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Article

Revisiting Vaidya Horizons

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Abstract: In this study, we located and compared different types of horizons in the spherically symmetric Vaidya solution. The horizons we found were trapping horizons, which can be null, timelike, or spacelike, null surfaces with constant area change and also conformal Killing horizons. The conformal Killing horizons only exist for certain choices of the mass function. Under a conformal transformation, the conformal Killing horizons can be mapped into true Killing horizons. This allows conclusions drawn in the dynamical Vaidya spacetime to be related to known properties of static spacetimes. We found the conformal factor that performs this transformation and wrote the new metric in explicitly static coordinates. Using this construction we found that the tunneling argument for Hawking radiation does not umabiguously support Hawking radiation being associated with the trapping horizon. We also used this transformation to derive the form of the surface gravity for a class of physical observers in Vaidya spacetimes.

Keywords: black holes; horizons; Vaidya; conformal transformations; Hawking radiation

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1. Introduction

That quantum effects are relevant at the horizons of black holes is shown by the Hawking effect. While the energy flux due to Hawking radiation is widely expected to be small, a longstanding debate has existed about the nature of quantum fields in the vicinity of black hole horizons and the true nature of the expectation value of the renormalized stress energy tensor. The question of what happens at the horizon of a black hole has received fresh attention due to the recently reignited debate about the firewall model. Two main approaches exist for locating the horizon of a black hole. The causal approach, familiar especially in the form of the global event horizon [1] and the quasi-local approach, typically

based on marginally outer trapped surfaces [2]. The horizon locations given by these two approaches coincide in globally static spacetimes such as the Schwarzschild solution, but differ often in dynamical spacetimes. The differences are small in many cases but can also become appreciable in physically realistic situations [3].

A key difference between the causal horizons and those based on marginally trapped surfaces is that the former are always by definition null surfaces generated by null vector fields, while the latter can be null but also spacelike and timelike. Another key difference between these two types of horizons is their behavior under a conformal transformation of the spacetime metric. A conformal transformation does not change the local causal structure, so event horizons do not change location provided the asymptotic structure remains the same. On the other hand, because a conformal transformation changes the metric area of surfaces, it will change the behaviors of null expansions and therefore the location of marginally outer trapped surfaces. In fact, surfaces that were trapped in one conformal frame can become untrapped in another conformal frame.

A natural physical phenomenon to associate with the boundary of a black hole is Hawking radiation. This is particularly relevant in the context of entropy microstates and information retention but may also have relevance to renormalized stress-energy and the quantum vacuum. A popular technique to investigate the Hawking effect is the Hamilton-Jacobi method (for a review, see [4]) which has the benefit not only of being simple to use but also of providing a direct realization of the relevant horizon surface as the location of the tunneling amplitude pole. The form of the Hamilton-Jacobi equation is manifestly conformally invariant and therefore its association with the trapping horizon might appear problematic [5].

In the following we will use the Vaidya solution to investigate some of these issues. The Vaidya solution [6] describes a radiating star or black hole. The Vaidya solution is an exact solution of the Einstein equations for a pressureless null dust and has a long history of being used as a tool to investigate dynamical processes such as Hawking radiation in dynamical black hole spacetimes [7]. While the solution cannot give a complete picture of the entire lifetime of a black hole, it can give insight to near horizon phenomena in the presence of a growing or shrinking horizon. A detailed discussion of the difference between the event horizon and the marginally trapped tubes of the Vaidya solution can be found in [8]. A collection of useful results relating to the Vaidya solution can be found amongst other places in [9].

We will focus on investigating what is required to locate the pole of the Hamilton-Jacobi tunneling equation to the dynamical trapping horizon. We will also address how null horizons can be defined in the Vaidya solution in a quasi-local way and study some of their properties. And finally we will find a static conformal cousin of the Vaidya solution and use this to calculate the energy spectrum of Hawking radiation in the Vaidya spacetime.

The first section gives relevant background and defines notation. The second section defines conformally static coordinates for the Vaidya solution and uses these to study again the Hamilton-Jacobi tunneling argument. The third section considers the subclass of Vaidya spacetimes for which a conformal transformation can be found to a static spacetime and uses standard results for static spacetimes to derive the surface gravity and Hawking temperature for these spacetimes. The conformal transformation is then

used to map these back to physically measurable energy values for observers in the Vaidya spacetime. We conclude with a discussion.

2. Horizons in Vaidya

The spherically symmetric Vaidya line element in advanced Eddington Finkelstein null coordinates is:

$$\mathrm{d}s^2 = -\left(1 - \frac{2m(v)}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}^2\Omega \tag{1}$$

The r coordinate is the areal coordinate such that the surfaces of spherical isometry have area $A = 4\pi r^2$ and $\mathrm{d}\Omega^2$ is the metric on these spheres. Future-directed, ingoing, radial null vectors, n, can be parameterized by the null coordinate v and are given by $n = -\mathrm{d}v$. We will assume in the following $m(v) > 0$ but otherwise the function $m(v)$ can take any form, except where we will require the spacetime to admit local conformal Killing horizons. Our focus will be on the local and quasi-local properties of this line element as a model of near-horizon physics. For notational simplicity we introduce the function $\triangle(v, r) = 1 - \frac{2m(v)}{r}$. The radial outgoing null expansions, l, are:

$$\theta_l = \frac{\triangle}{r} \tag{2}$$

and the ingoing expansions:

$$\theta_n = -\frac{2}{r} \tag{3}$$

The second derivatives are:

$$n^a\nabla_a\theta_l = \frac{1}{r^2}\left(1 - \frac{4m(v)}{r}\right) \tag{4}$$

and:

$$l^a\nabla_a\theta_n = \frac{\triangle}{r^2} \tag{5}$$

Thus there is a future outer trapping horizon (FOTH) in the language of [10] at $\triangle = 0$, equivalently $r = 2m(v)$. This is a well known, standard result. The location of the global event horizon of course cannot be located without knowledge of the full global spacetime and as we wish to pursue a quasi-local approach here, we will instead consider to what extent strictly null horizons can be defined quasi-locally.

2.1. Radial Null Vectors

In a spherically symmetric spacetime the causal past or future of any spherically symmetric region will itself be spherically symmetric and generated by radial null vectors. In the spherically symmetric Vaidya solution any outgoing radial null vector, parametrised by a parameter λ, satisfies:

$$\frac{\mathrm{d}r}{\mathrm{d}\lambda} = \frac{\triangle}{2}\frac{\mathrm{d}v}{\mathrm{d}\lambda} \tag{6}$$

which is future-directed outgoing for $\triangle > 0$. The second variation of this relation is:

$$\frac{\mathrm{d}^2r}{\mathrm{d}\lambda^2} = \frac{\triangle}{2}\frac{\mathrm{d}^2v}{\mathrm{d}\lambda^2} + \frac{m}{r^2}\frac{\mathrm{d}v}{\mathrm{d}\lambda}\frac{\mathrm{d}r}{\mathrm{d}\lambda} - \frac{\dot{m}}{r}\left(\frac{\mathrm{d}v}{\mathrm{d}\lambda}\right)^2 \tag{7}$$

where a dot denotes derivative with respect to the coordinate v. For the case $\frac{\mathrm{d}^2 v}{\mathrm{d}\lambda^2} = 0$ (which describes a linear relation between v and λ) this reduces to:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\lambda^2} = \left(\frac{m\triangle}{2r^2} - \frac{\dot{m}}{r}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}\lambda}\right)^2 \tag{8}$$

Outgoing null radial curves for which $\frac{\mathrm{d}^2 r}{\mathrm{d}\lambda^2} = 0$ will then satisfy:

$$\frac{2\dot{m}}{r} - \frac{m(v)\triangle}{r^2} = \frac{2\dot{m}}{r} - \frac{m(v)}{r^2} + \frac{2m^2}{r^3} = 0 \tag{9}$$

with solutions for r of:

$$r = \infty \;\text{ or }\; r = \frac{m(v)}{4\dot{m}}\left(1 \pm \sqrt{1 - 16\dot{m}}\right) \tag{10}$$

The finite solutions are then, to order $\dot{m}^2$:

$$r = 2m(v)\left(1 + 4\dot{m} + 32\dot{m}^2 + O(\dot{m}^3)\right) \tag{11}$$

or:

$$r = \frac{m(v)}{2\dot{m}} - 2m(v)\left(1 + 4\dot{m} + 32\dot{m}^2 + O(\dot{m}^3)\right) \tag{12}$$

This condition does not exactly locate the global event horizon, but for a Vaidya black hole that eventually settles down to a Schwarzschild black hole, this condition gives a strictly null surface that is a very close approximation to the location of the event horizon [3]. In a similar vein, it is also interesting to consider null surfaces whose parameter rate of area change is constant. The change in the isometry surface area ($A = 4\pi r^2$) along a set of outgoing null rays is given by:

$$\frac{\mathrm{d}A}{\mathrm{d}\lambda} = 4\pi r \triangle \frac{\mathrm{d}v}{\mathrm{d}\lambda} \tag{13}$$

For any parameterization, all radial null rays will satisfy:

$$\frac{\mathrm{d}r}{\mathrm{d}v} = \frac{\triangle}{2} \tag{14}$$

The second derivative is then:

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\lambda^2} = 4\pi\left[\left(\frac{\triangle^2}{2} - 2\dot{m} + \frac{m\triangle}{r}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}\lambda}\right)^2 + r\triangle\frac{\mathrm{d}^2 v}{\mathrm{d}\lambda^2}\right] \tag{15}$$

For $\frac{\mathrm{d}^2 v}{\mathrm{d}\lambda^2} = 0$, this reduces to:

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\lambda^2} = 2\pi\left(1 - \frac{2m}{r} - 4\dot{m}\right) \tag{16}$$

Radial null vectors for which $\frac{\mathrm{d}^2 A}{\mathrm{d}\lambda^2} = 0$ will then be situated at:

$$r = \frac{2m(v)}{1 - 4\dot{m}} \tag{17}$$

To order $\dot{m}^2$ this is just:

$$r = 2m(v)\left(1 + 4\dot{m} + 16\dot{m}^2 + O(\dot{m}^3)\right) \tag{18}$$

For small values of $\dot{m}$ these two constraints of constant radial coordinate change and constant area change select out null surfaces that are close to, but not collocated with, the trapping horizon at $r = 2m(v)$.

2.2. Conformal Killing Field

The vector field given in these coordinates by:

$$k^a = (v, r, 0, 0) \tag{19}$$

satisfies:

$$\nabla_a k_b + \nabla_b k_a = 2g_{ab} + 2(m - \dot{m}v)\delta^v_a \delta^v_b \tag{20}$$

Thus for $m = \dot{m}v$ this satisfies the requirements of a conformal Killing vector field, $\nabla_a k_b + \nabla_b k_a \propto g_{ab}$. For this to hold $m(v)$ must be a linear function of the advanced time, $m(v) = \mu v$, which holds for a subclass of Vaidya solutions. The norm of the vector field is:

$$k^a k_a = -\triangle v^2 + 2rv \tag{21}$$

This is null at $v = 0$ or:

$$r = \frac{v}{4}\left(1 \pm \sqrt{1 - 16m(v)/v}\right) \tag{22}$$

When k^a is a conformal Killing vector field (for $m(v) = \mu v$ and $\mu < 1/16$), there is a conformal Killing horizon at the same locations as the $\frac{\mathrm{d}^2 r}{\mathrm{d}\lambda^2} = 0$ null surfaces.

2.3. Hamilton-Jacobi Equation

The Hamilton-Jacobi method has been widely studied in the context of Hawking radiation, both for its simplicity and it applicability to dynamical situations. The massless Hamilton-Jacobi equation is:

$$g^{ab}\partial_a I \partial_b I = 0 \tag{23}$$

for a tunneling action I. This expresses the fact that the characteristic curves of the massless field are null geodesics, with tangents $\partial_a I$. If we assume a form of the tunneling amplitude as:

$$\mathrm{d}I = \partial_v I \mathrm{d}v + \partial_r I \mathrm{d}r = \omega \mathrm{d}v - k\mathrm{d}r \tag{24}$$

then for the Hamilton-Jacobi equation in the Vaidya spacetime we get:

$$-2\omega k + \triangle k^2 = 0 \tag{25}$$

with solutions $k = 0$ for the ingoing mode and:

$$k = \frac{2\omega}{\triangle} \tag{26}$$

for the outgoing mode. The outgoing mode has a pole at $\triangle = 0$, the location of the trapping horizon. This has been interpreted in various approaches as indicating that the Hawking radiation should be associated with this $\triangle = 0$ surface [11].

3. Vaidya in Conformally Static Coordinates

Consider now the coordinate transformation:

$$R^2 = c_o^2 \frac{r}{v} \tag{27}$$

and:

$$T = t_o \ln\left(\frac{v}{v_o}\right) \tag{28}$$

Here, c_o, t_o and v_o are dimensionful constants that ensure that both R and T have the usual dimensions of length. For simplicty, their values can be set to dimensionful unity, but we retain them explicitly for future use. Under this coordinate transformation, the usual Vaidya line element:

$$\mathrm{d}s^2 = -\left(1 - \frac{2m(v)}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}^2\Omega \tag{29}$$

takes the form:

$$\mathrm{d}s^2 = \frac{v_o^2}{c_o^2} e^{2T/t_o}\left[\left(-c_o^2\triangle + 2R^2\right)\frac{\mathrm{d}T^2}{t_o^2} + 4R\frac{\mathrm{d}T}{t_o}\mathrm{d}R + \frac{R^4}{c_o^2}\mathrm{d}\Omega^2\right] \tag{30}$$

The R coordinate is not the areal coordinate associated to the Kodama vector, but the T coordinate is still a null coordinate since $g^{-1}(\mathrm{d}T, \mathrm{d}T) = 0$. In these coordinates the radial null rays are given by $\frac{\mathrm{d}T}{\mathrm{d}\lambda} = 0$ (ingoing) or:

$$\frac{\mathrm{d}R}{\mathrm{d}\lambda} = \frac{1}{4R}\left(c_o^2\triangle - 2R^2\right)\frac{1}{t_o}\frac{\mathrm{d}T}{\mathrm{d}\lambda} \tag{31}$$

for the outgoing rays. The expansion of the outgoing null direction is:

$$\theta_l = \frac{c_o^2 e^{-T/t_o}}{v_o R^2}\triangle \tag{32}$$

which vanishes at $R = \infty$, $T = \infty$ or $\triangle = 0$. The conformal Killing vector field is now given by:

$$v\frac{\partial}{\partial v} + r\frac{\partial}{\partial r} = t_o\frac{\partial}{\partial T} \tag{33}$$

The norm of this vector is just:

$$g\left(\frac{\partial}{\partial T}, \frac{\partial}{\partial T}\right) = \frac{v_o^2}{t_o^2 c_o^2} e^{2T/t_o}\left(-c_o^2\triangle + 2R^2\right) \tag{34}$$

which vanishes at $T = -\infty$ or $-c_o^2\triangle + 2R^2 = 0$. Now suppose that we can write the tunneling action for the Hamilton-Jacobi equation as:

$$I = \int W\mathrm{d}T - \int K\mathrm{d}R \tag{35}$$

This is a different coordinate split compared to the version in Equation (24). In this case the Hamilton-Jacobi equation is:

$$g^{ab}\partial_a I\partial_b I = -\frac{t_o}{R}WK + \frac{c_o^2\triangle - 2R^2}{R^2}K^2 = 0 \tag{36}$$

which has solutions at $K/R = 0$ or:

$$K = \frac{W t_o R}{c_o^2 \triangle - 2R^2} \tag{37}$$

This last solution for K has a pole at the conformal Killing horizon $c_o^2 \triangle - 2R^2 = 0$, not at the trapping horizon $\triangle = 0$. The two different choices of coordinates systems, v, r or R, T give different answers for the location of the relevant pole and thus it would seem for the location of the relevant tunneling horizon.

4. Static Conformal Vaidya

The following metric is static:

$$\mathrm{d}s^2 = -\frac{\triangle}{vr}\mathrm{d}v^2 + \frac{2}{vr}\mathrm{d}v\mathrm{d}r + \frac{r}{v}\mathrm{d}\Omega^2 \tag{38}$$

for $\triangle = 1 - 2\mu v/r$ and μ a constant less than $1/16$, since there exists a Killing vector, k^a, given in these coordinates by:

$$k^a = (v, r, 0, 0) \tag{39}$$

which satisfies Killing's equation, $\nabla_a k_b + \nabla_b k_a = 0$, and whose norm:

$$k^a k_a = 2 - \frac{\triangle v}{r} \tag{40}$$

is negative, indicating a timelike vector, in the region bounded by:

$$r = \frac{v}{4}\left(1 \pm \sqrt{1 - 16\mu}\right) \tag{41}$$

At the boundaries of this region k^a is null and these are thus Killing horizons. This is entirely consistent with what was found above in Equation (22) since the metric under consideration is just the usual Vaidya metric conformally transformed by a conformal factor $1/vr$ and these Killing horizons are located at the same location as the conformal Killing horizons of the Vaidya spacetime when $m(v) = \mu v$. In this static spacetime these Killing horizons also coincide with the trapping horizons which are now given by:

$$\theta_l = -\frac{2}{v} + \frac{\triangle}{r} = 0 \tag{42}$$

Static coordinates exist for this solution, in which case it takes the form:

$$\mathrm{d}s^2 = c_o^2\left(-\frac{c_o^2}{R^2} + 2\mu\frac{c_o^4}{R^4} + 2\right)\frac{\mathrm{d}T^2}{t_o^2} + 4\frac{c_o^2}{R}\frac{\mathrm{d}T}{t_o}\mathrm{d}R + R^2\mathrm{d}\Omega^2 \tag{43}$$

where we have included a constant factor of c_o^2. In the dynamical v, r coordinates, the Hamilton-Jacobi equation with $\mathrm{d}I = \partial_v I \mathrm{d}v + \partial_r I \mathrm{d}r$ gives:

$$\partial_r I = \frac{2\partial_v I}{\triangle} \tag{44}$$

with a pole at $\triangle = 0$. In the static T, R coordinates the Hamilton-Jacobi equation gives:

$$\partial_R I = \frac{\partial_T I \; t_o R}{c_o^2 \triangle - 2R^2} \tag{45}$$

with a pole at $\triangle = 2R^2/c_o^2$. Both results give apparent poles for the spatial part of I but at different locations. These locations are however both conformally invariant when transforming to the dynamical Vaidya spacetime by a time dependent conformal transformation, *provided one keeps the same coordinates*. A coordinate choice must be made when determining where the pole in I (which should be a conformally invariant quantity) lies. In the static spacetime both the trapping horizon and event horizon are collocated with the Killing horizon. This is not true in the Vaidya spacetime. Conformal invariance of the relevant physics suggests that Hawking radiation should be associated with the conformal Killing horizon, not the trapping horizon (or apparent horizon) even in the Vaidya spacetime, which is a full solution of the Einstein equations with reasonable matter content formulated in the canonical conformal frame. For this static metric in static coordinates, a standard calculation [12] gives the surface gravity as:

$$\kappa = \frac{2}{t_0}\left(\frac{\sqrt{1-16\mu}}{1-\sqrt{1-16\mu}}\right) \tag{46}$$

The physical interpretation of this result is the surface gravity as would be measured by a static observer at infinity. However, the static Vaidya metric is not static asymptotically, only static in the region bounded by the two Killing horizons. A more relevant surface gravity is that related to the temperature measured by a static, constant R observer located between the two horizons. This is obtained by normalizing the Killing vector $\partial/\partial T$ such that its norm is -1 at the observer's location. For the case that the observer's location is given by $R^2 = R_o^2$ the observer's measured surface gravity is:

$$\kappa_{static} = \frac{2}{c_o\sqrt{\frac{c_0^2}{R_0^2} - 2 - 2\mu\frac{c_0^4}{R_0^4}}}\left(\frac{\sqrt{1-16\mu}}{1-\sqrt{1-16\mu}}\right) \tag{47}$$

While this result has been obtained using the standard techniques valid in static spacetimes, its immediate utility is that it can be transformed via the conformal transformation to the surface gravity that would be measured by observers following the same trajectory in the Vaidya spacetime. This is similar in vein to the conformal invariance of the temperature [13], now corrected for observers not located at infinity. In this case, observers of constant R coordinate are mapped to observers of constant v/r coordinate and we have:

$$\kappa_{Vaidya} = \frac{c_0^2}{vR_0}\kappa_{static} \tag{48}$$

Although these observers are not Kodama observers, having trajectories that keep them forever between the two conformal Killing horizons, this κ_{Vaidya} can be simply related to the energy of detected Hawking radiation particles and hence by a standard boost relation, can be related to the corresponding energy that would be measured by Kodama observers or any other set of observers, fully consistent with Dicke's conformal invariance under transformation of units [14]. In this way one can reconstruct the full energy spectrum of detected Hawking radiation particles for any set of observers outside a fully dynamical Vaidya black hole.

5. Discussion and Conclusions

The problems associated with global event horizons have been widely discussed. Their teleological nature is perhaps the most troublesome feature, but also the fact that event horizons cannot form

within other event horizons with respect to the same asymptotic region. The problems with marginally outer trapped surfaces and trapping horizons have also been examined. They are non-unique, depending on a choice of foliation [9], or equivalently a choice of null normals, they may intersect a given spatial hypersurface multiple times [1], they may extend partially into flat space regions [15], they have difficulties with the generalised second law [16] and are not invariant under conformal transformations [17].

Several of the problems with event horizons would be alleviated if they were not strictly related to null infinity, although what should replace this, is not clear. Killing horizons are examples of null surfaces and are quasi-locally defined. As an extension of Killing horizons, conformal Killing horizons are also null by definition and exist in several dynamical spacetimes. These give a test ground for investigating quasi-local null horizons in dynamical spacetimes. While a realistic spacetime may locally admit a timelike Killing vector field or a conformal Killing vector, it is unlikely that they would exist globally in a full solution and in certain cases the Killing horizons or conformal Killing horizons will only approximate the location of the global event horizon [3]. More generally we have also seen that there are quasi-locally defined, strictly null surfaces in the full class of dynamical Vaidya spacetimes. In certain cases these coincide with conformal Killing horizons, but can be found more generally.

The conformal invariance of the Hamilton-Jacobi equation used for tunneling Hawking radiation requires careful attention to how the tunneling amplitude is split into temporal and spatial parts. For different choices of coordinates the pole occurs in different positions. In the static conformal Vaidya solution it seems reasonable that the relevant horizon is the Killing horizon and this is indeed where the coordinate invariant trapping horizon is located. But the trapping horizon location is not conformally invariant and so it becomes difficult to map back to the Vaidya solution. This suggests that the trapping horizon as the source of Hawking radiation is not compatible with the semi-classical Weyl invariance of gravitational physics [18].

Finally, we have seen how a conformal transformation can be used to define the surface gravity and related energy spectrum of Hawking radiation in the Vaidya spacetime. The correct form of the surface gravity is still unknown [19] but these results should provide an interesting test of recent proposals. We leave this matter for future studies.

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Conflicts of Interest

The author declares no conflicts of interest.

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