



Article Robust IMM Filtering Approach with Adaptive Estimation of Measurement Loss Probability for Surface Target Tracking

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Abstract: A suitable jump Markov system (JMS) filtering approach provides an efficient technique for tracking surface targets. In complex surface target tracking situations, due to the joint influences of lost measurements with an unknown probability and heavy-tailed measurement noise (HTMN), the estimation accuracy of conventional interacting multiple model (IMM) methods may be seriously degraded. Aiming to address the filtering issues in JMSs with HTMNs and random measurement losses, this paper presents an IMM filtering approach with the adaptive estimation of unknown measurement loss probability. In this study, we assumed that the measurement noises obey student's t-distributions and then proposed Bernoulli random variables (BRVs) to characterize the random measurement loss. Notably, by converting the two likelihood functions from the weighted sum form to exponential multiplication, we established hierarchical Gaussian state space models to directly utilize the variational inference method. The system state vectors, unknown distribution parameters, BRVs, and unknown measurement loss probabilities were estimated simultaneously according to the variational Bayesian inference in the IMM framework. The results of maneuvering target tracking simulations verified that the presented filtering approach demonstrated superior estimation accuracy compared to existing IMM filters.

Keywords: variational Bayesian; interacting multiple model; random measurement loss; surface target tracking; heavy-tailed measurement noises

1. Introduction

State estimation holds significant importance for the maneuvering target tracking of conventional vessels and surface autonomous vessels [1]. Aiming to estimate the states of hidden dynamic systems from noisy measurements according to proper criteria, state estimation has attracted wide attention in signal processing communities. For linear Gaussian dynamic systems, the optimal solution for real-time online state estimation can be obtained by the Kalman filter (KF) (for abbreviations, see Table 1). However, the uncertainty of the system model in a jump Markov system (JMS) [2,3] may cause the typical KF's estimation accuracy to decline dramatically. In addition, the conventional KF requires Gaussian noise, and all measurements need to be obtained quickly. These assumptions are usually not satisfied in real applications, leading to limited filtering performance. Consequently, the expansion of the KF under different assumptions has garnered significant interest due to its extensive utilization in engineering [4–9].

Tracking surface targets can be considered a state estimation issue in JMSs. There is no existing optimal Bayesian solution to estimate the states of JMSs, since they introduce problems of nondeterministic polynomial difficulties and computational intractability [10]. In recent decades, a series of sub-optimal estimators have been proposed, such as particle filters, the generalized pseudo-Bayesian method, and the interacting multiple model (IMM)



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Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). approach, etc. [11–13]. Among these solutions, the IMM approach stands out as an efficient algorithm due to its ability to strike a balance between acceptable filtering performance and reasonable computational costs [13,14]. The theoretical details of the IMM approach can be found in [15]. Performance analyses of the IMM filter were provided by [16,17]. Typical IMM filters run KFs in parallel to update the state estimate of each mode and then fuse the sub-filter outputs through moment matching theory to obtain the system estimation results. An efficient way to improve the model-matched filters' performance is variational Bayesian (VB) theory, which executes approximation on conjugate exponential models under a reasonable calculation complexity [18,19]. By combining VB inference with the KF, many scholars have achieved remarkable breakthroughs [20–24]. In the past few years, VB-type IMM filters have been introduced to address the filtering issue of JMSs. The authors of [25] designed an adaptive IMM filter for handling unknown process and measurement noise covariances in JMSs. By utilizing the VB technique and the weighted Kullback–Leibler (KL) average method, the system state vectors and the noise covariances were jointly estimated. This technique achieved significant estimation performance in scenarios of uncertain noise covariances. However, this method required Gaussian-distributed measurement noise, and all measurement data had to be obtained quickly, which may not be possible in real application situations under the joint influences of measurement outliers and interrupted communication channels.

Table 1. Notations and definitions used in this article.

Notation	Definition	Notation	Definition	
CV	Constant velocity	UMLP	Unknown measurement loss probability	
CT	Coordinated turn	HGSSM	Hierarchical Gaussian state space model	
KF	Kalman filter	$St(\cdot; \mu, P, \lambda)$	Student's t-distribution,	
KL	Kullback–Leibler		μ -mean vector, <i>P</i> -scale matrix,	
VB	Variational Bayesian		λ -degree of freedom parameter	
BRV	Bernoulli random variable	$N(\cdot; \mu, P)$	Gaussian distribution,	
IMM	Interacting multiple model		μ -mean vector, <i>P</i> -scale matrix	
JMS	Jump Markov system	$G(\cdot; s, k)$	Gamma distribution,	
PDF	Probability density function		s-shape parameter,	
STD	Student's t-distribution		<i>k</i> -rate parameter	
SSIT	Single-step implementation time	$Be(\cdot; \alpha, \beta)$	Beta distribution	
HTMN	Heavy-tailed measurement noise		α - and β -shape parameters	

Considering the adverse effects of measurement outliers on state estimation in JMSs, the authors of [26] utilized Student's t-distribution (STD) to model the heavy-tailed measurement noise (HTMN). By introducing the VB approach, the posterior probability density functions (PDFs) of the state and noise parameters were approximated, which overcame the restrictions of the Gaussian assumption in real applications. In [27], the authors proposed a KF with a fading factor to cope with the state transition model mismatch and non-zero mean value statistical characteristics caused by outliers. The measurement noises were modeled as skew STDs, and the system state vectors were inferred by VB techniques. Additionally, the authors of [28] improved an IMM filter by modeling the measurement noises, scale covariance matrices, and freedom degree parameter as STDs, inverse Wishart distributions, and gamma distributions, respectively. According to the VB approach, system state vectors and unknown parameters were inferred simultaneously. This algorithm showed superior estimation performance when HTMN was present. However, the abovementioned algorithms may be not able to achieve satisfactory estimation accuracy in the presence of random measurement loss. To deal with this problem, the authors of [29] presented an adaptive IMM filter to address the unknown measurement loss probability (UMLP). The prior distribution of UMLP was selected rationally. The system state vectors, Bernoulli random variable (BRV), and UMLP were inferred. However, this algorithm assumed Gaussian-distributed measurement noise, and this assumption resulted in restricted estimation accuracy due to the sensitivity of Gaussian distributions to measurement outliers [30]. In a complex surface target tracking environment, HTMN and UMLP often coexist, and the generalized IMM algorithm for HTMN and UMLP needs to be further studied.

To address the JMS filtering issue under HTMN and UMLP effectively, this study designed a VB-based robust IMM Gaussian approximate filter (IMM-VBRGAF). The HTMN was modeled rationally, and the measurement loss in JMSs was characterized appropriately. By converting the forms of the measurement likelihood functions, a new hierarchical Gaussian state space model (HGSSM) could be constructed. Then, the state vectors, noise parameters, BRV, and UMLP were estimated. The surface target tracking simulation results verified the superiority of the presented approach. Table 2 outlines the existing algorithms currently in place.

Algorithm Technique and Methodology		Deficiencies (Compared with Proposed Filter)		
VBAKF [24]	(1) Gaussian noise modeling(2) VB inference(3) UMLP adaptive estimation	(1) Ignores the measurement outliers(2) Incapable of addressing the system-model mismatch		
VBRSF [23]	(1) Student's t noise modeling(2) VB inference(3) Multi-sensor fusion	(1) Ignores the measurement loss(2) Incapable of addressing the system–model mismatch		
IMM-CKF [13]	(1) Gaussian noise modeling(2) IMM filtering framework(3) Cubature Kalman filter	(1) Large estimation errors(2) Incapable of addressing measurement loss(3) Incapable of addressing measurement outliers		
IMM-VBKF [25]	(1) Gaussian noise modeling(2) VB inference(3) Noise covariance estimation	(1) Limited estimation accuracy(2) Limited to Gaussian noise		
IMM-VBAKF [29]	(1) Gaussian noise modeling(2) VB inference(3) UMLP adaptive estimation	(3) Sensitive to outliers in measurement noise		
IMM-VBSTDF [26]	(1) Student's t noise modeling(2) VB inference(3) IMM filtering framework	(1) Limited estimation accuracy(2) Incapable of addressing measurement loss		
IMM-ORSRCKF [28]	(1) Student's t noise modeling(2) VB inference(3) Noise parameter estimation			
Proposed IMM-VBRGAF	(1) Student's t noise modeling(2) VB inference(3) UMLP adaptive estimation	(1) Increased computational burden		

Table 2. Outlines of existing algorithms.

The main contributions of this paper are summarized below:

- The HTMN and one-step predicted PDF in this paper were modeled accurately as STDs and Gaussian distributions, respectively. A BRV was proposed to describe the measurement loss. To directly include the VB theory in this method, the form of two measurement likelihood functions was changed from a weighted summation to exponential multiplication, and a new HGSSM was therefore established.
- Considering the coupling of the state vector with HTMN, this paper proposes an approach that combines VB inference with the IMM method. The posterior PDFs conditioned on each mode were recursively approximated by a measurement updating process. Based on VB theory, the state vector, noise parameters, BRV, and UMLP could be jointly obtained. The final estimate was obtained by performing a weighted summation of the outputs from the sub-filters.

• The simulation results of the surface maneuvering target tracking validated that the proposed filtering approach surpassed existing IMM methods in terms of estimation accuracy. The proposed approach provides a solution for the filtering issues arising from the coexistence of HTMN and UMLP.

The remainder of this paper is arranged as follows. Section 2 provides the problem statement. The proposed HGSSM and IMM-VBRGAF are summarized in Section 3. In Section 4, the estimation performance of the designed filter is compared with that of existing filters. The conclusion of this article can be found in Section 5.

2. Problem Statement

Consider the following state space model of a JMS [29]:

$$x_{s}^{j} = f_{s-1}^{j} \left(x_{s-1}^{j} \right) + g_{s-1}^{j} \tag{1}$$

$$z_s^{\prime} = \tau_s^{\prime} h_s^{\prime} \left(x_s^{\prime} \right) + r_s^{\prime} \tag{2}$$

where *s* denotes the discrete time index; $x_s^j \in \mathbb{R}^{n \times 1}$ and $z_s^j \in \mathbb{R}^{m \times 1}$ are the system state vector and measurement vector, respectively, conditioned on the system mode index $M_s = j$; and *m* and *n* are their dimension numbers. M_s denotes a discrete variable indicating the state of the Markov chain, and its value is selected from $\{1, 2, 3, \ldots, l\}$ by the transition probability matrix $\Pi = [\pi_{ij}]_{l \times l'}$, where π_{ij} represents the transition probability from model *i* to *j* satisfying the following formulas:

$$\pi_{ij} \stackrel{\Delta}{=} P\{M_s = j | M_s = i\} \tag{3}$$

$$\sum_{j=1}^{l} \pi_{ij} = 1 \tag{4}$$

where $f_{s-1}^{j}(\cdot)$ and $h_{s}^{j}(\cdot)$ are the process and measurement functions, respectively, of the *j*-th mode. The process function $f_{s-1}^{j}(\cdot)$ in this study was switched between two different kinematic models (the coordinated turn (CT) model $f_{s}^{1}(\cdot)$ and the constant velocity (CV) model $f_{s}^{2}(\cdot)$ [3,29,31,32]) to describe the problem of the system–model mismatch in JMSs. g_{s-1}^{j} refers to the white Gaussian process noise vector with a mean value of zero and nominal covariance matrix Q_{s}^{j} , while r_{s}^{j} represents the white HTMN vector caused by measurement outliers, and the random variable τ_{s}^{j} was utilized to characterize the phenomenon of received or lost measurements. Note that the random variables x_{s}^{j} , g_{s-1}^{j} , r_{s}^{j} , and τ_{s}^{j} were assumed to be mutually independent.

 τ_s^j in Equation (2) was assumed to obey the Bernoulli distribution with two optional values, 1 and 0. In this paper, $\tau_s^j = 1$ denotes that the measurement was available from the sensor, and $\tau_s^j = 0$ means that the measurement was lost, i.e., the measurement only contained the noise vector r_s^j . The probabilities of τ_s^j can be defined as follows:

$$\Pr\left(\tau_s^j = 1\right) = 1 - \varphi_s^j \tag{5}$$

$$\Pr\left(\tau_s^j = 0\right) = \varphi_s^j \tag{6}$$

where $Pr(\cdot)$ represents the random event probability, and $\varphi_s^j \in [0, 1]$ denotes the UMLP.

According to Equations (1)–(6), due to the frequent changes in the motion models of JMSs and the fact that the measurements can be easily affected by measurement outliers and interrupted communication channels, the accuracy of traditional Kalman-type filtering may be insufficient or even very poor. Therefore, the goal of this work was to design a recursive filter to estimate the state of a JMS with HTMN and UMLP.

3. Main Results

3.1. The Proposed HGSSM

According to Equations (2), (5) and (6), the conditional likelihood PDF could be formulated as a weighted sum:

$$p\left(z_{s}^{j} \middle| x_{s}^{j}, \varphi_{s}^{j}, M_{s} = j\right) = \sum_{\tau_{s}^{j}=0}^{1} p\left(z_{s}^{j}, \tau_{s}^{j} \middle| x_{s}^{j}, \varphi_{s}^{j}, M_{s} = j\right)$$

$$= \Pr\left(\tau_{s}^{j} = 0\right) p\left(z_{s}^{j} \middle| x_{s}^{j}, \tau_{s}^{j} = 0, M_{s} = j\right) + \Pr\left(\tau_{s}^{j} = 1\right) p\left(z_{s}^{j} \middle| x_{s}^{j}, \tau_{s}^{j} = 1, M_{s} = j\right)$$
(7)
$$= \left(1 - \varphi_{s}^{j}\right) p_{v_{s}^{j}}\left(z_{s}^{j} - h_{s}^{j}\left(x_{s}^{j}\right)\right) + \varphi_{s}^{j} p_{v_{s}^{j}}\left(z_{s}^{j}\right)$$

where $p_{v_{\alpha}^{j}}(\cdot)$ is the PDF of measurement noise.

Remark 1. The document text continues here. From Equation (7), $p(z_s^j | x_s^j, \varphi_s^j, M_s = j)$ in a sum form has neither closed nor conjugate properties. It is unfeasible to employ the VB method directly to estimate the system state and unknown parameters. To solve this problem, by utilizing Equations (5) and (6), the probability of the BRV in Equation (7) is converted into the probability mass function of τ_s^j , i.e.,

$$p\left(\tau_{s}^{j}\middle|\varphi_{s}^{j}\right) = \left(1 - \varphi_{s}^{j}\right)^{\tau_{s}^{j}}\left(\varphi_{s}^{j}\right)^{\left(1 - \tau_{s}^{j}\right)} \tag{8}$$

Based on Equations (7) and (8), the conditional likelihood PDF can be reformulated as

$$p\left(z_{s}^{j}\left|x_{s}^{j},\varphi_{s}^{j},M_{s}=j\right)=\sum_{\tau_{s}^{j}=0}^{1}p\left(\tau_{s}^{j}\left|\varphi_{s}^{j}\right)p\left(z_{s}^{j}\left|x_{s}^{j},\tau_{s}^{j},M_{s}=j\right)\right.$$

$$=\sum_{\tau_{s}^{j}=0}^{1}\left(1-\varphi_{s}^{j}\right)^{\tau_{s}^{j}}p_{\upsilon_{s}^{j}}\left(z_{s}^{j}-h_{s}^{j}\left(x_{s}^{j}\right)\right)^{\tau_{s}^{j}}\left(\varphi_{s}^{j}\right)^{\left(1-\tau_{s}^{j}\right)}p_{\upsilon_{s}^{j}}\left(z_{s}^{j}\right)^{\left(1-\tau_{s}^{j}\right)}$$

$$=\sum_{\tau_{s}^{j}=0}^{1}p\left(\tau_{s}^{j}\left|\varphi_{s}^{j}\right)p_{\upsilon_{s}^{j}}\left(z_{s}^{j}-h_{s}^{j}\left(x_{s}^{j}\right)\right)^{\tau_{s}^{j}}p_{\upsilon_{s}^{j}}\left(z_{s}^{j}\right)^{\left(1-\tau_{s}^{j}\right)}$$
(9)

According to Equation (9), in order to further use the VB method to obtain an approximate solution, the conditional likelihood function is rewritten in exponential multiplication form as follows:

$$p(z_{s}^{j}|x_{s}^{j},\tau_{s}^{j},M_{s}=j) = p_{r_{s}^{j}}(z_{s}^{j}-h_{s}^{j}(x_{s}^{j}))^{\tau_{s}^{j}}p_{r_{s}^{j}}(z_{s}^{j})^{(1-\tau_{s}^{j})}$$
(10)

Remark 2. In VB inference, selecting suitable conjugate prior distributions of random variables is necessary. Therefore, a new HGSSM was established by choosing a reasonable series of prior PDFs.

In terms of noise processing, rationally modeling the process and measurement noise is critical to estimation accuracy. In this article, the process noise in Equation (2) is white Gaussian noise. Therefore, the one-step predictive PDF could be formulated by Gaussian distribution:

$$p\left(x_{s}^{j}\left|z_{1:s-1}^{j}\right.\right) = N\left(x_{s}^{j}; \hat{x}_{s|s-1}^{j}, P_{s|s-1}^{j}\right)$$
(11)

To handle the HTMN caused by measurement outliers, since STD is more robust to outliers than Gaussian distribution [22,30,33], the likelihood PDF was modeled with STD. Based on the

hierarchical characteristics of STD, the likelihood PDF in Equation (10) could be rewritten in hierarchical Gaussian form as follows:

$$p\left(z_{s}^{j}\left|x_{s}^{j},\tau_{s}^{j},M_{s}=j\right.\right)=N\left(z_{s}^{j};h_{s}^{j}\left(x_{s}^{j}\right),R_{s}^{j}/\lambda_{s}^{j}\right)^{\tau_{s}^{j}}N\left(z_{s}^{j};0,R_{s}^{j}/\lambda_{s}^{j}\right)^{\left(1-\tau_{s}^{j}\right)}$$

$$(12)$$

$$p\left(\lambda_{s}^{j}\right) = G\left(\lambda_{s}^{j}; \frac{\sigma_{s}^{j}}{2}, \frac{\sigma_{s}^{j}}{2}\right)$$

$$\tag{13}$$

Considering that τ_s^j selects a value between 0 and 1 through Bernoulli distribution, the probability mass functions of τ_s^j are obtained as follows:

$$p\left(\tau_{s}^{j}\middle|\varphi_{s}^{j}\right) = \left(1 - \varphi_{s}^{j}\right)^{\tau_{s}^{j}}\left(\varphi_{s}^{j}\right)^{\left(1 - \tau_{s}^{j}\right)}$$
(14)

Since the conjugate prior distributions guarantee that the posterior and prior distributions can be in the same functional form, which can further ensure the utilization of the VB approach, the analysis is greatly simplified when the conjugate prior distributions are used. In this paper, the unknown time-varying mixing coefficient φ_s^j was assumed to obey beta distribution:

$$p\left(\varphi_{s}^{j}\right) = Be\left(\varphi_{s}^{j}; \hat{\alpha}_{s|s-1}^{j}, \hat{\beta}_{s|s-1}^{j}\right)$$

$$(15)$$

With the determination of the the conjugate prior distributions, the HGSSM is established by Equations (11)–(15), and a novel VB-based robust IMM filtering approach is presented based on the constructed HGSSM.

3.2. Robust IMM Filter with UMLP

Similar to the conventional IMM method, the proposed IMM-VBRGAF approach also comprises the following four main recursive processes: (1) the interacting/mixing process, (2) the mode-conditioned filtering process, (3) the mode probabilities update process, and (4) the combination process [28,29,32,34].

Step 1: Interacting/Mixing Process

The posterior PDF at time s - 1 is obtained as follows:

$$p(x_{s-1}, \varphi_{s-1}|z_{1:s-1}, M_s = j)$$

$$= \sum_{i=1}^{l} p(x_{s-1}, \varphi_{s-1}|z_{1:s-1}, M_{s-1} = i) P\{M_{s-1} = i|M_s = j, z_{1:s-1}\}$$

$$= \sum_{i=1}^{l} u_{s-1}^{ij} N(x_{s-1}; \hat{x}_{s-1}^i, P_{s-1}^i) Be(\varphi_{s-1}; \hat{a}_{s-1}^i, \hat{\beta}_{s-1}^i)$$
(16)

Here, the mixing probability u_{s-1}^{ij} is obtained by utilizing the transition probability π_{ij} :

$$u_{s-1}^{ij} = \frac{1}{\bar{\varepsilon}_j} \pi_{ij} u_{s-1}^i \tag{17}$$

where the normalization constant $\bar{\varepsilon}_i$ can be calculated by

$$\bar{\varepsilon}_j = \sum_{i=1}^l \pi_{ij} u_{s-1}^i \tag{18}$$

and u_{s-1}^i denotes the mode probability when the time index is $s - 1, i, j \in \{1, 2, ..., l\}$.

The sum term in Equation (16) can be approximated with the following formula:

$$p(x_{s-1},\varphi_{s-1}|z_{1:s-1},M_s=j) \approx N\left(x_{s-1};\hat{x}_{s-1|s-1}^{0j},P_{s-1|s-1}^{0j}\right)Be\left(\varphi_{s-1};\hat{\alpha}_{s-1}^{0j},\hat{\beta}_{s-1}^{0j}\right)$$
(19)

where the system state $\hat{x}_{s-1|s-1}^{0j}$ and covariance matrix $P_{s-1|s-1}^{0j}$ can be combined by the standard IMM approach, i.e.,

$$\begin{cases} \hat{x}_{s-1|s-1}^{0j} = \sum_{i=1}^{l} u_{s-1|s-1}^{ij} x_{s-1|s-1}^{i} \\ P_{s-1|s-1}^{0j} = \sum_{i=1}^{l} u_{s-1|s-1}^{ij} \left(P_{s-1|s-1}^{i} + \left(\hat{x}_{s-1|s-1}^{i} - \hat{x}_{s-1|s-1}^{0j} \right) \left(\hat{x}_{s-1|s-1}^{i} - \hat{x}_{s-1|s-1}^{0j} \right)^{T} \right) \end{cases}$$
(20)

and the shape parameters $\hat{\alpha}_{s-1}^{0j}$ and $\hat{\beta}_{s-1}^{0j}$ can also be mixed by the standard IMM method:

$$\begin{cases} \hat{\alpha}_{s-1}^{0j} = \sum_{i=1}^{l} u_{s-1}^{ij} \hat{\alpha}_{s-1|s-1}^{i} \\ \hat{\beta}_{s-1}^{0j} = \sum_{i=1}^{l} u_{s-1}^{ij} \hat{\beta}_{s-1|s-1}^{i} \end{cases}$$
(21)

Step 2: Mode-Conditioned Filtering Process

The mixing state vector $\hat{x}_{s-1|s-1}^{0j}$, covariance $P_{s-1|s-1}^{0j}$, and shape parameters of beta distribution $\hat{\alpha}_{s-1}^{0j}$ and $\hat{\beta}_{s-1}^{0j}$ obtained in the previous process are used as the inputs of the mode-conditioned filter. The one-step predictive PDF is written as follows:

$$p(x_s, \varphi_s | z_{1:s-1}, M_s = j) \simeq N\left(x_s; \hat{x}_{s|s-1}^j, P_{s|s-1}^j\right) Be\left(\varphi_s; \hat{\alpha}_{s|s-1}^j, \hat{\beta}_{s|s-1}^j\right)$$
(22)

Based on the posterior PDF $N(x_{s-1}; \hat{x}_{s-1|s-1}^{j}, P_{s-1|s-1}^{j}), \hat{x}_{s|s-1}^{j}$ and $P_{s|s-1}^{j}$ can be computed through the time-update step of the standard Gaussian approximate filter [20,35], i.e.,

$$\begin{cases} \hat{x}_{s|s-1}^{j} = \int f_{s-1}^{j} \left(x_{s-1}^{j}\right) N\left(x_{s-1}; \hat{x}_{s-1|s-1}^{j}, P_{s-1|s-1}^{j}\right) dx_{s-1}^{j} \\ P_{s|s-1}^{j} = \int f_{s-1}^{j} \left(x_{s-1}^{j}\right) f_{s-1}^{(j)T} \left(x_{s-1}^{j}\right) N\left(x_{s-1}; \hat{x}_{s-1|s-1}^{j}, P_{s-1|s-1}^{j}\right) dx_{s-1}^{j} \\ -\hat{x}_{s|s-1}^{j} \left(\hat{x}_{s|s-1}^{j}\right)^{T} + Q_{s-1}^{j} \end{cases}$$
(23)

The shape parameters $\hat{\alpha}_{s|s-1}^{j}$ and $\hat{\beta}_{s|s-1}^{j}$ are spread by a forgetting factor $\rho \in (0, 1]$:

$$\begin{cases} \hat{\alpha}_{s|s-1}^{j} = \rho \hat{\alpha}_{s-1}^{j} \\ \hat{\beta}_{s|s-1}^{j} = \rho \hat{\beta}_{s-1}^{j} \end{cases}$$
(24)

In the measurement update steps, the aim is to obtain the conditional PDFs. However, an analytical solution does not exist due to the coupled conditional state vector and HTMN. Aiming to solve this difficulty, the VB method was introduced to obtain the approximated PDF $q(\cdot)$ based on the constructed HGSSM, i.e., [36–38]:

$$\begin{cases} p(\Xi|z_{1:s}, M_s = j) \approx q(x_s^j)q(\lambda_s^j)q(\tau_s^j)q(\varphi_s^j) \\ \Xi \stackrel{\Delta}{=} \left\{ x_s^j, \lambda_s^j, \tau_s^j, \varphi_s^j \right\} \end{cases}$$
(25)

By introducing the VB technique, the approximated PDF $q(\theta)$ could be obtained by minimizing the KL divergence as follows:

$$q\left(x_{s}^{j}\right)q\left(\lambda_{s}^{j}\right)q\left(\tau_{s}^{j}\right)q\left(\varphi_{s}^{j}\right) = \arg\min KL\left(q\left(x_{s}^{j}\right)q\left(\lambda_{s}^{j}\right)q\left(\tau_{s}^{j}\right)q\left(\varphi_{s}^{j}\right)\left\|p\left(\Xi\left|z_{s}^{j},M_{s}=j\right)\right.\right)$$
(26)

Here, $KL(q((\cdot)||p(\cdot))) \triangleq \int q(t) \log\left(\frac{q(t)}{p(t)}\right) dt$ denotes the KL divergence, and the optimal approximation can be obtained as follows:

$$\ln q(\theta) = E_{\Xi^{(-\theta)}}[logp(\Xi, z_{1:s}|M_s = j)] + const_{\theta}$$
(27)

where $E[\cdot]$ stands for the operation of expectation, θ represents one of the elements in Ξ , $\Xi^{(-\theta)}$ refers to the remaining elements in Ξ after removing the element θ , and const denotes the constant for the variable.

The joint PDF $p(\Xi, z_{1:s}|M_s = j)$ in Equation (27) is rewritten below:

$$p(\Xi, z_{1:s}|M_{s} = j)$$

$$=p(z_{s}|x_{s}, \lambda_{s}, \tau_{s}, \varphi_{s}, M_{s} = j)p(x_{s}|z_{1:s-1}, M_{s} = j)p(z_{1:s-1}|M_{s} = j)$$

$$\cdot p(\lambda_{s}|M_{s} = j)p(\tau_{s}|\varphi_{s}, M_{s} = j)p(\varphi_{s}|z_{1:s-1}, M_{s} = j)$$

$$=N(z_{s}; h(x_{s}^{j}), R_{s}^{j}/\lambda_{s}^{j})^{\tau_{s}^{j}}N(z_{s}; 0, R_{s}^{j}/\lambda_{s}^{j})^{(1-\tau_{s}^{j})}N(x_{s}; \hat{x}_{s|s-1}^{j}, P_{s|s-1}^{j})$$

$$\cdot G(\lambda_{s}; \frac{\sigma_{s}^{j}}{2}, \frac{\sigma_{s}^{j}}{2})(1-\varphi_{s}^{j})^{\tau_{s}^{j}}(\varphi_{s}^{j})^{(1-\tau_{s}^{j})}Be(\varphi_{s}; \hat{x}_{s|s-1}^{j}, \beta_{s|s-1}^{j})p(z_{1:s-1})$$
(28)

Logarithmically, Equation (28) is further obtained in the following form:

$$\begin{split} \log p\left(\Theta, z_{1:s}^{j} | M_{s} = j\right) &= -\frac{1}{2} \left(x_{s} - \hat{x}_{s|s-1}^{j}\right)^{T} \left(P_{s|s-1}^{j}\right)^{-1} \left(x_{s} - \hat{x}_{s|s-1}^{j}\right) \\ &- \frac{1}{2} \lambda_{s}^{j} \tau_{s}^{j} \left(1 - \tau_{s}^{j}\right) \left(z_{s} - h_{s}^{j}(x_{s})\right)^{T} \left(R_{s}^{j}\right)^{-1} \left(z_{s}^{j} - h_{s}^{j}(x_{s})\right) \\ &- \frac{1}{2} \lambda_{s}^{j} \left(1 - \tau_{s}^{j}\right) z_{s}^{T} \left(R_{s}^{j}\right)^{-1} z_{s} + \left(\frac{m + \sigma_{s}^{j}}{2} - 1\right) \log \lambda_{s}^{j} - \frac{\sigma_{s}^{j}}{2} \lambda_{s}^{j} \end{split}$$
(29)
$$&+ \left(\hat{\alpha}_{s|s-1}^{j} - 1\right) \log \varphi_{s}^{j} + \left(\hat{\beta}_{s|s-1}^{j} - 1\right) \log \left(1 - \varphi_{s}^{j}\right) \\ &+ \left(1 - \tau_{s}^{j}\right) \log \varphi_{s}^{j} + \tau_{s}^{j} \log \left(1 - \varphi_{s}^{j}\right) + c_{\Theta} \end{split}$$

Based on variational Bayesian theory, the posterior distribution of each variable can be approximated by fixed-point iteration.

Proposition 1. Let $\theta = x_{s}^{j}$; utilizing Equations (28) and (25), $q^{(D+1)}(x_{s}^{j})$ is updated by the following formula:

$$q^{(D+1)}\left(x_{s}^{j}\right) = N\left(x_{s}^{j}; \hat{x}_{s|s}^{(D+1)}, P_{s|s}^{(D+1)}\right)$$
(30)

Here, $q^{(D+1)}(\cdot)$ stands for the approximated PDFs after (D+1) iterations, and the mean $\hat{x}_{s|s}^{(D+1)}$ and covariance $P_{s|s}^{(D+1)}$ are updated using Gaussian approximate filters:

$$G_{s}^{(D+1)j} = P_{xz,s|s-1}^{(D+1)j} \left(P_{zz,s|s-1}^{(D+1)j} \right)^{-1}$$
(31)

$$\hat{x}_{s|s}^{(D+1)j} = \hat{x}_{s|s-1}^{j} - G_{s}^{(D+1)j} \left(z_{s}^{j} - \hat{z}_{s|s-1}^{(D+1)j} \right)$$
(32)

$$P_{s|s}^{(D+1)j} = P_{s|s-1}^{j} - G_{s}^{(D+1)j} P_{zz,s|s-1}^{(D+1)j} \left(G_{s}^{(D+1)j}\right)^{T}$$
(33)

$$z_{s|s-1}^{(D+1)j} = \int h_s^j \left(x_s^j \right) N\left(x_s^j; \hat{x}_{s|s-1}^j, P_{s|s-1}^j \right) dx_s^j$$
(34)

$$P_{zz,s|s-1}^{(D+1)j} = \int h_s^j (x_s^j) h_s^{(j)T} (x_s^j) N(x_s^j; \hat{x}_{s|s-1}^j, P_{s|s-1}^j) dx_s^j - \hat{z}_{s|s-1}^{(D+1)j} (\hat{z}_{s|s-1}^{(D+1)j})^T + \hat{R}_s^{(D+1)}$$
(35)

$$P_{xz,s|s-1}^{(D+1)j} = \int x_s^j \left(h_s^j \left(x_s^j \right) \right)^T N\left(x_s^j; \hat{x}_{s|s-1}^j, P_{s|s-1}^j \right) dx_s^j - \hat{x}_{s|s-1}^j \left(z_{s|s-1}^{(D+1)j} \right)^T$$
(36)

Here, $G_s^{(D+1)j}$ *denotes the Kalman gain, and the modified measurement noise covariances are computed below:*

$$\hat{R}_{s}^{(D+1)j} = R_{s}^{j} / E^{(D+1)} \left[\lambda_{s}^{j} \right] E^{(D+1)} \left[\tau_{s}^{j} \right]$$
(37)

Proposition 2. Let $\theta = \lambda_s^j$; utilizing Equations (25) and (28), $q^{(D+1)}(\lambda_s^j)$ is updated by the following formula:

$$q^{(D+1)}\left(\lambda_s^j\right) = G\left(\lambda_s; a_s^{(D+1)j}, b_s^{(D+1)j}\right)$$
(38)

 $a_s^{(D+1)j}$ and $b_s^{(D+1)j}$ are calculated by:

$$a_s^{(D+1)j} = 0.5\left(m + \sigma_s^j\right) \tag{39}$$

$$b_{s}^{(D+1)j} = 0.5 \left\{ E^{(D+1)} \left[\tau_{s}^{j} \right] tr \left(A_{s}^{(D+1)} E^{(D)} \left[R_{s}^{-1} \right] \right) + E^{(D+1)} \left[1 - \tau_{s}^{j} \right] tr \left(B_{s}^{(D+1)} E^{(D)} \left[R_{s}^{-1} \right] \right) + \sigma_{s}^{j} \right\}$$
(40)

where $A_s^{(D+1)}$ and $B_s^{(D+1)}$ are defined as

$$A_{s}^{(D+1)j} = \int \left(z_{s} - h_{s}^{j}(x_{s}) \right) \left(z_{s} - h_{s}^{j}(x_{s}) \right)^{T} N\left(x_{s}; \hat{x}_{s|s-1}^{j}, P_{s|s-1}^{j} \right) dx_{s}^{j}$$
(41)

$$B_{s}^{(D+1)j} = \int z_{s}(z_{s})^{T} N\left(x_{s}; \hat{x}_{s|s-1}^{j}, P_{s|s-1}^{j}\right) dx_{s}^{j}$$
(42)

Proposition 3. Let $\theta = \tau_s^j$; utilizing Equations (28) and (25), $\log q^{(D+1)}(\tau_s^j)$ is obtained as follows:

$$\ln q^{(D+1)} \left(\tau_{s}^{j}\right) = -0.5\lambda_{s}^{j}\tau_{s}^{j} \left(1-\tau_{s}^{j}\right) \left(z_{s}-h_{s}^{j}(x_{s})\right)^{T} \left(R_{s}^{j}\right)^{-1} \left(z_{s}^{j}-h_{s}^{j}(x_{s})\right) -0.5\lambda_{s}^{j} \left(1-\tau_{s}^{j}\right) z_{s}^{T} \left(R_{s}^{j}\right)^{-1} z_{s} + \left(1-\tau_{s}^{j}\right) \log \varphi_{s}^{j} + \tau_{s}^{j} \log \left(1-\varphi_{s}^{j}\right) + c_{\tau}$$
(43)

According to Equation (43), $q^{(D+1)}(\tau_s^j)$ can be updated by BRV switching values between 1 and 0, and the event probabilities can be calculated by the following formulas:

$$Pr^{(D+1)}\left(\tau_{s}^{j}=1\right) = \Delta^{(D+1)j} \exp\left\{S_{1}^{(D+1)j}\right\}$$
(44)

$$Pr^{(D+1)}\left(\tau_{s}^{j}=0\right) = \Delta^{(D+1)j} \exp\left\{S_{2}^{(D+1)j}\right\}$$
(45)

Here, $\Delta^{(D+1)j}$ *is the normalization constant, and* $S_1^{(D+1)j}$ *and* $S_2^{(D+1)j}$ *is calculated as follows:*

$$S_{1}^{(D+1)j} = E^{(D+1)} \left[\log \left(1 - \varphi_{s}^{j} \right) \right] - 0.5tr \left(A_{s}^{(D+1)j} E^{(D+1)} \left[\lambda_{s}^{j} \right] R_{s}^{-1} \right)$$
(46)

$$S_{2}^{(D+1)j} = E^{(D+1)} \left[\log\left(\varphi_{s}^{j}\right) \right] - 0.5tr \left(B_{s}^{(D+1)j} E^{(D+1)} \left[\lambda_{s}^{j}\right] R_{s}^{-1} \right)$$
(47)

Proposition 4. Let $\theta = \varphi_s^j$; utilizing Equations (25) and (28), $q^{(D+1)}(\varphi_s^j)$ is updated as follows:

$$q^{(D+1)}\left(\varphi_{s}^{j}\right) = Beta\left(\varphi_{s}; \hat{\alpha}_{s}^{(D+1)j}, \hat{\beta}_{s}^{(D+1)j}\right)$$

$$\tag{48}$$

where

$$\hat{\alpha}_{s}^{(D+1)j} = \hat{\alpha}_{s|s-1}^{j} - E^{(D+1)} \left[\tau_{s}^{j} \right] + 1$$
(49)

$$\hat{\beta}_{s}^{(D+1)j} = \hat{\beta}_{s|s-1}^{j} + E^{(D+1)} \left[\tau_{s}^{j} \right]$$
(50)

The necessary expectations calculated in VB iterations are given as follows:

$$E^{(D+1)}\left[\tau_{s}^{j}\right] = \frac{p^{(D+1)}\left(\tau_{s}^{j}=1\right)}{p^{(D+1)}\left(\tau_{s}^{j}=1\right) + p^{(D+1)}\left(\tau_{s}^{j}=0\right)}$$
(51)

$$E^{(D+1)}\left[\lambda_{s}^{j}\right] = S_{1}^{(D+1)j} / S_{2}^{(D+1)j}$$
(52)

$$E^{(D+1)}\left[\varphi_{s}^{j}\right] = \hat{\alpha}_{s}^{(D+1)j} / \left(\hat{\alpha}_{s}^{(D+1)j} + \hat{\beta}_{s}^{(D+1)j}\right)$$
(53)

$$E^{(D+1)}\left[\log\varphi_s^j\right] = \psi\left(\hat{\alpha}_s^{(D+1)j}\right) - \psi\left(\hat{\alpha}_s^{(D+1)j} + \hat{\beta}_s^{(D+1)j}\right)$$
(54)

$$E^{(D+1)}\left[\log\left(1-\varphi_s^j\right)\right] = \psi\left(\hat{\beta}_s^{(D+1)j}\right) - \psi\left(\hat{\alpha}_s^{(D+1)j} + \hat{\beta}_s^{(D+1)j}\right)$$
(55)

where $\psi(\cdot)$ *stands for a digamma function.*

Step 3: Mode Probabilities Update Process

In this process, in order to update the mode probability μ_s^j , the likelihood function of the designed IMM-VBRGAF can be jointly inferred by the VB approach, i.e.,

$$u_s^j = \frac{1}{e} \Lambda_{j,s} \bar{e}_j \tag{56}$$

where

$$e = \sum_{j=1}^{l} \Lambda_{j,s} \bar{e}_j \tag{57}$$

Here, \bar{e}_j represents the normalization constant, and $\Lambda_{j,s} = p(z_s | z_{1:s-1}, M_s = j)$ denotes the measurement likelihood function. The logarithmic form is obtained as follows:

$$\ln p(z_s|z_{1:s-1}, M_s = j) = L(\Omega) + KL(\Omega|p(\Xi|z_{1:s}, M_s = j))$$
(58)

where the $L(\Omega)$ indicates the evidence lower bound and

$$\Omega(x_s, \lambda_s, \tau_s, \varphi_s | z_{1:s}, M_s = j) = q\left(x_s^j\right) q\left(\lambda_s^j\right) q\left(\tau_s^j\right) q\left(\varphi_s^j\right)$$
(59)

By utilizing the VB inference iterations, the term $KL(\cdot)$ tends to be zero, and $\Lambda_{j,s}$ can be calculated by

$$\Lambda_{i,s} \simeq \exp\{L(\Omega)\}\tag{60}$$

and

$$L(\Omega) = \ln p(z_s | z_{1:s-1}, M_s = j)$$

= $\ln p(z_{1:s} | M_s = j) - \ln p(z_{1:s} | M_s = j)$ (61)
= $-\ln p(z_{1:s-1} | M_s = j) + E_{\Omega} \left[\ln \frac{p(z_{1:s}, x_s, \lambda_s, \tau_s, \varphi_s | M_s = j)}{p(x_s, \lambda_s, \tau_s, \varphi_s | z_{1:s}, M_s = j)} \right]$

where $E[\cdot]$ represents an expectation term, and

$$p(x_s, \lambda_s, \tau_s, \varphi_s | z_{1:s}, M_s = j) \simeq q\left(x_s^j\right) q\left(\lambda_s^j\right) q\left(\tau_s^j\right) q\left(\varphi_s^j\right)$$
(62)

 $L(\Omega)$ can be rewritten in the following form:

$$L(\Omega) = E_{\Omega}[\ln p(z_{1:s}, x_s, \lambda_s, \tau_s, \varphi_s | M_s = j)] - E_{\Omega}\left[\ln q\left(x_s^j\right)q\left(\lambda_s^j\right)q\left(\tau_s^j\right)q\left(\varphi_s^j\right)\right] - \ln p(z_{1:s-1}|M_s = j)$$
(63)

where

$$p(z_{1:s}, x_s, \lambda_s, \tau_s, \varphi_s | M_s = j)$$

= $p(z_s | x_s, \lambda_s, \tau_s, \varphi_s, M_s = j) p(x_s | z_{1:s-1}, M_s = j) p(z_{1:s-1} | M_s = j)$
 $\cdot p(\lambda_s | M_s = j) p(\tau_s | \varphi_s, M_s = j) p(\varphi_s | z_{1:s-1}, M_s = j)$ (64)

According to Equation (59), $L(\Omega)$ can be rewritten as follows:

$$L(\Omega) = E_{\Omega}[\ln p(z_s|x_s, \lambda_s, \tau_s, \varphi_s, M_s = j)] + E_{\Omega}[\ln p(x_s|z_{1:s-1}, M_s = j)] + E_{\Omega}[\ln p(\lambda_s|M_s = j)] + E_{\Omega}[\ln p(\tau_s|\varphi_s, M_s = j)] + E_{\Omega}[\ln p(\varphi_s|z_{1:s-1}, M_s = j)] - E_{\Omega}\left[\ln q\left(x_s^j\right)\right] - E_{\Omega}\left[\ln q\left(\lambda_s^j\right)\right] - E_{\Omega}\left[\ln q\left(\tau_s^j\right)\right] - E_{\Omega}\left[\ln q\left(\varphi_s^j\right)\right]$$
(65)

Step 4: Combination Process

Based on the Bayes' theories, the system state vector $\hat{x}_{s|s}$ and covariance $\hat{P}_{s|s}$ are computed by the total probability formula below:

$$\begin{cases} \hat{x}_{s|s} = \sum_{i=1}^{l} u_{s}^{i} \hat{x}_{s|s}^{i} \\ P_{s|s} = \sum_{i=1}^{l} u_{s}^{i} \left(P_{s|s}^{i} + \left(\hat{x}_{s|s}^{i} - \hat{x}_{s|s} \right) \left(\hat{x}_{s|s}^{i} - \hat{x}_{s|s} \right)^{T} \right) \end{cases}$$
(66)

The proposed IMM-VBRGAF algorithm combines the time update in Equations (23) and (24) with the measurement update in Equations (30)–(55) and operates recursively based on the IMM approach. A detailed flow chart and pseudocode are provided in Figure 1 and Algorithm 1, respectively.

Algorithm 1: One pseudocode implementation cycle for the proposed IMM-VBRGAF model. **Input:** $f_s^j(\cdot), h_s^j(\cdot), z_s, \hat{x}_{s-1|s-1}^j, P_{s-1|s-1}^j, u_0, \pi_{ij}, \hat{x}_{s-1|s-1}^j, \hat{\beta}_{s-1|s-1}^j, n, m, \rho, N_i,$ Q_{s-1}, R_s **Process 1: Interacting/Mixing** Combine probability u_{s-1}^{ij} using Equations (17) and (18) Combine state $\hat{x}_{s-1|s-1}^{0j}$ and covariance $\hat{P}_{s-1|s-1}^{0j}$ using Equation (20) Combine $\hat{\alpha}_{s-1|s-1}^{0j}$ and $\hat{\beta}_{s-1|s-1}^{0j}$ using Equation (21) **Process 2: Mode-Conditioned Filtering** Time updating: Predict system state $\hat{x}_{s|s-1}^{j}$ and covariance $\hat{P}_{s|s-1}^{j}$ using Equation (23) Predict shape parameters $\hat{\alpha}_{s|s-1}^{j}$ and $\hat{\beta}_{s|s-1}^{j}$ using Equation (24) Measurement updating: Obtain initial expectations using Equations (51)–(55) for $D = 0, 1, 2, \cdots, N_i - 1$ do Calculate $\hat{R}_{s}^{(D+1)j}$ using Equation (37) Update $q^{(D+1)}(x_s^j)$ as a Gaussian distribution Calculate $x_{s|s}^{(D+1)j}$, $P_{s|s}^{(D+1)j}$ using Equations (32) and (33) Calculate $A_s^{(D+1)j}$ and $B_s^{(D+1)j}$ using Equations (41) and (42) Update $q^{(D+1)}(\tau_s^j)$ as a Bernoulli distribution, given $q^{(D+1)}(x_s^j)$ Calculate $E^{(D+1)}\left[\tau_s^j\right]$ using Equation (51) Update $q^{(D+1)}\left(\lambda_s^{j}\right)$ as a Gamma distribution, given $q^{(D+1)}\left(\tau_s^{j}\right)$ and $q^{(D+1)}(x_s^j)$ Calculate $E^{(D+1)}\left[\lambda_s^j\right]$ using Equation (52) Update $q^{(D+1)}(q_s^j)$ as a beta distribution, given $q^{(D+1)}(x_s^j)$, $q^{(D+1)}(\lambda_s^j)$, and $q^{(D+1)}(\tau_s^j)$ Calculate $E^{(D+1)} \left[\log \varphi_s^j \right]$ and $E^{(D+1)} \left[\log \left(1 - \varphi_s^j \right) \right]$ using Equations (54) and (55) Sub-filter outputs $\hat{x}_{s|s}^j = \hat{x}_{s|s}^{(D+1)j}$, $\hat{P}_{s|s}^j = \hat{P}_{s|s}^{(D+1)j}$ Process 3: Mode Probabilities Update Calculate u_s^{j} using Equations (56)–(65) **Process 4: Combination** Calculate $\hat{x}_{s|s}$ and $P_{s|s}$ utilizing Equation (66) **Output:** $\hat{x}_{s|s}$, $P_{s|s}$



Figure 1. Flow chart for the proposed IMM-VBRGAF method.

4. Maneuvering Target Tracking Simulation

To demonstrate the effectiveness and advantages of the designed IMM-VBRGAF method, a transformation dynamic maneuvering target tracking simulation was utilized. A constant-velocity (CV) model and a coordinate turn (CT) model were used alternately for describing the maneuvering target dynamics. The mode indexes $M_s = 1$ and $M_s = 2$ represent the current CV and CT models, respectively, which are described in detail as follows:

$$\mathbf{x}_{s,CV} = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_{s-1,CV} + g_{s-1,CV}$$
(67)

$$\mathbf{x}_{s,CT} = \begin{bmatrix} 1 & \frac{\sin(\xi T)}{\xi} & 0 & -\frac{1-\cos(\xi T)}{\xi} & 0\\ 0 & \cos(\xi T) & 0 & -\sin(\xi T) & 0\\ 0 & \frac{1-\cos(\xi T)}{\xi} & 1 & \frac{\sin(\xi T)}{\xi} & 0\\ 0 & \sin(\xi T) & 0 & \cos(\xi T) & 0\\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_{s-1,CT} + g_{s-1,CT}$$
(68)

The system state vectors of the two models are defined as follows:

$$\mathbf{x}_{s,CV} = \left(x_s \dot{x}_s y_s \dot{y}_s\right)^T \tag{69}$$

$$\mathbf{x}_{s,CT} = (x_s \dot{x}_s y_s \dot{y}_s \xi_s)^T \tag{70}$$

where x_s and y_s denote the horizontal position, \dot{x}_s and \dot{y}_s denote the horizontal velocity, ξ_s represents the unknown turn rate, and the sampling period *T* was set to be 1 s. The process noise covariances were set as follows:

$$Q_{s,CV} = \begin{bmatrix} q_1 M_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & q_1 M_{2\times 2} \end{bmatrix}$$
(71)

$$Q_{s,CT} = \begin{bmatrix} q_1 M_{2\times 2} & 0_{2\times 2} & 0\\ 0_{2\times 2} & q_1 M_{2\times 2} & 0\\ 0_{1\times 2} & 0_{1\times 2} & q_2 T \end{bmatrix}$$
(72)

where

$$M_{2\times 2} = \begin{bmatrix} \frac{T^3}{3} & \frac{T^2}{2} \\ \frac{T^2}{2} & T \end{bmatrix}$$
(73)

Here, the power spectral densities q_1 and q_2 related to \mathbf{x}_s were set to be 0.1 m²s⁻³ and 1.75×10^{-4} s⁻³.

The switching between the CV model and the CT model was governed by a Markov chain, and the matrix of transition probabilities was initialized as $\Pi = \begin{bmatrix} 0.95 & 0.05 \\ 0.05 & 0.95 \end{bmatrix}$.

A sensor located at (p_x, p_y) obtained range and bearing measurements according to [39]:

$$h(x_{s}) = \begin{bmatrix} \sqrt{(x_{s} - p_{x})^{2} + (y_{s} - p_{y})^{2}} \\ \arctan((y_{s} - p_{y})/(x_{s} - p_{x})) \end{bmatrix}$$
(74)

Referring to [30,33], the HTMN is formulated below:

$$r_{s} \sim \begin{cases} N(0, 100R) & u.p. \ 0.1 \\ N(0, R) & u.p. \ 0.9 \end{cases}$$
(75)

where u.p. stands for "under a probability", and the nominal covariance matrix of measurement noise is $R = diag((10 \text{ m})^2, (0.1^\circ)^2)$.

In this simulation, the target moved alternately according to the CV model and the CT model from 0 s to 1000 s, i.e., from 0 s to 200 s, 401 s to 600 s, and 801 s to 1000 s, the target moved based on the CV model, whereas it implemented the CT model with a turn rate of 5° /s from 201 s to 400 s and from 601 s to 800 s. The system state vector x_0 and covariance matrix P_0 were initialized as (10,000 m, 10 m/s, 10,000 m, 10 m/s, 5° /s)^T and diag (100 m², 10 m²/s², 100 m², 10 m rad²/s²), respectively. The position of the sensor was (0 m, 0 m).

Referring to [24,29], the time-varying UMLP was set as follows:

$$p(\varphi_s) = \begin{cases} 0.1 & \text{for} \quad 1 \le s \le 200 \\ 0.3 & \text{for} \quad 201 \le s \le 600 \\ 0.1 & \text{for} \quad 601 \le s \le 1000 \end{cases}$$
(76)

To describe the received or lost measurements in the simulation, Figure 2 shows the binary flag sequence in one Monte Carlo cycle. From 1 s to 200 s the measurement loss probability was assumed to be 0.1; from 201 s to 600 s, the measurement loss probability increased to 0.3 and then decreased to 0.1 until 1000 s. When random measurement loss occurs, the system state vector information contained in the measurement cannot be collected.

Four IMM filters, namely the VB-based KF in the IMM framework with unknown noise covariances (IMM-VBKF) [25], the IMM variational Bayesian-based adaptive KF with UMLP (IMM-VBAKF) [29], the outlier-robust STD-based IMM filter utilizing the square-root cubature KF to address nonlinear problems (IMM-ORSRCKF) [28], and the designed IMM filter considering HTMN and UMLP, were compared.



Figure 2. The binary flag sequence in one Monte Carlo cycle.

In order to verify the estimation accuracy of IMM-VBRGAF and existing methods, we selected two evaluation indicators, namely root mean square error (RMSE) and averaged RMSE (ARMSE), and these indicators of position were defined similarly to [40,41], as follows:

$$RMSE_{pos}(s) = \sqrt{\frac{1}{N_{mc}} \sum_{k=1}^{N_{mc}} \left(\left(\hat{x}_{s}^{k} - x_{s}^{k} \right)^{2} + \left(\hat{y}_{s}^{k} - y_{s}^{k} \right)^{2} \right)}$$
(77)

$$ARMSE_{pos} = \frac{1}{T_s} \sum_{s=1}^{T_s} RMSE_{pos}(s)$$
(78)

where $N_{mc} = 1000$ is the Monte Carlo run times; (x_s^k, y_s^k) and $(\hat{x}_s^k, \hat{y}_s^k)$ denote the true and estimated position in the *k*-th Monte Carlo run, respectively; and T_s refers to the total sampling number. The RMSE and ARMSE of the turn rate or velocity were defined in a similar form of position.

The whole simulation could be divided into four experimental steps. In the first step, the estimation accuracy of IMM-VBRGAF and the state-of-the-art methods was evaluated. Figure 3 shows the tracking trajectory of the proposed method compared with the other techniques. The tracking trajectory of the proposed algorithm was closer to the real moving trajectory of the surface maneuvering target than that of the existing methods. Although the IMM-ORSRCKF and IMM-VBAKF methods could complete the target tracking task, the tracking trajectory showed large biases. Due to the joint action of HTMN and UMLP, filter divergence occurred in the tracking process of the IMM-VBKF method, which failed to complete the target task (thus, it is not included in the figure).

Figures 4–6 show the RMSEs of the positions, velocities, and turn rates of these methods. The RMSE curves of the IMM-VBKF algorithm were not plotted completely in order to clearly display the other curves in Figures 4–6. The reason for the poor performance of the IMM-VBKF method was that the coexistence of random measurement loss with HTMN had a significant impact on the filtering accuracy. As shown in Figures 4–6, the designed IMM-VBRGAF method had smaller RMSEs than the existing algorithms. The proposed filtering approach demonstrated increased robustness to measurement outliers and adapted effectively to random measurement loss. As illustrated in Figures 4–6, the proposed method had great advantages in estimation accuracy compared to the existing filters.



Figure 3. Tracking trajectories of different filters.



Figure 4. RMSEs of position for different filters.



Figure 5. RMSEs of velocity for different filters.

Table 3 lists in detail the ARMSEs of all filters for different time periods, and the singlestep implementation time (SSIT) with an iteration number of $N_i = 8$ is also presented for comparison. The computational complexity of the IMM-VBKF, IMM-ORSRCKF, IMM-VBAKF, and proposed IMM-VBRGAF methods was approximated as $O((12n^3 + 8n^2m + 12nm^2 + 3m^3)lN_i + 4n^3l)$, $O((2n^3/3 + 20n^2m + 14nm^2 + 4m^3)lN_i + (14n^3 + 8m^3/3)l)$, $O((13n^3/3 + 4n^2m + 14nm^2 + 8m^3)lN_i + 4n^3l)$, and $O((26n^3/3 + 4n^2m + 18nm^2 + 12m^3) \cdot lN_i + 4n^3l)$, respectively. Based on the theoretical computational complexity of the proposed filter and existing algorithms and the ARMSEs presented in Table 3, the proposed filter achieved a significant improvement in estimation accuracy, although the computational complexity increased slightly. The SSITs in Table 3 are consistent with this conclusion. Figure 7 provides the true and estimated loss probabilities of the designed filter. The results in Figure 7 indicate that when the random measurement loss probability changed, the designed IMM-VBRGAF could achieve the adaptive estimation of UMLP.



Figure 6. RMSEs of turn rate for different filters.

Table 3. ARMSE and SSITs of all filters for distinct periods of time. Method A, Method B, and Method C represent the IMM-VBKF [25], IMM-ORSRCKF [28], and IMM-VBAKF [29] methods, respectively. Method D represents the designed IMM-VBRGAF method.

	Time (s)	Method A	Method B	Method C	Method D
ARMSE _{pos} (m)	1~200	25.12	13.70	22.80	10.10
	$201 \sim 400$	299.98	39.55	30.85	16.15
	$401 \sim 600$	1291.14	32.50	27.45	12.69
	$601 \sim 800$	2480.05	27.99	26.58	13.66
	801~1000	3188.77	20.51	23.50	11.08
ARMSE _{vel} (m/s)	1~200	2.43	1.95	3.69	1.51
	$201 \sim 400$	13.01	5.79	6.36	3.64
	$401 \sim 600$	17.24	3.15	4.49	2.08
	$601 \sim 800$	17.99	5.09	5.79	3.27
	$801 \sim 1000$	18.35	2.69	4.03	1.90
ARMSE _{omg} (deg/s)	1~200	0.76	0.67	1.20	0.55
	$201 \sim 400$	4.48	3.04	3.35	2.42
	$401 \sim 600$	3.69	1.39	1.87	1.15
	$601 \sim 800$	6.10	2.84	3.21	2.34
	801~1000	3.38	1.33	1.89	1.18
SSIT (ms)	1~1000	1.45	1.29	1.14	1.42

In the second step of the simulation, the influence of different UMLPs on the filtering performance was tested. The UMLP in Equation (76) was set to different values, ranging from 0.1 to 0.3, throughout the simulation. Figures 8–10 show the ARMSEs of IMM-VBRGAF and the other methods under various UMLP values. With an increase in the UMLP, the performance of IMM-VBKF and IMM-ORSRCKF was affected significantly. The main

reason for this was that these two filters could not adaptively address the measurement loss. Although the IMM-VBAKF method could address the random measurement losses, it performed worse than the designed filter because it ignored the HTMN induced by measurement outliers. Since the IMM-VBRGAF could address the HTMN and UMLP at the same time, it exhibited superior adaptability to various random UMLPs and achieved a higher filtering accuracy than the existing filters.



Figure 7. Estimate of measurement loss probabilities for the proposed filtering approach.



Figure 8. ARMSE_{pos} for each filter with different loss probabilities.



Figure 9. ARMSE vel for each filter with different loss probabilities.



Figure 10. ARMSE_{omg} for each filter with different loss probabilities.

The third simulation step verified the filtering accuracy under different probabilities of measurement outliers. Figures 11–13 exhibit the curves of the ARMSEs with outlier probabilities ranging from 0.05 to 0.15. Figures 11–13 show that the IMM-VBKF model did not exhibit a satisfactory performance, further corroborating the conclusion regarding IMM-VBKF drawn from the previous two experimental steps. As the probability of measurement outliers increased, both IMM-VBKF and IMM-VBAKF exhibited a noticeable decline in estimation accuracy. This decline could be attributed to their reliance on the assumption that the measurement noise followed a Gaussian distribution, which is known to be sensitive to outliers. The performance of IMM-ORSRCKF was worse than that of IMM-VBRGAF, since this model assumed that all measurement loss. The results in Figures 11–13 validated that the designed filter had superior adaptive estimation performance than the existing filters under different measurement outlier probabilities.



Figure 11. ARMSE_{pos} for each filter with different outlier probabilities.

In the fourth experimental step, the impact of variations in the iteration number N_i on the filtering performance was assessed for each model. Considering the trend of divergence displayed by IMM-VBKF, only IMM-ORSRCKF, IMM-VBAKF, and the designed filter were compared in this step. Figures 14–16 show the ARMSE curves for a series of iteration numbers: $N_i = 1, 2, 3, \dots, 15$. The results indicated that IMM-VBRGAF began to show superior estimation accuracy when $N_i \ge 2$ and converged when the iteration number was fixed at 3. In comparison with the other algorithms, the proposed filter converged faster. While the N_i of IMM-VBRGAF gradually increased, the filtering accuracy improved.

However, it is essential to take computational efficiency into account. To strike a balance between computing costs and estimation accuracy, the recommended variational iteration number is between 4 and 8.



Figure 12. ARMSE *vel* for each filter with different outlier probabilities.



Figure 13. ARMSE_{omg} for each filter with different outlier probabilities.



Figure 14. ARMSE_{pos} for each filter with $N_i = 1, 2, \dots, 15$.



Figure 15. ARMSE_{*vel*} for each filter with $N_i = 1, 2, \dots, 15$.



Figure 16. ARMSE_{*omg*} for each filter with $N_i = 1, 2, \dots, 15$.

5. Conclusions

The coexistence of measurement outliers and random measurement losses results from the combination of unreliable sensors and interrupted communication channels, posing significant challenges in state estimation for surface maneuvering targets. As a result, surface maneuvering target tracking becomes highly challenging. Aiming to efficiently deal with measurement outliers and adaptively estimate the UMLP in JMSs, we designed an effective and robust IMM approach. Firstly, we modeled the HTMN using STD, and the likelihood function was transformed into an exponential product form by introducing BRVs. Secondly, to directly utilize the VB technique, we established a new HGSSM. The state vector, distribution parameters, BRV, and UMLP were inferred simultaneously according to the VB technique, and a Gaussian approximate filter was utilized to address the system nonlinearity. Finally, the results from the simulation of maneuvering target tracking validated the superiority of the IMM-VBRGAF technique over existing filters in terms of estimation accuracy, despite a minor increase in computational costs. Furthermore, the proposed approach exhibited enhanced robustness across various measurement outlier probabilities and achieved a higher filtering accuracy across different UMLPs. This algorithm could be utilized to address the state estimation issues in JMSs with the coexistence of HTMN and UMLP. In future research, the proposed tracking approach will be extended to multi-sensor networks, which are more in line with actual application scenarios. Real data surface target tracking experiments will also be considered based on the extended theoretical contents.

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