Axioms **2015**, *4*, 345-364; doi:10.3390/axioms4030345

OPEN ACCESS

axioms

ISSN 2075-1680

www.mdpi.com/journal/axioms

Article

Almost Periodic Solutions of Nonlinear Volterra Difference Equations with Unbounded Delay

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Academic Editor: Constantin Corduneanu

Received: 27 June 2015 / Accepted: 18 August 2015 / Published: 24 August 2015

Abstract: In order to obtain the conditions for the existence of periodic and almost periodic solutions of Volterra difference equations, $x(n+1) = f(n, x(n)) + \sum_{s=-\infty}^{n} F(n, s, x(n+s), x(n))$, we consider certain stability properties, which are referred to as (K, ρ)-weakly uniformly-asymptotic stability and (K, ρ)-uniformly asymptotic stability. Moreover, we discuss the relationship between the ρ-separation condition and the uniformly-asymptotic stability property in the ρ sense.

Keywords: almost periodic solutions; Volterra difference equations; (K, ρ)-relatively weakly uniformly-asymptotically stable; ρ-separation conditions

AMS (MOS) 2000 Subject classifications: 39A10, 39A11

1. Introduction

For ordinary and functional differential equations, the existence of almost periodic solutions of almost periodic systems has been studied by many authors. One of the most popular methods is to assume certain stability properties [1–8]. Song and Tian [9] showed the existence of periodic and almost periodic solutions for nonlinear Volterra difference equations by means of the (K, ρ)-stability condition.

Their results were extended to discrete Volterra equations by Hamaya [3]. For the existence theorem of almost periodic solutions in ordinary differential equations, Sell [10] introduced a new stability concept referred to as the weakly uniformly-asymptotic stability. This stability property is weaker than the uniformly-asymptotic stability (*cf.* [8]). The existence of almost periodic solutions of ordinary difference equation by using globally quasi-uniformly asymptotic stability has been recently studied [11].

In this paper, we discuss the relationship between weakly uniformly-asymptotic stability and uniformly-asymptotic stability of periodic and almost periodic Volterra difference equations. We also show that for periodic Volterra difference equations, (K, ρ)-weakly uniformly-asymptotic stability and (K, ρ)-uniformly-asymptotic stability are equivalent. Moreover, we obtain the conditions for the existence of almost periodic solutions of Volterra difference equations by using this (K, ρ)-weakly uniformly-asymptotically-stable in the hull. The relationship between our weakly uniformly-asymptotic stability and globally quasi-uniformly-asymptotic stability described in [11] is very complicated; however, the definition of our stability is clearer and simpler than that in [11]. In the next section, as an application, we show the existence of almost periodic solutions for a Ricker-type Volterra difference equation with infinite delay by using the technique of an invariant set and luxury Lyapunov functionals. For the finite delay case, Xu [12] showed sufficient conditions for determining the invariant and attracting sets and the globally uniformly-asymptotic stability of Volterra difference equations, as well as providing useful examples to illustrate the results obtained above. Finally, we consider the relationship between the ρ-separation condition and (K, ρ)-uniformly-asymptotic stability property. It can be seen that the results of our theorem hold for the integrodifferential equations described in [3–5].

Let R^m denote Euclidean m-space; $\mathbf{Z}$ is the set of integers; $\mathbf{Z}^+$ is the set of nonnegative integers; and $|\cdot|$ will denote the Euclidean norm in R^m. For any interval $I \subset \mathbf{Z}$, we denote by $BS(I)$ the set of all bounded functions mapping I into R^m and set $|\phi|_I = \sup\{|\phi(s)| : s \in I\}$.

Now, for any function $x : (-\infty, a) \to R^m$ and $n < a$, define a function $x_n : \mathbf{Z}^- = \{s|s \in \mathbf{Z}, -\infty < s \leq 0\} \to R^m$ by $x_n(s) = x(n+s)$ for $s \in \mathbf{Z}^-$. Let BS be a real linear space of functions mapping $\mathbf{Z}^-$ into R^m with sup-norm:

$$BS = \{\phi| \ \phi : \mathbf{Z}^- \to R^m \quad \text{with} \quad |\phi| = \sup_{s \in \mathbf{Z}^-} |\phi(s)| < \infty\}$$

We introduce an almost periodic function $f(n,x) : \mathbf{Z} \times D \to R^m$, where D is an open set in R^m.

Definition 1. *$f(n,x)$ is said to be almost periodic in n uniformly for $x \in D$, if for any $\epsilon > 0$ and any compact set K in D, there exists a positive integer $L^*(\epsilon, K)$, such that any interval of length $L^*(\epsilon, K)$ contains an integer τ for which:*

$$|f(n+\tau, x) - f(n,x)| \leq \epsilon$$

for all $n \in \mathbf{Z}$ and all $x \in K$. Such a number τ in the above inequality is called an ϵ-translation number of $f(n,x)$.

In order to formulate a property of almost periodic functions (this is equivalent to Definition 1), we discuss the concept of the normality of almost periodic functions. Namely, let $f(n,x)$ be almost periodic in n uniformly for $x \in D$. Then, for any sequence $\{h_k'\} \subset \mathbf{Z}$, there exist a subsequence $\{h_k\}$ of $\{h_k'\}$ and a function $g(n,x)$, such that:

$$f(n+h_k,x) \to g(n,x) \tag{1}$$

uniformly on $\mathbf{Z} \times K$ as $k \to \infty$, where K is a compact set in D. There are many properties of the discrete almost periodic functions [13], which are corresponding properties of the continuous almost periodic functions $f(t,x) \in C(R \times D, R^m)$ [2,8]. We shall denote by $T(f)$ the function space consisting of all translates of f, that is $f_\tau \in T(f)$, where:

$$f_\tau(n,x) = f(n+\tau,x), \qquad \tau \in \mathbf{Z} \tag{2}$$

Let $H(f)$ denote the uniform closure of $T(f)$ in the sense of (2). $H(f)$ is called the hull of f. In particular, we denote by $\Omega(f)$ the set of all limit functions $g \in H(f)$, such that for some sequence $\{n_k\}$, $n_k \to \infty$ as $k \to \infty$ and $f(n+n_k,x) \to g(n,x)$ uniformly on $\mathbf{Z} \times S$ for any compact subset S in R^m. By (1), if $f : \mathbf{Z} \times D \to R^m$ is almost periodic in n uniformly for $x \in D$, so is a function in $\Omega(f)$. The following concept of asymptotic almost periodicity was introduced by Frechet in the case of continuous functions (*cf.* [2,8]).

Definition 2. *$u(n)$ is said to be asymptotically almost periodic if it is a sum of an almost periodic function $p(n)$ and a function $q(n)$ defined on $I^* = [a,\infty) \subset \mathbf{Z}^+ = \{l \in \mathbf{Z} |\ 0 \le l < +\infty\}$, which tends to zero as $n \to \infty$, that is,*

$$u(n) = p(n) + q(n)$$

$u(n)$ is asymptotically almost periodic if and only if for any sequence $\{n_k\}$, such that $n_k \to \infty$ as $k \to \infty$, there exists a subsequence $\{n_{k_j}\}$ for which $u(n+n_{k_j})$ converges uniformly on $a \le n < \infty$.

2. Preliminaries

We consider a system of Volterra difference equations:

$$x(n+1) = f(n,x(n)) + \sum_{s=-\infty}^{0} F(n,s,x(n+s),x(n)) \tag{3}$$

where $f : \mathbf{Z} \times R^m \to R^m$ is continuous in the second variable $x \in R^m$ and $F : \mathbf{Z} \times \mathbf{Z}^- \times R^m \times R^m$ is continuous for $x \in R^m$ and $y \in R^m$.

We impose the following assumptions on Equation (3):

(H1) $f(n,x)$ and $F(n,s,x,y)$ are ω-periodic functions, such that there is an $\omega > 0$, such that $f(n+\omega,x) = f(n,x)$ for all $n \in \mathbf{Z}$, $x \in R^m$ and $F(n+\omega,s,x,y) = F(n,s,x,y)$ for all $n \in \mathbf{Z}$, $s \in \mathbf{Z}^-$, $x \in R^m$ and $y \in R^m$.

(H2) $f(n,x)$ is almost periodic in n uniformly for $x \in R^m$, and $F(n,s,x,y)$ is almost periodic in n uniformly for $(s,x,y) \in K^*$, that is for any $\epsilon > 0$ and any compact set K^*, there exists an integer $L^* = L^*(\epsilon,K) > 0$, such that any interval of length L^* contains a τ for which:

$$|F(n+\tau,s,x,y)-F(n,s,x,y)|\leq\epsilon$$

for all $n\in\mathbf{Z}$ and all $(s,x,y)\in K^*$.

(H3) For any $\epsilon>0$ and any $r>0$, there exists an $S=S(\epsilon,r)>0$, such that:

$$\sum_{s=-\infty}^{-S}|F(n,s,x(n+s),x(n))|\leq\epsilon$$

for all $n\in\mathbf{Z}$, whenever $|x(\sigma)|\leq r$ for all $\sigma\leq n$.

(H4) Equation (3) has a bounded unique solution $u(n)$ defined on $\mathbf{Z}^+$, which passes through $(0,u_0)$, that is $\sup_{n\geq 0}|u(n)|<\infty$ and $u_0\in BS$.

Now, we introduce ρ-stability properties with respect to the compact set K.

Let K be the compact set in R^m, such that $u(n)\in K$ for all $n\in\mathbf{Z}$, where $u(n)=\phi^0(n)$ for $n\leq 0$. For any $\theta,\psi\in BS$, we set:

$$\rho(\theta,\psi) \;=\; \sum_{j=1}^{\infty}\rho_j(\theta,\psi)/[2^j(1+\rho_j(\theta,\psi))]$$

where:

$$\rho_j(\theta,\psi) \;=\; \sup_{-j\leq s\leq 0}|\theta(s)-\psi(s)|$$

Clearly, $\rho(\theta_n,\theta)\to 0$ as $n\to\infty$ if and only if $\theta_n(s)\to\theta(s)$ uniformly on any compact subset of $\mathbf{Z}^-$ as $n\to\infty$.

We denote by (BS,ρ) the space of bounded functions $\phi:\mathbf{Z}^-\to R^m$ with ρ.

In what follows, we need the following 10 definitions of stability.

Definition 3. *The bounded solution $u(n)$ of Equation (3) is said to be:*

(i) (K, ρ)-uniformly stable (in short, (K, ρ)-US) if for any $\epsilon>0$, there exists a $\delta(\epsilon)>0$, such that if $n_0\geq 0$, $\rho(x_{n_0},u_{n_0})<\delta(\epsilon)$, then $\rho(x_n,u_n)<\epsilon$ for all $n\geq n_0$, where $x(n)$ is a solution of (3) through (n_0,ϕ), such that $x_{n_0}(s)=\phi(s)\in K$ for all $s\leq 0$. In the case above where $\delta(\epsilon)$ depends on the initial time $n_0\geq 0$, this only gives the definition of the (K, ρ)-stable of $u(n)$ (in short, (K, ρ)-S).

(ii) (K, ρ)-equi-asymptotically stable (in short, (K, ρ)-EAS) if it is (K, ρ)-S and for any $\epsilon>0$, there exists a $\delta_0(n_0)>0$ and a $T(n_0,\epsilon)>0$, such that if $n_0\geq 0$, $\rho(x_{n_0},u_{n_0})<\delta_0(n_0)$, then $\rho(x_n,u_n)<\epsilon$ for all $n\geq n_0+T(n_0,\epsilon)$, where $x(n)$ is a solution of (3) through (n_0,ϕ), such that $x_{n_0}(s)=\phi(s)\in K$ for all $s\leq 0$.

(iii) (K, ρ)-weakly uniformly-asymptotically stable (in short, (K, ρ)-WUAS) if it is (K, ρ)-US and there exists a $\delta_0>0$, such that if $n_0\geq 0$, $\rho(x_{n_0},u_{n_0})<\delta_0$, then $\rho(x_n,u_n)\to 0$ as $n\to\infty$, where $x(n)$ is a solution of (3) through (n_0,ϕ), such that $x_{n_0}(s)=\phi(s)\in K$ for all $s\leq 0$.

(iv) (K, ρ)-uniformly-asymptotically stable (in short, (K, ρ)-UAS) if it is (K, ρ)-US and is (K, ρ)-quasi-uniformly-asymptotically stable, that is, if the δ_0 and the T in the above (iii) are independent of n_0: for any $\epsilon>0$ there exists a $\delta_0>0$ and a $T(\epsilon)>0$, such that if $n_0\geq 0$, $\rho(x_{n_0},u_{n_0})<\delta_0$, then $\rho(x_n,u_n)<\epsilon$ for all $n\geq n_0+T(\epsilon)$, where $x(n)$ is a solution of (3) through (n_0,ϕ), such that $x_{n_0}(s)=\phi(s)\in K$ for all $s\leq 0$.

(v) (K, ρ)-globally equi-asymptotically-stable (in short, (K, ρ)-GEAS) if it is (K, ρ)-S and for any $\epsilon > 0$ and any $\alpha > 0$, there exists a $T(n_0, \epsilon, \alpha) > 0$, such that if $n_0 \geq 0$, $\rho(x_{n_0}, u_{n_0}) < \alpha$, then $\rho(x_n, u_n) < \epsilon$ for all $n \geq n_0 + T(n_0, \epsilon, \alpha)$, where $x(n)$ is a solution of (3) through (n_0, ϕ), such that $x_{n_0}(s) = \phi(s) \in K$ for all $s \leq 0$.

(vi) (K, ρ)-globally weakly uniformly-asymptotically stable (in short, (K, ρ)-GWUAS) if it is (K, ρ)-US and $\rho(x_n, u_n) \to 0$ as $n \to \infty$, where $x(n)$ is a solution of (3) through (n_0, ϕ), such that $x_{n_0}(s) = \phi(s) \in K$ for all $s \leq 0$.

(vii) (K, ρ)-globally uniformly-asymptotically stable (in short, (K, ρ)-GUAS) if it is (K, ρ)-US and is (K, ρ)-globally quasi-uniformly-asymptotically stable, that is, if the T in the above (vi) are independent of n_0: for any $\epsilon > 0$ and $\alpha > 0$, there exists a $T(\epsilon, \alpha) > 0$, such that if $n_0 \geq 0$, $\rho(x_{n_0}, u_{n_0}) < \alpha$, then $\rho(x_n, u_n) < \epsilon$ for all $n \geq n_0 + T(\epsilon, \alpha)$, where $x(n)$ is a solution of (3) through (n_0, ϕ), such that $x_{n_0}(s) = \phi(s) \in K$ for all $s \leq 0$.

(viii) (K, ρ)-totally stable (in short, (K, ρ)-TS) if for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ and such that if $n_0 \geq 0$, $\rho(x_{n_0}, u_{n_0}) < \delta(\epsilon)$ and $h \in BS([n_0, \infty))$, which satisfies $|h|_{[n_0,\infty)} < \delta(\epsilon)$, then $\rho(x_n, u_n) < \epsilon$ for all $n \geq n_0$, where $x(n)$ is a solution of:

$$x(n+1) = f(n, x(n)) + \sum_{s=-\infty}^{0} F(n, s, x(n+s), x(n)) + h(n)$$

through (n_0, ϕ), such that $x_{n_0}(s) = \phi(s) \in K$ for all $s \leq 0$. In the case where $h(n) \equiv 0$, this gives the definition of the (K, ρ)-US of $u(n)$.

(ix) (K, ρ)-attracting in $\Omega(f, F)$ (*cf.* [12], in short, (K, ρ)-A in $\Omega(f, F)$), if there exists a $\delta_0 > 0$, such that if $n_0 \geq 0$ and any $(v, g, G) \in \Omega(u, f, F)$, $\rho(x_{n_0}, v_{n_0}) < \delta_0$, then $\rho(x_n, v_n) \to 0$ as $n \to \infty$, where $x(n)$ is a solution of:

$$x(n+1) = g(n, x(n)) + \sum_{s=-\infty}^{0} G(n, s, x(n+s), x(n)) \tag{4}$$

through (n_0, ψ), such that $x_{n_0}(s) = \psi(s) \in K$ for all $s \leq 0$.

(x) (K, ρ)-weakly uniformly-asymptotically stable in $\Omega(f, F)$ (in short, (K, ρ)-WUAS in $\Omega(f, F)$), if it is (K, ρ)-US in $\Omega(f, F)$, that is if for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, such that if $n_0 \geq 0$ and any $(v, g, G) \in \Omega(u, f, F)$, $\rho(x_{n_0}, v_{n_0}) < \delta(\epsilon)$, then $\rho(x_n, v_n) < \epsilon$ for all $n \geq n_0$, where $x(n)$ is a solution of (4) through (n_0, ψ), such that $x_{n_0}(s) = \psi(s) \in K$ for all $s \leq 0$ and (K, ρ)-A in $\Omega(f, F)$.

When we restrict the solutions x to the ones in K, *i.e.*, $x(n) \in K$ for all $n \geq n_0$, then we can say that $u(n)$ is (K, ρ)-relatively weakly uniformly-asymptotically stable in $\Omega(f, F)$ (*i.e.*, (K, ρ)-RWUAS in $\Omega(f, F)$, and so on). For (iii) and (iv) in Definition 3, (K, ρ)-WUAS is weaker than (K, ρ)-UAS, as shown in Example 3.1 in [8].

3. Stability of Bounded Solutions in Periodic and Almost Periodic Systems

Theorem 1. *Under the Assumptions (H3) and (H4), if the bounded solution $u(n)$ of Equation (3) is (K, ρ)-WUAS, then it is (K, ρ)-EAS.*

Proof. Since solution $u(n)$ of Equation (3) is (K, ρ)-US, $u(n)$ is (K, ρ)-S. Suppose that there is no such T in (ii) of Definition 3. Then, there exist some $\epsilon > 0, n_0 \geq 0$ and sequences $\{x^k\}, \{n_k\}$, such that $\rho(u_{n_0}, x^k_{n_0}) \leq \delta_0, n_k \to \infty$ as $k \to \infty$ and:

$$\rho(u_{n_k}, x^k_{n_k}) \geq \epsilon \tag{5}$$

where $x^k(n)$ is a solution of Equation (3) through $(n_0, x^k_{n_0})$. On any interval $[n_0, n_0 + l]$, $l \in \mathbf{Z}^+$, the sequence $\{x^k(n)\}$ is uniformly bounded, since $x^k_{n_0}(s) \to x_{n_0}(s)$ uniformly on any compact set in $\{s \in \mathbf{Z}^- | -\infty < s \leq n_0\}$ as $k \to \infty$ if necessary taking a subsequence of $\{x^k\}$, and hence, we can find a solution $x(n)$ through (n_0, x_{n_0}) of (3) defined for all $n \geq n_0$ by Assumptions (H3) and (H4), where $\rho(x_{n_0}, u_{n_0}) \leq \delta_0$. Moreover, there exists a subsequence of $\{x^k(n)\}$, such that $\{x^{k_j}(n)\}$ tends to $x(n)$ as $j \to \infty$ through (n_0, x_{n_0}) uniformly on any compact interval. Since, for every solution $\{x(n)\}$, $\rho(u_n, x_n) \to 0$ as $n \to \infty$, we have at some $n_1 \geq n_0$:

$$\rho(u_{n_1}, x_{n_1}) < \frac{1}{2}\delta(\epsilon) \tag{6}$$

where $\delta(\epsilon)$ is the one for (K, ρ)-US in (i) of Definition 3. Denoting by $\{x^k(n)\}$ through $(n_0, x^k_{n_0})$ the subsequence again, if k is sufficiently large, we have:

$$\rho(x^k_{n_1}, x_{n_1}) < \frac{1}{2}\delta(\epsilon) \tag{7}$$

From (7) and (6), it follows that $\rho(u_{n_1}, x^k_{n_1}) < \delta(\epsilon)$. Therefore, by the (K, ρ)-US of $u(n)$, we have:

$$\rho(x^k_n, u_n) < \epsilon$$

for all $n \geq n_1$, which contradicts (5). This proves the theorem.

For the periodic system, we have the following theorem.

Theorem 2. *Under Assumptions (H1), (H3) and (H4), if the bounded solution $u(n)$ of Equation (3) is (K, ρ)-WUAS, then it is (K, ρ)-UAS.*

Proof. Since $u(n)$ is (K, ρ)-US, there exists a $\delta_0^* > 0$, such that n_0 is a positive integer and $\rho(x_{n_0}, u_{n_0}) \leq \delta_0^*$ implies $\rho(u_n, x_n) < \frac{\delta_0}{2}$ for all $n \geq n_0$, where δ_0 is the one in (iii) of Definition 3. Suppose that for this δ_0^*, solution $u(n)$ is not (K, ρ)-UAS. Then, for some $\epsilon > 0$, there exist sequences $\{k_j\}, \{x^{k_j}\}$ and $\{\tau_{k_j}\}$, such that $k_j \to \infty$, $\tau_{k_j} \to \infty$ as $j \to \infty$, where k_j is a positive integer, and:

$$\rho(u_{k_j\omega}, x^{k_j}_{k_j\omega}) < \frac{\delta_0}{2} \tag{8}$$

and:

$$\rho(u_{k_j\omega+\tau_{k_j}}, x^{k_j}_{k_j\omega+\tau_{k_j}}) \geq \epsilon \tag{9}$$

where $x^{k_j}(n)$ is a solution of (3) through $(k_{j\omega}, x^{k_j}_{k_j\omega})$. Clearly, by (H4), $u(n)$ is a bounded solution of (3) passing through $(k_j\omega, u_{k_j\omega})$, and hence, there is a subsequence $\{m_j\}$ of $\{k_j\}$ and u_{n_0}, such that $m_j \to \infty$

monotonically as $j \to \infty$ and $u_{m_j\omega} \to u_{n_0}$ as $j \to \infty$. Then, there exists an integer $p > 0$, such that if $j \geq p$, we have $\rho(u_{m_j\omega}, u_{n_0}) < \frac{\delta_0}{4}$. Thus, for any $j \geq p$, we have:

$$\rho(u_{m_j\omega}, u_{m_p\omega}) < \frac{\delta_0}{2} \tag{10}$$

From (8) with $k_j = m_j$ and (10), it follows that:

$$\rho(x^{m_j}_{m_j\omega}, u_{m_p\omega}) < \delta_0$$

By Theorem 1, there exists a $T(m_p\omega, \frac{\epsilon}{4}) > 0$, such that:

$$\rho(x_n, u_n) < \frac{\epsilon}{4}$$

for all $n \geq m_p\omega + T(m_p\omega, \frac{\epsilon}{4})$ and where $x(n)$ is a solution of (3) through $(m_p\omega, u_{m_j\omega})$ and:

$$\rho(\tilde{x}_n, u_n) < \frac{\epsilon}{4}$$

for all $n \geq m_p\omega + T(m_p\omega, \frac{\epsilon}{4})$ and where $\tilde{x}(n)$ is a solution of (3) through $(m_p\omega, x^{m_j}_{m_p\omega})$. This implies that:

$$\rho(\tilde{x}_n, x_n) < \frac{\epsilon}{2} \tag{11}$$

for all $n \geq m_p\omega + T(m_p\omega, \frac{\epsilon}{4})$. Since ω is the period and m_j, m_p are integers, it follows from (11) that for any $j \geq p$:

$$\rho(x_n, u_n) < \rho(\tilde{x}_n, x_n) + \rho(\tilde{x}_n, u_n) < \epsilon$$

for all $n \geq m_j\omega + T(m_p\omega, \frac{\epsilon}{4})$. This contradicts (9), because $T(m_p\omega, \frac{\epsilon}{4})$ depends only on ϵ. This completes the proof.

The following lemma is needed for the proofs of Theorems 3, 5 and 8.

Lemma 1. *When* $(v, g, G) \in \Omega(u, f, F)$*,* $v(n)$ *is a solution defined on* $\mathbf{Z}$ *of:*

$$x(n+1) = g(n, x(n)) + \sum_{s=-\infty}^{0} G(n, s, x(n+s), x(n))$$

and $v(n) \in K$ *for all* $n \in \mathbf{Z}$*.*

Proof. Since $(v, g, G) \in \Omega(u, f, F)$, there exists a sequence $\{n_k\}$, $n_k \to \infty$ as $k \to \infty$, such that:

$$f(n + n_k, x) \to g(n, x)$$

uniformly on $\mathbf{Z} \times K$ for any compact set $K \subset R^m$:

$$F(n + n_k, s, x, y) \to G(n, s, x, y)$$

uniformly on $\mathbf{Z} \times \mathbf{Z}^* \times K \times K$ for any compact subset $\mathbf{Z}^*$ in $\mathbf{Z}^-$ and:

$$u(n + n_k) \to v(n)$$

uniformly on any compact subset in $\mathbf{Z}$ as $k \to \infty$. Set $u^k(n) = u(n + n_k)$. Then, $u^k(n)$ is a solution defined for $n \geq -n_k$ of:

$$x(n+1) = f(n+n_k, x(n)) + \sum_{s=-\infty}^{0} F(n+n_k, s, x(n+s), x(n)) \tag{12}$$

through $(0, u^k_{n_k})$, $u^k_{n_k}(s) \in K$, $s \leq 0$. There exists an $r > 0$, such that $|u^k(n)| \leq r$ and $|v(n)| \leq r$ for all $n \in \mathbf{Z}$, $k \geq 1$. Then, by Assumption (H3), for this r and any $\epsilon > 0$, there exists an integer $S = S(\epsilon, r) > 0$, such that:

$$\sum_{s=-\infty}^{-S} |F(n, s, u^k(n+s), u^k(n))| \leq \epsilon \quad \text{and} \quad \sum_{s=-\infty}^{-S} |G(n, s, v(n+s), v(n))| \leq \epsilon$$

Then, we have:

$$\begin{aligned}
& |\sum_{s=-\infty}^{0} F(n, s, u^k(n+s), u^k(n)) - \sum_{s=-\infty}^{0} G(n, s, v(n+s), v(n))| \\
\leq & \sum_{s=-\infty}^{-S} |F(n, s, u^k(n+s), u^k(n))| + \sum_{s=-\infty}^{-S} |G(n, s, v(n+s), v(n))| \\
+ & \sum_{s=-S}^{0} |F(n, s, u^k(n+s), u^k(n)) - G(n, s, v(n+s), v(n))| \\
\leq & 2\epsilon + \sum_{s=-S}^{0} |F(n, s, u^k(n+s), u^k(n)) - G(n, s, v(n+s), v(n))|
\end{aligned}$$

Since $F(n, s, x, y)$ and $G(n, s, x, y)$ are continuous for x, y and $u^k(s)$ converges to $v(s)$ on discrete interval $\{s \in \mathbf{Z}^-|, -S \leq s \leq 0\}$ as $k \to \infty$, there exists an integer $k_0(\epsilon) > 0$, such that:

$$\sum_{s=-S}^{0} |F(n, s, u^k(n+s), u^k(n)) - G(n, s, v(n+s), v(n))| \leq \epsilon$$

when $k \geq k_0(\epsilon)$. Thus, we have:

$$\sum_{s=-\infty}^{0} F(n, s, u^k(n+s), u^k(n)) \to \sum_{s=-\infty}^{0} G(n, s, v(n+s), v(n))$$

as $k \to \infty$, because $u^k(n) \to v(n)$ uniformly on any compact set in $\mathbf{Z}$. Therefore, by letting $k \to \infty$ in (12), $v(n)$ is a solution of (4) on $\mathbf{Z}$ and $G \in \Omega(F)$.

For the almost periodic System (3), we have the following theorem.

Theorem 3. *Under the above Assumptions (H2), (H3) and (H4), if the zero solution $u(n) \equiv 0$ of Equation (3) is (K, ρ)-WUAS, then it is (K, ρ)-UAS.*

Proof. Since the zero solution is (K, ρ)-US, there exists a $\delta(\delta_0) > 0$, such that $\rho(x_{n_0}, 0) \leq \delta(\delta_0)$ implies $\rho(x_n, 0) < \delta_0$ for all $n \geq n_0 \geq 0$, where $x(n)$ is a solution of (3) through (n_0, x_{n_0}) and δ_0 is the number given in (iii) of Definition 3. Let $\epsilon > 0$ be given. We shall now show that there exists a number $T(\epsilon) > 0$, such that $x_{n_0}(s) \in K$, $s \leq 0$, $\rho(x_{n_0}, 0) \leq \delta(\delta_0)$, and for any $n_0 \geq 0$, there exists an n_1, $n_0 \leq n_1 \leq n_0 + T(\epsilon)$, such that $\rho(x_{n_1}, 0) < \delta(\epsilon)$, where $\delta(\epsilon)$ is the one for the (K, ρ)-US of $u(n) \equiv 0$. Then, clearly it will follow that $\rho(x_n, 0) < \epsilon$ for $n \geq n_0 + T(\epsilon)$, which shows that the zero solution is (K, ρ)-UAS.

Suppose that there is no $T(\epsilon)$. Then, for each integer $k \geq 1$, there exist a function $x^k_{n_k}(s) \in K$, $s \leq 0$ and an $n_k \geq 0$, such that $\rho(x^k_{n_k}, 0) \leq \delta(\delta_0)$ and $\rho(x^k_n, 0) \geq \delta(\epsilon)$ for all $n_k \leq n \leq n_k + k$, where $x^k(n)$ is a solution of (3) through $(n_k, x^k_{n_k})$. Letting $y^k(n) = x^k(n + n_k)$, $y^k(n)$ is a solution of:

$$x(n+1) = f(n + n_k, x(n)) + \sum_{s=-\infty}^{0} F(n + n_k, s, x(n+s), x(n))$$

through $(0, x^k_{n_k})$, $x^k_{n_k}(s) \in K$, $s \leq 0$ and $\rho(y^k_n, 0) \leq \delta(\epsilon)$ on $0 \leq n \leq k$. Since $\rho(x^k_{n_k}, 0) \leq \delta(\delta_0)$, $\rho(y^k_n, 0) \leq \delta_0$, $f(n, x)$ is almost periodic in n uniformly for $x \in R^m$ and $F(n, s, x, y)$ is almost periodic in n uniformly for $(s, x, y) \in K^*$ for any compact set $K^* \subset \mathbf{Z}^- \times R^m \times R^m$, there exist an initial function x_{n_0}, functions $g(n, x)$, $G(n, s, x, y)$, $z(n)$ and a subsequence $\{k_j\}$ of $\{k\}$, such that:

$$x^{k_j}_{n_{k_j}}(s) \to x_{n_0}(s)$$

uniformly on any compact interval in $\mathbf{Z}^-$,

$$f(n + n_{k_j}, x) \to g(n, x)$$

uniformly on $\mathbf{Z} \times K$ for any compact set:

$$K = \{x \in R^m | |x| \leq \delta_0\}$$

$$F(n + n_{k_j}, s, x, y) \to G(n, s, x, y)$$

uniformly on any compact set on:

$$\mathbf{Z}^- \times K \times K$$

and:

$$y^{k_j}(n) \to z(n)$$

uniformly on any compact interval in $\mathbf{Z}^+$ as $j \to \infty$. By Lemma 1, $z(n)$ is a solution of:

$$x(n+1) = g(n, x(n)) + \sum_{s=-\infty}^{0} G(n, s, x(n+s), x(n))$$

which is defined on $n \in \mathbf{Z}^+$, $x_0(s) \in K$ for $s \leq 0$ and passes through $(0, x_0)$. For fixed $n \geq 0$, there is a j sufficiently large, so that:

$$\rho(y^{k_j}_n, 0) - \rho(y^{k_j}_n, z_n) \leq \rho(z_n, 0)$$

Since $\rho(y_n^{k_j}, 0) \geq \delta(\epsilon)$ and $\rho(y_n^{k_j}, z_n) < \frac{\delta(\epsilon)}{2}$ for large j, we have:

$$\rho(z_n, 0) > \frac{\delta(\epsilon)}{2} \quad \text{for all} \quad n \geq 0 \tag{13}$$

Moreover, clearly:

$$\rho(z_n, 0) \leq \delta_0 \quad \text{for all} \quad n \geq 0 \tag{14}$$

Since (g, G) is in $\Omega(f, F)$, (f, F) is in $\Omega(g, G)$ and, hence, there exists a sequence $\{\tau_k\}$, such that $\tau_k \to \infty$ as $k \to \infty$ and $g(n+\tau_k, x) \to f(n, x)$ uniformly for $n \in \mathbf{Z}$ and $x \in K$ and $G(n+\tau_k, s, x, y) \to F(n, s, x, y)$ uniformly for $n \in \mathbf{Z}$ and $(s, x, y) \in K^*$ as $k \to \infty$. If we set $v^k(n) = z(n + \tau_k)$, $v^k(n)$ is a solution through $(0, z_{\tau_k})$ of:

$$x(n+1) = g(n + \tau_k, x(n)) + \sum_{s=-\infty}^{0} G(n + \tau_k, s, x(n+s), x(n))$$

Since $\rho(z_n, 0) \leq \delta_0$ for all $n \geq 0$, $\{v^k(n)\}$ is uniformly bounded. Hence, there exists a subsequence $\{\tau_{k_j}\}$ of $\{\tau_k\}$, such that:

$$g(n + \tau_{k_j}, x) \to f(n, x)$$

uniformly for $n \in \mathbf{Z}$ and $x \in K$,

$$G(n + \tau_{k_j}, s, x, y) \to F(n, s, x, y) :$$

uniformly for $n \in \mathbf{Z}$, $s \in \mathbf{Z}^-$,

$$x \in K and y \in K$$

and:

$$v^{k_j}(n) \to w(n)$$

on any compact interval in $\mathbf{Z}$ as $j \to \infty$. Here, we can see that $w(n)$ is a solution of (3), by Lemma 1. For fixed $n \geq 0$, there exists a j so large that:

$$\rho(w_n, 0) \geq \rho(v_n^{k_j}, 0) - \rho(v_n^{k_j}, w_n) \geq \frac{\delta(\epsilon)}{2} - \frac{\delta(\epsilon)}{4} = \frac{\delta(\epsilon)}{4} \tag{15}$$

because $\tau_{k_j} > 0$ for j sufficiently large and $\rho(v_n^{k_j}, 0) = \rho(z_{n+\tau_{k_j}}, 0) \geq \frac{\delta(\epsilon)}{2}$ by (13). Moreover, by (14), we have $\rho(w_0, 0) \leq \delta_0$. However, this implies that $\rho(w_n, 0) \to 0$ as $n \to \infty$; this contradicts (15). This proves the theorem.

The following corollary can be proven by the same argument as in the proof of Theorem 1.

Corollary 1. *Under Assumptions (H3) and (H4), if the bounded solution $u(n)$ of Equation (3) is (K, ρ)-GWUAS, then it is (K, ρ)-GEAS.*

Theorem 4. *Assume Conditions (H1), (H3) and (H4). If the solution $u(n)$ of Equation (3) is (K, ρ)-GWUAS, then the solution $u(n)$ of Equation (3) is (K, ρ)-GUAS.*

Proof. Since we have a bounded solution $u(n)$ of Equation (3) by (H4), let $B > 0$ be such that $|u(n)| \leq B$ for all $n \geq 0$ and $|u_0(s)| \leq B$ for all $s \leq 0$. Then, we can take $\rho(u_n, 0) \leq B/(1+B) =: B^*$ for all $n \geq 0$ and $\rho(u_s, 0) \leq B^*$ for all $s \leq 0$ from the definition of ρ. Since $|u(n)| \leq B$ and $u(n)$ is (K, ρ)-GEAS by Corollary 1, we can show that the solution of (3) is (K, ρ)-equi-bounded. Therefore, for any $\alpha > 0$ and $n_0 \geq 0$, we can find a $\beta(\alpha) > 0$, such that if $n_0 \geq 0$ and $\rho(x_{n_0}, u_{n_0}) \leq \alpha$, then $\rho(x_n, u_n) < \beta(\alpha)$ for all $n \geq n_0$.

By the assumption of (K, ρ)-GWUAS, $u(n)$ is (K, ρ)-US, and hence, it is sufficient to show that for any $\epsilon > 0$ and $\alpha > 0$, there exists a $T(\epsilon, \alpha) > 0$, such that if $\rho(x_{n_0}, u_{n_0}) \leq \alpha$, then:

$$\rho(x_n, u_n) < \epsilon \quad \text{for all} \quad n \geq n_0 + T(\epsilon, \alpha)$$

To do this, given $\alpha > 0$, if $0 \leq n_0 < \omega$ and $\rho(x_{n_0}, u_{n_0}) \leq 2B^* + \alpha$, then:

$$\rho(x_\omega, u_\omega) < \beta(2B^* + \alpha)$$

By (K, ρ)-GEAS, there exists a $T_1(\omega, \frac{\epsilon}{2}, \alpha) > 0$, such that if $\rho(x_\omega, u_\omega) < \beta(2B^* + \alpha)$, then $\rho(x_n, u_n) < \frac{\epsilon}{2}$ for all $n \geq \omega + T_1(\omega, \frac{\epsilon}{2}, \alpha)$.

Now, consider a solution $x(n)$ of (3), such that $\rho(x_{n_0}, u_{n_0}) \leq \alpha$ and $k\omega \leq n_0 < (k+1)\omega$, where $k = 0, 1, 2, \cdots$. Since System (3) is periodic in n of period ω by (H1), we have:

$$x(n) = x(n - k\omega), \quad n \geq n_0 \tag{16}$$

and $x_{n_0}(s) = x_{n_0 - k\omega}(s) \in K$ for all $s \leq 0$. Moreover, $u(n + k\omega)$ also is a solution of (3), such that $u_0^k(s) = u_{k\omega}(s) \in K$ for all $s \leq 0$, which we shall denote by $v(n) = u(n + k\omega) := u^k(n)$ through $(0, u_0^k)$. Then, we have:

$$\rho(v_\omega, u_\omega) \leq 2B^*(< \beta(2B^* + \alpha))$$

and hence, we have:

$$\rho(v_n, u_n) < \frac{\epsilon}{2} \quad \text{for all} \quad n \geq \omega + T_1(\omega, \frac{\epsilon}{2}, \alpha) \tag{17}$$

Since $\rho(x_{n_0}, u_{n_0}) \leq \alpha$ and $u(n_0) = v(n_0 - k\omega)$ through $(0, u_{n_0}^k)$, it follows from (16) that:

$$\rho(x_{n_0 - k\omega}, v_{n_0 - k\omega}) \leq \alpha$$

which implies that $\rho(x_{n_0 - k\omega}, u_{n_0 - k\omega}) \leq 2B^* + \alpha$, because $\rho(v_{n_0 - k\omega}, u_{n_0 - k\omega}) \leq 2B^*$. Therefore, we have:

$$\rho(x_{n - k\omega}, u_{n - k\omega}) < \frac{\epsilon}{2} \tag{18}$$

for all $n \geq (k+1)\omega + T_1(\omega, \frac{\epsilon}{2}, \alpha)$ since $0 \leq n_0 - k\omega < \omega$. From (17), it follows that:

$$\rho(v_{n - k\omega}, u_{n - k\omega}) < \frac{\epsilon}{2} \tag{19}$$

for all $n \geq (k+1)\omega + T_1(\omega, \frac{\epsilon}{2}, \alpha)$. Thus, by (18) and (19):

$$\rho(x_{n - k\omega}, v_{n - k\omega}) < \epsilon$$

for all $n \geq (k+1)\omega + T_1(\omega, \frac{\epsilon}{2}, \alpha)$, which implies that:

$$\rho(x_n, u_n) < \epsilon$$

for all $n \geq n_0 + T(\epsilon, \alpha)$, where $T(\epsilon, \alpha) = \omega + T_1(\omega, \frac{\epsilon}{2}, \alpha)$, because $n_0 \geq k\omega$. Thus, we see that the solution $u(n)$ is (K, ρ)-UAS.

For the ordinary differential equation, it is well known that an example in ([8], pp. 81) is of a scalar almost periodic equation, such that the zero solution is GWUAS, but is not GUAS.

We say that Equation (3) is regular, if the solutions of every limiting Equation (4) of (3) are unique for the initial value problem.

Theorem 5. *Under Assumptions (H2), (H3) and (H4), if Equation (3) is regular and the unique solution $u(n)$ of Equation (3) is (K, ρ)-RWUAS in $\Omega(f, F)$, then the solution $u(n)$ of Equation (3) is (K, ρ)-RTS.*

Proof. Suppose that $u(n)$ is not (K, ρ)-RTS. Then, there exists a small $\epsilon > 0$, $0 < 1/k < \epsilon < \delta_0$, where δ_0 is the number for (K, ρ)-A in $\Omega(f, F)$ of (ix) in Definition 3, and sequences $\{s_k\} \subset \mathbf{Z}^+$, $\{r_k\}, r_k > 0$, $\{h_k\}$ and $\{\phi^k\}$, such that $\phi^k : (-\infty, s_k] \to R^m$ and $h_k : [s_k, +\infty) \to R^m$ are bounded functions satisfying $|h_k(n)| < 1/k$ for $n \geq s_k$ and:

$$\begin{aligned} &\rho(u_{s_k}, x^k_{s_k}) < \frac{1}{k}, \quad \rho(u_n, x^k_n) < \epsilon, \quad n \in [s_k, s_k + r_k - 1) \\ &\text{and} \quad \rho(u_{s_k + r_k}, x^k_{s_k + r_k}) = \epsilon, \end{aligned} \tag{20}$$

for sufficient large k, where $x^k(n)$ is a solution of:

$$x(n+1) = f(n, x(n)) + \sum_{s=-\infty}^{0} F(n, s, x(n+s), x(n)) + h_k(n), \quad n \geq s_k$$

passing through (s_k, ϕ^k), such that $x^k_n \in \bar{K}$ for all $n \geq s_k$ and $k \geq 1$, where $\bar{K}$ is the compact set, such that $\bar{K} = \overline{N(\epsilon_0, K)}$ for some $\epsilon_0 > 0$; here, $\overline{N(\epsilon_0, K)}$ denotes the closure of the ϵ_0 -neighborhood $N(\epsilon_0, K)$ of K, and we let this $\bar{K}$ denote K again. Since K is a compact set of R^m, it follows that for $k \geq 1$, $\{x^k(s_k + r_k + n)\}$ and $\{x^k(s_k + n)\}$ are uniformly bounded for all s_k and $n \geq -\infty$. We first consider the case where $\{r_k\}_{k \geq 1}, r_k \to \infty$ as $k \to \infty$. Taking a subsequence if necessary, we may assume from (H2) and the properties of almost periodic functions in $\Omega(\cdot)$ that there exists a $(v, g, G) \in \Omega(u, f, F)$, such that $f(n + s_k + r_k, x) \to g(n, x)$ uniformly on $\mathbf{Z}^+ \times K$, $F(n + s_k + r_k, s, x, y) \to G(n, s, x, y)$ uniformly on $\mathbf{Z}^+ \times K^*$, $x^k(n + s_k + r_k) \to z(n)$ uniformly on $\mathbf{Z}^+$ and $u(n + s_k + r_k) \to v(n)$ uniformly on $\mathbf{Z}^+$, as $k \to \infty$, where $z, v : Z^+ \to R^m$ are some bounded functions. Since:

$$\begin{aligned} & x^k(n + s_k + r_k + 1) = f(n + s_k + r_k, x^k(n + s_k + r_k)) \\ + & \sum_{s=-\infty}^{0} F(n + s_k + r_k, s, x^k(n + s_k + r_k + s), x^k(n + s_k + r_k)) \\ + & h_k(n + s_k + r_k) \end{aligned}$$

such that $x_0^k(s) \in K$ for all $s \leq 0$, passing to the limit as $k \to \infty$, by Lemma 1, we conclude that $z(n)$, for $n \geq 0$, is the solution of the following equation of:

$$x(n+1) = g(n, x(n)) + \sum_{s=-\infty}^{0} G(n, s, x(n+s), x(n)), \quad n \in \mathbf{Z}^+ \tag{21}$$

Similarly, $v(n)$ for $n \geq 0$ is also a solution of (21). By $(v, g, G) \in \Omega(u, f, F)$, $x^k_{s_k+r_k} \to z_0$ and $u_{s_k+r_k} \to v_0$ in BS as $k \to \infty$. It follows from (20) that we have:

$$\rho(v_0, z_0) = \lim_{k\to\infty} \rho(u_{s_k+r_k}, x^k_{s_k+r_k}) = \epsilon < \delta_0 \tag{22}$$

Notice that $v(n)$, for $n \geq 0$, is a solution of (21) passing through $(0, v_0)$, and $v(n)$ is RWUAS of limiting Equation (21) by $(v, g, G) \in \Omega(u, f, F)$ and the similar result of Lemma 3 in [4]. Then, we obtain $\rho(v_n, z_n) \to 0$ as $n \to \infty$. This is a contradiction to (22). Thus, the sequence $\{r_k\}$ must be bounded. We can assume that, taking a subsequence if necessary, $0 < r_k \to r_0 < \infty$ as $k \to \infty$. Moreover, we may assume that $x^k(s_k + n) \to \tilde{z}(n)$ and $u(s_k + n) \to \tilde{v}(n)$ for each $n \in \mathbf{Z}$, and $f(n + s_k, \phi^k) \to \tilde{g}(n, \phi)$ uniformly on $\mathbf{Z}^+ \times K$, $F(n + s_k, s, \phi^k, \phi^k) \to \tilde{G}(n, s, \phi, \phi)$ uniformly on $\mathbf{Z}^+ \times K^*$, for $(v, g, G) \in \Omega(u, f, F)$. Since $u_{s_k} \to \tilde{v}_0$ and $x^k_{s_k} \to \tilde{z}_0 = \phi(s)$ in BS as $k \to \infty$, we have:

$$\rho(\tilde{v}_0, \tilde{z}_0) = \lim_{k\to\infty} \rho(u_{s_k}, x^k_{s_k}) = 0$$

by (20), and hence, we have $\tilde{v}_0 \equiv \tilde{z}_0$, that is $\tilde{v}(s) = \tilde{z}(s)$ for all $s \in (-\infty, 0]$. Moreover, $\tilde{v}(n)$ and $\tilde{z}(n)$ satisfy the same equation of:

$$x(n+1) = \tilde{g}(n, x(n)) + \sum_{s=-\infty}^{0} \tilde{G}(n, s, x(n+s), x(n))$$

The uniqueness of the solutions for the initial value problems implies that $\tilde{v}(n) \equiv \tilde{z}(n)$ for $n \in Z^+$, and hence, we have $\rho(\tilde{v}_{r_0}, \tilde{z}_{r_0}) = 0$. On the other hand, and again from (20), we have:

$$\rho(\tilde{v}_{r_0}, \tilde{z}_{r_0}) = \lim_{k\to\infty} \rho(u_{s_k+r_k}, x^k_{s_k+r_k}) = \epsilon$$

This is a contradiction. This shows that $u(n)$ is (K, ρ)-RTS.

We have the following existence theorem of an almost periodic solution for Equation (3).

Theorem 6. *Under Assumptions (H2), (H3) and (H4), if Equation (3) is regular and the unique solution $u(n)$ of Equation (3) is (K, ρ)-RWUAS in $\Omega(f, F)$, then Equation (3) has an almost periodic solution.*

Proof. From Theorem 5, the unique solution $u(n)$ of Equation (3) is (K, ρ)-RTS. Thus, by Theorem 1 and 2 in [14], we have an almost periodic solution.

4. Applications in a Prey-Predator Model

We consider the existence of an almost periodic solution of a system with a strictly positive component of Volterra difference equation:

$$\begin{cases} x_1(n+1) = x_1(n) \exp\{b_1(n) - a_1(n)x_1(n) - c_2(n) \sum_{s=-\infty}^{n} K_2(n-s)x_2(s)\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (E) \\ x_2(n+1) = x_2(n) \exp\{-b_2(n) - a_2(n)x_2(n) + c_1(n) \sum_{s=-\infty}^{n} K_1(n-s)x_1(s)\} \end{cases}$$

which describes a model for the dynamics of a prey-predator discrete system in mathematical ecology. We can regard Equation (3) as the following functional difference equation with axiomatic phase space B and (K, ρ) topology (*cf.* [14]):

$$x(n+1) = h(n, x_n), \quad n \in \mathbf{Z}^+ \tag{23}$$

where $h : \mathbf{Z}^+ \times B \to R^m$. Then, we also hold Theorems 1, 3, 5 and 6 for (23), and we can treat System (E) as an application of Equation (3). In (E), setting $a_i(n)$ and $b_i(n)$ are R-valued bounded almost periodic function in $\mathbf{Z}$, $a_i = \inf_{n\in\mathbf{Z}} a_i(n), A_i = \sup_{n\in\mathbf{Z}} a_i(n), b_i = \inf_{n\in\mathbf{Z}} b_i(n)$, $B_i = \sup_{n\in\mathbf{Z}} b_i(n), c_i = \inf_{n\in\mathbf{Z}} c_i(n)$ and $C_i = \sup_{n\in\mathbf{Z}} c_i(n)$ $(i = 1, 2)$, and $K_i : \mathbf{Z}^+ \to R^+ (i = 1, 2)$ denote delay kernels, such that:

$$K_i(s) \geq 0, \quad \sum_{s=0}^{\infty} K_i(s) = 1 \quad \text{and} \quad \sum_{s=0}^{\infty} sK_i(s) < \infty (i = 1, 2)$$

We set:

$$\alpha_1 = \exp\{B_1 - 1\}/a_1, \ \ \alpha_2 = \exp\{-b_2 + C_1\alpha_1 - 1\}/a_2,$$
$$\beta_1 = \min\{\exp\{b_1 - A_1\alpha_1 - C_2\alpha_2\}(b_1 - C_2\alpha_2)/A_1, \ \ \{b_1 - C_2\alpha_2\}/A_1\}$$

$and:$

$$\beta_2 = \min\{\exp\{-B_2 - A_2\alpha_2 + c_1\beta_1\}(-B_2 + c_1\beta_1)/A_2, \ \ \{-B_2 + c_1\beta_1\}/A_2\}$$

(*cf.* [4], and 4 Applications in population dynamic systems in [11]). We now make the following assumptions:

(i) $a_i > 0, b_i > 0 \ \ (i = 1, 2)$ and $c_1 > 0, c_2 \geq 0$,
(ii) $b_1 > C_2\alpha_2$ and $B_2 < c_1\beta_1$,
(iii) there exists a positive constant m, such that:

$$a_i > C_i + m \quad (i = 1, 2)$$

Then, we have $0 < \beta_i < \alpha_i$ for each $i = 1, 2$. If $u(n) = (u_1(n), u_2(n))$ is a solution of (E) through $(0, \phi)$, such that $\beta_i \leq \phi(s) \leq \alpha_i (i = 1, 2)$ for all $s \leq 0$, then we have $\beta_i \leq u_i(n) \leq \alpha_i (i = 1, 2)$ for all $n \geq 0$. Let K be the closed bounded set in R^2, such that:

$$K = \{(x_1, x_2) \in R^2; \beta_i \leq x_i \leq \alpha_i \ \text{ for each } \ i = 1, 2\}$$

Then, K is invariant for System (E), that is we can see that for any $n_0 \in \mathbf{Z}$ and any φ, such that $\varphi(s) \in K$, $s \leq 0$, every solution of (E) through (n_0, φ) remains in K for all $n \geq n_0$. Hence, K is invariant for its limiting equations. Now, we shall see that the existence of a strictly positive almost periodic solution of (E) can be obtained under Conditions (i), (ii) and (iii). For System (E), we first introduce the change of variables:

$$u_i(n) = \exp\{v_i(n)\}, \ \ x_i(n) = \exp\{y_i(n)\}, \ \ i = 1, 2$$

Then, System (E) can be written as:

$$\begin{cases} y_1(n+1) - y_1(n) = b_1(n) - a_1(n)\exp\{y_1(n)\} - c_2(n)\sum_{s=-\infty}^{n} K_2(n-s)\exp\{y_2(s)\} \\ \hfill (E_0) \\ y_2(n+1) - y_2(n) = -b_2(n) - a_2(n)\exp\{y_2(n)\} + c_1(n)\sum_{s=-\infty}^{n} K_1(n-s)\exp\{y_1(s)\} \end{cases}$$

We now consider the Lyapunov functional:

$$V(v(n), y(n)) = \sum_{i=1}^{2}\{|v_i(n) - y_i(n)| + \sum_{s=0}^{\infty} K_i(s)\sum_{l=n-s}^{n-1} c_i(s+l)|\exp\{v_i(l)\} - \exp\{y_i(l)\}|\}$$

where $y(n)$ and $v(n)$ are solutions of (E_0), which remains in K. Calculating the differences, we have:

$$\Delta V(v(n), y(n)) \leq -mD\sum_{i=1}^{2}|v_i(n) - y_i(n)|$$

where set $D = \max\{\exp\{\beta_1\}, \exp\{\beta_2\}\}$, and let $x_i(n)$ be solutions of (E), such that $x_i(n) \geq \beta_i$ for $n \geq n_0 \;\; (i = 1, 2)$. Thus, $\sum_{i=1}^{2}|v_i(n) - y_i(n)| \to 0$ as $n \to \infty$, and hence, $\rho(v_n, y_n) \to 0$ as $n \to \infty$. Thus, we have that $v(n)$ is (K, ρ)-A in Ω of (E_0). Moreover, by using this Lyapunov functional, we can show that $v(n)$ is (K, ρ)-RUS in Ω of (E_0), that is (K, ρ)-RWUAS in Ω of (E_0). Thus, from Theorem 5, $v(n)$ is (K, ρ)-RTS, because K is invariant. By the equivalence between (E) and (E_0), the solution $u(n)$ of (E) is (K, ρ)-RWUAS in Ω, and hence, it is (K, ρ)-RTS. Therefore, it follows from Theorem 6 that System (E) has an almost periodic solution $p(n)$, such that $\beta_i \leq p_i(n) \leq \alpha_i$, $(i = 1, 2)$, for all $n \in \mathbf{Z}$.

5. Stability Property and Separation Condition

In order to discuss the conditions for the existence of an almost periodic solution in a Volterra integrodifferential equation with infinite delay, we discussed the relationship between the total stability with respect to a certain metric ρ and the separation condition with respect to ρ (*cf.* [5]). In this final section, we discuss a new approach of a relationship between the ρ-separation condition and (K, ρ)-uniformly-asymptotic stability property in a metric ρ sense for a nonlinear Volterra difference equation with infinite delay.

Let K be a compact set in R^m, such that $u(n) \in K$ for all $n \in \mathbf{Z}$, where $u(n) = \phi^0(n)$ for $n < 0$. If $x(n)$ is a solution, such that $x(n) \in K$ for all $n \in \mathbf{Z}$, we say that x is in K.

Definition 4. *We say that Equation (3) satisfies the ρ-separation condition in K, if for each $(g, G) \in \Omega(f, F)$, there exists a $\lambda(g, G) > 0$, such that if x and y are distinct solution of (4) in K, then we have:*

$$\rho(x_n, y_n) \geq \lambda(g, G) \quad \textit{for all}, \quad n \in \mathbf{Z}$$

If Equation (3) satisfies the ρ-separation condition in K, then we can choose a positive constant λ_0 independent of (g, G) for which $\rho(x_n, y_n) \geq \lambda_0$ for all $n \in \mathbf{Z}$, where x and y are a distinct solution of (4) in K. We call λ_0 the ρ-separation constant in K (e.g., [8], pp. 189–190).

Definition 5. *A solution $x(n)$ of (3) in K is said to be (K, ρ)-relatively totally stable (in short, (K, ρ)-RTS), if for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, such that $\rho(x_n, y_n) < \epsilon$ for all $n \geq n_0$ whenever*

$\rho(x_{n_0}, y_{n_0}) < \delta(\epsilon)$ at some $n_0 \in Z$ and $h \in BS([n_0, \infty))$, which satisfies $|h|_{[n_0,\infty)} < \delta(\epsilon)$, for $n \geq n_0$. Here, $y(n)$ is a solution through (n_0, y_{n_0}) of:

$$x(n+1) = f(n, x(n)) + \sum_{s=-\infty}^{0} F(n, s, x(n+s), x(n)) + h(n)$$

such that $y_{n_0}(s) \in K$ for all $s \leq 0$ and $y(n) \in K$ for $n \geq n_0$. In the case where $h(n) \equiv 0$, this gives the definition of the (K, ρ)-relatively uniform stability of $x(n)$ (in short, (K, ρ)-RUS).

The following Proposition 1 can be proven by the same argument as in the proof for integrodifferential equations by Hamaya and Yoshizawa [5].

Proposition 1. *Under Assumptions (H2), (H3) and (H4), if Equation (3) satisfies the ρ-separation condition in K, then for any $(g, G) \in \Omega(f, F)$, any solution x of (4) in K is (K, ρ)-RTS. Moreover, we can choose the number $\delta(\cdot)$ in Definition 5, so that $\delta(\epsilon)$ depends only on ϵ and is independent of (g, G) and solutions.*

Theorem 7. *Under Assumptions (H2), (H3) and (H4), suppose that Equation (3) satisfies the ρ-separation condition in K. If $w(n)$ is a solution of (3), such that $w(n) \in K$ for all $n \in$ **Z**, then $w(n)$ is almost periodic.*

Proof. By Proposition 1, solution $w(n)$ of (3) is (K, ρ)-RTS, because $(f, F) \in \Omega(f, F)$. Then, $w(n)$ is asymptotically almost periodic on $[0, \infty)$ by Theorem 1 in [14]. Thus, it has the decomposition $w(n) = p(n) + q(n)$, where $p(n)$ is almost periodic in n, $q(n)$ is bounded function and $q(n) \to 0$ as $n \to \infty$. Since $w(n) \in K$ for all $n \in$ **Z**, $p(n)$ is a solution of (3) in K. If $w(n_1) \neq p(n_1)$ at some n_1, we have two distinct solutions of (3) in K. Thus, we have $\rho(w_n, p_n) \geq \lambda_0 > 0$ for all $n \in$ **Z**, where λ_0 is the ρ-separation constant. However, $w(n) - p(n) \to 0$ as $n \to \infty$, and hence, $\rho(w_n, p_n) \to 0$ as $n \to \infty$. This contradiction shows $w(n) \equiv p(n)$ for all $n \in$ **Z**.

Definition 6. *A solution $x(n)$ of (3) in K is said to be (K, ρ)-relatively uniformly-asymptotically stable (in short, (K, ρ)-RUAS), if it is (K, ρ)-RUS and if there exists a $\delta_0 > 0$ and for any $\epsilon > 0$ there exists a $T(\epsilon) > 0$, such that if $\rho(x_{n_0}, y_{n_0}) < \delta_0$ at some $n_0 \in$ **Z**, then $\rho(x_n, y_n) < \epsilon$ for all $n \geq n_0 + T(\epsilon)$, where $y(n)$ is a solution of (3) through (n_0, y_{n_0}), such that $y_{n_0}(s) \in K$ for all $s \leq 0$ and $y(n) \in K$ for all $n \geq n_0$.*

We show that the ρ-separation condition will be characterized in terms of (K, ρ) uniformly-asymptotic stability of solutions in K of limiting equations. For ordinary differential equations, this kind of problem has been discussed by Nakajima [15].

Theorem 8. *Under Assumptions (H2), (H3) and (H4), Equation (3) satisfies the ρ-separation condition in K if and only if for any $(g, G) \in \Omega(f, F)$, any solution x of (4) in K is (K, ρ)-RUAS with common triple $(\delta_0, \delta(\cdot), T(\cdot))$.*

Proof. We suppose that Equation (3) satisfies the ρ-separation condition in K. Then, it follows from Proposition 1 that for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, such that for any $(g, G) \in \Omega(f, F)$ and any solution $x(n)$ of (4) in K, if $\rho(x_{n_0}, y_{n_0}) < \delta(\epsilon)$ at some $n_0 \in \mathbf{Z}$, then $\rho(x_n, y_n) < \epsilon$ for all $n \geq n_0$, where $y(n)$ is a solution of (4), such that $y_{n_0}(s) \in K$ for all $s \leq 0$ and $y(n) \in K$ for $n \geq n_0$. Now, let δ_0 be a positive constant, such that $\delta_0 < \delta(\lambda_0/2)$, where λ_0 is the ρ-separation constant. For this δ_0, we shall show that for any $\epsilon > 0$, there exists a $T(\epsilon) > 0$, such that for any $(g, G) \in \Omega(f, F)$ and any solution $x(n)$ of (4) in K, $\rho(x_n, y_n) < \epsilon$ for all $n \geq n_0 + T(\epsilon)$, whenever $\rho(x_{n_0}, y_{n_0}) < \delta_0$ at some $n_0 \in \mathbf{Z}$, where $y(n)$ is a solution of (4), such that $y_{n_0}(s) \in K$ for all $s \leq 0$ and $y(n) \in K$ for all $n \geq n_0$.

Suppose not. Then, there exist an $\epsilon, 0 < \epsilon < \delta_0/2$ and sequences $\{(g_k, G_k)\}, \{x^k\}, \{y^k\}, \{s_k\}$ and $\{n_k\}$, such that $(g_k, G_k) \in \Omega(f, F), x^k(n)$ is a solution in K of:

$$x(n+1) = g_k(n, x(n)) + \sum_{s=-\infty}^{0} G_k(n, s, x(n+s), x(n)) \tag{24}$$

and that $n_k \geq s_k + k$,

$$\rho(x^k_{s_k}, y^k_{s_k}) < \delta_0 < \delta(\lambda_0/2) \tag{25}$$

and:

$$\rho(x^k_{n_k}, y^k_{n_k}) \geq \epsilon$$

where $y^k(n)$ is a solution of (24), such that $y^k_{s_k}(s) \in K$ for all $s \leq 0$ and $y^k(n) \in K$ for all $n \geq s_k$. Since (25) implies $\rho(x^k_n, y^k_n) < \lambda_0/2$ for all $n \geq s_k$, we have:

$$\epsilon \leq \rho(x^k_{n_k}, y^k_{n_k}) \leq \lambda_0/2 \tag{26}$$

If we set $w^k(n) = x^k(n + n_k)$ and $z^k(n) = y^k(n + n_k)$, then $w^k(n)$ is a solution in K of:

$$x(n+1) = g_k(n + n_k, x(n)) + \sum_{s=-\infty}^{0} G_k(n + n_k, s, x(n+s), x(n)) \tag{27}$$

and $z^k(n)$ is defined for $n \geq -k$ and is a solution of (27), such that $z^k_{-k}(s) \in K$ for all $s \leq 0$ and $z^k(n) \in K$ for all $n \geq -k$. Since $(g_k(n + n_k, x), G_k(n + n_k, s, x, y)) \in \Omega(f, F)$, taking a subsequence if necessary, we can assume that $w^k(n) \to w(n)$ uniformly on any compact interval in $\mathbf{Z}$, $z^k(n) \to z(n)$ uniformly on any compact interval in $\mathbf{Z}$, $g_k(n + n_k, x) \to h(n, x)$ uniformly on $\mathbf{Z} \times K$ and $G(n + n_k, s, x, y) \to H(n, s, x, y)$ uniformly on $\mathbf{Z} \times S^* \times K \times K$ for any compact set S^* in $(-\infty, 0]$ as $k \to \infty$, where $(h, H) \in \Omega(f, F)$. Then, by the similar argument as in the proof of Lemma 1 (*cf.* Lemma 5 in [5]), $w(n)$ and $z(n)$ are solutions in K of:

$$x(n+1) = h(n, x(n)) + \sum_{s=-\infty}^{0} H(n, s, x(n+s), x(n)) \tag{28}$$

On the other hand, we have:

$$\rho(w_0, z_0) = \lim_{k\to\infty} \rho(w^k_0, z^k_0) = \lim_{k\to\infty} \rho(x^k_{n_k}, y^k_{n_k})$$

Thus, it follows from (26) that:

$$\epsilon \leq \rho(w_0, z_0) \leq \lambda_0/2 \tag{29}$$

Since $w(n)$ and $z(n)$ are distinct solutions of (28) in K, (29) contradicts the ρ-separation condition. This shows that for any $(g, G) \in \Omega(f, F)$, any solution x of (4) in K is (K, ρ)-RUAS with a common triple $(\delta_0, \delta(\cdot), T(\cdot))$.

Now, we assume that for any $(g, G) \in \Omega(f, F)$, any solution of (4) in K is (K, ρ)-RUAS with a common triple $(\delta_0, \delta(\cdot), T(\cdot))$. First of all, we shall see that any two distinct solutions $x(n)$ and $y(n)$ in K of a limiting equation of (3) satisfy:

$$\liminf_{n\to-\infty} \rho(x_n, y_n) \geq \delta_0 \tag{30}$$

Suppose not. Then, for some $(g, G) \in \Omega(f, F)$, there exist two distinct solutions $x(n)$ and $y(n)$ of (4) in K that satisfy:

$$\liminf_{n\to-\infty} \rho(x_n, y_n) < \delta_0 \tag{31}$$

Since $x(n) \not\equiv y(n)$, we have $|x(n_0) - y(n_0)| = \epsilon > 0$ at some $n_0 \in \mathbf{Z}$. Thus, we have $\rho(x_{n_0}, y_{n_0}) \geq \epsilon/2(1+\epsilon)$. By (31), there exists an $n_1 \in \mathbf{Z}$, such that $\rho(x_{n_1}, y_{n_1}) < \delta_0$ and $n_1 < n_0 - T(\epsilon/4(1+\epsilon))$, where $T(\cdot)$ is the number for (K, ρ)-RUAS. Since $x(n)$ is (K, ρ)-RUAS, we have $\rho(x_{n_0}, y_{n_0}) < \epsilon/4(1+\epsilon)$, which contradicts $\rho(x_{n_0}, y_{n_0}) \geq \epsilon/2(1+\epsilon)$. Thus, we have (30).

For any solution $x(n)$ in K, there exists a positive constant c, such that $|x(n)| \leq c$ for all $n \in \mathbf{Z}$. Denote by $O^+(x)$ the set of the closure of positive orbit of x, that is,

$$O^+(x) := \overline{\{x_n | n \in \mathbf{Z}^+\}}$$

such that $|\phi(s)| \leq c$ for $s \in (-\infty, 0]$. Then, $O^+(x)$ is compact in (BS, ρ). Thus, there is a finite number of coverings, which consist of m_0 balls with a diameter of $\delta_0/4$. We shall see that the number of distinct solutions of (4) in K is at most m_0. Suppose that there are m_0+1 distinct solutions $x^{(j)}(n)$ $(j = 1, 2, \cdots, m_0+1)$. By (30), there exists an $n_2 \in \mathbf{Z}$, such that:

$$\rho(x^{(i)}_{n_2}, x^{(j)}_{n_2}) \geq \delta_0/2 \text{ for } i \neq j \tag{32}$$

Since $x^{(j)}_{n_2}$, $j = 1, 2, \cdots, m_0+1$ are in $O^+(x)$, some two of these, say $x^{(i)}_{n_2}, x^{(j)}_{n_2}$, $(i \neq j)$, are in one ball, and hence, $\rho(x^{(i)}_{n_2}, x^{(j)}_{n_2}) < \delta_0/4$, which contradicts (32). Therefore the number of solutions of (4) in K is $m \leq m_0$. Thus, we have the set of solutions of (4) in K:

$$\{x^{(1)}(n), x^{(2)}(n), \cdots, x^{(m)}(n)\}$$

and

$$\liminf_{n\to-\infty} \rho(x^{(i)}_n, x^{(j)}_n) \geq \delta_0 \text{ for } i \neq j. \tag{33}$$

Consider a sequence $\{n_k\}$, such that $n_k \to -\infty$, $g(n+n_k, x) \to g(n, x)$ uniformly on $\mathbf{Z} \times K$ and $G(n+n_k, s, x, y) \to G(n, s, x, y)$ uniformly on $\mathbf{Z} \times S^* \times K \times K$ for any compact set S^* in $(-\infty, 0]$ as $k \to \infty$. Since the sequences $\{x^{(j)}(n+n_k)\}$, $1 \leq j \leq m$, are uniformly bounded, there exists a subsequence of $\{n_k\}$, which will be denoted by $\{n_k\}$ again, and functions $y^{(j)}(n)$, such that $x^{(j)}(n+n_k) \to y^{(j)}(n)$, uniformly on any compact interval in $\mathbf{Z}$ as $k \to \infty$. Clearly, $y^{(j)}(n)$ is the solution of (4) in K. Since we have:

$$\rho(y^{(i)}_n, y^{(j)}_n) = \lim_{k\to\infty} \rho(x^{(i)}_{n+n_k}, x^{(j)}_{n+n_k}) \text{ for } n \in \mathbf{Z}$$

it follows from (33) that:

$$\rho(y_n^{(i)}, y_n^{(j)}) \geq \delta_0 \text{ for all } n \in \mathbf{Z} \text{ and } i \neq j \tag{34}$$

Since we have (34), distinct solutions of (4) in K are $y^{(1)}(n), y^{(2)}(n), \cdots, y^{(m)}(n)$. This shows that Equation (3) satisfies the ρ-separation condition in K with the ρ-separation constant δ_0.

Acknowledgments

The authors would like to express their gratitude to the referees and the editors for their many helpful comments.

Author Contributions

Yoshihiro Hamaya contributes theorems, others and their proofs to our article, Tomomi Itokazu was a M. Sc. student of first author and also "4. Applications in a Prey-Predator Model" in our paper is the summary of her M. Sc. article and Kaori Saito is a PhD. student of first author and she contributes the proofreading of English and others for our paper.

Conflicts of Interest

The authors declare no conflict of interest.

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