D' \quad (38)$$

Conflicts of Interest

The author declares no conflict of interest.

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