

Article

What if Newton's Gravitational Constant Was Negative?

Ismael Ayuso [†], José P. Mimoso ^{*,†} and Nelson J. Nunes [†]

Departamento de Física and Instituto de Astrofísica e Ciências do Espaço, Faculdade de Ciências, Universidade de Lisboa, Edifício C8, Campo Grande, 1769-016 Lisboa, Portugal; iayuso@fc.ul.pt (I.A.); njnunes@fc.ul.pt (N.J.N.)

* Correspondence: jpmimoso@fc.ul.pt

† These authors contributed equally to this work.

Received: 30 January 2019; Accepted: 11 March 2019; Published: 18 March 2019



Abstract: In this work, we seek a cosmological mechanism that may define the sign of the effective gravitational coupling constant, G . To this end, we consider general scalar-tensor gravity theories as they provide the field theory natural framework for the variation of the gravitational coupling. We find that models with a quadratic potential naturally stabilize the value of G into the positive branch of the evolution and further, that de Sitter inflation and a relaxation to General Relativity is easily attained.

Keywords: gravitation; modified gravity; cosmology; variation of constants

1. Introduction

In Newton's law of gravitation, the gravitational constant, G , is assumed to be positive. This is a question of choice, and apparently it was P. S. Laplace who introduced the constant for the first time in his *Traité de Mécanique Céleste*, in 1799 [1] as

$$F = -k^2 \frac{m_1 m_2}{r^2}. \quad (1)$$

Originally, Newton had put forward both the proportionality of the gravitational centripetal force (in his words) to the quantity of matter of the two bodies in interaction, as well as the inverse proportionality to the square of their separation [2]. However, he did not explicitly introduce G [3]¹, presumably due to the lack of an internationally accepted system of units. Of course, this is required since, after all, G adjusts the dimensions of both sides of the defining equation for the strength of the gravitational interaction and the sign of this denotes the attractive or repulsive character of the force. Odd enough, it was in 1798, one year before the publication of Laplace's treaty, that H. Cavendish measured G with a torsion balance, but just as a necessary step, of secondary importance, to weigh the density of the Earth [4]. This measure was made with the remarkable accuracy of 1%.

The subsequent success of the gravitational law in tackling the motion of the celestial bodies of the solar system is well known, and, at the beginning of the 20th century, the only major problem was the anomaly in the precession of Mercury's perihelium, a mismatch first revealed by Le Verrier in 1855.

It was Einstein's General theory of Relativity which not only solved this puzzle with flying colours, but also revolutionized our understanding of gravitation. One of the pillars of the theory is that we

¹ Quoting Clifford Will [3], "It is interesting to notice that the term "*gravitational constant*" never occurs in the *Principia*. In fact it seems that the universal constant of proportionality that we now call G does not make an appearance until well in the eighteenth century in Laplace's "*Mécanique Céleste*".

should recover Newtonian gravity when considering weak fields and bodies moving with low speeds when compared to the speed of light. Thus, G whose role in the theory is to couple the geometry to the matter content of the Universe, is taken to be positive and is a constant under this framework.

In 1938, Dirac made an astounding proposal, dubbed the Large Number Hypothesis, according to which any *dimensionless* ratio between two fundamental quantities of nature should be of the order unity (for a more detailed account of the motivations see [5]). This led him to put forward that if G were to evolve with the Hubble rate of expansion of the Universe this would account for the present disparity of about 40 order of magnitude between gravitational and electromagnetic forces at the atomic level.

This was the first time the variation of some fundamental constant was explicitly and seriously envisaged. Dirac's proposal was given a field theoretical realization, first by P. Jordan within a Kaluza-Klein type approach (thus involving extra-dimensions), and then, in 1961, by Brans-Dicke theory [3,6] motivated by Mach's principle. In both cases a dynamical scalar field couples to the spacetime curvature and thus plays itself a gravitational role. In the suite, a plethora of extended gravity theories that affect the coupling between the space-time geometry and the matter sector also prescribe the variation of this fundamental "constant" G [7,8].

In principle, within this framework, it becomes possible for G to change sign, trading, attracting into repulsive gravity, and conversely. This might happen either during the cosmological time evolution, or even conceivably it might happen at spatially separated regions of space-time. The concern about the sign of the gravitational constant has been envisaged as a constraint to be respected by the spectrum of modified gravity theories, but the focus has never been directed to devise a mechanism to assure its positiveness. For instance, Barrow [9] proposed that the formation of primordial black holes during the early stages of the Universe might retain "memory" of the value of the gravitational constant at the time of their formation, and hence exhibit diverse values of the latter depending on the instant of their formation, around $t_{\text{prim}} \sim 10^{-25}$ s.

In the present work we investigate the, somewhat heretical, possibility that the effective gravitational coupling might be negative within the general class of scalar-tensor (ST) gravity theories. We analyze a cosmological mechanism that determines the positiveness of the sign of G , even though it may exhibit transient periods in the negative region. We show that this cosmological device relies on the role of a cosmological potential, which reproduces a positive cosmological constant in the so-called Einstein frame. From this latter viewpoint it can be understood as another role of paramount importance of this remarkable constant. In Refs. [10,11] I. Roxburgh analysed the issues of the sign and magnitude of the gravitational constant, based on Einstein's correspondence principle which demands that Newtonian gravity be recovered in the weak field limit of the theory. His analysis is done in the framework of GR and is somewhat motivated by Mach's principle, leading him to conclude that G must be positive. Other studies which carry some relation to the present work are [12–19].

In this work we shall start by briefly looking at the implications of having $G < 0$ in cosmology, namely showing that inflation arises for a considerably large set of parameters, and that we obtain bouncing solutions that avoid the initial singularity when a cosmological constant is also considered. Then we analyze the cosmological behaviour of scalar-tensor theories to show how a subset of the solutions exhibit negative G , and how a cosmological potential provides us with a mechanism that favours positive G and eventually stabilizes its sign. In essence we will show that the presence of a cosmological constant in the Einstein frame provides such a mechanism for an extended set of varying G theories, which represents a relevant feature for the existence of a non-vanishing cosmological constant in the Einstein frame (and of a corresponding cosmological potential in the Jordan frame).

2. Negative G in GR

It must be said that if we envisage the trading of a positive G into a negative one within Einstein's General Relativity, we will be mutating its attractive nature into a repulsive one, and this avoids the need to rely on exotic matter, violating the strong energy condition, to produce inflationary stages.

This is therefore an alternative ad-hoc device, akin to the Albrecht and Magueijo's varying speed of light to avoid the perplexing complications of the inflationary scenarios [20]. The down side of this way of producing repulsive gravity, is that once assumed, it is for ever. There would be no way of exiting inflation with canonical matter sources. The scalar-tensor scenario that we consider afterwards avoids the latter problem, and present us with a natural, and theoretically consistent framework for exploring the possible negativeness of G .

2.1. Friedmann Models with a Single Fluid

Consider the usual FLRW universes of the standard cosmological model, and take $G = -|G|$ in Einstein's GR. We then have the following field equations

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -\frac{8\pi|G|}{3}\rho \quad (2)$$

$$\frac{\ddot{a}}{a} = \frac{8\pi|G|}{6}(\rho + 3p), \quad (3)$$

where dots denote derivatives with respect to the time, a is the scale factor of the Universe and ρ and p are the energy density and pressure, respectively. The signs on the right hand side are the opposite with respect to the usual ones. However, the Bianchi contracted identities are immune to this change of sign and the energy conservation equation is preserved

$$\dot{\rho} = -3H(\rho + p). \quad (4)$$

Thus, when the matter content satisfies the weak and strong energy conditions, $\rho > 0$, $\rho + p \geq 0$, and $\rho + 3p \geq 0$, we see from the Raychaudhuri Equation, (3), that the expansion is accelerated, $\ddot{a} \geq 0$. Yet, this inflationary behaviour is constrained by the Friedmann Equation (2). It can be easily verified that the single fluid solutions are forbidden when $k = 0, +1$, and are restricted to $\rho \leq 3/8\pi|G|a^2$ when $k = -1$.

Further, notice that the transformation $|G| \rightarrow -|G|$ which is performed in the Einstein field equations of the FLRW models, produces a system which mimics phantom matter provided the equation of state relating the pressure and the energy density of matter is such that $p(\rho) \rightarrow -p(-\rho)$ when $\rho \rightarrow -\rho$, preserving the field equations (we remark that this happens to be the case for the barotropic equations $p = (\gamma - 1)\rho$ which are usually considered; in addition the cosmography framework, as exposed in [21,22], also absorbs this transformation and is left unchanged).

2.2. Model with a Cosmological Constant

Consider a cosmological constant in addition to the perfect fluid

$$T_{ab} = -\Lambda g_{ab} \quad (5)$$

for a metric with the signature $-+++$. The field equations now read

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \lambda - \frac{8\pi|G|}{3}\rho \quad (6)$$

$$\frac{\ddot{a}}{a} = \lambda + \frac{8\pi|G|}{6}(\rho + 3p), \quad (7)$$

where $\lambda = \Lambda/3$. Recasting the latter equations in conformal time η defined by $d\eta = dt/a(t)$ we get

$$(a')^2 + ka^2 = \lambda a^4 - \frac{8\pi|G|}{3}\rho a^4 \quad (8)$$

$$\frac{a''}{a} = 2\lambda a^2 - k - \frac{8\pi|G|}{6}(\rho - 3p)a^2, \quad (9)$$

where $'$ denotes derivative with respect to the conformal time. Assuming that the matter content is a perfect fluid with equation of state (EOS) $p = (\gamma - 1)\rho$ where γ is a constant that takes values in the range $0 < \gamma \leq 2$, we derive the exact solutions from Equation (8)

$$\int \frac{da}{(\lambda a^4 - (8\pi|G|/3)\rho_0 a^{4-3\gamma} - k a^2)^{1/2}} = \pm(\eta - \eta_0) \quad (10)$$

which yields Jacobi elliptic functions. Naturally, in the latter equation η_0 is an arbitrary integration constant that sets the origin of time. There are four cases that are of special interest: (i) Radiation, i.e., $\gamma = 4/3$, (ii) Dust, i.e., $\gamma = 1$, (iii) Stiff matter, i.e., $\gamma = 2$, and (iv) The coasting model $\gamma = 2/3$.

A case which is of great interest is the case where we have a combination of pressureless matter and radiation together with a cosmological constant, since these are the 3 major components that best fit the expansion history of the universe (Λ CDM model) [23].

The corresponding dynamical system, from (9) reads

$$a' = b \quad (11)$$

$$b' = -ka - \frac{8\pi|G|}{6}\rho_d^0 + 2\lambda a^3 = -ka - \frac{\Omega_m^0 H_0^2}{2} + 2\Omega_\Lambda^0 H_0^2 a^3, \quad (12)$$

where ρ_d^0 is the current density of dust. In addition, the Friedmann constraint equation becomes:

$$ka^2 + b^2 = -\frac{8\pi|G|\rho_d^0}{3}a - \frac{8\pi|G|\rho_r^0}{3} + \lambda a^4. \quad (13)$$

In Figure 1, we represent the phase diagrams depicting the qualitative behaviour of these negative G models for some choices of matter content (For a recent review of the methods of dynamical systems in cosmology see [24]). Analyzing the existence and nature of the fixed points, we classify the possible dynamical behaviours. Please note that we have compactified the phase diagrams using the transformation $x = \operatorname{arctanh} a$ and $y = \operatorname{arctanh} b$, so that the boundary lines $x = \pm 1$ and $y = \pm 1$, respectively, correspond to $a \rightarrow \pm\infty$ and $b \rightarrow \pm\infty$. This allows us to devise the asymptotic solutions at infinity.

The number and position of the fixed points in the finite region of the phase plane (a, b) is defined by the roots of Equation (12) when $b = 0$. Therefore, there will be at most three fixed points on the a axis (plus the fixed points at infinity which will not be on the a axis). In Figure 1 we display the qualitative behaviour of the model for the three spatial curvatures and use reasonable values for the parameters in Equation (12), which take into consideration the Λ CDM model². We adopt $\Omega_\Lambda^0 H_0^2 / \Omega_m^0 H_0^2 \approx 2$, except in Figure 1d.

² One must though be wary that in the phase-diagrams of Figure 1 the half-plane corresponding to negative values of a is not physical, as it corresponds to $a < 0$. Yet its representation is useful, because it illustrates the complete behavior of the mathematical dynamical system underlying the physical scenario, regardless of the physical consistency of some of its parts. Moreover, in the present case it also allows comparison with the phase-diagrams of the scalar-tensor models.

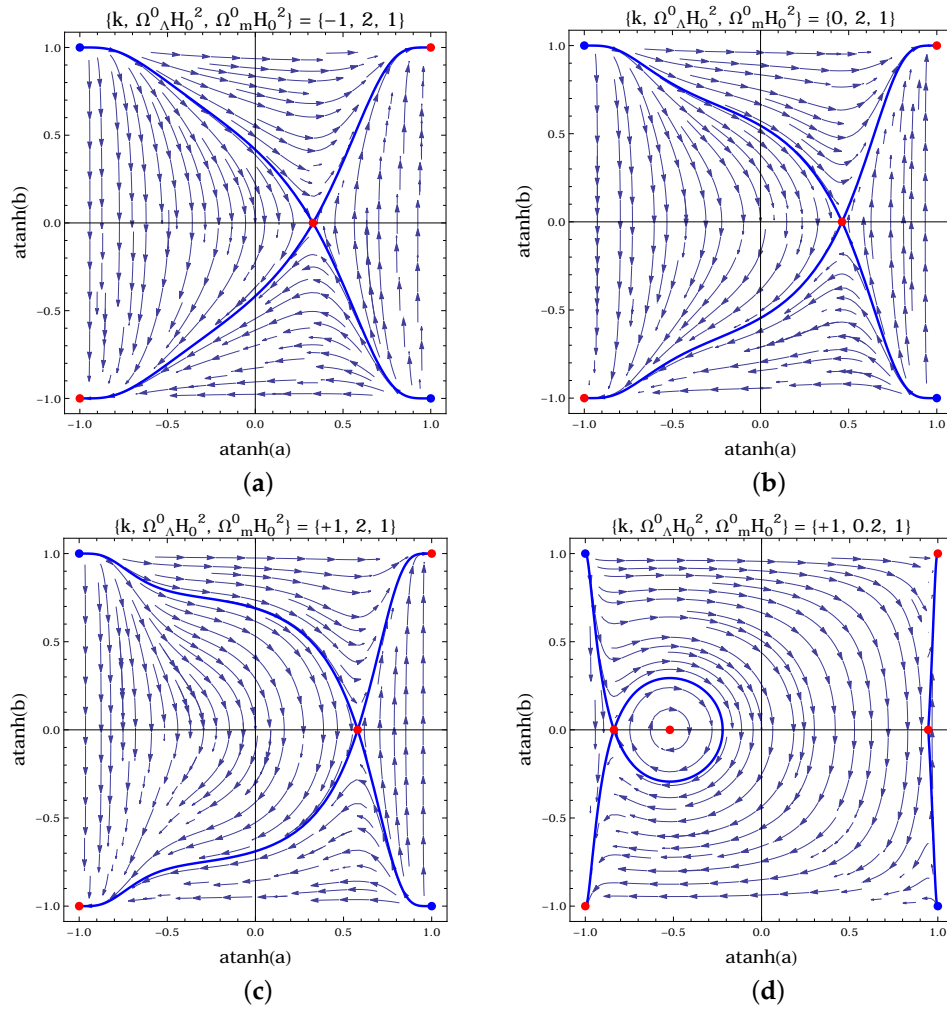


Figure 1. Phase-diagrams displaying the behaviour of the vector field (a, a') for GR with a negative G . From left to right and from top to down: (a) represents the $k = -1$ models, (b) the $k = 0$ models, and (c,d) correspond both to the $k = +1$ models.

In Figure 1a,b, $k = -1$ and $k = 0$ models are represented, and Figure 1c,d correspond both to the $k = +1$ models. The lower half-planes of the various diagrams are mirror reflections from the upper ones under the $t \rightarrow -t$, $(a, b) \rightarrow (a, -b)$ symmetry. In Figure 1a, there are three fixed points, one at $\{a, b\} = \{a_*, 0\}$, where a_* is a finite root of the right-hand side of Equation (12) (which for our choice of parameters corresponds to $\{a, b\} = \{0.341164, 0\}$), and two at infinity. The former fixed point is a saddle point denoting an unstable static solution; the latter points at $b = \pm\infty$ are contracting and expanding de Sitter solutions related by the time reversal symmetry. Although the model is open, there are re-collapsing solutions at the left of the finite fixed point, and solutions collapsing from infinity and bouncing back to it. In Figure 1b there also are three fixed points, one at $\{a, b\} = \{0.5, 0\}$, and the other two at $b = \pm\infty$. The qualitative behaviour is alike the one found for $k = -1$ case. The Figure 1c,d phase diagrams display the two possible behaviours of the $k = +1$ which translate the existence of a bifurcation associated with two different subsets in what regards the balance of parameters (the bifurcation occurs for $8\pi|G|\rho_d^0 = 4/\sqrt{6\lambda}$). In the left one, Figure 1c, once again there are three fixed points at $\{a, b\} = \{0.662359, 0\}$ plus the two fixed points at infinity. From a qualitative viewpoint we find the same behaviour as in the previous open models. However, in Figure 1d there are three fixed points on the x -axis, two saddle points and a center in between, the latter of which corresponds to a basin of oscillatory behaviour. Yet being located in the left-half plane, i.e., $a < 0$, its impact on the physical right-hand side is not qualitatively noticeable, apart from reducing the proportion of solutions which evolve towards the deS points at $b = \pm\infty$.

Analyzing the fixed points of the compactified phase-diagrams, we find that for $k = -1$ and $k = 0$, there are only three fixed points, while for $k = +1$ there could be either three or five fixed points, counting the two critical points at infinity. In the former cases, when there are only three fixed points, the qualitative behaviour is the same independently of the value of k . Different spatial curvature indexes only distinguish through a horizontal shift of the location of the critical point on the horizontal axis. These fixed points are saddle points which correspond to unstable static solutions. In all k cases, there are solutions where a expands to infinity. This fact was foreseeable, because of the repulsive character of gravity when G is changed to $-|G|$. They correspond to asymptotic de Sitter solutions (deS), both in the future and in the past, upon time reversal. They reflect the eventual domination of the cosmological λ -term, overcoming the impact of the sign of G , yet the swapping of the sign of G enhances this domination. Obviously, according to this behaviour, the current accelerated expansion of the Universe is not a problem, but we have a gravity which is inconsistent at small scales with the weak field limit, and thus would be at odds with the Solar System behaviour [10,11]. However, the analysis pursued in this section is merely a previous step to assess the cosmological impact of a negative gravity. In the following section we shall consider the issue within the more appropriate framework of modified metric gravity theories which assume the variation of G .

3. Scalar-Tensor Gravity Theories

We now consider general scalar-tensor gravity theories given by the action

$$S = \int \sqrt{-g} d^4x \left[\phi R - \frac{\omega(\phi)}{\phi} \phi_{;\mu} \phi^{;\mu} - 2U(\phi) \right] + S_m, \quad (14)$$

where a potential term $U(\phi)$ of cosmological nature is considered. (We shall also use $U(\phi) = \phi \lambda(\phi)$). The archetypal feature of this class of theories is the fact the Newton's gravitational coupling is $G = 1/\phi$, and generically varies. The scalar field ϕ may be seen as the gravitational permittivity of the space-time [25].

The field equations are

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \lambda(\phi) g_{\alpha\beta} = \frac{\omega(\phi)}{\phi^2} \left[\phi_{;\alpha} \phi_{;\beta} - \frac{1}{2} g_{\alpha\beta} \phi_{;\gamma} \phi^{;\gamma} \right] + \frac{1}{\phi} [\phi_{;\alpha\beta} - g_{\alpha\beta} \phi_{;\gamma} \phi^{;\gamma}] + 8\pi \frac{T_{\alpha\beta}}{\phi} \quad (15)$$

$$\square\phi - \frac{2\phi^2\lambda'(\phi) - 2\phi\lambda(\phi)}{2\omega(\phi) + 3} = \frac{1}{2\omega(\phi) + 3} [8\pi T - \omega'(\phi) \phi_{;\gamma} \phi^{;\gamma}] \quad (16)$$

where $T \equiv T^c_c$ is the trace of the energy-momentum tensor, $T_\alpha{}^\beta$. When applied to the FLRW models we obtain

$$3 \left(\frac{\dot{a}}{a} \right)^2 + 3 \frac{\dot{a}\dot{\phi}}{a\phi} + 3 \frac{k}{a^2} = \lambda(\phi) + \frac{\omega(\phi)}{2} \frac{\dot{\phi}^2}{\phi^2} + 8\pi \frac{\rho}{\phi}, \quad (17)$$

$$2 \frac{d}{dt} \left(\frac{\dot{a}}{a} \right) + 3 \left(\frac{\dot{a}}{a} \right)^2 + 2 \frac{\dot{a}\dot{\phi}}{a\phi} + \frac{k}{a^2} = \lambda(\phi) - \frac{\omega(\phi)}{2} \frac{\dot{\phi}^2}{\phi^2} - 8\pi \frac{p}{\phi} - \frac{\ddot{\phi}}{\phi}, \quad (18)$$

$$\ddot{\phi} + 3 \frac{\dot{a}\dot{\phi}}{a\phi} + \frac{2\phi^2\lambda'(\phi) - 2\phi\lambda(\phi)}{2\omega(\phi) + 3} = -\frac{1}{2\omega(\phi) + 3} [8\pi(3p - \rho) + \omega'(\phi)\dot{\phi}^2] \quad (19)$$

Please note that the cosmological potential $U(\phi) = \phi \lambda(\phi)$ effectively reduces to a cosmological constant when $\lambda(\phi) = \lambda_0 = \text{constant}$ in this frame.

We introduce the redefined variables

$$X = \frac{\phi}{\phi_0} a^2, \quad Y' = \sqrt{\frac{2\omega(\phi) + 3}{3}} \frac{\phi'}{\phi}, \quad (20)$$

and use conformal time $d\eta = dt/a = d\tilde{t}/\sqrt{X}$. Observe that in the definition of X , ϕ_0 is the value of ϕ at some initial condition which we shall normalize $\phi_0 = 1$ without loss of generality. More importantly, observe that $X < 0$ when $\phi < 0$, i.e., when $G < 0$. This is in fact the crucial detail which allows us to extend the study of the dynamics into the region where $\phi = 1/G$ is negative.

The FLRW equations are then recast as

$$(X')^2 + 4kX^2 - (Y'X)^2 = 4MX \left(\frac{X}{\phi}\right)^{\frac{4-3\gamma}{2}} + \frac{4}{3} \left(\frac{\lambda(\phi)}{\phi}\right) X^3, \quad (21)$$

The scalar-field equation

$$[Y'X]' = M(4-3\gamma) \sqrt{\frac{3}{2\omega+3}} \left(\frac{X}{\phi}\right)^{\frac{4-3\gamma}{2}} - \frac{2X^2}{\sqrt{2\omega(\phi)+3}} \left(\frac{d\lambda}{d\phi} - \frac{\lambda(\phi)}{\phi}\right), \quad (22)$$

The generalized Raychaudhuri equation

$$X'' + 4kX = 3M(2-\gamma) \left(\frac{X}{\phi}\right)^{\frac{4-3\gamma}{2}} + 2X^2 \left(\frac{\lambda(\phi)}{\phi}\right), \quad (23)$$

where M is a constant defined by $M \equiv 8\pi\rho_0/3$.

When the potential $U(\phi) = \lambda_0 \phi^2$, the latter equations reduce to

$$(X')^2 + 4kX^2 - (Y'X)^2 = 4MX \left(\frac{X}{\phi}\right)^{\frac{4-3\gamma}{2}} + \frac{4}{3} \lambda_0 X^3, \quad (24)$$

$$[Y'X]' = M(4-3\gamma) \sqrt{\frac{3}{2\omega+3}} \left(\frac{X}{\phi}\right)^{\frac{4-3\gamma}{2}}, \quad (25)$$

$$X'' + 4kX = 3M(2-\gamma) \left(\frac{X}{\phi}\right)^{\frac{4-3\gamma}{2}} + 2\lambda_0 X^2. \quad (26)$$

These equations can be exactly integrated for the cases where the variables decouple, which are vacuum, radiation and stiff matter [26–28]. Indeed, from Equation (25) we see that in these cases

$$Y' = \frac{f_0}{X(\eta)}, \quad (27)$$

where f_0 is an arbitrary integration constant which fixes the initial value of Y . However, for our present purposes, we only need to assess the qualitative behaviour of the dynamical system [19,29–32].

The crucial point in our analysis of the sign of the gravitational coupling ϕ is that instead of choosing either the original so-called Jordan frame or the conformally transformed Einstein frame arising from rescaling the metric with a factor ϕ/ϕ_0 , where ϕ_0 is an initial value of ϕ , say, at present, we consider the variables (X, Y) , where $X = \phi a^2/\phi_0$ reflects the actual sign of ϕ , as $a^2 \geq 0$.

An inspection of Equations (24)–(26) shows that when $X \rightarrow 0$, Equation (24) is dominated by the scalar field term $(Y'X)^2$, which is constant for radiation, so that we expect the phase space trajectories to cross the $X = 0$ axes from right to left when $X' < 0$, and from left to right when $X' > 0$. Interestingly, Equation (26) shows that when $X \rightarrow 0$, the dominant term is $3M(2-\gamma) \lim_{X \rightarrow 0} (X/\phi)^{(4-3\gamma)/2}$, so that it is actually the matter term the responsible for the turning around of the trajectories towards the positive side of X , and hence of $\phi > 0$. Finally, the presence of a quadratic cosmological potential eventually dominates for large values of X and consequently stabilizes the sign of ϕ . In the following subsections we perform a qualitative analysis confirming this behaviour.

Studies of scalar-tensor theories have been performed by one of the present authors using these techniques [14,16], and similar and complementary analysis can be found in the literature

that followed [18,19,30,31,33,34]. Some of these works focus their analysis to the case of Brans-Dicke theory [30,33,34], while others investigations consider more general scalar-tensor theories. In most of the cases, the qualitative studies rely on choices of variables that make it difficult or even impossible to discuss the sector of the phase space where ϕ is negative, e.g., [33,34] (for a more detailed discussion of the use of the qualitative analysis of dynamical systems applied to scalar-tensor theories see [24,29] and references therein).

3.1. Models without a Cosmological Potential

We begin considering the case where the Brans-Dicke (BD) like scalar field ϕ is massless, i.e., the cosmological potential is absent. This will be contrasted with the case where there is a quadratic potential. By the same token it will enable us to assess whether there is any effect due to a variation of the coupling $\omega(\phi)$ with regard to the issue of determining the sign of ϕ .

For this case, the previous Equation (26) can be written for vacuum ($M = 0$) and a stiff fluid ($\gamma = 2$) as:

$$\begin{aligned} X' &= W \\ W' &= -4kX \end{aligned} \quad (28)$$

and for radiation ($\gamma = 4/3$) as:

$$\begin{aligned} X' &= W \\ W' &= 2M - 4kX. \end{aligned} \quad (29)$$

The plots represented in Figure 2 show the phase diagrams of this systems (X, W) for the case without potential, i.e., $\lambda_0 = 0$. This is again done resorting to a phase space compactification where the points at $X, W \rightarrow \pm\infty$ are located at the boundaries $x = \operatorname{arctanh} X = \pm 1$, $y = \operatorname{arctanh} W = \pm 1$. One realizes that there are trajectories which cross the $X = 0$ dividing line in both directions, thus promoting the transition between a negative ϕ into a positive ϕ , and conversely (and hence of a swapping of the sign of G , as $\phi = G^{-1}$). Please note that as in the GR case previously considered, there is a mirror reflection between the top and lower half of the phase diagrams, arising from time reversal.

We have used the following color scheme depending on the beginning and the end of the trajectory: (i) Blue: The trajectory starts and finishes with positive value of ϕ ; (ii) Yellow: It begins and finishes with a positive value, but passes through negative values; (iii) Green: It starts with negative values, but finishes with positive values; (iv) White: The trajectory oscillates between positive and negative values; (v) Orange: It starts with positive values, but finishes with negative value; (vi) Red: It starts and finishes with a negative value of ϕ (and hence of G).

In Figure 2, the top three phase diagrams represent vacuum and stiff fluid models, whereas the lower three correspond to radiation models. From left to right we have $k = -1, 0$ and $k = +1$ models, respectively. It is immediately apparent that in the vacuum models the number of trajectories that cross in one direction, say from left to right, is the same as that of those which cross in the opposite direction. Once again there is a mirror symmetry with time reversal between the top half and the lower one. Therefore, assuming that a measure of the probability of the model to have a positive gravitational constant, or otherwise, is proportional to the phase space area, we realise that both signs occur with the same probability. In this sense, the same behaviour occurs for the three cases of vacuum or stiff fluid. In addition, in the case of a positive curvature, i.e., Figure 2c the sign oscillates forever, while in the open models the trajectories evolve towards the Milne solution, $x = y = 1$, in the $k = \pm 1$ case, and are characterized by $X' = \pm f_0$ in the $k = 0$ models, which actually correspond to the solutions found in [35].

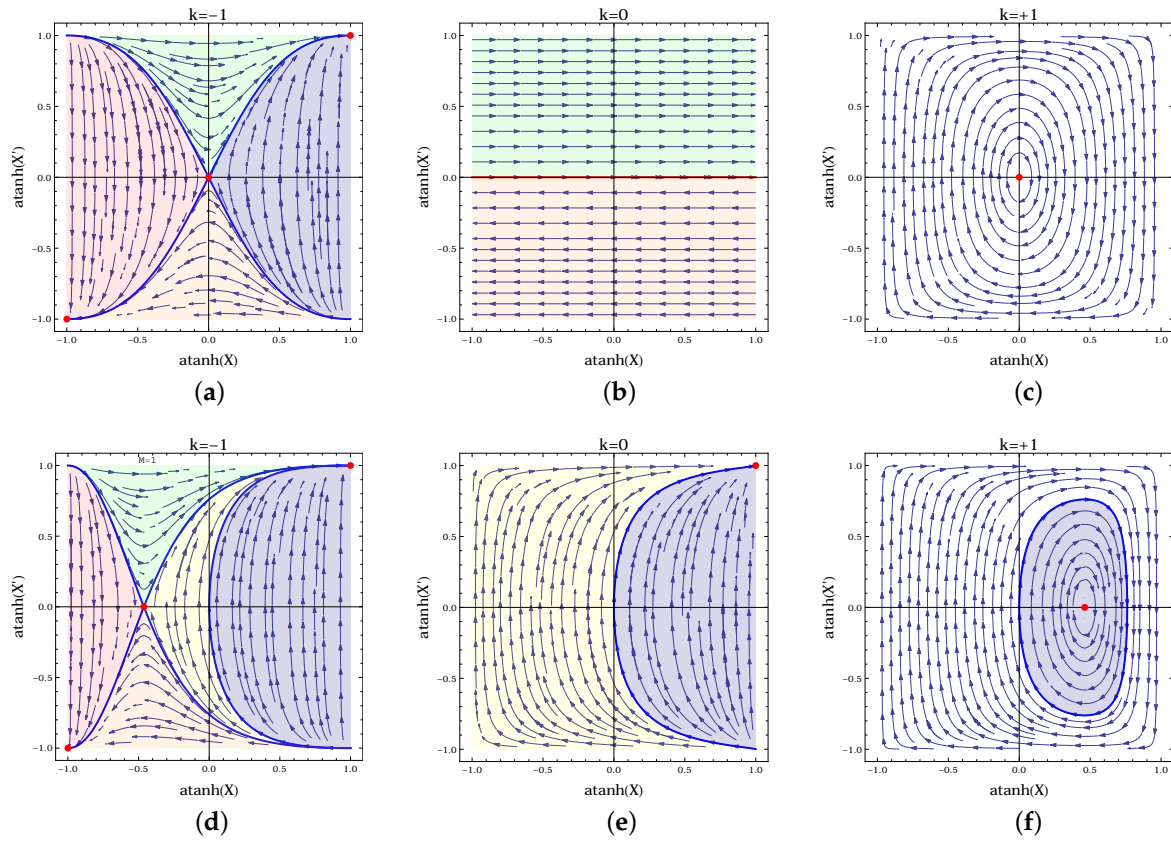


Figure 2. Behaviour of X and X' for a scalar-tensor theory without potential. The upper three phase diagrams, (a–c), respectively, correspond to the $k = -1, 0, +1$ cases for vacuum and stiff fluid. The lower three phase diagrams, (d–f), respectively, correspond to the $k = -1, 0, +1$ cases for radiation.

On the other hand, for radiation [26], there is a preference for the positive sign, or at least, to finish with a positive gravitational coupling, as represented by the blue and yellow areas. This is a reflection of the fact that the solutions are late time dominated by the matter component [36,37]. This is more apparent in the $k = -1, 0$ cases, but in the $k = +1$ cases now exhibits a subset of trajectories with oscillatory behavior, confined in the $\phi > 0$ region. The impact of matter is dependent on the scalar-tensor coupling $\omega(\phi)$. When $1/\sqrt{2\omega + 3} \rightarrow 0$ the scalar-tensor theories approach GR, and this is implicit in Equation (27).

We thus see that the dynamics of the vacuum FLRW models in ST gravity does not favour positive values of G with regard to the alternative possibility of a negative G . The dynamics is such that the upper half of the phase space corresponds to $G > 0$ and the other half to $G < 0$, and mirror reflections of one another. Thus both possibilities in what concerns the sign of G have the same probability. However, when matter is present, in the case of open models (including $k = 0$) there is higher probability for a positive value of G , following from a larger proportion of solutions which evolve to become matter dominated.

3.2. Models with a Cosmological Potential

When we allow for a potential with a positive λ_0 , we see that a quite different picture emerges. Indeed, the phase diagrams corresponding to this case are represented in Figure 3, and we see that now there is an equilibrium point at $x = 1, x' = 1$, corresponding to $X = +\infty, X' = +\infty$, which attracts almost all trajectories of the phase plane, and this happens for all spatial curvature cases. This attractor at infinity corresponds to a de Sitter attractor, and thus to exponential behaviour of X in cosmic time. The only trajectories which do not end at this critical point are found in the closed $k = +1$ and open models $k = -1$, circling the center equilibrium point.

This is illustrated by the vacuum and radiation which were envisaged in the massless ST models in the previous subsection.

- Vacuum ($M = 0$) or stiff fluid $\gamma = 2$ with a cosmological potential

Recalling that $U(\phi) = \lambda_0 \phi^2$, we derive for these two cases:

$$\begin{aligned} X' &= W \\ W' &= 2\lambda_0 X^2 - 4kX \end{aligned} \quad (30)$$

Please note that the case $\gamma = 2$ corresponding to stiff matter can be shown to be reducible to the vacuum case of a theory with a different coupling strength $\omega(\phi)$ (see [27], and the companion paper [28] to the present work).

Now, the fixed points $\{X, X'\}$ within a finite locus will be positioned at $\{0, 0\}$ and $\{2k/\lambda_0, 0\}$. To show the graphics of the phase diagrams, λ_0 has been taken equal to 4, 5 with the purpose to show both points sufficiently separated.

- Radiation case $\gamma = 4/3$ with a cosmological potential

In this case the system is:

$$\begin{aligned} X' &= W \\ W' &= 2M + 2\lambda_0 X^2 - 4kX \end{aligned} \quad (31)$$

and the fixed points are:

$$X = \frac{k \pm \sqrt{k^2 - M\lambda_0}}{\lambda_0}, \quad W = 0.$$

Therefore, for the cases $k = \pm 1$, at the fixed points we require $M\lambda_0 < 1$. When this is not satisfied there are no fixed points, as illustrated in Figure 3d. For the case $k = 0$ there are no fixed points within the finite region of the phase plane.

The qualitative behaviour depicted in Figure 3 reveals that with the exception of the oscillatory solutions, confined to a closed patch, all other solutions emerge from a collapsing deS solution at $X = \infty, X' = -\infty$ and end in the deS solution at $X = \infty, X' = +\infty$, thus revealing the domination of the cosmological potential term [14,30,32–34,38,39]. More importantly, we see that the solutions are attracted to the positive G half plane. We thus conclude that the consideration of a cosmological potential³ has the power to induce the dynamics of FLRW models to favour positive values of G instead of negative G s. Thus this provides a cosmological mechanism to stabilize G in the positive sector. In addition, it happens that the deS asymptotic behaviour is accompanied with a relaxation towards GR [14–16,18,41]

³ One possible origin for such a potential might be found from a mechanism similar to the dark fluid model of [40].

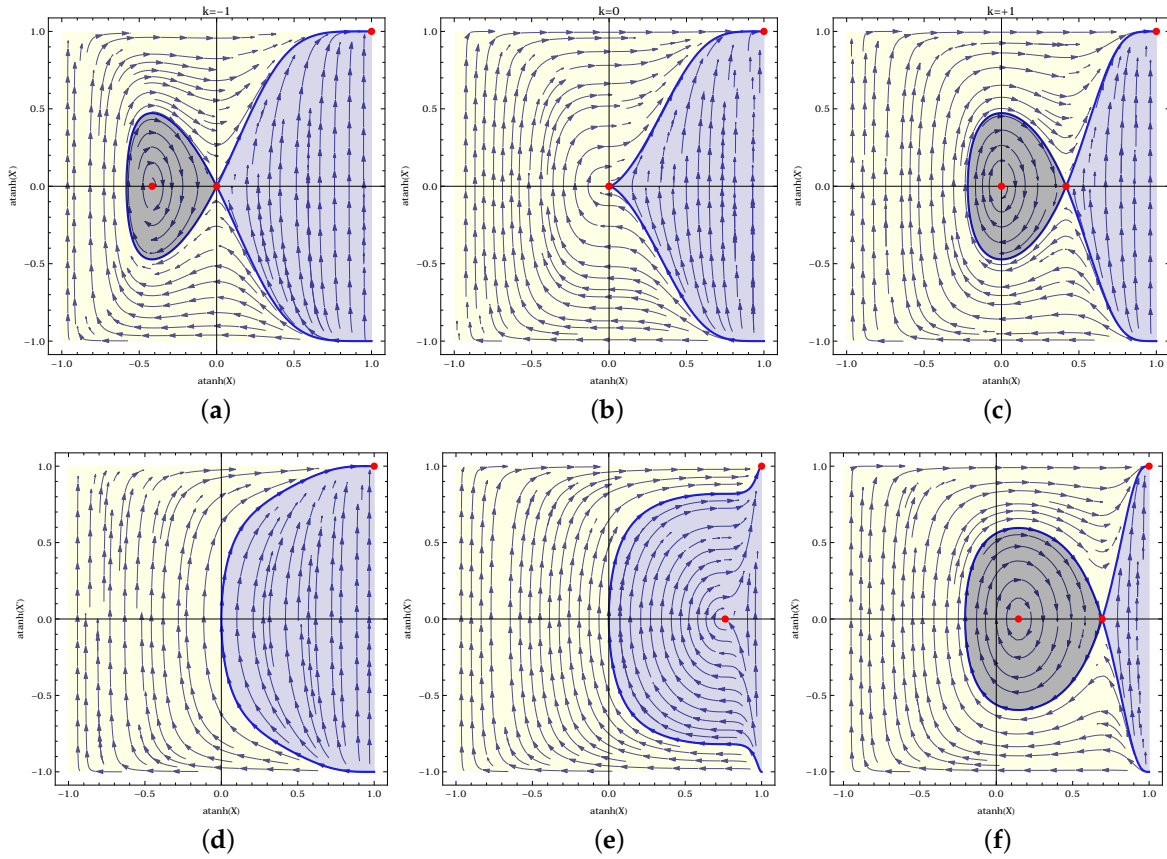


Figure 3. Behaviour of X and X' for a scalar-tensor theory with a potential λ_0 . The upper three phase diagrams, (a–c), respectively, correspond to the $k = -1, 0, +1$ cases for vacuum and stiff fluid. The lower three phase diagrams, (d–f), depict the qualitative behaviour of radiation models, with: (d) When $M\lambda_0 > 1$ (e) When $M\lambda_0 = 1$ (f) When $M = 1/4$ and $\lambda_0 = 2$.

4. Observational Features

ST gravity theories have to satisfy several observational bounds, namely, the so-called Parametrized Post-Newtonian (PPN) weak field, solar system tests, bounds stemming from the cosmic microwave background (CMB), from baryonic acoustic oscillations (BAO), and from primordial Big-Bang Nucleosynthesis (BBN), as well as bounds on the time variation of the gravitational “constant” \dot{G}/G [3,42–44].

The present state of the art tell us that at present $\omega_p \gtrsim 4 \times 10^4$, $\dot{G}/G \lesssim 10^{-10} \text{ yr}^{-1}$, and from BBN we have $\alpha_0^2 \lesssim 10^{-6.5} \beta^{-a} (\Omega_{\text{matter}} h_0)^{-3/2}$ when $\beta > 0.5$ [45–49], where α and β correspond to the Damour and Nordtvedt (DN) [50] parametrization $(\sqrt{2\omega + 3})^{-1} = \alpha(\tilde{\phi})$, with $\alpha(\tilde{\phi}) = \alpha_0 + \beta_0 (\tilde{\phi} - \tilde{\phi}_0)$, $\tilde{\phi}(\phi) = \int \sqrt{(2\omega + 3)/2} d \ln \phi$, being the redefined BD scalar field in the so-called Einstein frame. Indeed, as shown by DN, it is possible to associate a second internal potential $A(\tilde{\phi})$ to the running coupling $\omega(\phi)$ such $\alpha(\tilde{\phi}) = dA/d\tilde{\phi}$ and whenever it has a minimum it drives the relaxation to GR.

The local weak field bounds can be somewhat alleviated if some chameleon or Vainshtein mechanism applies, but it is difficult to evade the other bounds on wider scales. Yet, the vast majority of these bounds pertain to models where the scalar field has no cosmological potential (see though [49]).

These bounds, therefore, imply that a primordial variation of the gravitational coupling must have been severely damped before the time of BBN such that a positive coupling, satisfying mild deviations from GR [47] is not only compatible with the observations on light elements abundances, but also solves the so-called ^7Li problem. In the suite, during the following radiation and matter epochs, the cosmological approach to GR is achieved, implying that G is positive.

The mechanism investigated in this work fulfills this scenario. It tells us that the solutions approach a de Sitter behaviour, which corresponds to $X \propto (\eta_0 - \eta)^{-2}$ and $X' \propto (\eta_0 - \eta)^{-3}$, so that $X(t) \propto \exp(\sqrt{\lambda_0/3} \, \tilde{t})$. When we consider the radiation epoch of the universe, we have

$$Y' = \frac{f_0}{X(\eta)}, \quad (32)$$

and hence, $Y' \rightarrow 0$, asymptotically with $X \rightarrow +\infty$. As $Y' = \sqrt{(2\omega(\phi) + 3)/3} \, \phi'/\phi$, we obtain in cosmic time (in the so-called Einstein frame)

$$\frac{\dot{G}}{G} = \pm \sqrt{\frac{3}{2\omega + 3}} \frac{f_0}{\sqrt{X}} \equiv \pm \frac{\sqrt{3} f_0 \alpha(\phi)}{\sqrt{X}}. \quad (33)$$

As $Y' = f_0/X \rightarrow 0$, and the PPN parameter $\alpha \propto (\sqrt{2\omega(\phi) + 3})^{-1}$ satisfies the bound $\alpha \lesssim 10^{-8}$, as $\omega_p \gtrsim 4 \times 10^4$, we have that $\dot{G}/G \lesssim 10^{-8}/\sqrt{X(\tilde{t})}$, where $X(\tilde{t}) \rightarrow \infty$ in an exponential way, thus easily meeting the bounds on \dot{G}/G . Please note that in this limit, $G \rightarrow G_N$ and $X(t)$ also becomes exponential in the Jordan frame.

5. Summary and Conclusions

In this work, we have investigated a cosmological mechanism that induces the value of the gravitational effective coupling “constant” to be positive. This is naturally done in the framework of scalar-tensor (ST) gravity theories, where this coupling varies, and which thus allow for the possibility of a negative coupling. We have considered the cosmological evolution of ST models both with and without the presence of a cosmological potential. We have resorted to a dynamical systems analysis which enable us to put in evidence the relevant qualitative features of the models. In the absence of the cosmological potential, the presence of matter or radiation favours a positive value of the gravitational “constant”, when the evolution enters a phase of matter domination. This is a mild effect and it is a consequence of Damour and Nordtvedt’s relaxation mechanism towards GR [50]. However, it is when a quadratic cosmological potential, $U(\phi) = \lambda_0 \phi^2$, is present that an attracting mechanism towards a positive value of the gravitational running “constant” becomes manifest. This is accompanied by an asymptotic de Sitter behaviour.

By the same token, this system produces two additional effects: a de Sitter inflation and a relaxation towards general relativity. The latter effect allows, in particular, the fulfilment of the observational bounds on $|\dot{G}/G|$, when the potential is exactly quadratic in the Jordan frame. It effectively acts as a cosmological constant in the Einstein frame and the stabilization of the gravitational constant in the positive sector, may be seen as a by-product of the cosmic no-hair theorem. This mechanism of stabilization of the sign of G should take place early enough, in the primordial stages of the universe, consistently with the latest assessments of observational constraints on ST theories [44,51]

Author Contributions: These authors contributed equally to this work.

Funding: This research was funded by Fundação para a Ciência e a Tecnologia (FCT) grant number PD/BD/114435/2016 under the IDPASC PhD Program and also grant number UID/FIS/04434/2013. This research was also funded by the projects PTDC/FIS- OUT/29048/2017 and IF/00852/2015. of Fundação para a Ciência e a Tecnologia <https://www.fct.pt/index.phtml.en>.

Acknowledgments: JPM expresses his gratitude to Norbert Van den Bergh for kindly bringing the references [10,11] to his attention. In the suite all of the authors are also indebted to both Prof. Ian Roxburgh and George Rideout of the Gravity Research Foundation for most kindly enabling the access to these manuscripts.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
GR	General Relativity
ST	Scalar-Tensor
FLRW	Friedmann-Lemaître-Robertson-Walker
BD	Brans-Dicke
deS	de Sitter
DN	Damour and Nordtvedt
PPN	Parametrised Post-Newtonian
CMB	Cosmic microwave background
BBN	Big-Bang nucleosynthesis

References

1. Laplace, P.S. *Traité de Mécanique Céleste*; Hillard, Gray, Little, and Wilkins (Little and Brown): Boston, MA, USA, 1799; pp. 1829–1839.
2. Newton, I. *Philosophiæ Naturalis Principia Mathematica*, 3rd ed.; Koyré, A., Cohen, I.B., Whitman, A., Eds.; Harvard UP: Cambridge, MA, USA, 1972.
3. Will, C.M. Experimental Gravitation From Newton's Principia To Einstein's General Relativity. In *300 Hundred Years of Gravitation*; Hawking, S., Israel, W., Eds.; Cambridge University Press: Cambridge, UK, 1987; pp. 98–99.
4. The Eöt-Wash Group. Available online: <https://www.npl.washington.edu/eotwash/gravitational-constant> (accessed on 23 August 2016).
5. Barrow, J.D.; Tipler, F.J.L. *The Anthropic Cosmological Principle*; Oxford University Press: Oxford, UK, 1988; pp. 32–58.
6. Brans, C.; Dicke, R.H. Mach's principle and a relativistic theory of gravitation. *Phys. Rev.* **1961**, *124*, 925–935. [[CrossRef](#)]
7. Clifton, T.; Ferreira, P.G.; Padilla, A.; Skordis, C. Modified Gravity and Cosmology. *Phys. Rept.* **2012**, *513*, 1–189. [[CrossRef](#)]
8. Avelino, P.; Barreiro, T.; Carvalho, C.S.; da Silva, A.; Lobo, F.S.N.; Martin-Moruno, P.; Mimoso, J.P.; Nunes, N.J.; Rubiera-Garcia, D.; Saez-Gomez, D.; et al. Unveiling the Dynamics of the Universe. *Symmetry* **2016**, *8*, 79. [[CrossRef](#)]
9. Barrow, J.D. Gravitational memory? *Phys. Rev. D* **1993**, *47*, 1730. [[CrossRef](#)]
10. Roxburgh, I. *The Sign and Magnitude of the Constant of Gravity in General Relativity*; Honorable Mention of the Gravity Prize of the Gravity Research Foundation: Wellesley Hills, MA, USA, 1980. Available online: https://static1.squarespace.com/static/5852e579be659442a01f27b8/t/5c4b36c0aa4a99d762417ca5/1548433090438/roxburgh_sign_magnit_G.pdf (accessed on 23 August 2016).
11. Roxburgh, I. *The Constant of Gravity in General Relativity*; Queen Mary College: London, UK, 1980. Available online: <https://static1.squarespace.com/static/5852e579be659442a01f27b8/t/5c4b36dfcd8366a71b9a7520/1548433120589/roxburgh.pdf> (accessed on 23 August 2016).
12. Khuri, N.N. Sign of the induced gravitational constant. *Phys. Rev. D* **1982**, *26*, 2664. [[CrossRef](#)]
13. Barker, B.M. General scalar-tensor theory of gravity with constant G. *Astr. J.* **1978**, *219*, 5–11. [[CrossRef](#)]
14. Mimoso, J.P.; Nunes, A. General relativity as a cosmological attractor of scalar tensor gravity theories. *Phys. Lett. A* **1998**, *248*, 325–331. [[CrossRef](#)]
15. Mimoso, J.P.; Nunes, A. General relativity as an attractor to scalar tensor gravity theories. *Astrophys. Space Sci.* **1999**, *261*, 327–330. [[CrossRef](#)]
16. Mimoso, J.P.; Nunes, A. A Qualitative Analysis of the Attractor Mechanism of General relativity. *Astrophys. Space Sci.* **2003**, *283*, 661–149. [[CrossRef](#)]
17. Mimoso, J.P. The Dynamics of Scalar Fields in Cosmology. In *Dynamics, Games and Science II*; Peixoto, M.M., Pinto, A.A., Rands, D.A., Eds.; Springer: Berlin, Germany, 2011; pp. 543–547.

18. Jarv, L.; Kuusk, P.; Saal, M. Potential dominated scalar-tensor cosmologies in the general relativity limit: Phase space view. *Phys. Rev. D* **2010**, *81*, 15. [[CrossRef](#)]
19. Jarv, L.; Kuusk, P.; Saal, M. Scalar-tensor cosmologies with a potential in the general relativity limit: Time evolution. *Phys. Lett. B* **2010**, *694*, 1–5. [[CrossRef](#)]
20. Albrecht, A.; Magueijo, J. A Time varying speed of light as a solution to cosmological puzzles. *Phys. Rev. D* **1999**, *59*, 043516. [[CrossRef](#)]
21. Dunsby, P.K.S.; Luongo, O. On the theory and applications of modern cosmography. *Int. J. Geom. Methods Mod. Phys.* **2016**, *13*, 1630002. [[CrossRef](#)]
22. Aviles, A.; Gruber, C.; Luongo, O.; Quevedo, H. Cosmography and constraints on the equation of state of the Universe in various parametrizations. *Phys. Rev. D* **2012**, *86*, 123516. [[CrossRef](#)]
23. Aghanim, N. Planck 2018 results. VI. Cosmological parameters. *arXiv* **2018**, arXiv:1807.06209.
24. Bahamonde, S.; Böhmer, C.G.; Carloni, S.; Copeland, E.J.; Fang, W.; Tamanini, N. Dynamical systems applied to cosmology: Dark energy and modified gravity. *Phys. Rep.* **2018**, 775–777. [[CrossRef](#)]
25. Cembranos, J.A.R.; Coma Díaz, M.; Martín-Moruno, P. Modified gravity as a diagravitational medium. *Phys. Lett. B* **2019** 788, 336. [[CrossRef](#)]
26. Barrow, J.D. Scalar-tensor cosmologies. *Phys. Rev. D* **1993**, *47*, 5329. [[CrossRef](#)]
27. Mimoso, J.P.; Wands, D. Massless fields in scalar-tensor cosmologies. *Phys. Rev. D* **1995**, *51*, 477. [[CrossRef](#)]
28. Raposo, D.; Ayuso, I.; Lobo, F.S.; Mimoso, J.P.; Nunes, N.J. Scalar-tensor cosmologies with a quadratic potential. **2018**, In preparation; Unpublished.
29. Faraoni, V. Phase space geometry in scalar-tensor cosmology. *Ann. Phys.* **2005**, *317*, 366. [[CrossRef](#)]
30. Santos, C.; Gregory, R. Cosmology in Brans-Dicke theory with a scalar potential. *Ann. Phys.* **1997**, *258*, 111. [[CrossRef](#)]
31. Carloni, S.; Capozziello, S.; Leach, J.A.; Dunsby, P.K.S. Cosmological dynamics of scalar-tensor gravity. *Class. Quant. Grav.* **2008**, *25*, 035008. [[CrossRef](#)]
32. Charters, T.C.; Nunes, A.; Mimoso, J.P. Stability analysis of cosmological models through Liapunov's method. *Class. Quant. Grav.* **2001**, *18*, 1703. [[CrossRef](#)]
33. Hrycyna, O.; Szydlowski, M. Dynamical complexity of the Brans-Dicke cosmology. *J. Cosmol. Astropart. Phys.* **2013**, 1312, 016. [[CrossRef](#)]
34. García-Salcedo, R.; González, T.; Quiros, I. Brans-Dicke cosmology does not have the Λ CDM phase as a universal attractor. *Phys. Rev. D* **2015**, *92*, 124056. [[CrossRef](#)]
35. O'Hanlon, J.; Tupper, B.O.J. Vacuum-field solutions in the Brans-Dicke theory. *Il Nuovo Cimento* **1972**, *7*, 305. [[CrossRef](#)]
36. Nariai, H. On the Green's function in an expanding universe and its role in the problem of Mach's principle. *Prog. Theor. Phys.* **1968**, *40*, 49. [[CrossRef](#)]
37. Barrow, J.D.; Mimoso, J.P. Perfect fluid scalar-tensor cosmologies. *Phys. Rev. D* **1994**, *50*, 3746. [[CrossRef](#)]
38. Capozziello, S.; de Ritis, R.; Scudellaro, P. Nonminimal coupling and cosmic no hair theorem. *Phys. Lett. A* **1994**, *188*, 130–136. [[CrossRef](#)]
39. Capozziello, S.; de Ritis, R.; Rubano, C.; Scudellaro, P. Nonminimal coupling, no hair theorem and matter cosmologies. *Phys. Lett. A* **1995**, *201*, 145–150. [[CrossRef](#)]
40. Luongo, O.; Muccino, M. Speeding up the universe using dust with pressure. *Phys. Rev. D* **2018** 98, 103520. [[CrossRef](#)]
41. Kuusk, P.; Jarv, L.; Saal, M. Scalar-tensor cosmologies: General relativity as a fixed point of the Jordan frame scalar field. *Int. J. Mod. Phys. A* **2009**, *24*, 1631. [[CrossRef](#)]
42. Will, C.M. The Confrontation between General Relativity and Experiment. *Living Rev. Rel.* **2014**, *17*, 4–117. [[CrossRef](#)]
43. Koyama, K. Cosmological Tests of Modified Gravity. *Rep. Prog. Phys.* **2016**, *4*, 046902. [[CrossRef](#)]
44. Ooba, J.; Ichiki, K.; Chiba, T.; Sugiyama, N. Cosmological constraints on scalar-tensor gravity and the variation of the gravitational constant. *Prog. Theor. Exp. Phys.* **2017**, *4*, 043E03. [[CrossRef](#)]
45. Mimoso, J.P. Primordial Cosmology in Jordan-Brans-Dicke Theory: Nucleosynthesis. In *Classical and Quantum Gravity* Bento, M.C., Bertolami, O., Mourao, J.M., Picken, R., Eds.; World Scientific: Singapore, 1993.
46. Damour, T.; Pichon, B. Big bang nucleosynthesis and tensor-scalar gravity. *Phys. Rev. D* **1999**, *59*, 123502. [[CrossRef](#)]

47. Larena, J.; Alimi, J.M.; Serna, A.T. Big Bang nucleosynthesis in scalar tensor gravity: The key problem of the primordial Li-7 abundance. *Astrophys. J.* **2007**, *658*, 1–10. [[CrossRef](#)]
48. Iocco, F.; Mangano, G.; Miele, G.; Pisanti, O.; Serpico, P.D. Primordial Nucleosynthesis: From precision cosmology to fundamental physics. *Phys. Rep.* **2009**, *472*, 1–76. [[CrossRef](#)]
49. Coc, A.; Descouvemont, P.; Olive, K.A.; Uzan, J.P.; Vangioni, E. The variation of fundamental constants and the role of A=5 and A=8 nuclei on primordial nucleosynthesis. *Phys. Rev. D* **2012**, *86*, 043529. [[CrossRef](#)]
50. Damour, T.; Nordtvedt, K. Tensor-scalar cosmological models and their relaxation toward general relativity. *Phys. Rev. D* **1993**, *48*, 3436. [[CrossRef](#)]
51. Lee, S. Constraints on scalar-tensor theories of gravity from observations. *J. Cosmol. Astropart. Phys.* **2011**, *1103*, 021. [[CrossRef](#)]



© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).