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Euclidean Space Controllability Conditions for Singularly Perturbed Linear Systems with Multiple State and Control Delays

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Received: 3 February 2019; Accepted: 17 March 2019; Published: 21 March 2019

Abstract: A singularly perturbed linear time-dependent controlled system with multiple point-wise delays and distributed delays in the state and control variables is considered. The delays are small, of order of a small positive multiplier for a part of the derivatives in the system. This multiplier is a parameter of the singular perturbation. Two types of the considered singularly perturbed system, standard and nonstandard, are analyzed. For each type, two much simpler parameter-free subsystems (the slow and fast ones) are associated with the original system. It is established in the paper that proper kinds of controllability of the slow and fast subsystems yield the complete Euclidean space controllability of the original system for all sufficiently small values of the parameter of singular perturbation. Illustrative examples are presented.

Keywords: singularly perturbed system; multiple state and control delays; controllability

MSC: 34K26; 93B05; 93C23

1. Introduction

Differential systems with a small positive multiplier for a part of the highest order derivatives, called singularly perturbed differential systems, are adequate mathematical models for real-life processes with two-time-scale dynamics. In real-life problems, the small multiplier (a parameter of singular perturbation) can be a time constant, a mass, a capacitance, a geotropic reaction, and some other parameters in physics, chemistry, engineering, biology, medicine, etc (see e.g., [1–3] and references therein). An important class of singularly perturbed differential systems represents the systems with small time delays (of order the parameter of singular perturbation). Such systems arise in various real-life applications, for instance, in nuclear engineering [4], in botany [5], in physiology and medicine [6,7], in control engineering [8], and in communication engineering [9,10]. Distributed small delays appear, for instant, in stabilizing controls of singularly perturbed systems with small delays (either point-wise, or distributed, or point-wise and distributed) [11]. In such a case, a closed-loop system contains a distributed small delay. The stabilizing property of a distributed small delay also is used in the present paper (see Sections 3.3, 3.4 and 4.2).

Various topics in theory and applications of singularly perturbed controlled systems, without and with delays in state and control variables, were extensively investigated in the literature (see e.g., [1,12–14] and references therein).

Controllability of a system is one of its basic properties. This property means the ability to transfer the system from any position of a given set of initial positions to any position of a given set of terminal positions in a finite time by a proper choice of the control function. Different types of controllability for systems without or with delays were extensively studied in the literature (see e.g., [15–18] and references therein). To check whether a singularly perturbed system is controllable in a proper sense, the corresponding controllability conditions can be directly applied for any specified value of the small parameter $\varepsilon > 0$ of singular perturbation. However, the stiffness, as well as a possible high dimension of the singularly perturbed system, can considerably complicate this application. Moreover, such an application depends on the value of ε, and it should be repeated if this parameter changes. Furthermore, in most of real-life problems the current value of ε is unknown. These circumstances are crucial in the analysis of the controllability of singularly perturbed systems. They motivate the derivation of conditions, which being independent of ε, guarantee the controllability of a singularly perturbed system for all sufficiently small values of this parameter, i.e., robustly with respect to ε.

Controllability of singularly perturbed systems was analyzed in a number of works. Thus, in [19–22], the complete controllability of some linear and nonlinear undelayed systems was studied using the separation of time scales concept (see e.g., [1]). In [23] the robust complete Euclidean space controllability, as well as the controllability with respect to the slow state variable and with respect to the fast state variable, were studied for a linear standard singularly perturbed time-invariant system with a single nonsmall pointwise state delay. In [24,25], using the separation of time scales concept, parameter free conditions of complete Euclidean space controllability were obtained for linear standard singularly perturbed systems with pointwise and distributed small state delays. In [26], this result was extended to nonstandard singularly perturbed systems with multiple pointwise and distributed small delays in the state variables. In [27], parameter-free complete Euclidean space controllability conditions, which are not based on the separation of time-scales concept, were derived for a class of linear singularly perturbed systems with small state delays. In [28], a singularly perturbed linear time-dependent controlled system with a single small pointwise delay in the state and control variables was considered. Parameter-free conditions of the complete Euclidean space controllability were established for standard and nonstandard types of this system. In [29], a singularly perturbed linear time-dependent system with small state delays (multiple point-wise and distributed) was studied. Along with the set of time delay differential equations describing the dynamics of this system, a set of delay-free algebraic equations, describing the system's output, also was considered. Based on the separation of time-scales concept, different parameter-free sufficient conditions for the Euclidean space output controllability of this system were established. In [30], the complete Euclidean space controllability for one class of singularly perturbed systems with nonsmall delays (point-wise and distributed) in the state variables was studied. In [31], the defining equations method was used for analysis of the complete Euclidean space controllability of a linear singularly perturbed neutral type system with a single nonsmall pointwise delay. The particular cases of the Euclidean space output controllability, the controllability with respect to the slow state variable and the controllability with respect to the fast state variable, also were studied.

In the present paper, we consider a singularly perturbed linear time-varying system with multiple small point-wise delays and with small distributed delays in the state and control variables. The complete Euclidean space controllability of this system, robust with respect to ε, is studied. This study is based on a transformation of the complete Euclidean space controllability of the original system with delays in the state and the control to an equivalent output controllability of a new singularly perturbed system with only state delays. In the new system, the original control variable becomes an additional fast state variable. The Euclidean dimension of the slow mode equation in the new system is the same as in the original system, while the Euclidean dimension of the fast mode equation is larger than such a dimension in the original system. Further analysis is carried out based on the asymptotic decomposition of the original and transformed systems. Each system is decomposed into two much simpler ε-free subsystems, slow and fast ones. Equivalence of proper kinds of controllability of the slow subsystems, corresponding to the original and transformed systems, is established. Also, it is established the equivalence of proper kinds of controllability of the fast subsystems. Assuming the controllability of the slow and fast subsystems, associated with the transformed system, the Euclidean space output controllability of the latter is established for all sufficiently small values of $\varepsilon > 0$. Then, using the above mentioned equivalence of the controllability of the original and transformed systems, as well as of their slow and fast subsystems, the complete Euclidean space controllability of the original system, robust with respect to ε, is deduced from the assumption on proper kinds of controllability of its slow and fast subsystems. Note that the original system of the present paper is much more general than the original system of [28]. Moreover, in the present paper we propose another, more general, approach to the analysis of the nonstandard case of the original system. Also, we propose here much simpler proof of the Euclidean space output controllability of the transformed system.

The paper is organized as follows. In the next section, the rigorous problem statement, the main definitions and the objective of the paper are formulated. Some auxiliary results, including the transformation of the original system, are presented in Section 3. Section 4 is devoted to main results of the paper. An illustrative example is solved in Section 5. Conclusions are placed in Section 6.

The following main notations are applied in the paper:

1. E^n is the n-dimensional real Euclidean space.
2. The Euclidean norm of either a vector or a matrix is denoted by $\|\cdot\|$.
3. The upper index T denotes the transposition either of a vector x (x^T) or of a matrix A (A^T).
4. I_n denotes the identity matrix of dimension n.
5. The notation $O_{n_1\times n_2}$ is used for the zero matrix of the dimension $n_1 \times n_2$, excepting the cases where the dimension of zero matrix is obvious. In such cases, we use the notation 0 for the zero matrix.
6. $L^2[t_1,t_2;E^n]$ denotes the linear space of all vector-valued functions $x(\cdot):[t_1,t_2] \to E^n$ square integrable in the interval $[t_1,t_2]$; for any $x(\cdot) \in L^2[t_1,t_2;E^n]$ and $y(\cdot) \in L^2[t_1,t_2;E^n]$, the inner product in this space is defined as:
$$\langle x(\cdot), y(\cdot)\rangle_{L^2} = \int_{t_1}^{t_2} x^T(t)y(t)dt;$$
the norm of any $x(\cdot) \in L^2[t_1,t_2;E^n]$ is defined as:
$$\|x(\cdot)\|_{L^2} = \left(\int_{t_1}^{t_2} x^T(t)x(t)dt\right)^{1/2}.$$
7. $L^2_{\mathrm{loc}}[\bar{t},+\infty;E^n]$ denotes the linear space of all vector-valued functions $x(\cdot):[\bar{t},+\infty) \to E^n$ square integrable in any subinterval $[t_1,t_2] \subset [\bar{t},+\infty)$.
8. $W^{1,2}[t_1,t_2;E^n]$ denotes the corresponding Sobolev space, i.e., the linear space of all vector-valued functions $x(\cdot):[t_1,t_2] \to E^n$ square integrable in the interval $[t_1,t_2]$ with the first derivatives (generalized) square integrable in this interval.
9. $\mathrm{col}(x,y)$, where $x \in E^n$, $y \in E^m$, denotes the column block-vector of the dimension $n+m$ with the upper block x and the lower block y, i.e., $\mathrm{col}(x,y) = (x^T,y^T)^T$.
10. $\mathrm{Re}\lambda$ denotes the real part of a complex number λ.

2. Problem Formulation and Main Definitions

2.1. Original System

Consider the controlled system
$$\begin{aligned}\frac{dx(t)}{dt} &= \textstyle\sum_{j=0}^{N}\left[A_{1j}(t,\varepsilon)x(t-\varepsilon h_j) + A_{2j}(t,\varepsilon)y(t-\varepsilon h_j)\right]\\ &\quad+ \textstyle\int_{-h}^{0}\left[G_1(t,\eta,\varepsilon)x(t+\varepsilon\eta) + G_2(t,\eta,\varepsilon)y(t+\varepsilon\eta)\right]d\eta\\ &\quad+ \textstyle\sum_{j=0}^{N} B_{1j}(t,\varepsilon)u(t-\varepsilon h_j) + \int_{-h}^{0} H_1(t,\eta,\varepsilon)u(t+\varepsilon\eta)d\eta, \quad t \geq 0,\end{aligned} \tag{1}$$

$$
\begin{aligned}
\varepsilon \tfrac{dy(t)}{dt} &= \textstyle\sum_{j=0}^{N} \left[A_{3j}(t,\varepsilon)x(t-\varepsilon h_j) + A_{4j}(t,\varepsilon)y(t-\varepsilon h_j)\right] \\
&\quad + \textstyle\int_{-h}^{0} \left[G_3(t,\eta,\varepsilon)x(t+\varepsilon\eta) + G_4(t,\eta,\varepsilon)y(t+\varepsilon\eta)\right]d\eta \\
&\quad + \textstyle\sum_{j=0}^{N} B_{2j}(t,\varepsilon)u(t-\varepsilon h_j) + \int_{-h}^{0} H_2(t,\eta,\varepsilon)u(t+\varepsilon\eta)d\eta, \quad t \geq 0,
\end{aligned} \tag{2}
$$

where $x(t) \in E^n$, $y(t) \in E^m$, $u(t) \in E^r$ ($u(t)$ is a control); $\varepsilon > 0$ is a small parameter; $N \geq 1$ is an integer; $0 = h_0 < h_1 < h_2 < \cdots < h_N = h$ are some given constants independent of ε; $A_{ij}(t,\varepsilon)$, $G_i(t,\eta,\varepsilon)$, $B_{kj}(t,\varepsilon)$, $H_k(t,\eta,\varepsilon)$, $(i = 1,\ldots,4;\ j = 0,\ldots,N;\ k = 1,2)$ are matrix-valued functions of corresponding dimensions, given for $t \geq 0$, $\eta \in [-h,0]$ and $\varepsilon \in [0,\varepsilon^0]$, ($\varepsilon^0 > 0$); the functions $A_{ij}(t,\varepsilon)$ and $B_{kj}(t,\varepsilon)$, $(i = 1,\ldots,4;\ j = 0,\ldots,N;\ k = 1,2)$ are continuous in $(t,\varepsilon) \in [0,+\infty) \times [0,\varepsilon^0]$; the functions $G_i(t,\eta,\varepsilon)$ and $H_k(t,\eta,\varepsilon)$, $(i = 1,\ldots,4;\ k = 1,2)$ are piecewise continuous in $\eta \in [-h,0]$ for any $(t,\varepsilon) \in [0,+\infty) \times [0,\varepsilon^0]$; the functions $G_i(t,\eta,\varepsilon)$ and $H_k(t,\eta,\varepsilon)$, $(i = 1,\ldots,4;\ k = 1,2)$ are continuous with respect to $(t,\varepsilon) \in [0,+\infty) \times [0,\varepsilon^0]$ uniformly in $\eta \in [-h,0]$.

For any given $\varepsilon \in (0,\varepsilon^0]$ and $u(\cdot) \in L^2_{\mathrm{loc}}[-\varepsilon h,+\infty; E^r]$, the system (1)-(2) is a linear time-dependent nonhomogeneous functional-differential system. It is infinite-dimensional with the state variables $\big(x(t), x(t+\varepsilon\eta)\big)$ and $\big(y(t), y(t+\varepsilon\eta)\big)$, $\eta \in [-h,0]$. Moreover, (1)-(2) is a singularly perturbed system. The Equation (1) is the slow mode of this system, while the Equation (2) is its fast mode.

Definition 1. *For a given $\varepsilon \in (0,\varepsilon^0]$, the system (1)-(2) is said to be completely Euclidean space controllable at a given time instant $t_c > 0$ if for any $x_0 \in E^n$, $y_0 \in E^m$, $u_0 \in E^r$, $\varphi_x(\cdot) \in L^2[-\varepsilon h,0;E^n]$, $\varphi_y(\cdot) \in L^2[-\varepsilon h,0;E^m]$, $\varphi_u(\cdot) \in L^2[-\varepsilon h,0;E^r]$, $x_c \in E^n$ and $y_c \in E^m$ there exists a control function $u(\cdot) \in W^{1,2}[0,t_c;E^r]$ satisfying $u(0) = u_0$, for which the system (1)-(2) with the initial and terminal conditions*

$$x(\tau) = \varphi_x(\tau), \quad y(\tau) = \varphi_y(\tau), \quad u(\tau) = \varphi_u(\tau), \quad \tau \in [-\varepsilon h, 0), \tag{3}$$

$$x(0) = x_0, \quad y(0) = y_0, \tag{4}$$

$$x(t_c) = x_c, \quad y(t_c) = y_c, \tag{5}$$

has a solution.

2.2. Asymptotic Decomposition of the Original System

For the sake of further analysis, let us decompose asymptotically the original singularly perturbed system (1)-(2) into two much simpler ε-free subsystems, the slow and fast ones. The slow subsystem is obtained from (1)-(2) by setting formally $\varepsilon = 0$ in these controlled functional-differential equations, which yields

$$\frac{dx_s(t)}{dt} = A_{1s}(t)x_s(t) + A_{2s}(t)y_s(t) + B_{1s}(t)u_s(t), \quad t \geq 0, \tag{6}$$

$$0 = A_{3s}(t)x_s(t) + A_{4s}(t)y_s(t) + B_{2s}(t)u_s(t), \quad t \geq 0, \tag{7}$$

where $x_s(t) \in E^n$ and $y_s(t) \in E^m$ are state variables; $u_s(t) \in E^r$ is a control;

$$A_{is}(t) = \sum_{j=0}^{N} A_{ij}(t,0) + \int_{-h}^{0} G_i(t,\eta,0)d\eta, \quad i = 1,\ldots,4, \tag{8}$$

$$B_{ks}(t) = \sum_{j=0}^{N} B_{kj}(t,0) + \int_{-h}^{0} H_k(t,\eta,0)d\eta, \quad k = 1,2. \tag{9}$$

The slow subsystem (6)-(7) is a descriptor (differential-algebraic) system, and it is delay-free and ε-free.

If

$$\det A_{4s}(t) \neq 0, \quad t \geq 0, \tag{10}$$

we can eliminate the state variable $y_s(t)$ from the slow subsystem (6)-(7). Such an elimination yields the differential equation with respect to $x_s(t)$

$$\frac{dx_s(t)}{dt} = \bar{A}_s(t)x_s(t) + \bar{B}_s(t)u_s(t), \quad t \geq 0, \tag{11}$$

where

$$\bar{A}_s(t) = A_{1s}(t) - A_{2s}(t)A_{4s}^{-1}(t)A_{3s}(t), \ \bar{B}_s(t) = B_{1s}(t) - A_{2s}(t)A_{4s}^{-1}(t)B_{2s}(t). \tag{12}$$

The differential Equation (11) also is called the slow subsystem, associated with the original system (1)-(2).

The fast subsystem is derived from (2) in the following way: (a) the terms containing the state variable $\big(x(t), x(t+\varepsilon\eta)\big)$, $\eta \in [-h, 0]$ are removed from (2); (b) the transformations of the variables $t = t_1 + \varepsilon\xi$, $y(t_1 + \varepsilon\xi) \stackrel{\triangle}{=} y_f(\xi)$, $u(t_1 + \varepsilon\xi) \stackrel{\triangle}{=} u_f(\xi)$ are made in the resulting system, where $t_1 \geq 0$ is any fixed time instant.

Thus, we obtain the system

$$\begin{aligned} \frac{dy_f(\xi)}{d\xi} &= \sum_{j=0}^{N} A_{4j}(t_1 + \varepsilon\xi, \varepsilon)y_f(\xi - h_j) + \int_{-h}^{0} G_4(t_1 + \varepsilon\xi, \eta, \varepsilon)y_f(\xi + \eta)d\eta \\ &+ \sum_{j=0}^{N} B_{2j}(t_1 + \varepsilon\xi, \varepsilon)u_f(\xi - h_j) + \int_{-h}^{0} H_2(t_1 + \varepsilon\xi, \eta, \varepsilon)u_f(\xi + \eta)d\eta. \end{aligned}$$

Finally, setting formally $\varepsilon = 0$ in this system and replacing t_1 with t yield the fast subsystem

$$\begin{aligned} \tfrac{dy_f(\xi)}{d\xi} &= \textstyle\sum_{j=0}^{N} A_{4j}(t, 0)y_f(\xi - h_j) + \int_{-h}^{0} G_4(t, \eta, 0)y_f(\xi + \eta)d\eta \\ &+ \textstyle\sum_{j=0}^{N} B_{2j}(t, 0)u_f(\xi - h_j) + \int_{-h}^{0} H_2(t, \eta, 0)u_f(\xi + \eta)d\eta, \quad \xi \geq 0, \end{aligned} \tag{13}$$

where $t \geq 0$ is a parameter; $y_f(\xi) \in E^m$, $u_f(\xi) \in E^r$; $\big(y_f(\xi), y_f(\xi + \eta)\big)$, $\eta \in [-h, 0]$ is a state variable, while $\big(u_f(\xi), u_f(\xi + \eta)\big)$, $\eta \in [-h, 0]$ is a control variable.

The new independent variable ξ is called the stretched time, and it is expressed by the original time t in the form $\xi = (t - t_1)/\varepsilon$. Thus, for any $t > t_1$, $\xi \to +\infty$ as $\varepsilon \to +0$.

The fast subsystem (13) is a differential equation with state and control delays. It is of a lower Euclidean dimension than the original system (1)-(2), and it is ε-free.

Definition 2. *Subject to (10), the system (11) is said to be completely controllable at a given time instant $t_c > 0$ if for any $x_0 \in E^n$ and $x_c \in E^n$ there exists a control function $u_s(\cdot) \in L^2[0, t_c; E^r]$, for which (11) has a solution $x_s(t)$, $t \in [0, t_c]$, satisfying the initial and terminal conditions*

$$x_s(0) = x_0, \quad x_s(t_c) = x_c. \tag{14}$$

Definition 3. *The system (6)-(7) is said to be impulse-free controllable with respect to $x_s(t)$ at a given time instant $t_c > 0$ if for any $x_0 \in E^n$ and $x_c \in E^n$ there exists a control function $u_s(\cdot) \in L^2[0, t_c; E^r]$, for which (6)-(7) has an impulse-free solution $\mathrm{col}(x_s(t), y_s(t))$, $t \in [0, t_c]$, satisfying the initial and terminal conditions (14).*

Definition 4. *For a given $t \geq 0$, the system (13) is said to be completely Euclidean space controllable if for any $y_0 \in E^m$, $u_0 \in E^r$, $\varphi_{yf}(\cdot) \in L^2[-h, 0; E^m]$, $\varphi_{uf}(\cdot) \in L^2[-h, 0; E^r]$ and $y_c \in E^m$ there exist a number $\xi_c > 0$, independent of y_0, u_0, $\varphi_{yf}(\cdot)$, $\varphi_{uf}(\cdot)$ and y_c, and a control function $u_f(\cdot) \in W^{1,2}[0, \xi_c; E^r]$ satisfying $u_f(0) = u_0$, for which the system (13) with the initial and terminal conditions*

$$y_f(\eta) = \varphi_{yf}(\eta), \quad u_f(\eta) = \varphi_{uf}(\eta), \quad \eta \in [-h, 0); \quad y_f(0) = y_0, \tag{15}$$

$$y_f(\xi_c) = y_c, \tag{16}$$

has a solution.

2.3. Objective of the Paper

The objective of the paper is the following: using the ε-independent assumptions on the controllability of the systems (11) and (13), as well as (6)-(7) and (13), to establish the complete Euclidean space controllability of the original singularly perturbed system (1)-(2) for all sufficiently small values of $\varepsilon > 0$, i.e., robustly with respect to this parameter.

3. Auxiliary Results

In this section, some properties of systems with state and control delays are studied. Based on these results, in the next section different parameter-free conditions for the complete Euclidean space controllability of the original singularly perturbed system are derived.

3.1. Auxiliary System with Delay-Free Control

Consider the differential system, consisting of the Equations (1), (2) and the equation

$$\varepsilon \frac{du(t)}{dt} = -u(t) + v(t), \quad t \geq 0. \tag{17}$$

In this new system, $\big(x(t), x(t+\varepsilon\eta)\big)$, $\big(y(t), y(t+\varepsilon\eta)\big)$, $\big(u(t), u(t+\varepsilon\eta)\big)$, $\eta \in [-h, 0]$ are state variables, while $v(t) \in E^r$ is a control. Thus, in the system (1), (2), (17) only the state variables have delays, while the control is delay-free. Moreover, in contrast with the original system (1)-(2), the new system contains two fast modes, the Equations (2) and (17).

For the new system (1), (2), (17), we consider the algebraic output equation

$$\zeta(t) = Z\mathrm{col}\big(x(t), y(t), u(t)\big), \quad t \geq 0, \tag{18}$$

where the $(n+m) \times (n+m+r)$-matrix Z has the block form

$$Z = \Big(I_{n+m}, 0\Big). \tag{19}$$

Let us rewrite the system (1), (2), (17), (18) in a new form, more convenient for the further analysis.

For a given $\varepsilon \in (0, \varepsilon^0]$, let us introduce into the consideration the block vector $\omega(t) = \mathrm{col}\big(y(t), u(t)\big)$, $t \geq -\varepsilon h$, and the block matrices

$$\mathcal{A}_{1j}(t,\varepsilon) = A_{1j}(t,\varepsilon), \ \ \mathcal{A}_{2j}(t,\varepsilon) = \Big(A_{2j}(t,\varepsilon), B_{1j}(t,\varepsilon)\Big), \ \ j = 0,1,\ldots,N, \ \ t \geq 0, \tag{20}$$

$$\mathcal{A}_{3j}(t,\varepsilon) = \begin{pmatrix} A_{3j}(t,\varepsilon) \\ O_{r\times n} \end{pmatrix}, \quad j = 0,1,\ldots,N, \quad t \geq 0, \tag{21}$$

$$\mathcal{A}_{40}(t,\varepsilon) = \begin{pmatrix} A_{40}(t,\varepsilon) & B_{20}(t,\varepsilon) \\ O_{r\times m} & -I_r \end{pmatrix}, \quad t \geq 0, \tag{22}$$

$$\mathcal{A}_{4j}(t,\varepsilon) = \begin{pmatrix} A_{4j}(t,\varepsilon) & B_{2j}(t,\varepsilon) \\ O_{r\times m} & O_{r\times r} \end{pmatrix}, \quad j = 1,\ldots,N, \quad t \geq 0, \tag{23}$$

$$\mathcal{G}_1(t,\eta,\varepsilon) = G_1(t,\eta,\varepsilon), \ \mathcal{G}_2(t,\eta,\varepsilon) = \Big(G_2(t,\eta,\varepsilon), H_1(t,\eta,\varepsilon)\Big), \ t \geq 0, \ \eta, \in [-h, 0], \tag{24}$$

$$\mathcal{G}_3(t,\eta,\varepsilon) = \begin{pmatrix} G_3(t,\eta,\varepsilon) \\ O_{r\times n} \end{pmatrix}, \quad \mathcal{G}_4(t,\eta,\varepsilon) = \begin{pmatrix} G_4(t,\eta,\varepsilon) & H_2(t,\eta,\varepsilon) \\ O_{r\times m} & O_{r\times r} \end{pmatrix}, \quad t \geq 0, \quad \eta \in [-h,0], \tag{25}$$

$$\mathcal{B}_1 = O_{n\times r}, \quad \mathcal{B}_2 = \begin{pmatrix} O_{m\times r} \\ I_r \end{pmatrix}. \tag{26}$$

Based on the above introduced vector and matrices, we can rewrite the auxiliary system (1), (2), (17), (18) in the equivalent form

$$\frac{dx(t)}{dt} = \sum_{j=0}^{N} \Big[\mathcal{A}_{1j}(t,\varepsilon)x(t-\varepsilon h_j) + \mathcal{A}_{2j}(t,\varepsilon)\omega(t-\varepsilon h_j)\Big]$$
$$+ \int_{-h}^{0} \Big[\mathcal{G}_1(t,\eta,\varepsilon)x(t+\varepsilon\eta) + \mathcal{G}_2(t,\eta,\varepsilon)\omega(t+\varepsilon\eta)\Big]d\eta, \quad t \geq 0, \tag{27}$$

$$\varepsilon\frac{d\omega(t)}{dt} = \sum_{j=0}^{N} \Big[\mathcal{A}_{3j}(t,\varepsilon)x(t-\varepsilon h_j) + \mathcal{A}_{4j}(t,\varepsilon)\omega(t-\varepsilon h_j)\Big]$$
$$+ \int_{-h}^{0} \Big[\mathcal{G}_3(t,\eta,\varepsilon)x(t+\varepsilon\eta) + \mathcal{G}_4(t,\eta,\varepsilon)\omega(t+\varepsilon\eta)\Big]d\eta + \mathcal{B}_2 v(t), \quad t \geq 0, \tag{28}$$

$$\zeta(t) = Z\mathrm{col}\big(x(t),\omega(t)\big), \quad t \geq 0. \tag{29}$$

Definition 5. *For a given $\varepsilon \in (0,\varepsilon^0]$, the system (27)-(28), (29) is said to be Euclidean space output controllable at a given time instant $t_c > 0$ if for any $x_0 \in E^n$, $\omega_0 \in E^{m+r}$, $\varphi_x(\cdot) \in L^2[-\varepsilon h,0;E^n]$, $\varphi_\omega(\cdot) \in L^2[-\varepsilon h,0;E^{m+r}]$ and $\zeta_c \in E^{n+m}$ there exists a control function $v(\cdot) \in L^2[0,t_c;E^r]$, for which the solution $\mathrm{col}\big(x(t),\omega(t)\big)$, $t \in [0,t_c]$ of the system (27)-(28) with the initial conditions*

$$x(\tau) = \varphi_x(\tau), \;\; \omega(\tau) = \varphi_\omega(\tau), \;\; \tau \in [-\varepsilon h,0); \;\; x(0) = x_0, \;\; \omega(0) = \omega_0$$

satisfies the terminal condition $Z\mathrm{col}\big(x(t_c),\omega(t_c)\big) = \zeta_c$.

Proposition 1. *For a given $\varepsilon \in (0,\varepsilon^0]$, the system (1)-(2) is completely Euclidean space controllable at a given time instant $t_c > 0$, if and only if the system (27)-(28), (29) is Euclidean space output controllable at this time instant.*

Proof. The proposition is proven similarly to [28] (Lemma 1). □

Now, let us decompose asymptotically the system (27)-(28), (29) into the slow and fast subsystems. We start with the slow subsystem. The dynamic part of this subsystem is obtained from (27)-(28) by setting there formally $\varepsilon = 0$. The output part of the slow subsystem is obtained from (29) by removing formally the term with the Euclidean part $\omega(t)$ of the fast state variable $\big(\omega(t),\omega(t+\varepsilon\eta)\big)$, $\eta \in [-h,0]$. Thus, the slow subsystem has the form

$$\frac{dx_s(t)}{dt} = \mathcal{A}_{1s}(t)x_s(t) + \mathcal{A}_{2s}(t)\omega_s(t), \quad t \geq 0, \tag{30}$$

$$0 = \mathcal{A}_{3s}(t)x_s(t) + \mathcal{A}_{4s}(t)\omega_s(t) + \mathcal{B}_2 v_s(t), \quad t \geq 0, \tag{31}$$

$$\zeta_s(t) = x_s(t), \quad t \geq 0, \tag{32}$$

where $x_s(t) \in E^n$ and $\omega_s(t) \in E^{m+r}$ are state variables; $v_s(t) \in E^r$ is a control; $\zeta_s(t) \in E^n$ is an output; $\omega_s(t) = \mathrm{col}\big(y_s(t),u_s(t)\big)$, $y_s \in E^m$, $u_s(t) \in E^r$;

$$\mathcal{A}_{is}(t) = \sum_{j=0}^{N} \mathcal{A}_{ij}(t,0) + \int_{-h}^{0} \mathcal{G}_i(t,\eta,0)d\eta, \quad i = 1,\ldots,4, \tag{33}$$

or using (8)-(9), (20)–(25)

$$\begin{aligned} \mathcal{A}_{1s}(t) = A_{1s}(t), \quad \mathcal{A}_{2s}(t) = \Big(A_{2s}(t), B_{1s}(t)\Big), \\ \mathcal{A}_{3s}(t) = \begin{pmatrix} A_{3s}(t) \\ O_{r\times n} \end{pmatrix}, \quad \mathcal{A}_{4s}(t) = \begin{pmatrix} A_{4s}(t) & B_{2s}(t) \\ O_{r\times m} & -I_r \end{pmatrix}. \end{aligned} \tag{34}$$

From the expression for $\mathcal{A}_{4s}(t)$ we have that $\det \mathcal{A}_{4s}(t) = (-1)^r \det A_{4s}(t)$. Thus, $\det \mathcal{A}_{4s}(t) \neq 0$, $t \geq 0$ if and only if $\det A_{4s}(t) \neq 0$, $t \geq 0$. Therefore, subject to (10), the differential-algebraic system (30)-(31) can be converted to the differential equation

$$\frac{dx_s(t)}{dt} = \bar{\mathcal{A}}_s(t)x_s(t) + \bar{\mathcal{B}}_s(t)v_s(t), \quad t \geq 0, \tag{35}$$

where

$$\bar{\mathcal{A}}_s(t) = \mathcal{A}_{1s}(t) - \mathcal{A}_{2s}(t)\mathcal{A}_{4s}^{-1}(t)\mathcal{A}_{3s}(t), \quad \bar{\mathcal{B}}_s(t)v_s(t) = -\mathcal{A}_{2s}(t)\mathcal{A}_{4s}^{-1}(t)\mathcal{B}_2. \tag{36}$$

Using the Equations (34) and (36), the Equation (35) can be rewritten as:

$$\frac{dx_s(t)}{dt} = \bar{A}_s(t)x_s(t) + \bar{B}_s(t)v_s(t), \quad t \geq 0, \tag{37}$$

where the matrix-valued coefficients $\bar{A}_s(t)$ and $\bar{B}_s(t)$ are given in (12). Hence, subject to (10), the slow subsystem associated with (27)-(28), (29) consists of the differential Equation (37) and the output Equation (32).

Remark 1. *Comparison of the differential Equations (37) and (11) directly yields that the former can be obtained from the latter by replacing in it $u_s(t)$ with $v_s(t)$, and vice versa. Moreover, the output in the system (37), (32) coincides with $x_s(t)$. Hence, the output controllability of this system means its controllability with respect to $x_s(t)$. Therefore, the output controllability of (37), (32) coincides with the complete controllability of (37) and, thus, it is equivalent to the complete controllability of the system (11).*

Remark 2. *Similarly to Remark 1, since the output in the system (30)-(31), (32) coincides with $x_s(t)$, then an output controllability of this system coincides with a proper controllability of its dynamic part (30)-(31) with respect to $x_s(t)$.*

Definition 6. *The system (30)-(31) is said to be impulse-free controllable with respect to $x_s(t)$ at a given time instant $t_c > 0$ if for any $x_0 \in E^n$ and $x_c \in E^n$ there exists a control function $v_s(\cdot) \in L^2[0,t_c;E^r]$, for which (30)-(31) has an impulse-free solution $\mathrm{col}(x_s(t),\omega_s(t))$, $t \in [0,t_c]$, satisfying the initial and terminal conditions $x_s(0) = x_0$ and $x_s(t_c) = x_c$.*

Proposition 2. *The system (6)-(7) is impulse-free controllable with respect to $x_s(t)$ at a given time instant $t_c > 0$ if and only if the system (30)-(31) is impulse-free controllable with respect to $x_s(t)$ at this time instant.*

Proof. Eliminating the component $u_s(t)$ of the state variable $\omega_s(t)$ from the system (30)-(31), we convert the latter to the equivalent system consisting of the equation $u_s(t) = v_s(t)$ and the system

$$\frac{dx_s(t)}{dt} = A_{1s}(t)x_s(t) + A_{2s}(t)y_s(t) + B_{1s}(t)v_s(t), \quad t \geq 0, \tag{38}$$

$$0 = A_{3s}(t)x_s(t) + A_{4s}(t)y_s(t) + B_{2s}(t)v_s(t), \quad t \geq 0, \tag{39}$$

where $A_{is}(t)$, $B_{ks}(t)$, $(i = 1, \ldots, 4;\ k = 1, 2)$ are given in (8)-(9).

Therefore, the impulse-free controllability with respect to $x_s(t)$ of the system (30)-(31) is equivalent to such a controllability of the system (38)-(39). Now, the comparison of the latter with the system (6)-(7) directly yields the statement of the proposition. □

Proceed to the fast subsystem, associated with the system (27)-(28), (29). The dynamic part of this subsystem is constructed similarly to the fast subsystem (13), associated with the original system (1)-(2). The output part of the fast subsystem is obtained from (29) by removing formally the term with the Euclidean part $x(t)$ of the state variable $\big(x(t), x(t+\varepsilon\eta)\big)$, $\eta \in [-h, 0]$. Thus the fast subsystem, associated with the auxiliary system (27)-(28), (29), consists of the differential equation

$$\frac{d\omega_f(\xi)}{d\xi} = \sum_{j=0}^{N} \mathcal{A}_{4j}(t,0)\omega_f(\xi - h_j) + \int_{-h}^{0} \mathcal{G}_4(t,\eta,0)\omega_f(\xi+\eta)d\eta + \mathcal{B}_2 v_f(\xi),\ \xi \geq 0, \tag{40}$$

and the output equation

$$\zeta_f(\xi) = \Omega_f \omega_f(\xi), \quad \xi \geq 0, \quad \Omega_f = \big(I_m, O_{m\times r}\big), \tag{41}$$

where $t \geq 0$ is a parameter; $\omega_f(\xi) \in E^{m+r}$; $\big(\omega_f(\xi), \omega_f(\xi+\eta)\big)$ is a state variable; $v_f(\xi) \in E^r$ is a control; $\zeta_f(\xi) \in E^m$ is an output.

Note that in contrast with the system (13), in the differential system (40) only the state variable has delays, while the control is undelayed.

Definition 7. *For a given $t \geq 0$, the system (40)-(41) is said to be Euclidean space output controllable if for any $\omega_0 \in E^{m+r}$, $\varphi_{\omega f}(\cdot) \in L^2[-h, 0; E^{m+r}]$ and $\zeta_{fc} \in E^m$ there exist a number $\xi_c > 0$, independent of ω_0, $\varphi_{\omega f}(\cdot)$ and ζ_{fc}, and a control function $v_f(\cdot) \in L^2[0, \xi_c; E^r]$, for which the solution $\omega_f(\xi)$, $\xi \in [0, \xi_c]$ of the differential Equation (40) with the initial conditions*

$$\omega_f(\eta) = \varphi_{\omega f}(\eta), \quad \eta \in [-h, 0); \quad \omega_f(0) = \omega_0 \tag{42}$$

satisfies the terminal condition

$$\Omega_f \omega_f(\xi_c) = \zeta_{fc}. \tag{43}$$

Lemma 1. *For a given $t \geq 0$, the system (13) is completely Euclidean space controllable if and only if the system (40)-(41) is Euclidean space output controllable.*

Proof. *Sufficiency.* Let us assume that, for some given $t \geq 0$, the system (40)-(41) is Euclidean space output controllable. Let $\omega_0 \in E^{m+r}$, $\varphi_{\omega f}(\cdot) \in L^2[-h, 0; E^{m+r}]$ and $\zeta_{fc} \in E^m$ be arbitrary given. Then, there exists a number $\xi_c > 0$, independent of ω_0, $\varphi_{\omega f}(\cdot)$ and ζ_{fc}, and a control function $v_f(\cdot) \in L^2[0, \xi_c; E^r]$, for which the differential Equation (40) with the initial (42) and terminal (43) conditions has a solution $\omega_f(\xi)$, $\xi \in [0, \xi_c]$. Let us represent the vector ω_0 and the vector-valued function $\omega_f(\xi)$ in the block form as: $\omega_0 = \mathrm{col}\big(y_0, u_0\big)$, $y_0 \in E^m$, $u_0 \in E^r$; $\omega_f(\xi) = \mathrm{col}\big(y_f(\xi), u_f(\xi)\big)$, $y_f(\xi) \in E^m$, $u_f(\xi) \in E^r$, $\xi \in [0, \xi_c]$. Also, we represent the vector-valued function $\varphi_{\omega f}(\eta)$ in the block form as: $\varphi_{\omega f}(\eta) = \mathrm{col}\big(\varphi_{yf}(\eta), \varphi_{uf}(\eta)\big)$, $\eta \in [-h, 0]$. Note, that the component $u_f(\xi)$ of the above mentioned solution $\omega_f(\xi)$ to the boundary-valued problem (40), (42), (43) satisfies the conditions $u_f(\eta) = \varphi_{uf}(\eta)$, $\eta \in [-h, 0)$ and $u_f(0) = u_0$. Moreover, since $v_f(\cdot) \in L^2[0, \xi_c; E^r]$, then $u_f(\xi) \in W^{1,2}[0, \xi_c; E^r]$. Thus, for the control function $u_f(\xi)$, the vector-valued function $y_f(\xi)$, $\xi \in [0, \xi_c]$ is a solution of the system (13) satisfying the initial condition (15) and the terminal conditions $y_f(\zeta_c) = \zeta_{fc}$. Hence, re-denoting ζ_{fc} as y_c and using Definition 4, we directly obtain that, for the given $t \geq 0$, the system (13) is completely

Euclidean space controllable. This completes the proof of the sufficiency.
Necessity. The necessity is proven similarly to the sufficiency.

Thus, the lemma is proven. □

3.2. Output Controllability of the Auxiliary System and its Slow and Fast Subsystems: Necessary and Sufficient Conditions

3.2.1. Output Controllability of the Auxiliary System

For a given $\varepsilon \in (0, \varepsilon^0]$, let us consider the block vector $z(t) = \mathrm{col}\big(x(t), \omega(t)\big)$, $t \geq -\varepsilon h$, and the block matrices

$$\mathcal{A}_j(t,\varepsilon) = \begin{pmatrix} \mathcal{A}_{1j}(t,\varepsilon) & \mathcal{A}_{2j}(t,\varepsilon) \\ \frac{1}{\varepsilon}\mathcal{A}_{3j}(t,\varepsilon) & \frac{1}{\varepsilon}\mathcal{A}_{4j}(t,\varepsilon) \end{pmatrix}, \quad j = 0, 1, \ldots, N, \tag{44}$$

$$\mathcal{G}(t,\eta,\varepsilon) = \begin{pmatrix} \mathcal{G}_1(t,\eta,\varepsilon) & \mathcal{G}_2(t,\eta,\varepsilon) \\ \frac{1}{\varepsilon}\mathcal{G}_3(t,\eta,\varepsilon) & \frac{1}{\varepsilon}\mathcal{G}_4(t,\eta,\varepsilon) \end{pmatrix}, \ \mathcal{B}(\varepsilon) = \begin{pmatrix} \mathcal{B}_1 \\ \frac{1}{\varepsilon}\mathcal{B}_2 \end{pmatrix} = \begin{pmatrix} O_{n\times r} \\ \frac{1}{\varepsilon}\mathcal{B}_2 \end{pmatrix}. \tag{45}$$

Thus, the auxiliary system (27)-(29), can be rewritten in the equivalent form

$$\frac{dz(t)}{dt} = \sum_{j=0}^{N} \mathcal{A}_j(t,\varepsilon) z(t - \varepsilon h_j) + \int_{-h}^{0} \mathcal{G}(t,\eta,\varepsilon) z(t + \varepsilon\eta) d\eta + \mathcal{B}(\varepsilon) v(t), \ \ t \geq 0, \tag{46}$$

$$\zeta(t) = Zz(t), \quad t \geq 0. \tag{47}$$

It is clear that the system (46)-(47) is equivalent to the auxiliary system (27)-(29).

Definition 8. *For a given $\varepsilon \in (0, \varepsilon^0]$, the system (46)-(47) is said to be Euclidean space output controllable at a given time instant $t_c > 0$ if for any $z_0 \in E^{n+m+r}$, $\varphi_z(\cdot) \in L^2[-\varepsilon h, 0; E^{n+m+r}]$, and $\zeta_c \in E^{n+m}$ there exists a control function $v(\cdot) \in L^2[0, t_c; E^r]$, for which the solution $z(t)$, $t \in [0, t_c]$ of the system (46) with the initial conditions $z(\tau) = \varphi_z(\tau)$, $\tau \in [-h, 0)$, $z(0) = z_0$ satisfies the terminal condition $Zz(t_c) = \zeta_c$.*

Let, for a given $\varepsilon \in (0, \varepsilon^0]$, the $(n+m+r) \times (n+m+r)$-matrix-valued function $\Psi(\sigma, \varepsilon)$, $\sigma \in [0, t_c]$ be a solution of the terminal-value problem

$$\begin{gathered} \tfrac{d\Psi(\sigma,\varepsilon)}{d\sigma} = -\sum_{j=0}^{N} \big(\mathcal{A}_j(\sigma + \varepsilon h_j, \varepsilon)\big)^T \Psi(\sigma + \varepsilon h_j, \varepsilon) \\ -\int_{-h}^{0} \big(\mathcal{G}(t - \varepsilon\eta, \eta, \varepsilon)\big)^T \Psi(\sigma - \varepsilon\eta, \varepsilon) d\eta, \quad \sigma \in [0, t_c), \\ \Psi(t_c, \varepsilon) = I_{n+m+r}; \quad \Psi(\sigma, \varepsilon) = 0, \quad \sigma > t_c, \end{gathered} \tag{48}$$

where it is assumed that $\mathcal{A}_{ij}(t,\varepsilon) = \mathcal{A}_{ij}(t_c,\varepsilon)$, $\mathcal{G}_i(t,\eta,\varepsilon) = \mathcal{G}_i(t_c,\eta,\varepsilon)$, $t > t_c$, $\eta \in [-h, 0]$, $\varepsilon \in [0, \varepsilon^0]$, $(i = 1, \ldots, 4;\ j = 1, \ldots, N)$. Due to the results of [32] (Section 4.3), $\Psi(\sigma, \varepsilon)$ exists and is unique for $\sigma \in [0, t_c]$, $\varepsilon \in (0, \varepsilon^0]$.

Consider the following two matrices of the dimensions $(n+m+r) \times (n+m+r)$ and $(n+m) \times (n+m)$, respectively:

$$W(t_c, \varepsilon) = \int_0^{t_c} \Psi^T(\sigma,\varepsilon)\mathcal{B}(\varepsilon)\mathcal{B}^T(\varepsilon)\Psi(\sigma,\varepsilon) d\sigma \tag{49}$$

and

$$W_Z(t_c, \varepsilon) = ZW(t_c, \varepsilon)Z^T. \tag{50}$$

Proposition 3. *For a given $\varepsilon \in (0, \varepsilon^0]$, the auxiliary system (27)-(29) is Euclidean space output controllable at a given time instant $t_c > 0$ if and only if the matrix $W_Z(t_c, \varepsilon)$ is nonsingular, i.e.,* $\det W_Z(t_c, \varepsilon) \neq 0$.

Proof. By virtue of the results of [29] (Corollary 1), the system (46)-(47) is Euclidean space output controllable at the time instant t_c if and only if $\det W_Z(t_c, \varepsilon) \neq 0$. Since this system is equivalent to the auxiliary system (27)-(29), then, due to Definitions 5 and 8, the auxiliary system also is Euclidean space output controllable at t_c if and only if $\det W_Z(t_c, \varepsilon) \neq 0$. This completes the proof of the proposition. □

3.2.2. Output Controllability of the Slow and Fast Subsystems Associated with the Auxiliary System

We start with the slow subsystem (37).

Let, for a given $t_c > 0$, the $n \times n$-matrix-valued function $\Psi_s(\sigma)$, $\sigma \in [0, t_c]$ be the unique solution of the terminal-value problem

$$\frac{d\Psi_s(\sigma)}{d\sigma} = -\big(\bar{A}_s(\sigma)\big)^T \Psi_s(\sigma), \quad \sigma \in [0, t_c), \quad \Psi_s(t_c) = I_n. \tag{51}$$

Consider the $n \times n$-matrix

$$W_s(t_c) = \int_0^{t_c} \Psi_s^T(\sigma)\bar{B}_s(\sigma)\bar{B}_s^T(\sigma)\Psi_s(\sigma)d\sigma. \tag{52}$$

By virtue of the results of [15], we have the following proposition.

Proposition 4. *Let the condition (10) be fulfilled in the interval $[0, t_c]$. Then, the slow subsystem (37), associated with the auxiliary system (27)-(29), is completely controllable at the time instant t_c, if and only if the matrix $W_s(t_c)$ is nonsingular, i.e.,* $\det W_s(t_c) \neq 0$.

Proceed to the fast subsystem (40)-(41).

Let, for any given $t \geq 0$, the $(m + r) \times (m + r)$-matrix-valued function $\Psi_f(\xi, t)$ be the unique solution of the following initial-value problem:

$$\begin{aligned} \frac{d\Psi_f(\xi)}{d\xi} &= \sum_{j=0}^{N} \big(\mathcal{A}_{4j}(t, 0)\big)^T \Psi_f(\xi - h_j) \\ &+ \int_{-h}^{0} \big(\mathcal{G}_4(t, \eta, 0)\big)^T \Psi_f(\xi + \eta)d\eta, \quad \xi > 0, \\ \Psi_f(\xi) &= 0, \quad \xi < 0, \quad \Psi_f(0) = I_{m+r}. \end{aligned} \tag{53}$$

Consider the $m \times m$-matrix-valued function

$$W_f(\xi, t) = \Omega_f \int_0^{\xi} \Psi_f^T(\rho, t)\mathcal{B}_2\mathcal{B}_2^T\Psi_f(\rho, t)d\rho\Omega_f^T, \quad \xi \geq 0, \quad t \geq 0. \tag{54}$$

By virtue of the results of [29] (Corollary 1), we have the following assertion.

Proposition 5. *For a given $t \geq 0$, the fast subsystem (40)-(41) of the auxiliary system (27)-(29) is Euclidean space output controllable if and only if there exists a number $\xi_c > 0$ such that the matrix $W_f(\xi_c, t)$ is nonsingular, i.e.,* $\det W_f(\xi_c, t) \neq 0$.

3.3. Linear Control Transformation in the Auxiliary System

Let us transform the control $v(t)$ in the auxiliary system (27)-(28), (29) as follows:

$$v(t) = K_1(t)\omega(t) + \int_{-h}^{0} K_2(t, \eta)\omega(t + \varepsilon\eta)d\eta + w(t), \tag{55}$$

where $w(t) \in E^r$ is a new control; $K_1(t)$ and $K_2(t, \eta)$ are any specified matrix-valued functions of the dimension $r \times (m + r)$ given for $t \geq 0$, $\eta \in [-h, 0]$; $K_1(t)$ is continuous for $t \geq 0$; $K_2(t, \eta)$ is continuous with respect to $t \geq 0$ uniformly in $\eta \in [-h, 0]$, and this function is piecewise continuous in $\eta \in [-h, 0]$ for any $t \geq 0$.

Due to this transformation, the dynamic part (27)-(28) of the system (27)-(28), (29) becomes as:

$$\frac{dx(t)}{dt} = \sum_{j=0}^{N} \Big[\mathcal{A}_{1j}(t,\varepsilon)x(t-\varepsilon h_j) + \mathcal{A}_{2j}(t,\varepsilon)\omega(t-\varepsilon h_j)\Big]$$
$$+ \int_{-h}^{0} \Big[\mathcal{G}_1(t,\eta,\varepsilon)x(t+\varepsilon\eta) + \mathcal{G}_2(t,\eta,\varepsilon)\omega(t+\varepsilon\eta)\Big]d\eta, \quad t \geq 0, \tag{56}$$

$$\varepsilon\frac{d\omega(t)}{dt} = \sum_{j=0}^{N} \Big[\mathcal{A}_{3j}(t,\varepsilon)x(t-\varepsilon h_j) + \mathcal{A}_{4j}^{K}(t,\varepsilon)\omega(t-\varepsilon h_j)\Big]$$
$$+ \int_{-h}^{0} \Big[\mathcal{G}_3(t,\eta,\varepsilon)x(t+\varepsilon\eta) + \mathcal{G}_4^{K}(t,\eta,\varepsilon)\omega(t+\varepsilon\eta)\Big]d\eta + \mathcal{B}_2 w(t), \quad t \geq 0, \tag{57}$$

where

$$\mathcal{A}_{40}^{K}(t,\varepsilon) = \mathcal{A}_{40}(t,\varepsilon) + \mathcal{B}_2 K_1(t), \quad \mathcal{A}_{4j}^{K}(t,\varepsilon) = \mathcal{A}_{4j}(t,\varepsilon), \quad j = 1,\ldots,N, \tag{58}$$

$$\mathcal{G}_4^{K}(t,\eta,\varepsilon) = \mathcal{G}_4(t,\eta,\varepsilon) + \mathcal{B}_2 K_2(t,\eta). \tag{59}$$

Proposition 6. *For a given $\varepsilon \in (0, \varepsilon^0]$, the system (27)-(28), (29) is Euclidean space output controllable at a given time instant $t_c > 0$, if and only if the system (56)-(57), (29) is Euclidean space output controllable at this time instant.*

Proof. The proposition is proven similarly to [29] (Lemma 3). □

As a direct consequence of Propositions 1 and 6, we obtain the following assertion.

Corollary 1. *For a given $\varepsilon \in (0, \varepsilon^0]$, the system (1)-(2) is completely Euclidean space controllable at a given time instant $t_c > 0$, if and only if the system (56)-(57), (29) is Euclidean space output controllable at this time instant.*

Now, let us decompose asymptotically the singularly perturbed system (56)-(57), (29) into the slow and fast subsystems. This decomposition is carried out similarly to that for the system (27)-(28), (29). Thus, the slow subsystem, associated with (56)-(57), (29), consists of the differential-algebraic system

$$\frac{dx_s(t)}{dt} = \mathcal{A}_{1s}(t)x_s(t) + \mathcal{A}_{2s}(t)\omega_s(t), \quad t \geq 0, \tag{60}$$

$$0 = \mathcal{A}_{3s}(t)x_s(t) + \mathcal{A}_{4s}^{K}(t)\omega_s(t) + \mathcal{B}_2 w_s(t), \quad t \geq 0, \tag{61}$$

and the output Equation (32). In (60)-(61), (32), $x_s(t) \in E^n$ and $\omega_s(t) \in E^{m+r}$ are state variables; $w_s(t) \in E^r$ is a control; $\zeta_s(t) \in E^n$ is an output; $\mathcal{A}_{ls}(t)$, $(l = 1,2,3)$ are given in (33);

$$\mathcal{A}_{4s}^{K}(t) = \sum_{j=0}^{N} \mathcal{A}_{4j}^{K}(t,0) + \int_{-h}^{0} \mathcal{G}_4^{K}(t,\eta,0)d\eta. \tag{62}$$

If

$$\det \mathcal{A}_{4s}^{K}(t) \neq 0, \quad t \geq 0, \tag{63}$$

the differential-algebraic system (60)-(61) can be reduced to the differential equation with respect to $x_s(t)$

$$\frac{dx_s(t)}{dt} = \bar{\mathcal{A}}_s^{K}(t)x_s(t) + \bar{\mathcal{B}}_s^{K}(t)w_s(t), \quad t \geq 0, \tag{64}$$

where

$$\bar{\mathcal{A}}_s^K(t) = \mathcal{A}_{1s}(t) - \mathcal{A}_{2s}(t)\big(\mathcal{A}_{4s}^K(t)\big)^{-1}\mathcal{A}_{3s}(t),$$

$$\bar{\mathcal{B}}_s^K(t) = -\mathcal{A}_{2s}(t)\big(\mathcal{A}_{4s}^K(t)\big)^{-1}\mathcal{B}_2.$$

Thus, subject to (63), the slow subsystem associated with (56)-(57), (29) is (64), (32).

The fast subsystem, associated with (56)-(57), (29), consists of the differential equation with state delays

$$\frac{d\omega_f(\xi)}{d\xi} = \sum_{j=0}^{N} \mathcal{A}_{4j}^K(t,0)\omega_f(\xi - h_j) + \int_{-h}^{0} \mathcal{G}_4^K(t,\eta,0)\omega_f(\xi+\eta)d\eta \\ +\mathcal{B}_2 w_f(\xi), \quad \xi \geq 0, \tag{65}$$

and the output Equation (41). Note, that in (65), (41), $t \geq 0$ is a parameter, while ξ is an independent variable. Moreover, in this system, $\omega_f(\xi) \in E^{m+r}$; $\big(\omega_f(\xi), \omega_f(\xi+\eta)\big)$, $\eta \in [-h, 0)$ is a state variable; $w_f(\xi) \in E^r$, ($w_f(\xi)$ is a control); $\zeta_f(\xi) \in E^m$, ($\zeta_f(\xi)$ is an output).

Remark 3. *Since the output in the slow subsystem in both forms, (60)-(61), (32) and (64), (32), coincides with the state variable $x_s(t)$, then an output controllability of the slow subsystem is a controllability of its dynamic part with respect to $x_s(t)$. Namely, for the slow subsystem in the form (60)-(61), (32) such a controllability is the impulse-free controllability of the system (60)-(61) with respect to $x_s(t)$. For the slow subsystem in the form (64), (32), the controllability with respect to $x_s(t)$ is the complete controllability of the system (64).*

Proposition 7. *The system (30)-(31) is impulse-free controllable with respect to $x_s(t)$ at a given time instant $t_c > 0$ if and only if the system (60)-(61) is impulse-free controllable with respect to $x_s(t)$ at this time instant.*

Proof. The proposition is proven similarly to [26] (Lemma 3). □

Based on Propositions 2 and 7, we directly obtain the following corollary.

Corollary 2. *The system (6)-(7) is impulse-free controllable with respect to $x_s(t)$ at a given time instant $t_c > 0$ if and only if the system (60)-(61) is impulse-free controllable with respect to $x_s(t)$ at this time instant.*

Proposition 8. *Let the condition (63) be satisfied. Then, the system (60)-(61) is impulse-free controllable with respect to $x_s(t)$ at a given time instant $t_c > 0$, if and only if the system (64) is completely controllable at this time instant.*

Proof. The proposition is proven similarly to [26] (Theorem 2). □

Proposition 9. *Let the conditions (10) and (63) be valid. Then, the system (37) (and therefore, the system (11)) is completely controllable at a given time instant $t_c > 0$ if and only if the system (64) is completely controllable at this time instant.*

Proof. The proposition is proven similarly to [25] (Lemma 3.6). □

By virtue of the results of [29] (Lemma 6), we have the following assertion.

Proposition 10. *For a given $t \geq 0$, the system (40)-(41) is Euclidean space output controllable if and only if the system (65), (41) is Euclidean space output controllable.*

Based on Lemma 1 and Propositions 10, we directly have the following corollary.

Corollary 3. *For a given $t \geq 0$, the system (13) is completely Euclidean space controllable if and only if the system (65), (41) is Euclidean space output controllable.*

3.4. Hybrid Set of Riccati-Type Matrix Equations

Let us denote

$$\mathcal{S}_{22} \stackrel{\triangle}{=} \mathcal{B}_2\mathcal{B}_2^T. \tag{66}$$

Consider the following set, consisting of one algebraic and two differential equations (ordinary and partial) for matrices $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$:

$$\mathcal{P}(t)\mathcal{A}_{40}(t,0) + \mathcal{A}_{40}^T(t,0)\mathcal{P}(t) - \mathcal{P}(t)\mathcal{S}_{22}\mathcal{P}(t) + \mathcal{Q}(t,0) + \mathcal{Q}^T(t,0) + I_{m+r} = 0, \tag{67}$$

$$\begin{aligned} \frac{d\mathcal{Q}(t,\eta)}{d\eta} = {} & \Big(\mathcal{A}_{40}^T(t,0) - \mathcal{P}(t)\mathcal{S}_{22}\Big)\mathcal{Q}(t,\eta) + \mathcal{P}(t)\mathcal{G}_4(t,\eta,0) \\ & + \textstyle\sum_{j=1}^{N-1}\mathcal{P}(t)\mathcal{A}_{4j}(t,0)\delta(\eta+h_j) + \mathcal{R}(t,0,\eta), \end{aligned} \tag{68}$$

$$\begin{aligned} \left(\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\chi}\right)\mathcal{R}(t,\eta,\chi) = {} & \mathcal{G}_4^T(t,\eta,0)\mathcal{Q}(t,\chi) \\ & + \mathcal{Q}^T(t,\eta)\mathcal{G}_4(t,\chi,0) + \sum_{j=1}^{N-1}\mathcal{A}_{4j}^T(t,0)\mathcal{Q}(t,\chi)\delta(\eta+h_j) \\ & + \sum_{j=1}^{N-1}\mathcal{Q}^T(t,\eta)\mathcal{A}_{4j}(t,0)\delta(\chi+h_j) - \mathcal{Q}^T(t,\eta)\mathcal{S}_{22}(t,0)\mathcal{Q}(t,\chi), \end{aligned} \tag{69}$$

where $t \geq 0$ is a parameter; $\eta \in [-h,0]$ and $\chi \in [-h,0]$ are independent variables; $\delta(\cdot)$ is the Dirac delta-function.

The set of the Equations (67)-(69) is subject to the boundary conditions

$$\begin{aligned} & \mathcal{Q}(t,-h) = \mathcal{P}(t)\mathcal{A}_{4N}(t,0), \\ \mathcal{R}(t,-h,\eta) = \mathcal{A}_{4N}^T(t,0)\mathcal{Q}(t,\eta), \quad & \mathcal{R}(t,\eta,-h) = \mathcal{Q}^T(t,\eta)\mathcal{A}_{4N}(t,0). \end{aligned} \tag{70}$$

Let $t_c > 0$ be a given time instant.

In what follows of this subsection, we assume:

- **(I)** The matrix-valued functions $\mathcal{A}_{4j}(t,0)$, $(j = 0,1,\ldots,N)$ are continuously differentiable in the interval $[0,t_c]$.
- **(II)** The matrix-valued function $\mathcal{G}_4(t,\eta,0)$ is continuously differentiable with respect to $t \in [0,t_c]$ uniformly in $\eta \in [-h,0]$.
- **(III)** The matrix-valued function $\mathcal{G}_4(t,\eta,0)$ is piece-wise continuous with respect to $\eta \in [-h,0]$ for each $t \in [0,t_c]$.

For the sake of the further analysis of the set (67)-(70), we introduce the following definition.

For a given $t \in [0,t_c]$, consider the state-feedback control in the fast subsystem (40)

$$\tilde{v}_f(\omega_{f,\xi}) = \widetilde{K}_{1f}(t)\omega_f(\xi) + \int_{-h}^{0}\widetilde{K}_{2f}(t,\eta)\omega_f(\xi+\eta)d\eta, \tag{71}$$

where $\omega_{f,\xi} = \{\omega_f(\xi+\eta), \eta \in [-h,0]\}$, $\widetilde{K}_{1f}(t)$ and $\widetilde{K}_{2f}(t,\eta)$ are an $r \times m$-matrix and an $r \times m$-matrix-valued function of η, respectively; $\widetilde{K}_{2f}(t,\eta)$ is piece-wise continuous in the interval $[-h,0]$.

Definition 9. *For a given $t \in [0,t_c]$, the fast subsystem (40) is called L^2-stabilizable if there exists the state-feedback control (71) such that for any given $\omega_0 \in E^{m+r}$, $\varphi_{\omega f}(\cdot) \in L^2[-h,0;E^{m+r}]$, the solution $\tilde{\omega}_f(\xi)$*

of (40) with $v_f(\xi) = \tilde{v}_f(\omega_{f,\xi})$ *and subject to the initial conditions (42) satisfies the inclusion* $\widetilde{\omega}_f(\xi) \in L^2[0, +\infty; E^{m+r}]$.

The following proposition is a direct consequence of the results of [33] (Theorems 5.9 and 6.1).

Proposition 11. *Let the assumption (III) be valid. Let, for any* $t \in [0, t_c]$, *the fast subsystem (40) be* L^2*-stabilizable. Then, for any* $t \in [0, t_c]$, *the set of the Equations (67)-(69) subject to the boundary conditions (70) has the unique solution* $\{\mathcal{P}(t),\ \mathcal{Q}(t,\eta),\ \mathcal{R}(t,\eta,\chi),\ (\eta,\chi) \in [-h,0] \times [-h,0]\}$ *such that:*

(a) $\mathcal{P}^T(t) = \mathcal{P}(t)$;

(b) *the matrix-valued function* $\mathcal{Q}(t,\eta)$ *is piece-wise absolutely continuous in* $\eta \in [-h,0]$ *with the bounded jumps at* $\eta = -h_j$, $(j = 1,\ldots,N-1)$;

(c) *the matrix-valued function* $\mathcal{R}(t,\eta,\chi)$ *is piece-wise absolutely continuous in* $\eta \in [-h,0]$ *and in* $\chi \in [-h,0]$ *with the bounded jumps at* $\eta = -h_{j_1}$ *and* $\chi = -h_{j_2}$, $(j_1 = 1,\ldots,N-1;\ j_2 = 1,\ldots,N-1)$, *moreover,* $\mathcal{R}^T(t,\eta,\chi) = \mathcal{R}(t,\chi,\eta)$;

(d) *all roots* $\lambda(t)$ *of the equation*

$$\det\Big[\lambda I_m - \Big(\mathcal{A}_{40}(t,0) - \mathcal{S}_{22}\mathcal{P}(t)\Big) - \textstyle\sum_{j=1}^{N} \mathcal{A}_{4j}(t,0)\exp(-\lambda h_j) \\ - \textstyle\int_{-h}^{0} \Big(\mathcal{G}_4(t,\eta,0) - \mathcal{S}_{22}\mathcal{Q}(t,\eta)\Big)\exp(\lambda\eta)d\eta\Big] = 0 \tag{72}$$

satisfy the inequality

$$\mathrm{Re}\lambda(t) < -2\gamma(t), \quad t \in [0, t_c], \tag{73}$$

where $\gamma(t) > 0$ *is some function of* t.

By virtue of the results of [34] (Lemmas 4.1, 4.2 and 3.2), we directly have the following three assertions.

Proposition 12. *Let the assumptions (I)-(III) be valid. Let, for any* $t \in [0, t_c]$, *the fast subsystem (40) be* L^2*-stabilizable. Then, the matrices* $\mathcal{P}(t)$, $\mathcal{Q}(t,\eta)$, $\mathcal{R}(t,\eta,\chi)$ *are continuous functions of* $t \in [0, t_c]$ *uniformly in* $(\eta,\chi) \in [-h,0] \times [-h,0]$.

Proposition 13. *Let the assumptions (I)-(III) be valid. Let, for any* $t \in [0, t_c]$, *the fast subsystem (40) be* L^2*-stabilizable. Then, the derivatives* $d\mathcal{P}(t)/dt$, $\partial\mathcal{Q}(t,\eta)/\partial t$, $\partial\mathcal{R}(t,\eta,\chi)/\partial t$ *exist and are continuous functions of* $t \in [0, t_c]$ *uniformly in* $(\eta,\chi) \in [-h,0] \times [-h,0]$.

Proposition 14. *Let the assumptions (I)-(III) be valid. Let, for any* $t \in [0, t_c]$, *the fast subsystem (40) be* L^2*-stabilizable. Then, there exists a positive number* $\bar{\gamma}$ *such that all roots* $\lambda(t)$ *of the Equation (72) satisfy the inequality* $\lambda(t) < -2\bar{\gamma}$, $t \in [0, t_c]$.

4. Parameter-Free Controllability Conditions

In this section, we derive ε-free sufficient conditions for the Euclidean space output controllability of the auxiliary system (27)-(28), (29) and ε-free sufficient conditions for the complete Euclidean space controllability of the original system (1)-(2).

Let $t_c > 0$ be a given time instant independent of ε.

4.1. Case of the Standard System (1)-(2)

In this subsection, we assume that *the condition (10) holds for all* $t \in [0, t_c]$. In the literature, singularly perturbed systems with such a feature are called standard (see e.g., [1,12]).

In what follows, we also assume:

(AI) The matrix-valued functions $A_{ij}(t,\varepsilon)$, $B_{kj}(t,\varepsilon)$, $(i=1,\ldots,4;\ j=0,1,\ldots,N;\ k=1,2)$, are continuously differentiable with respect to $(t,\varepsilon) \in [0,t_c] \times [0,\varepsilon^0]$.

(AII) The matrix-valued functions $G_i(t,\eta,\varepsilon)$, $(i=1,\ldots,4)$ are piece-wise continuous with respect to $\eta \in [-h,0]$ for each $(t,\varepsilon) \in [0,t_c] \times [0,\varepsilon^0]$, and they are continuously differentiable with respect to $(t,\varepsilon) \in [0,t_c] \times [0,\varepsilon^0]$ uniformly in $\eta \in [-h,0]$.

(AIII) The matrix-valued functions $H_k(t,\eta,\varepsilon)$, $(k=1,2)$ are piece-wise continuous with respect to $\eta \in [-h,0]$ for each $(t,\varepsilon) \in [0,t_c] \times [0,\varepsilon^0]$, and they are continuously differentiable with respect to $(t,\varepsilon) \in [0,t_c] \times [0,\varepsilon^0]$ uniformly in $\eta \in [-h,0]$.

(AIV) All roots $\lambda(t)$ of the equation

$$\det\left[\lambda I_m - \sum_{j=0}^{N} A_{4j}(t,0)\exp(-\lambda h_j) - \int_{-h}^{0} G_4(t,\eta,0)\exp(\lambda\eta)d\eta\right] = 0 \tag{74}$$

satisfy the inequality $\mathrm{Re}\lambda(t) < -2\beta$ for all $t \in [0,t_c]$, where $\beta > 0$ is some constant.

Lemma 2. *(Main Lemma) Let the assumptions (AI)-(AIV) be valid. Let the system (37) be completely controllable at the time instant t_c. Let, for $t = t_c$, the system (40)-(41) be Euclidean space output controllable. Then, there exists a positive number ε_1, $(\varepsilon_1 \le \varepsilon^0)$, such that for all $\varepsilon \in (0,\varepsilon_1]$, the singularly perturbed system (27)-(28), (29) is Euclidean space output controllable at the time instant t_c.*

Proof of the lemma is presented in Section 4.3.

Remark 4. *Note that the Euclidean space output controllability for singularly perturbed systems with small state delays was studied in [29]. In this paper, the case of the standard original system was treated in Theorems 1–3 where different ε-free sufficient conditions for the Euclidean space output controllability of the original system were formulated. These conditions depend considerably on relations between the Euclidean dimensions of the state and output variables of the system. However, due to the specific form (19) of the matrix of the coefficients Z in the output equation of the system (27)-(28), (29), only Theorem 1 of [29] and only in the very specific case $n \le r$ is applicable to this system. Therefore, in Section 4.3, we present the proof of Lemma 2 which is not based on the results of [29]. In particular, this proof is uniformly valid for all relations between the Euclidean dimensions of the state and output variables of the system (27)-(28), (29).*

Theorem 1. *Let the assumptions (AI)-(AIV) be valid. Let the system (11) be completely controllable at the time instant t_c. Let, for $t = t_c$, the system (13) be completely Euclidean space controllable. Then, for all $\varepsilon \in (0,\varepsilon_1]$, the singularly perturbed system (1)-(2) is completely Euclidean space controllable at the time instant t_c.*

Proof. Based on Proposition 1, Remark 1 and Lemma 1, the theorem directly follows from Lemma 2. □

4.2. Case of the Nonstandard System (1)-(2)

In this subsection, in contrast with the previous one, we consider the case where *the condition (10) does not hold at least for one value of $t \in [0,t_c]$*. In the literature, singularly perturbed systems with such a feature are called nonstandard (see e.g., [1,12]). Since the condition (10) is not satisfied for some $\bar{t} \in [0,t_c]$, then $\det A_{4s}(\bar{t}) = 0$. The latter, along with the Equation (8), means that one of the roots $\lambda(\bar{t})$ of the Equation (74) equals zero. Thus, in the case of the nonstandard system (1)-(2) the assumption (AIV) is not valid. Therefore, in this subsection, we replace this assumption as follows.

We assume:

(AV) For all $t \in [0,t_c]$ and any complex number λ with $\mathrm{Re}\lambda \ge 0$, the following equality is valid:

$$\mathrm{rank}\Big[F_A(t,\lambda) - \lambda I_m\,,\ F_B(t,\lambda)\Big] = m, \tag{75}$$

where

$$
\begin{aligned}
F_A(t,\lambda) &= \sum_{j=0}^{N} A_{4j}(t,0)\exp(-\lambda h_j) + \int_{-h}^{0} G_4(t,\eta,0)\exp(\lambda\eta)d\eta, \\
F_B(t,\lambda) &= \sum_{j=0}^{N} B_{2j}(t,0)\exp(-\lambda h_j) + \int_{-h}^{0} H_2(t,\eta,0)\exp(\lambda\eta)d\eta.
\end{aligned} \tag{76}
$$

Lemma 3. *Let the assumption (AV) be valid. Then, for all $t \in [0, t_c]$ and any complex number λ with* $\mathrm{Re}\lambda \geq 0$, *the following equality is valid:*

$$
\begin{aligned}
\mathrm{rank}\Big[&\sum_{j=0}^{N} \mathcal{A}_{4j}(t,0)\exp(-\lambda h_j) \\
&+ \int_{-h}^{0} \mathcal{G}_4(t,\eta,0)\exp(\lambda\eta)d\eta - \lambda I_{m+r}\,,\ \mathcal{B}_2 \Big] = m + r.
\end{aligned} \tag{77}
$$

Proof. Using the block form of the matrices $\mathcal{A}_{4j}(t,\varepsilon)$, $(j = 0,1,\ldots,N)$, $\mathcal{G}_4(t,\eta,\varepsilon)$, $\mathcal{B}_2$ (see the Equations (22), (23), (25), (26)), we can rewrite the block matrix in the left-hand side of (77) as follows:

$$
\begin{aligned}
&\left(\sum_{j=0}^{N} \mathcal{A}_{4j}(t,0)\exp(-\lambda h_j) + \int_{-h}^{0} \mathcal{G}_4(t,\eta,0)\exp(\lambda\eta)d\eta - \lambda I_{m+r}\,,\ \mathcal{B}_2 \right) = \\
&\qquad \begin{pmatrix} F_A(t,\lambda) - \lambda I_m & F_B(t,\lambda) & O_{m\times r} \\ O_{r\times m} & -(\lambda+1)I_r & I_r \end{pmatrix}.
\end{aligned} \tag{78}
$$

The Equation (78), along with the Equation (75), directly yields the Equation (77), which completes the proof of the lemma. □

Corollary 4. *Let the assumption (AV) be valid. Then, for any $t \in [0, t_c]$, the fast subsystem (40) is L^2-stabilizable.*

Proof. The corollary is a direct consequence of Lemma 3 and the results of [35] (Theorem 3.5). □

Theorem 2. *Let the assumptions (AI)-(AIII),(AV) be valid. Let the system (6)-(7) be impulse-free controllable with respect to $x_s(t)$ at the time instant t_c. Let, for $t = t_c$, the system (13) be completely Euclidean space controllable. Then, there exists a positive number ε_2, $(\varepsilon_2 \leq \varepsilon^0)$, such that for all $\varepsilon \in (0, \varepsilon_2]$, the singularly perturbed system (1)-(2) is completely Euclidean space controllable at the time instant t_c.*

Proof. Let us start with the auxiliary system (27)-(28), (29). Due to the assumptions (AI)-(AIII) and the Equations (20)-(26), the matrix-valued coefficients of this system satisfy the conditions similar to the assumptions (AI) and (AII) on the matrix-valued functions $A_{ij}(t,\varepsilon)$ and $G_i(t,\eta,\varepsilon)$, $(i = 1,\ldots,4;\ j = 0,1,\ldots,N)$.

For a given $\varepsilon \in (0, \varepsilon^0]$ in the auxiliary system (27)-(28), (29), let us make the control transformation (55), where

$$
K_1(t) = -\mathcal{B}_2^T \mathcal{P}(t), \quad K_2(t,\eta) = -\mathcal{B}_2^T \mathcal{Q}(t,\eta), \quad t \in [0, t_c], \ \eta \in [-h, 0], \tag{79}
$$

and $\mathcal{P}(t)$ and $\mathcal{Q}(t,\eta)$ are the components of the solution to the problem (67)-(69), (70) mentioned in Proposition 11. As a result of this transformation, we obtain the system (56)-(57), (29). By virtue of Corollary 4 and Propositions 11, 13, the matrix-valued coefficients of this system satisfy the conditions similar to the assumptions (AI) and (AII) on the matrix-valued functions $A_{ij}(t,\varepsilon)$ and $G_i(t,\eta,\varepsilon)$, $(i = 1,\ldots,4;\ j = 0,1,\ldots,N)$.

The slow and fast subsystems, associated with (56)-(57), (29), are (60)-(61) and (65), (41), respectively. Since the system (6)-(7) is impulse-free controllable with respect to $x_s(t)$ at the time instant t_c, then due to Corollary 2, the system (60)-(61) is impulse-free controllable with respect to

$x_s(t)$ at the time instant t_c. Furthermore, since, for $t = t_c$, the system (13) is completely Euclidean space controllable, then due to Corollary 3, the system (65), (41) for $t = t_c$ is Euclidean space output controllable. By virtue of Corollary 4 and Propositions 11, 14, the value $\lambda = 0$ is not a root of the Equation (72) for all $t \in [0, t_c]$. Hence, the matrix $\mathcal{A}_{4s}^{K}(t)$, given by (62), (79), is invertible for all $t \in [0, t_c]$. Thus, the slow subsystem (60)-(61) is reduced to the differential Equation (64). Therefore, due to Proposition 8, the above mentioned impulse-free controllability of the system (60)-(61) yields the complete controllability of the system (64) at the time instant t_c. Now, by application of Lemma 2 to the system (56)-(57), (29), we directly obtain the existence of a positive number ε_2, ($\varepsilon_2 \le \varepsilon^0$), such that for all $\varepsilon \in (0, \varepsilon_2]$, this system is Euclidean space output controllable at the time instant t_c. Finally, using Corollary 1 yields the complete Euclidean space controllability of the system (1)-(2) at the time instant t_c for all $\varepsilon \in (0, \varepsilon_2]$, which completes the proof of the theorem. □

4.3. *Proof of Main Lemma (Lemma 2)*

In the proof of Main Lemma, the following two auxiliary proposition are used.

4.3.1. Auxiliary Propositions

For any given $t \in [0, t_c]$ and any complex number μ, let us consider the matrix

$$\mathcal{W}(t, \mu) = \sum_{j=0}^{N} \mathcal{A}_{4j}(t, 0) \exp(-\mu h_j) + \int_{-h}^{0} \mathcal{G}_4(t, \eta, 0) \exp(\mu\eta) d\eta, \tag{80}$$

where $\mathcal{A}_{4j}(t, \varepsilon)$, $(j = 0, 1, \ldots, N)$ and $\mathcal{G}_4(t, \eta, \varepsilon)$ are given in (22)-(23) and (25), respectively.

Proposition 15. *Let the assumption (AIV) be valid. Then, all roots $\mu(t)$ of the equation*

$$\det\left[\mu I_{m+r} - \mathcal{W}(t, \mu)\right] = 0 \tag{81}$$

satisfy the inequality $\mathrm{Re}\mu(t) < -2\nu$ *for all* $t \in [0, t_c]$, *where* $\nu = \min\{\beta, 1/4\}$.

Proof. Using (22)-(23), (25) and (80), we obtain for all $t \in [0, t_c]$:

$$\det\left[\mu I_{m+r} - \mathcal{W}(t, \mu)\right] =$$
$$\det\left[\mu I_m - \sum_{j=0}^{N} A_{41}(t, 0) \exp(-\lambda h_j) - \int_{-h}^{0} G_4(t, \eta, 0) \exp(\mu\eta) d\eta\right] (\mu + 1)^r,$$

meaning that for any $t \in [0, t_c]$ the set of all roots $\mu(t)$ of the Equation (81) consists of all roots of the Equation (74) and the root $\mu(t) \equiv -1$ of the multiplicity r. This observation, along with the assumption (AIV), directly yields the statement of the proposition. □

Let us partition the matrix-valued function $\Psi(\sigma, \varepsilon)$, given by the terminal-value problem (48), into blocks as:

$$\Psi(\sigma, \varepsilon) = \begin{pmatrix} \Psi_1(\sigma, \varepsilon) & \Psi_2(\sigma, \varepsilon) \\ \Psi_3(\sigma, \varepsilon) & \Psi_4(\sigma, \varepsilon) \end{pmatrix}, \tag{82}$$

where the blocks $\Psi_1(\sigma, \varepsilon)$, $\Psi_2(\sigma, \varepsilon)$, $\Psi_3(\sigma, \varepsilon)$ and $\Psi_4(\sigma, \varepsilon)$ are of the dimensions $n \times n$, $n \times (m + r)$, $(m + r) \times n$ and $(m + r) \times (m + r)$, respectively.

Proposition 16. *Let the assumptions (AI)-(AIV) be valid. Then, there exists a positive number ε_0, ($\varepsilon_0 \le \varepsilon^0$), such that for all $\varepsilon \in (0, \varepsilon_0]$ the matrix-valued functions $\Psi_1(\sigma, \varepsilon)$, $\Psi_2(\sigma, \varepsilon)$, $\Psi_3(\sigma, \varepsilon)$, $\Psi_4(\sigma, \varepsilon)$ satisfy the inequalities:*

$$\left\|\Psi_1(\sigma,\varepsilon)-\Psi_{1s}(\sigma)\right\|\le a\varepsilon,\quad \left\|\Psi_2(\sigma,\varepsilon)\right\|\le a,\ \ \sigma\in[0,t_c], \tag{83}$$

$$\left\|\Psi_3(\sigma,\varepsilon)-\varepsilon\Psi_{3s}(\sigma)\right\|\le a\varepsilon\big[\varepsilon+\exp(-\nu(t_c-\sigma)/\varepsilon)\big],\quad \sigma\in[0,t_c], \tag{84}$$

$$\left\|\Psi_4(\sigma,\varepsilon)-\Psi_{4f}\big((t_c-\sigma)/\varepsilon\big)\right\|\le a\varepsilon,\quad \sigma\in[0,t_c], \tag{85}$$

where

$$\Psi_{1s}(\sigma)=\Psi_s(\sigma),\quad \Psi_{3s}(\sigma)=-\Big(\mathcal{A}_{4s}^T(\sigma)\Big)^{-1}\mathcal{A}_{2s}^T(\sigma)\Psi_s(\sigma),\quad \sigma\in[0,t_c],$$
$$\Psi_{4f}(\xi)=\Psi_f(\xi,t_c),\quad \xi\ge 0;$$

the matrix-valued functions $\Psi_s(\sigma)$ and $\Psi_f(\xi,t)$ are given by the terminal-value problem (51) and the initial-value problem (53), respectively; $a>0$ is some constant independent of ε.

Proof. Based on Proposition 15, the validity of the inequalities (83)-(85) is proven similarly to [25] (Lemma 3.2). □

Remark 5. *By virtue of Proposition 15 and the results of [36], we have the inequality*

$$\left\|\Psi_{4f}(\xi)\right\|\le a\exp(-2\nu\xi),\quad \xi\ge 0, \tag{86}$$

where $a>0$ is some constant.

4.3.2. Main Part of the Proof

Due to Proposition 3, in order to prove Main Lemma, it is necessary and sufficient to show the existence of a positive number ε_1 such that

$$\det W_Z(t_c,\varepsilon)\neq 0\quad \forall\varepsilon\in(0,\varepsilon_1], \tag{87}$$

where the $(n+m)\times(n+m)$-matrix $W_Z(t_c,\varepsilon)$ is defined by the Equations (49)-(50).

Let, for a given $\varepsilon\in(0,\varepsilon_0]$, the matrix $W_1(t_c,\varepsilon)$ of the dimension $n\times n$, the matrix $W_2(t_c,\varepsilon)$ of the dimension $n\times(m+r)$ and the matrix $W_3(t_c,\varepsilon)$ of the dimension $(m+r)\times(m+r)$ be the upper left-hand, upper right-hand and lower right-hand blocks, respectively, of the symmetric matrix $W(t_c,\varepsilon)$, given by the Equation (49). Thus,

$$W(t_c,\varepsilon)=\begin{pmatrix} W_1(t_c,\varepsilon) & W_2(t_c,\varepsilon)\\ W_2^T(t_c,\varepsilon) & W_3(t_c,\varepsilon)\end{pmatrix}. \tag{88}$$

Using (49), and the block representations of the matrices $\mathcal{B}(\varepsilon)$ and $\Psi(\sigma,\varepsilon)$ (see the Equations (45) and (82)), we obtain

$$\begin{aligned}W_1(t_c,\varepsilon)=\int_0^{t_c}\Big[&\Psi_1^T(\sigma,\varepsilon)\mathcal{S}_{11}\Psi_1(\sigma,\varepsilon)+(1/\varepsilon)\Psi_3^T(\sigma,\varepsilon)\mathcal{S}_{12}^T\Psi_1(\sigma,\varepsilon)\\ &+(1/\varepsilon)\Psi_1^T(\sigma,\varepsilon)\mathcal{S}_{12}\Psi_3(\sigma,\varepsilon)+(1/\varepsilon^2)\Psi_3^T(\sigma,\varepsilon)\mathcal{S}_{22}\Psi_3(\sigma,\varepsilon)\Big]d\sigma,\end{aligned} \tag{89}$$

$$\begin{aligned}W_2(t_c,\varepsilon)=\int_0^{t_c}\Big[&\Psi_1^T(\sigma,\varepsilon)\mathcal{S}_{11}\Psi_2(\sigma,\varepsilon)+(1/\varepsilon)\Psi_3^T(\sigma,\varepsilon)\mathcal{S}_{12}^T\Psi_2(\sigma,\varepsilon)\\ &+(1/\varepsilon)\Psi_1^T(\sigma,\varepsilon)\mathcal{S}_{12}\Psi_4(\sigma,\varepsilon)+(1/\varepsilon^2)\Psi_3^T(\sigma,\varepsilon)\mathcal{S}_{22}\Psi_4(\sigma,\varepsilon)\Big]d\sigma,\end{aligned} \tag{90}$$

$$W_3(t_c,\varepsilon) = \int_0^{t_c} \Big[\Psi_2^T(\sigma,\varepsilon)\mathcal{S}_{11}\Psi_2(\sigma,\varepsilon) + (1/\varepsilon)\Psi_4^T(\sigma,\varepsilon)\mathcal{S}_{12}^T\Psi_2(\sigma,\varepsilon) \\ +(1/\varepsilon)\Psi_2^T(\sigma,\varepsilon)\mathcal{S}_{12}\Psi_4(\sigma,\varepsilon) + (1/\varepsilon^2)\Psi_4^T(\sigma,\varepsilon)\mathcal{S}_{22}\Psi_4(\sigma,\varepsilon)\Big]d\sigma, \tag{91}$$

where, due to (45),

$$\mathcal{S}_{11} = \mathcal{B}_1\mathcal{B}_1^T = O_{n\times n}, \quad \mathcal{S}_{12} = \mathcal{B}_1\mathcal{B}_2^T = O_{n\times(m+r)}, \\ \mathcal{S}_{22} = \mathcal{B}_2\mathcal{B}_2^T = \begin{pmatrix} O_{m\times m} & O_{m\times r} \\ O_{r\times m} & I_r \end{pmatrix}. \tag{92}$$

The latter, along with (89)-(91), yields

$$W_1(t_c,\varepsilon) = (1/\varepsilon^2)\int_0^{t_c} \Psi_3^T(\sigma,\varepsilon)\mathcal{S}_{22}\Psi_3(\sigma,\varepsilon)d\sigma, \tag{93}$$

$$W_2(t_c,\varepsilon) = (1/\varepsilon^2)\int_0^{t_c} \Psi_3^T(\sigma,\varepsilon)\mathcal{S}_{22}\Psi_4(\sigma,\varepsilon)d\sigma, \tag{94}$$

$$W_3(t_c,\varepsilon) = (1/\varepsilon^2)\int_0^{t_c} \Psi_4^T(\sigma,\varepsilon)\mathcal{S}_{22}\Psi_4(\sigma,\varepsilon)d\sigma. \tag{95}$$

Let us estimate the matrices $W_1(t_c,\varepsilon)$, $W_2(t_c,\varepsilon)$ and $W_3(t_c,\varepsilon)$. We start with $W_1(t_c,\varepsilon)$. Denote

$$\Delta\Psi_3(\sigma,\varepsilon) \stackrel{\triangle}{=} \Psi_3(\sigma,\varepsilon) - \varepsilon\Psi_{3s}(\sigma). \tag{96}$$

Using this notation, we can rewrite the expression (93) for $W_1(t_c,\varepsilon)$ as:

$$W_1(t_c,\varepsilon) = (1/\varepsilon^2)\int_0^{t_c} \Big[\varepsilon^2\Psi_{3s}^T(\sigma)\mathcal{S}_{22}\Psi_{3s}(\sigma) + \varepsilon\Psi_{3s}^T(\sigma)\mathcal{S}_{22}\Delta\Psi_3(\sigma,\varepsilon) \\ +\varepsilon\big(\Delta\Psi_3(\sigma,\varepsilon)\big)^T\mathcal{S}_{22}\Psi_{3s}(\sigma) + \big(\Delta\Psi_3(\sigma,\varepsilon)\big)^T\mathcal{S}_{22}\Delta\Psi_3(\sigma,\varepsilon)\Big]d\sigma. \tag{97}$$

Due to Proposition 16 (see the Equation (84)) and the Equation (96), we have $\|\Delta\Psi_3(\sigma,\varepsilon)\| \le a\varepsilon\big[\varepsilon + \exp(-\nu(t_c-\sigma)/\varepsilon)\big]$, $\sigma \in [0,t_c]$, $\varepsilon \in (0,\varepsilon_0]$. Applying this inequality to the expression (97) for the matrix $W_1(t_c,\varepsilon)$, we obtain the inequality

$$\left\| W_1(t_c,\varepsilon) - \int_0^{t_c} \Psi_{3s}^T(\sigma)\mathcal{S}_{22}\Psi_{3s}(\sigma)d\sigma \right\| \le a\varepsilon, \quad \varepsilon \in (0,\varepsilon_0], \tag{98}$$

where $a > 0$ is some constant independent of ε.

Now, let us treat the integral in the left-hand side of (98). Using the Equation (86), we have

$$W_{1s} \stackrel{\triangle}{=} \int_0^{t_c} \Psi_{3s}^T(\sigma)\mathcal{S}_{22}\Psi_{3s}(\sigma)d\sigma \\ = \int_0^{t_c} \Psi_s^T(\sigma)\mathcal{A}_{2s}(\sigma)\mathcal{A}_{4s}^{-1}(\sigma)\mathcal{S}_{22}\Big(\mathcal{A}_{4s}^T(\sigma)\Big)^{-1}\mathcal{A}_{2s}^T(\sigma)\Psi_s(\sigma)d\sigma. \tag{99}$$

Taking into account the block form of the matrices $\mathcal{B}_2$, $\mathcal{A}_{2s}(\sigma)$ and $\mathcal{A}_{4s}(\sigma)$ (see the Equations (26), (34)) and the expression for $\bar{B}_s(\sigma)$ (see the Equation (12)), we obtain

$$
\begin{aligned}
&\mathcal{A}_{2s}(\sigma)\mathcal{A}_{4s}^{-1}(\sigma)\mathcal{B}_2 = \\
&\Big(A_{2s}(\sigma), B_{1s}(\sigma)\Big)\begin{pmatrix} A_{4s}(\sigma) & B_{2s}(\sigma) \\ O_{r\times m} & -I_r \end{pmatrix}^{-1}\begin{pmatrix} O_{m\times r} \\ I_r \end{pmatrix} = \\
&\Big(A_{2s}(\sigma), B_{1s}(\sigma)\Big)\begin{pmatrix} A_{4s}^{-1}(\sigma) & A_{4s}^{-1}(\sigma)B_{2s}(\sigma) \\ O_{r\times m} & -I_r \end{pmatrix}\begin{pmatrix} O_{m\times r} \\ I_r \end{pmatrix} = \\
&\Big(A_{2s}(\sigma)A_{4s}^{-1}(\sigma), A_{2s}(\sigma)A_{4s}^{-1}(\sigma)B_{2s}(\sigma) - B_{1s}(\sigma)\Big)\begin{pmatrix} O_{m\times r} \\ I_r \end{pmatrix} = \\
&-\Big(B_{1s}(\sigma) - A_{2s}(\sigma)A_{4s}^{-1}(\sigma)B_{2s}(\sigma)\Big) = -\bar{B}_s(\sigma).
\end{aligned}
\tag{100}
$$

Finally, using the expression for $\mathcal{S}_{22}$ (see the Equations (92)), as well as the Equations (52), (99) and (100), we obtain that $W_{1s} = W_s(t_c)$. The latter, along with (98), yields

$$
\big\|W_1(t_c,\varepsilon) - W_s(t_c)\big\| \le a\varepsilon, \quad \varepsilon \in (0, \varepsilon_0], \tag{101}
$$

where $a > 0$ is some constant independent of ε.

Similarly to (101), we obtain the existence of a positive number $\bar{\varepsilon}_0 \le \varepsilon_0$ such that the following inequalities are satisfied:

$$
\big\|W_2(t_c,\varepsilon)\big\| \le a, \quad \big\|\varepsilon W_3(t_c,\varepsilon) - W_{3f}(t_c)\big\| \le a\varepsilon, \quad \varepsilon \in (0, \bar{\varepsilon}_0], \tag{102}
$$

where $a > 0$ is some constant independent of ε;

$$
W_{3f}(t_c) = \int_0^{+\infty} \Psi_f^T(\rho, t_c)\mathcal{B}_2\mathcal{B}_2^T\Psi_f(\rho, t_c)d\rho. \tag{103}
$$

By virtue of the inequality (86), the integral in the expression for $W_{3f}(t_c)$ converges.

Now, let us proceed to analysis of the matrix $W_Z(t_c,\varepsilon)$. Using the Equations (19), (50) and (88), we obtains the following block representation of the matrix $W_Z(t_c,\varepsilon)$:

$$
W_Z(t_c,\varepsilon) = \begin{pmatrix} W_1(t_c,\varepsilon) & W_{21}(t_c,\varepsilon) \\ W_{21}^T(t_c,\varepsilon) & W_{31}(t_c,\varepsilon) \end{pmatrix}, \tag{104}
$$

where $W_{21}(t_c,\varepsilon)$ is the left-hand block of the dimension $n \times m$ of the matrix $W_2(t_c,\varepsilon)$, while $W_{31}(t_c,\varepsilon)$ is the upper left-hand block of the dimension $m \times m$ of the matrix $W_3(t_c,\varepsilon)$.

By virtue of (102), we immediately have that

$$
\|W_{21}(t_c,\varepsilon)\| \le a, \quad \|\varepsilon W_{31}(t_c,\varepsilon) - W_{3f,1}(t_c)\| \le a\varepsilon, \quad \varepsilon \in (0, \bar{\varepsilon}_0], \tag{105}
$$

where $W_{3f,1}(t_c)$ is the upper left-hand block of the dimension $m \times m$ of the matrix $W_{3f}(t_c)$.

Let us show that

$$
\det W_{3f,1}(t_c) \ge b, \tag{106}
$$

where $b > 0$ is some number.

Note that $W_{3f,1}(t_c)$ can be represented as:

$$
W_{3f,1}(t_c) = \Omega_f W_{3f}(t_c)\Omega_f^T, \tag{107}
$$

where Ω_f is given in (41).

Comparison of the expressions for $W_f(\xi, t)$ and $W_{3f,1}(t_c)$ (see the Equations (54) and (107)), and use of expression for $W_{3f}(t_c)$ (see the Equation (103)) yield that

$$
W_{3f,1}(t_c) = \lim_{\xi \to +\infty} W_f(\xi, t_c). \tag{108}
$$

Let us observe that, for any $\xi > 0$ and $t \in [0, t_c]$, the matrix $W_f(\xi, t)$ is positive semi-definite. Moreover, since the system (40)-(41) is Euclidean space output controllable for $t = t_c$, then by virtue of Proposition 5, $\det W_f(\xi_c, t_c) \neq 0$ with some $\xi_c > 0$. Therefore, $\det W_f(\xi_c, t_c) > 0$ and $W_f(\xi_c, t_c)$ is a positive definite matrix.

For any $\xi > \xi_c$, we have

$$W_f(\xi, t_c) = W_f(\xi_c, t_c) + \Omega_f \int_{\xi_c}^{\xi} \Psi_f^T(\rho, t)\mathcal{B}_2\mathcal{B}_2^T\Psi_f(\rho, t)d\rho\Omega_f^T,$$

and the second addend in the right-hand side of this equation is a positive semi-definite matrix. Hence, by use of the results of [37], we obtain that

$$\det W_f(\xi, t_c) \geq \det W_f(\xi_c, t_c) > 0, \quad \xi > \xi_c.$$

The latter, along with the equality (108), directly yields the inequality (106), where $b = \det W_f(\xi_c, t_c)$.

Now, we proceed to the proof of the inequality (87). Let us introduce into the consideration the matrix

$$L(\varepsilon) = \begin{pmatrix} I_n & O_{n\times m} \\ O_{m\times n} & \sqrt{\varepsilon} I_m \end{pmatrix}.$$

For any $\varepsilon > 0$, $\det L(\varepsilon) > 0$.

Using the Equation (104), we obtain

$$L(\varepsilon)W_Z(t_c, \varepsilon)L(\varepsilon) = \begin{pmatrix} W_1(t_c, \varepsilon) & \sqrt{\varepsilon} W_{21}(t_c, \varepsilon) \\ \sqrt{\varepsilon} W_{21}^T(t_c, \varepsilon) & \varepsilon W_{31}(t_c, \varepsilon) \end{pmatrix}.$$

Calculating the limit of the determinant of this matrix as $\varepsilon \to +0$, and using the inequalities (101), (105), (106) and Proposition 4, we obtain

$$\begin{aligned} \lim_{\varepsilon\to+0} \det \Big(L(\varepsilon)W_Z(t_c, \varepsilon)L(\varepsilon)\Big) &= \det \begin{pmatrix} W_s(t_c) & 0 \\ 0 & W_{3f,1}(t_c) \end{pmatrix} \\ &= \det W_s(t_c) \det W_{3f,1}(t_c) \neq 0. \end{aligned}$$

This inequality, along with the inequality $\det L(\varepsilon) > 0$, $\varepsilon > 0$, implies the existence of a positive number ε_1 such that the inequality (87) is valid. This completes the proof of Main Lemma.

5. Examples

5.1. Example 1

Consider the following system, a particular case of (1)-(2),

$$\begin{aligned} \frac{dx(t)}{dt} &= x(t) - 4y(t) + 5y(t-\varepsilon) + \int_{-2}^{0} \eta x(t + \varepsilon\eta)d\eta \\ &\quad + (t-5)u(t) - tu(t-\varepsilon), \quad t \geq 0, \\ \varepsilon\frac{dy(t)}{dt} &= 3x(t) + (t-5)y(t) - x(t-\varepsilon) - x(t-2\varepsilon) + y(t-\varepsilon) \\ &\quad + (t-2)u(t) + tu(t-\varepsilon), \quad t \geq 0, \end{aligned} \tag{109}$$

where $x(t)$, $y(t)$ and $u(t)$ are scalars, i.e., $n = m = r = 1$; $h_1 = 1$, $h_2 = h = 2$.

We study the complete Euclidean space controllability of the system (109) at the time instant $t_c = 2$ for all sufficiently small $\varepsilon > 0$. For this purpose, let us write down the slow and fast subsystems associated with (109). Begin with the slow subsystem. For the system (109), the matrix $A_{4s}(t)$, given in (8), becomes a scalar and has the form $A_{4s}(t) = t - 4$. Thus, the condition (10) is satisfied for all $t \in [0, 2]$, meaning that the slow subsystem associated with (109) can be reduced to the differential Equation (11), i.e.,

$$\frac{dx_s(t)}{dt} = \frac{t-3}{4-t}x_s(t) + \frac{7t-22}{4-t}u_s(t), \quad t \in [0, 2]. \tag{110}$$

Due to (13), the fast subsystem associated with the system (109) is

$$\frac{dy_f(\xi)}{d\xi} = (t-5)y_f(\xi) + y_f(\xi-1) + (t-2)u_f(\xi) + tu_f(\xi-1), \quad \xi \geq 0, \tag{111}$$

where $t \in [0, 2]$ is a parameter. It should be noted the following. Although the delay in the original system (109) is 2ε, the delay in the fast subsystem is 1 (but not 2), meaning that in this subsystem the coefficients for the terms with the delay 2 equal zero. Therefore, in what follows, it is sufficient to analyze the fast subsystem with the delay 1.

It is seen directly that the assumptions (AI)-(AIII) are satisfied for the system (109). Let us show the fulfillment of the assumption (AIV) for this system. Indeed, the Equation (74) becomes as:

$$\lambda - t + 5 - \exp(-\lambda) = 0. \tag{112}$$

For $\mathrm{Re}\lambda \geq -0.5$, one obtains the following:

$$\mathrm{Re}\big(\lambda - t + 5 - \exp(-\lambda)\big) \geq 2.85 - t > 0 \quad \forall t \in [0, 2],$$

meaning that all roots $\lambda(t)$ of the Equation (112) satisfy the inequality $\mathrm{Re}\lambda(t) < -0.5$, $t \in [0, 2]$. Thus, for the system (109) and $t_c = 2$, the assumption (AIV) is satisfied with $\beta = 0.25$. Since the assumptions (AI)-(AIII) also are satisfied for the system (109) and $t_c = 2$, one can try to use Theorem 1 in order to find out whether the system (109) is completely Euclidean space controllable at $t_c = 2$ for all sufficiently small values of $\varepsilon > 0$. For this purpose, proper kinds of controllability of the systems (110) and (111) should be analyzed. Let us start with the system (110). Since the coefficient for $u_s(t)$ in (110) differs from zero for $t \in [0, 2]$, this system is completely controllable at the time instant $t_c = 2$.

Proceed to the system (111). Due to Lemma 1, for the given $t = t_c = 2$, this system is completely Euclidean space controllable if for this value of t the auxiliary system (40)-(41) with the scalar control $v_f(\xi)$ is Euclidean space output controllable. For $t = 2$, this system becomes

$$\begin{aligned} \frac{d\omega_f(\xi)}{d\xi} &= \widetilde{A}\omega_f(\xi) + \widetilde{H}\omega_f(\xi-1) + \widetilde{B}v_f(\xi), \\ \zeta_f(\xi) &= \widetilde{Z}\omega_f(\xi), \quad \xi \geq 0, \end{aligned} \tag{113}$$

where

$$\widetilde{A} = -\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{H} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \widetilde{Z} = (1\ ,\ 0).$$

Note that the Euclidean dimension of the state variable in (113) is $n_f = 2$, while such dimensions of the control and the output are $r_f = 1$ and $q_f = 1$, respectively. To verify the Euclidean space output controllability of the system (113), we apply the algebraic criterion for such a controllability of a time-invariant differential-difference system (see [38,39]). Using this criterion, we are going to show that the system (113) is Euclidean space output controllable at any given instant $\xi_c \in (1, 2]$ of the stretched time ξ. For this purpose, we construct the following matrices:

$$\widetilde{A}_0 = \widetilde{A} = -\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{A}_1 = \begin{pmatrix} \widetilde{A}_0 & O_{2\times 2} \\ \widetilde{H} & \widetilde{A}_0 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\widetilde{E}_0 = I_2, \quad \widetilde{E}_1 = (O_{2\times 2}, I_2), \quad \widetilde{Z}_0 = \widetilde{Z}, \quad \widetilde{Z}_1 = \widetilde{Z}\widetilde{E}_1 = (0,0,1,0),$$
$$\widetilde{C}_0 = I_2, \quad \widetilde{B}_0 = \widetilde{B}, \quad \widetilde{B}_1 = \widetilde{C}_1\widetilde{B},$$

where

$$\widetilde{C}_1 = \begin{pmatrix} I_2 \\ \exp\left(\widetilde{A}_0\right)\widetilde{C}_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \exp(-3) & 0 \\ 0 & \exp(-1) \end{pmatrix}.$$

Hence,

$$\widetilde{B}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \exp(-1) \end{pmatrix}.$$

Due to the results of [38,39], the system (113) is Euclidean space output controllable at a given value $\xi_c \in (1,2]$ of the independent variable ξ, if and only if the rank of the following matrix equals to q_f:

$$\widetilde{D} = \left(\widetilde{Z}_0\widetilde{B}_0, \ldots, \widetilde{Z}_0\widetilde{A}_0^{n_f-1}\widetilde{B}_0, \widetilde{Z}_1\widetilde{B}_1, \ldots, \widetilde{Z}_1\widetilde{A}_1^{2n_f-1}\widetilde{B}_1\right).$$

Since each block of the matrix $\widetilde{D}$ is scalar and $q_f = 1$, then it is sufficient to show that at least one block in this matrix differs from zero. Remember that $n_f = 2$. Therefore, $\widetilde{Z}_1\widetilde{A}_1\widetilde{B}_1$ is a block of $\widetilde{D}$. Calculating this block, we obtain $\widetilde{Z}_1\widetilde{A}_1\widetilde{B}_1 = 2 \neq 0$, meaning that $\mathrm{rank}\widetilde{D} = q_f = 1$. Thus, the system (113) is Euclidean space output controllable with any given value $\xi_c \in (1,2]$ mentioned in Definition 7. Hence, the system (111) is completely Euclidean space controllable. Therefore, by virtue of Theorem 1, the system (109) is completely Euclidean space controllable at $t_c = 2$ robustly with respect to $\varepsilon > 0$ for all its sufficiently small values.

5.2. Example 2

Consider the following particular case of the system (1)-(2):

$$\begin{aligned} \frac{dx(t)}{dt} &= 2(t-1)x(t) + 4y(t) - 2tx(t-\varepsilon) - y(t-\varepsilon) \\ &\quad + tu(t) - u(t-\varepsilon) + \int_{-1}^{0} 2t\eta u(t+\varepsilon\eta)d\eta, \quad t \geq 0, \\ \varepsilon\frac{dy(t)}{dt} &= 4x(t) - y(t) - 2x(t-\varepsilon) + y(t-\varepsilon) + 2u(t) - u(t-\varepsilon), \quad t \geq 0, \end{aligned} \tag{114}$$

where $x(t)$, $y(t)$ and $u(t)$ are scalars, i.e., $n = m = r = 1; h = 1$.

In this example, like in the previous one, we study the complete Euclidean space controllability of the considered system. We study this controllability at the time instant $t_c = 2$ for all sufficiently small $\varepsilon > 0$.

The asymptotic decomposition of the system (114) yields the slow and fast subsystems, respectively,

$$\begin{aligned} \tfrac{dx_s(t)}{dt} &= -2x_s(t) + 3y_s(t) - u_s(t), \quad t \geq 0, \\ 0 &= 2x_s(t) + u_s(t), \quad t \geq 0, \end{aligned} \tag{115}$$

and

$$\frac{dy_f(\xi)}{d\xi} = -y_f(\xi) + y_f(\xi - 1) + 2u_f(\xi) - u_f(\xi - 1), \quad \xi \geq 0. \tag{116}$$

It is seen that the assumptions (AI)-(AIII) are satisfied for the system (114). The condition (10) is not satisfied for this system, meaning that (114) is a nonstandard system, and it does not satisfy the assumption (AIV). Indeed, for the system (114), the Equation (74) becomes as:

$$\lambda + 1 - \exp(-\lambda) = 0. \tag{117}$$

For this equation, $\lambda = 0$ is a single root with the nonnegative real part.

Let us show the fulfillment of the assumption (AV) for the system (114). The matrix in the Equation (75) becomes as:

$$\big[-1 + \exp(-\lambda) - \lambda\,, 2 - \exp(-\lambda) \big]. \tag{118}$$

For $\lambda = 0$, the rank of this matrix equals to the Euclidean dimension of the fast subsystem $m = 1$. Since $\lambda = 0$ is a single root with the nonnegative real part of the Equation (117), then the rank of the matrix (118) equals $m = 1$ for all complex λ with $\mathrm{R}e\lambda \geq 0$. Thus, the assumption (AV) is fulfilled for the system (114).

Now, let us find out whether the systems (115) and (116) are controllable in the sense mentioned in Theorem 2. We start with (115). Let x_0 and x_c be any given numbers. Let $\vartheta = (x_c - x_0)/6$. One can verify immediately that for the numbers x_0 and x_c, there exists a control $u_s(t) \in L^2[0, 2; E^1]$, namely,

$$u_s(t) = -2x_0 - 3\vartheta t^2,$$

such that the system (115), subject to the initial $x_s(0) = x_0$ and terminal $x_s(2) = x_c$ conditions, has an impulse-free solution, namely,

$$x_s(t) = x_0 + 1.5\vartheta t^2, \quad y_s(t) = \vartheta t.$$

Thus, the system (115) is impulse-free controllable with respect to $x_s(t)$ at the time instant $t_c = 2$. Proceed to (116). The complete Euclidean space controllability of this system is shown similarly to such a kind of controllability of the system (111) in the previous example. Now, using Theorem 2, we obtain the complete Euclidean space controllability of the system (114) at the time instant $t_c = 2$ robustly with respect to $\varepsilon > 0$ for all its sufficiently small values.

6. Conclusions

In this paper, a singularly perturbed linear time-dependent controlled differential system with time delays (multiple point-wise and distributed) in the state and control variables was analyzed. The case where the delays are small of the order of a small positive multiplier ε for a part of the derivatives in the differential equations was treated. The complete Euclidean space controllability of the considered system, robust with respect to the small parameter ε, was studied. This study uses the asymptotic decomposition of the original system into two lower dimensions ε-free subsystems, the slow and fast ones. The slow subsystem is a differential-algebraic delay-free system. This subsystem, subject to a proper assumption, can be converted to a differential equation. The fast subsystem is a differential system with multiple point-wise delays and distributed delays in the state and the control. It was shown that proper kinds of controllability of the slow and fast subsystems yield the complete Euclidean space controllability of the original system valid for all sufficiently small values of ε.

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

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