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Quotient Structures of BCK/BCI-Algebras Induced by Quasi-Valuation Maps

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Abstract: Relations between I-quasi-valuation maps and ideals in BCK/BCI-algebras are investigated. Using the notion of an I-quasi-valuation map of a BCK/BCI-algebra, the quasi-metric space is induced, and several properties are investigated. Relations between the I-quasi-valuation map and the I-valuation map are considered, and conditions for an I-quasi-valuation map to be an I-valuation map are provided. A congruence relation is introduced by using the I-valuation map, and then the quotient structures are established and related properties are investigated. Isomorphic quotient BCK/BCI-algebras are discussed.

Keywords: ideal; I-quasi-valuation map; I-valuation map; quasi-metric

MSC: 06F35; 03G25; 03C05

1. Introduction

BCK/BCI-algebras are an important class of logical algebras introduced by Imai and Iséki (see [1–4]), and have been extensively investigated by several researchers. It is known that the class of BCK-algebras is a proper subclass of BCI-algebras. Song et al. [5] introduced the notion of quasi-valuation maps based on a subalgebra and an ideal in BCK/BCI-algebras, and then they investigated several properties. They provided relations between a quasi-valuation map based on a subalgebra and a quasi-valuation map based on an ideal, and gave a condition for a quasi-valuation map based on an ideal to be a quasi-valuation map based on a subalgebra in BCI-algebras. Using the notion of a quasi-valuation map based on an ideal, they constructed (pseudo) metric spaces, and showed that the binary operation $*$ in BCK-algebras is uniformly continuous.

In this paper, we discuss relations between I-quasi-valuation maps and ideals in BCK/BCI-algebras. Using the notion of an I-quasi-valuation map of a BCK/BCI-algebra, we induce the quasi-metric space, and investigate several properties. We discuss relations between the I-quasi-valuation map and the I-valuation map. We provide conditions for an I-quasi-valuation map to be an I-valuation map. We use I-quasi-valuation maps to introduce a congruence relation, and then we construct the quotient structures and investigate related properties. We establish isomorphic quotient BCK/BCI-algebras.

2. Preliminaries

By a BCI-algebra, we mean a nonempty set X with a binary operation $*$ and a special element 0 satisfying the following axioms:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a BCK-algebra. Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)(x * 0 = x), \tag{1}$$
$$(\forall x, y, z \in X)(x * y = 0 \ \Rightarrow\ (x * z) * (y * z) = 0,\ (z * y) * (z * x) = 0), \tag{2}$$
$$(\forall x, y, z \in X)((x * y) * z = (x * z) * y), \tag{3}$$
$$(\forall x, y, z \in X)(((x * z) * (y * z)) * (x * y) = 0). \tag{4}$$

Any BCI-algebra X satisfies the following condition:

$$(\forall x, y \in X)(0 * (x * y) = (0 * x) * (0 * y)). \tag{5}$$

We can define a partial ordering $\leq$ on X as follows:

$$(\forall x, y \in X)\ (x \leq y \iff x * y = 0)\,.$$

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies the following conditions:

$$0 \in I, \tag{6}$$
$$(\forall x, y \in X)\ (x * y \in I,\ y \in I \ \Rightarrow\ x \in I)\,. \tag{7}$$

An ideal I of a BCI-algebra X is said to be *closed* if

$$(\forall x \in X)(x \in I \ \Rightarrow\ 0 * x \in I). \tag{8}$$

We refer the reader to the books [6,7] for further information regarding BCK/BCI-algebras.

3. Quasi-Valuation Maps on *BCK/BCI*-Algebras

In what follows, let X denote a BCK/BCI-algebra unless otherwise specified.

Definition 1 ([5])**.** *By a quasi-valuation map of X based on an ideal (briefly I-quasi-valuation map of X), we mean a mapping $f : X \to \mathbb{R}$ which satisfies the conditions*

$$f(0) = 0, \tag{9}$$
$$(\forall x, y \in X)\ (f(x) \geq f(x * y) + f(y))\,. \tag{10}$$

The I-quasi-valuation map f is called an I-valuation map of X if

$$(\forall x \in X)(f(x) = 0 \ \Rightarrow\ x = 0). \tag{11}$$

Lemma 1 ([5])**.** *For any I-quasi-valuation map f of X, we have the following assertions:*

(1) *f is order reversing.*
(2) *$f(x * y) + f(y * x) \leq 0$ for all $x, y \in X$.*
(3) *$f(x * y) \geq f(x * z) + f(z * y)$ for all $x, y, z \in X$.*

Corollary 1. *Every quasi-valuation map f of a BCK-algebra X satisfies:*

$$(\forall x \in X)(f(x) \leq 0).$$

Theorem 1. *For any ideal I of X, define a map*

$$f_I : X \to \mathbb{R},\ x \mapsto \begin{cases} 0 & \text{if } x \in I, \\ t & \text{otherwise} \end{cases},$$

where t is a negative number in $\mathbb{R}$. Then, f_I is an I-quasi-valuation map of X. Moreover, f_I is an I-valuation map of X if and only if I is the trivial ideal of X (i.e., $I = \{0\}$).

Proof. Straightforward. □

Theorem 2. *If f is an I-quasi-valuation map of X, then the set*

$$A_f := \{x \in X \mid f(x) \geq 0\}$$

is an ideal of X.

Proof. Obviously $0 \in A_f$. Let $x, y \in X$ be such that $x * y \in A_f$ and $y \in A_f$. Then, $f(x * y) \geq 0$ and $f(y) \geq 0$. It follows from (10) that $f(x) \geq f(x * y) + f(y) \geq 0$ and so that $x \in A_f$. Therefore A_f is an ideal of X. □

Note that if an ideal of a BCI-algebra X is of finite order, then it is a closed ideal of X, and every ideal of a BCK-algebra X is a closed ideal of X (see [6]). Hence, we have the following corollary.

Corollary 2. *Let X be a finite BCI-algebra or a BCK-algebra. If f is an I-quasi-valuation map of X, then the set A_f is a closed ideal of X.*

Theorem 3. *If I is an ideal of X, then $A_{f_I} = I$.*

Proof. We get $A_{f_I} = \{x \in X \mid f_I(x) \geq 0\} = \{x \in X \mid x \in I\} = I$. □

Definition 2. *A real-valued function d on $X \times X$ is called a quasi-metric if it satisfies:*

$$(\forall x, y \in X)\,(d(x, y) \leq 0,\ d(x, x) = 0)\,, \tag{12}$$

$$(\forall x, y \in X)\,(d(x, y) = d(y, x))\,, \tag{13}$$

$$(\forall x, y, z \in X)\,(d(x, z) \geq d(x, y) + d(y, z))\,. \tag{14}$$

The pair (X, d) is called the quasi-metric space.

Given a real-valued function f on X, define a mapping

$$d_f : X \times X \to \mathbb{R},\ (x, y) \mapsto f(x * y) + f(y * x).$$

Theorem 4. *If a real-valued function f on X is an I-quasi-valuation map of X, then d_f is a quasi-metric on $X \times X$.*

The pair (X, d_f) is called the quasi-metric space induced by f.

Proof. Using Lemma 1(2), we have $d_f(x,y) = f(x*y) + f(y*x) \leq 0$ for all $(x,y) \in X \times X$. Obviously, $d_f(x,x) = 0$ and $d_f(x,y) = d_f(y,x)$ for all $x, y \in X$. Using Lemma 1(3), we get

$$\begin{aligned} d_f(x,y) + d_f(y,z) &= (f(x*y) + f(y*x)) + (f(y*z) + f(z*y)) \\ &= (f(x*y) + f(y*z)) + (f(z*y) + f(y*x)) \\ &\leq f(x*z) + f(z*x) = d_f(x,z) \end{aligned}$$

for all $x, y, z \in X$. Therefore d_f is a quasi-metric on X. □

Proposition 1. *Let f be an I-quasi-valuation map of a BCK-algebra X such that*

$$(\forall x \in X)(x \neq 0 \ \Rightarrow \ f(x) \neq 0). \tag{15}$$

Then, the quasi-metric space (X, d_f) induced by f satisfies:

$$(\forall x, y \in X)(d_f(x,y) = 0 \ \Rightarrow \ x = y). \tag{16}$$

Proof. Assume that $d_f(x,y) = 0$ for $x, y \in X$. Then, $f(x*y) + f(y*x) = 0$, and so $f(x*y) = 0$ and $f(y*x) = 0$ by Corollary 1. It follows from (15) that $x*y = 0$ and $y*x = 0$. Hence $x = y$. □

We provide conditions for an I-quasi-valuation map to be an I-valuation map.

Theorem 5. *Let f be an I-quasi-valuation map of a BCI-algebra X such that A_f is a closed ideal of X. If the quasi-metric d_f induced by f satisfies the condition (16), then f is an I-valuation map of X.*

Proof. Assume that f does not satisfy the condition (11). Then, there exists $x \in X$ such that $x \neq 0$ and $f(x) = 0$. Thus, $x \in A_f$, and so $0 * x \in A_f$ since A_f is a closed ideal of X. Hence $f(0*x) \geq 0$, which implies that

$$0 = f(0) \geq f(0*x) + f(x) = f(0*x) \geq 0.$$

Thus, $f(0*x) = 0$, and so $d_f(x,0) = f(x*0) + f(0*x) = f(x) = 0$. It follows from (16) that $x = 0$. Therefore, f is an I-valuation map of X. □

Since every ideal is closed in a BCK-algebra, we have the following corollary.

Corollary 3. *Given an I-quasi-valuation map f of a BCK-algebra X, if the quasi-metric d_f induced by f satisfies the condition (16), then f is an I-valuation map of X.*

Consider the BCI-algebra $(\mathbb{Z}, -, 0)$ and define a map f on $\mathbb{Z}$ as follows:

$$f_k : \mathbb{Z} \to \mathbb{R}, \ x \mapsto \begin{cases} 0 & \text{if } x = 0, \\ k - x & \text{otherwise} \end{cases},$$

where k is a negative integer. For any $x \in \mathbb{Z} \setminus \{0\}$ and $y \in \mathbb{Z}$, we have $f_k(x) = k - x$ and

$$f_k(x-y) + f_k(y) = \begin{cases} k - x & \text{if either } y = 0 \text{ or } y = x, \\ 2k - x & \text{otherwise.} \end{cases}$$

It follows that $f_k(x) \geq f_k(x-y) + f_k(y)$ for all $x, y \in \mathbb{Z}$, and so f_k is an I-quasi-valuation map of $(\mathbb{Z}, -, 0)$. It is clear that the set

$$A_{f_k} = \{x \in \mathbb{Z} \mid f_k(x) \geq 0\} = \{x \in \mathbb{Z} \mid x \leq k\} \cup \{0\}$$

is an ideal of $(\mathbb{Z}, -, 0)$ which is not closed. Using Theorem 4, we know that d_{f_k} is a quasi-metric induced by f_k and satisfies:

$$(\forall x, y \in X)(d_{f_k}(x, y) = 0 \;\Rightarrow\; x = y).$$

However, f_k is not an I-valuation map of $(\mathbb{Z}, -, 0)$ since $f_k(k) = 0$ and $k \neq 0$. This shows that if A_f is not a closed ideal of X, then the conclusion of Theorem 5 is not true.

Proposition 2. *Given an I-quasi-valuation map f of X, the quasi-metric space (X, d_f) satisfies:*

(1) $d_f(x, y) \leq \min\{d_f(x * a, y * a), d_f(a * x), d_f(a * y)\}$,
(2) $d_f(x * y, a * b) \geq d_f(x * y, a * y) + d_f(a * y, a * b)$,

for all $x, y, a, b \in X$.

Proof. Let $x, y, a, b \in X$. Using (4), we have

$$(y * a) * (x * a) \leq y * x \text{ and } (x * a) * (y * a) \leq x * y.$$

Since f is order reversing, it follows that

$$f(y * x) \leq f((y * a) * (x * a)) \text{ and } f(x * y) \leq f((x * a) * (y * a)).$$

Thus,

$$\begin{aligned} d_f(x, y) &= f(x * y) + f(y * x) \\ &\leq f((y * a) * (x * a)) + f((x * a) * (y * a)) \\ &= d_f(x * a, y * a). \end{aligned}$$

Similarly, we get

$$d_f(x, y) \leq d_f(a * x, a * y).$$

Therefore, (1) is valid. Now, using Lemma 1(3) implies that

$$f((x * y) * (a * b)) \geq f((x * y) * (a * y)) + f((a * y) * (a * b))$$

and

$$f((a * b) * (x * y)) \geq f((a * b) * (a * y)) + f((a * y) * (x * y))$$

for all $x, y, a, b \in X$. Hence

$$\begin{aligned} d_f(x * y, a * b) &= f((x * y) * (a * b)) + f((a * b) * (x * y)) \\ &\geq f((x * y) * (a * y)) + f((a * y) * (a * b)) \\ &\quad\; + f((a * b) * (a * y)) + f((a * y) * (x * y)) \\ &\geq f((x * y) * (a * y)) + f((a * y) * (x * y)) \\ &\quad\; + f((a * b) * (a * y)) + f((a * y) * (a * b)) \\ &= d_f(x * y, a * y) + d_f(a * y, a * b) \end{aligned}$$

for all $x, y, a, b \in X$. Therefore, (2) is valid. $\square$

Definition 3. *Let f be an I-quasi-valuation map of X. Define a relation θ_f on X by*

$$(\forall x, y \in X) \left((x, y) \in \theta_f \iff f(x * y) + f(y * x) = 0 \right). \tag{17}$$

Theorem 6. *The relation θ_f on X which is given in (17) is a congruence relation on X.*

Proof. It is clear that θ_f is an equivalence relation on X. Let $x, y, u, v \in X$ be such that $(x, y) \in \theta_f$ and $(u, v) \in \theta_f$. Then, $f(x * y) + f(y * x) = 0$ and $f(u * v) + f(v * u) = 0$. It follows from Proposition 2 that

$$\begin{aligned} & f((x * u) * (y * v)) + f((y * v) * (x * u)) \\ & = d_f(x * u, y * v) \geq d_f(x, y) \\ & = f(x * y) + f(y * x) = 0. \end{aligned}$$

Hence, $f((x * u) * (y * v)) + f((y * v) * (x * u)) = 0$, and so $(x * u, y * v) \in \theta_f$. Therefore, θ_f is a congruence relation on X. $\square$

Definition 4. *Let f be an I-quasi-valuation map of X and θ_f be a congruence relation on X induced by f. Given $x \in X$, the set*

$$x_f := \{y \in X \mid (x, y) \in \theta_f\}$$

is called an equivalence class of x.

Denote by X_f the set of all equivalence classes; that is,

$$X_f := \{x_f \mid x \in X\}.$$

Theorem 7. *Let f be an I-quasi-valuation map of X. Then, $(X_f, \odot, 0_f)$ is a BCK/BCI-algebra where "$\odot$" is the binary operation on X_f which is defined as follows:*

$$(\forall x_f, y_f \in X_f) \left(x_f \odot y_f = (x * y)_f \right).$$

Proof. Let X be a BCI-algebra. The operation $\odot$ is well-defined since f is an I-quasi-valuation map of X. For any $x_f, y_f, z_f \in X_f$, we have

$$\begin{aligned} & ((x_f \odot y_f) \odot (x_f \odot z_f)) \odot (z_f \odot y_f) = (((x * y) * (x * z)) * (z * y))_f = 0_f, \\ & (x_f \odot (x_f \odot y_f)) \odot y_f = ((x * (x * y)) * y)_f = 0_f, \\ & x_f \odot x_f = (x * x)_f = 0_f. \end{aligned}$$

Assume that $x_f \odot y_f = 0_f$ and $y_f \odot x_f = 0_f$. Then, $(x * y)_f = 0_f$ and $(y * x)_f = 0_f$, which imply that $(x * y, 0) \in \theta_f$ and $(y * x, 0) \in \theta_f$. It follows from (1), (5), and (10) that

$$\begin{aligned} 0 &= f((x * y) * 0) + f(0 * (x * y)) \\ &= f(x * y) + f((0 * x) * (0 * y)) \\ &\leq f(x * y) + f(0 * x) - f(0 * y) \end{aligned}$$

and

$$\begin{aligned} 0 &= f((y * x) * 0) + f(0 * (y * x)) \\ &= f(y * x) + f((0 * y) * (0 * x)) \\ &\leq f(y * x) + f(0 * y) - f(0 * x). \end{aligned}$$

Hence, $f(x*y)+f(0*x)-f(0*y)=0$ and $f(y*x)+f(0*y)-f(0*x)=0$, which imply that $f(x*y)+f(y*x)=0$. Hence, $(x,y)\in\theta_f$; that is, $x_f=y_f$. Therefore, $(X_f,\odot,0_f)$ is a BCI-algebra. Moreover, if X is a BCK-algebra, then $0*x=0$ for all $x\in X$. Hence, $0_f\odot x_f=(0*x)_f=0_f$ for all $x_f\in X_f$. Hence, $(X_f,\odot,0_f)$ is a BCK-algebra. □

The following example illustrates Theorem 7.

Example 1. *Let $X=\{0,a,b,c,d\}$ be a set with the $*$-operation given by Table 1.*

Table 1. $*$-operation.

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	0
b	b	b	0	b	0
c	c	c	c	0	c
d	d	d	d	d	0

Then, $(X;*,0)$ is a BCK-*algebra (see [7]), and a real-valued function f on X defined by*

$$f=\begin{pmatrix} 0 & a & b & c & d \\ 0 & -4 & -9 & 0 & -11 \end{pmatrix}$$

is an I-quasi-valuation map of X (see [5]). It is routine to verify that

$$\theta_f=\{(0,0),(a,a),(b,b),(c,c),(d,d),(0,c),(c,0)\},$$

and $X_f=\{0_f,a_f,b_f,d_f\}$ is a BCK-algebra where $0_f=\{0,c\}$, $a_f=\{a\}$, $b_f=\{b\}$, and $d_f=\{d\}$.

Proposition 3. *Given an I-quasi-valuation map f of a BCI-algebra X, if A_f is a closed ideal of X, then $A_f\subseteq 0_f$.*

Proof. Let $x\in A_f$. Then, $0*x\in A_f$ since A_f is a closed ideal, and so $f(x)\geq 0$ and $f(0*x)\geq 0$. It follows from (1) that

$$f(0*x)+f(x*0)=f(0*x)+f(x)\geq 0,$$

and so that $f(0*x)+f(x*0)=0$ by using Lemma 1(2). Hence, $(0,x)\in\theta_f$; that is, $x\in 0_f$. Therefore, $A_f\subseteq 0_f$. □

Corollary 4. *If f is an I-quasi-valuation map of a BCK-algebra X, then $A_f\subseteq 0_f$.*

Proposition 4. *Let f be an I-quasi-valuation map of a BCI-algebra such that*

$$(\forall x\in X)(f(x)\leq 0). \tag{18}$$

Then, $0_f\subseteq A_f$.

Proof. Let $x\in 0_f$. Then, $(0,x)\in\theta_f$, and so

$$f(0*x)+f(x)=f(0*x)+f(x*0)=0.$$

It follows from (18) that $f(0*x)=0=f(x)$. Hence, $x\in A_f$, and therefore $0_f\subseteq A_f$. □

Let I be an ideal of X and let η_I be a relation on X defined as follows:

$$(\forall x, y \in X)((x, y) \in \eta_I \Leftrightarrow x * y \in I,\ y * x \in I).$$

Then, η_I is a congruence relation on X, which is called the ideal congruence relation on X induced by I (see [6]). Denote by X/I the set of all equivalence classes; that is,

$$X/I := \{[x]_I \mid x \in X\},$$

where $[x]_I = \{y \in X \mid (x, y) \in \eta_I\}$. If we define a binary operation $*_I$ on X/I by $[x]_I *_I [y]_I = [x * y]_I$ for all $[x]_I, [y]_I \in X/I$, then $(X, *_I, [0]_I)$ is a BCK/BCI-algebra (see [6]).

Proposition 5. *If f is an I-quasi-valuation map of X, then $\eta_{A_f} \subseteq \theta_f$.*

Proof. Let $x, y \in X$ be such that $(x, y) \in \eta_{A_f}$. Then, $x * y \in A_f$ and $y * x \in A_f$, which imply that $f(x * y) \geq 0$ and $f(y * x) \geq 0$. Hence, $f(x * y) + f(y * x) \geq 0$, and so $f(x * y) + f(y * x) = 0$ by using Lemma 1(2). Thus, $(x, y) \in \theta_f$. This completes the proof. □

Proposition 6. *If f is an I-quasi-valuation map of X such that $A_f = X$, then $\theta_f \subseteq \eta_{A_f}$.*

Proof. Let $x, y \in X$ be such that $(x, y) \in \theta_f$. Then, $f(x * y) + f(y * x) = 0$, and so $f(x * y) = 0$ and $f(y * x) = 0$ by the condition $A_f = X$. It follows that $x * y \in A_f$ and $y * x \in A_f$. Hence, $(x, y) \in \eta_{A_f}$, and therefore $\theta_f \subseteq \eta_{A_f}$. □

Theorem 8. *If I is an ideal of X, then $\eta_I = \theta_{f_I}$.*

Proof. Let $x, y \in X$ be such that $(x, y) \in \eta_I$. Then, $x * y \in I$ and $y * x \in I$. It follows that $f_I(x * y) = 0$ and $f_I(y * x) = 0$. Hence, $f_I(x * y) + f_I(y * x) = 0$, and thus $(x, y) \in \theta_{f_I}$.

Conversely, let $(x, y) \in \theta_{f_I}$ for $x, y \in X$. Then, $f_I(x * y) + f_I(y * x) = 0$, which implies that $f_I(x * y) = 0$ and $f_I(y * x) = 0$ since $f_I(x) \leq 0$ for all $x \in X$. Hence, $x * y \in I$ and $y * x \in I$; that is, $(x, y) \in \eta_I$. This completes the proof. □

Corollary 5. *If f is an I-quasi-valuation map of X, then $\eta_{A_f} = \theta_{f_{A_f}}$.*

Theorem 9. *For any two different I-quasi-valuation maps f and g of X, if $0_f = 0_g$, then θ_f and θ_g coincide, and so $X_f = X_g$.*

Proof. Let $x, y \in X$ be such that $(x, y) \in \theta_f$. Then, $(x * y, 0) = (x * y, y * y) \in \theta_f$, and so $x * y \in 0_f$. Similarly, we have $y * x \in 0_f$. It follows from $0_f = 0_g$ that $x_g \odot y_g = (x * y)_g = 0_g$ and $y_g \odot x_g = (y * x)_g = 0_g$. Hence, $x_g = y_g$, and so $(x, y) \in \theta_g$. Similarly, we can verify that if $(x, y) \in \theta_g$, then $(x, y) \in \theta_f$. Therefore, θ_f and θ_g coincide and so $X_f = X_g$. □

Theorem 10. *Let I be an ideal of X and let f be an I-quasi-valuation map of X such that $0_f \subseteq I$. If we denote*

$$I_f := \{x_f \mid x \in I\},$$

then the following assertions are valid.

(1) $(\forall x \in X)(x \in I \Leftrightarrow x_f \in I_f)$.
(2) *I_f is an ideal of X_f.*

Proof. (1) It is clear that if $x \in I$, then $x_f \in I_f$. Let $x \in X$ be such that $x_f \in I_f$. Then, there exists $y \in I$ such that $x_f = y_f$. Hence, $(x, y) \in \theta_f$, and so $(x * y, 0) = (x * y, y * y) \in \theta_f$. It follows that $x * y \in 0_f \subseteq I$ and so that $x \in I$.

(2) Clearly, $0_f \in I_f$ since $0 \in I$. Let $x, y \in X$ be such that $x_f \odot y_f \in I_f$ and $y_f \in I_f$. Then, $(x * y)_f = x_f \odot y_f \in I_f$, and so $x * y \in I$ and $y \in I$ by (1). Since I is an ideal of X, it follows that $x \in I$ and so that $x_f \in I_f$. Therefore, I_f is an ideal of X_f. □

Theorem 11. *For any I-quasi-valuation map f of X, if J^* is an ideal of X_f, then the set*

$$J := \{x \in X \mid x_f \in J^*\}$$

is an ideal of X containing 0_f.

Proof. It is obvious that $0 \in 0_f \subseteq J$. Let $x, y \in X$ be such that $x * y \in J$ and $y \in J$. Then, $y_f \in J^*$ and $x_f \odot y_f = (x * y)_f \in J^*$. Since J^* is an ideal of X_f, it follows that $x_f \in J^*$ (i.e., $x \in J$). Therefore, J is an ideal of X. □

Let $\mathcal{I}(X_f)$ denote the set of all ideals of X_f, and let $\mathcal{I}(X, f)$ denote the set of all ideals of X containing 0_f. Then, there exists a bijection between $\mathcal{I}(X_f)$ and $\mathcal{I}(X, f)$; that is, $\psi : \mathcal{I}(X_f) \to \mathcal{I}(X, f),\ I \mapsto I_f$ is a bijection.

Proposition 7. *Let $\varphi : X \to Y$ be a homomorphism of BCK/BCI-algebras. If f is an I-quasi-valuation map of Y, then the composition $f \circ \varphi$ of f and φ is an I-quasi-valuation map of X.*

Proof. We have $(f \circ \varphi)(0) = f(\varphi(0)) = f(0) = 0$. For any $x, y \in X$, we get

$$\begin{aligned}(f \circ \varphi)(x) &= f(\varphi(x)) \\ &\geq f(\varphi(x) * \varphi(y)) + f(\varphi(y)) \\ &= f(\varphi(x * y)) + f(\varphi(y)) \\ &= (f \circ \varphi)(x * y) + (f \circ \varphi)(y).\end{aligned}$$

Hence, $f \circ \varphi$ is an I-quasi-valuation map of X. □

Theorem 12. *Let $\varphi : X \to Y$ be an onto homomorphism of BCK/BCI-algebras. If f is an I-quasi-valuation map of Y, then $X_{f\circ\varphi}$ and Y_f are isomorphic.*

Proof. Define a map $\zeta : X_{f\circ\varphi} \to Y_f$ by $\zeta(x_{f\circ\varphi}) = \varphi(x)_f$ for all $x \in X$. If we let $x_{f\circ\varphi} = a_{f\circ\varphi}$ for $a, x \in X$, then

$$\begin{aligned}0 &= (f \circ \varphi)(x * a) + (f \circ \varphi)(a * x) \\ &= f(\varphi(x * a)) + f(\varphi(a * x)) \\ &= f(\varphi(x) + \varphi(a)) + f(\varphi(a) * \varphi(x)),\end{aligned}$$

which implies that $\zeta(x_{f\circ\varphi}) = \varphi(x)_f = \varphi(a)_f = \zeta(a_{f\circ\varphi})$. Hence, ζ is well-defined. For any $a, x \in X$, we have

$$\begin{aligned}\zeta(x_{f\circ\varphi} \odot a_{f\circ\varphi}) &= \zeta((x * a)_{f\circ\varphi}) = \varphi(x * a)_f \\ &= (\varphi(x) * \varphi(a))_f = \varphi(x)_f \odot \varphi(a)_f \\ &= \zeta(x_{f\circ\varphi}) \odot \zeta(a_{f\circ\varphi}).\end{aligned}$$

This shows that ζ is a homomorphism. For any y_f in Y_f, there exists $x \in X$ such that $\varphi(x) = y$, since φ is surjective. It follows that $\zeta(x_{f\circ\varphi}) = \varphi(x)_f = y_f$. Thus, ζ is surjective. Suppose that $\zeta(x_{f\circ\varphi}) = \zeta(a_{f\circ\varphi})$ for any $x_{f\circ\varphi}, a_{f\circ\varphi} \in X_{f\circ\varphi}$. Then, $\varphi(x)_f = \varphi(a)_f$, and so

$$\begin{aligned}(f\circ\varphi)(x*a) + (f\circ\varphi)(a*x) &= f(\varphi(x*a)) + f(\varphi(a*x)) \\ &= f(\varphi(x)*\varphi(a)) + f(\varphi(a)*\varphi(x)) = 0.\end{aligned}$$

Hence, $x_{f\circ\varphi} = a_{f\circ\varphi}$. This shows that ζ is injective, and therefore $X_{f\circ\varphi}$ and Y_f are isomorphic. □

Theorem 13. *Given an I-quasi-valuation map f of X, the following assertions are valid.*

(1) *The map $\pi : X \to X_f,\ x \mapsto x_f$ is an onto homomorphism.*
(2) *For each I-quasi-valuation map g^* of X_f, there exist an I-quasi-valuation map g of X such that $g = g^* \circ \pi$.*
(3) *If $A_f = X$, then the map*

$$f^* : X_f \to \mathbb{R},\ x_f \mapsto f(x)$$

is an I-quasi-valuation map of X_f.

Proof. (1) and (2) are straightforward.

(3) Assume that $x_f = y_f$ for $x, y \in X$. Then, $f(x*y) + f(y*x) = 0$, which implies from the assumption that $f(x*y) = 0 = f(y*x)$. Since $x*(x*y) \le y$ for all $x, y \in X$, we get $f(y) \le f(x*(x*y))$. It follows that

$$f(x) \ge f(x*(x*y)) + f(x*y) \ge f(x*y) + f(y) \ge f(y).$$

Similarly, we show that $f(x) \le f(y)$, and so $f(x) = f(y)$; that is, $f^*(x_f) = f^*(y_f)$. Therefore, f^* is well-defined. Now, we have $f^*(0_f) = f(0) = 0$ and

$$f^*(x_f) = f(x) \ge f(x*y) + f(y) = f^*((x*y)_f) + f^*(y_f) = f^*(x_f \odot y_f) + f^*(y_f).$$

Therefore, f^* is an I-quasi-valuation map of X_f. □

4. Conclusions

Quasi-valuation maps on BCK/BCI-algebras were studied by Song et al. in [5]. The aim of this paper was to study the quotient structures of BCK/BCI-algebras induced by quasi-valuation maps. We have described relations between I-quasi-valuation maps and ideals in BCK/BCI-algebras. We have induced the quasi-metric space by using an I-quasi-valuation map of a BCK/BCI-algebra, and have investigated several properties. We have considered relations between the I-quasi-valuation map and the I-valuation map, and have provided conditions for an I-quasi-valuation map to be an I-valuation map. We have used I-quasi-valuation maps to introduce a congruence relation, and then constructed the quotient structures with related properties. We have established isomorphic quotient BCK/BCI-algebras. In the future, from a purely mathematical standpoint, we will apply the concepts and results in this article to related algebraic structures, such as BCC-algebras (see [8]), pseudo BCI-algebras (see [9,10]), and so on. From an application standpoint, we will try to find the possibility of extending our proposed approach to some decision-making problem, mathematical programming, medical diagnosis, etc.

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