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Article

Boas' Formula and Sampling Theorem

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Abstract: In 1937, Boas gave a smart proof for an extension of the Bernstein theorem for trigonometric series. It is the purpose of the present note (i) to point out that a formula which Boas used in the proof is related with the Shannon sampling theorem; (ii) to present a generalized Parseval formula, which is suggested by the Boas' formula; and (iii) to show that this provides a very smart derivation of the Shannon sampling theorem for a function which is the Fourier transform of a distribution involving the Dirac delta function. It is also shows that, by the argument giving Boas' formula for the derivative $f'(x)$ of a function $f(x)$, we can derive the corresponding formula for $f'''(x)$, by which we can obtain an upperbound of $|f'''(x) + 3R^2 f'(x)|$. Discussions are given also on an extension of the Szegö theorem for trigonometric series, which Boas mentioned in the same paper.

Keywords: Shannon sampling theorem; Boas' formula; generalized Parseval formula; Bernstein theorem; Szegö theorem

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1. Introduction

Pólya and Szegö has taken up the Bernstein theorem for trigonometric series in their famous book [1]. In [2] ([Vol. II, p. 11]), the theorem is given as follows.

Theorem 1. *Let $f_T(x)$ be a trigonometric polynomial of order $R_T \in \mathbb{Z}_{>0}$, which is expressed as follows:*

$$f_T(x) = \frac{1}{2}a_0 + \sum_{n=1}^{R_T}(a_n \cos nx + b_n \sin nx), \tag{1}$$

$M_T \in \mathbb{R}_{>0}$, and $|f_T(x)| \leqq M_T$ hold for all $x \in \mathbb{R}$. Then $|f'_T(x)| \leqq M_T R_T$, with equality if and only if $f_T(x)$ is of the form $M_T \cos(nx + c)$ for $n \in \mathbb{Z}$ and $c \in \mathbb{R}$.

Here $\mathbb{R}$ and $\mathbb{Z}$ denote the sets of all real numbers and all integers, respectively, and $\mathbb{R}_{>0} := \{x \in \mathbb{R} | x > 0\}$, $\mathbb{Z}_{>0} := \{n \in \mathbb{Z} | n > 0\}$ and $\mathbb{Z}_{[a,b]} := \{n \in \mathbb{Z} | a \leq n \leq b\}$ for $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ satisfying $a \leq b$. We use also $\mathbb{C}$ which denotes the set of all complex numbers.

Boas [3] gave generalizations of this theorem and a related Szegö theorem for trigonometric series, which are Theorems 2 and 9 given below. The generalized theorems are concerned with a function $f(x)$ which can be expressed as follows:

$$f(x) = \int_{-R}^{R} e^{ixt} d\alpha(t). \tag{2}$$

Here and throughout the present paper, $\alpha(t)$ is a complex-valued function of bounded variation, and $R \in \mathbb{R}_{>0}$.

The generalized Bernstein theorem is the following.

Theorem 2. *Let a function $f(x)$ be given by (2), $M \in \mathbb{R}_{>0}$, and $|f(x)| \leqq M$ hold for all $x \in \mathbb{R}$. Then $|f'(x)| \leqq MR$.*

In a later paper [4], Boas said that the proofs in [3] are lengthy and complicated, and gave a very smart proof for this theorem. That proof was based on the following formula, which we shall call *Boas' formula*:

$$f'(x) = \frac{4R}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(2n+1)^2} f(x + \frac{\pi}{2R} + \frac{n\pi}{R}). \tag{3}$$

Proof of Theorem 2 By using $|f(x)| \leqq M$ and the well-known summation formula:

$$\sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{4}, \tag{4}$$

in (3), we obtain $|f'(x)| \leqq MR$. A derivations of (4) is given in Remark 4 in Section 3.1. ■

When we see the formula (3), we expect that there must exist a sampling theorem which is applicable to the function $f(x)$, and (3) must be obtained by its differentiation. This is the motivation of the present paper. To achieve this object, we present the following sampling theorem.

Theorem 3. *Let a function $f(x)$ be given by (2) in terms of $\alpha(t)$ which is continuous in a neighbourhood of $t = -R$ as well as of $t = R$. Then $f(x)$ is expressed as follows:*

$$f(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} f(y + \frac{n\pi}{R}) \frac{\sin(xR - yR - n\pi)}{xR - yR - n\pi}, \tag{5}$$

where y is any real number.

This theorem is proved in Section 3.

Remark 1. *If we put* $y = 0$, (5) *takes the form of the ordinary well-known Shannon sampling theorem* [5].

We confirm the following proposition in Section 3.1.

Proposition 1. *By taking the term-by-term differentiation of* (5) *with respect to* x, *and then putting* $y = x + \frac{\pi}{2R}$, *we obtain* (3).

In the formulas (3) and (5), $f'(x)$ and $f(x)$ are expressed in terms of an enumerable set $\{f(x_n)\}_{n\in\mathbb{Z}}$. We shall call such a formula a sampling formula (S-formula).

We can use the argument deriving Boas' formula (3) to derive the corresponding formula for $f'''(x)$. We prove the following theorem in Section 3.2.

Theorem 4. *Let the assumption in Theorem 2 be satisfied. Then* $|f'''(x) + 3R^2 f'(x)| \leqq 2MR^3$.

In Section 2, we give a generalized Parseval formula and the lemmas that provide the conditions under which the formula holds. By using these, we show that it readily provides a very simple derivation of the S-formulas (3) and (5) and of S-formulas for some functions defined similarly to (2), in Section 3. Some comments are given on the derivation of the Boas' formula (3), in Section 3.1. In Section 4, discussions are given on the generalized Szegö theorem. Concluding remarks are presented in Section 5.

Here we note that a function $f(x)$ expressed as (2) is continuous and bounded. In fact, if we denote the total variation of $\alpha(t)$ by T, then (2) shows $|f(x)| \leqq T$ for all $x \in \mathbb{R}$.

2. Generalized Parseval Formula

In the present paper, we are concerned with integrals of the form:

$$I = \langle h, g\rangle := \int_{-R}^{R} h(t)g(t)d\alpha(t) \tag{6}$$

Here we assume that $g(t)$ is continuous in $[-R, R]$, and $h(t)$ is integrable in $(-R, R)$ and has the Fourier series, so that $h(t)$ is expressed as follows:

$$h(t) = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{in\pi t/R} \tag{7}$$

for $t \in (-R, R)$ at which $h(t)$ is continuous.

When $\alpha(t)$ is absolutely continuous in $[-R, R]$, and its derivative $F(t)$ is defined by $F(t) = d\alpha(t)/dt$, I is expressed as $I = \int_{-R}^{R} h(t)g(t)F(t)dt$. If the squares of $h(t)$ and of $g(t)F(t)$ are integrable in $(-R, R)$, we have the Parseval formula:

$$I = \langle h, g\rangle = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n d_n \tag{8}$$

where c_n are the coefficients in (7) and d_n are given by $d_n = \int_{-R}^{R} e^{in\pi t/R} g(t)F(t)dt$.

We now present two lemmas which guarantee the validity of the formula (8) for the I defined by (6), assuming that d_n are defined by:

$$d_n = \int_{-R}^{R} e^{in\pi t/R} g(t) d\alpha(t). \tag{9}$$

Lemma 1. *Let the Fourier series* (7) *of* $h(t)$ *converge uniformly, and let* d_n *be defined by* (9)*. Then the formula* (8) *holds for* $I = \langle h, g\rangle$ *defined by* (6)*.*

Proof Substituting (7) in (6), we obtain (8) by term-by-term integration, which is allowed, since the convergence of the Fourier series (7) is uniform and $\alpha(t)$ is of bounded variation. ∎

Remark 2. *If* $h(t)$ *is of bounded variation and continuous in* $[-R, R]$ *and satisfies* $h(R) = h(-R)$*, then its Fourier series* (7) *converges uniformly, by Theorem* (8.6) *in* [2] (Vol. I, p. 58) *or by the Fejér theorem* [2] (Vol. I, p. 89).

Lemma 2. *Let* $h(t)$ *be of bounded variation and piecewise continuous in* $[-R, R]$*, and let* $\alpha(t)$ *be continuous at every discontinuous point of* $h(t)$*, as well as at* $t = -R$ *and* $t = R$ *when* $h(t)$ *is discountinuous at* $t = -R$*,* $t = R$ *or both, or when* $h(-R) \neq h(R)$*, and let* d_n *be defined by* (9)*. Then the formula* (8) *holds for* $I = \langle h, g\rangle$ *defined by* (6)*.*

Proof We use the notations that:

$$\tilde{h}_N(t) = \sum_{n=-N}^{N} c_n e^{in\pi t/R}, \quad \tilde{I}_N = \langle \tilde{h}_N, g\rangle = \int_{-R}^{R} \tilde{h}_N(t) g(t) d\alpha(t), \tag{10}$$

and D_ϵ is the sum of neighbourhoods of the points at which $\alpha(t)$ is not continuous. There exists such a $B \in \mathbb{R}_{>0}$ that $|h(t)g(t)| < B$ and $|\tilde{h}_N(t)g(t)| < B$ for $N \in \mathbb{Z}_{>0}$, since the partial sums $\tilde{h}_N(t)$ of the Fourier series are uniformly bounded [2] (Vol. I, p. 90, Theorem (3.7)). For an arbitrary $\epsilon \in \mathbb{R}_{>0}$, we choose D_ϵ such that the the total variation of $\alpha(t)$ in D_ϵ is less than ϵ, and then choose $N \in \mathbb{Z}_{>0}$ such that $|h(t) - \tilde{h}_N(t)| \cdot |g(t)| < \epsilon$ outside D_ϵ. This is possible since $\tilde{h}_N(t)$ converges uniformly outside D_ϵ, as seen in Remark 2. If we denote the total variation of $\alpha(t)$ by T, we have the inequality $|I - \tilde{I}_N| < 2B \cdot \epsilon + \epsilon \cdot T$. This shows that $\tilde{I}_N \to I$ as $N \to \infty$. ∎

The two generalized theorems in Boas' paper [4] are proved below with the aid of Lemma 1 and Remark 2.

3. Generalized Sampling Theorem

We consider four functions $h_l(t)$ of $t \in [-R, R]$ for $l \in \mathbb{Z}_{[1,4]}$, which are:

$$h_1(t) = 1, \quad h_2(t) = -i\frac{t}{|t|}, \quad h_3(t) = i\frac{t}{R}, \quad h_4(t) = \frac{|t|}{R}. \tag{11}$$

We define four functions $I_l(x)$ of $x \in \mathbb{R}$ for $l \in \mathbb{Z}_{[1,4]}$, by:

$$I_l(x) = \langle h_l, e^{ixt}\rangle = \int_{-R}^{R} h_l(t) e^{ixt} d\alpha(t). \tag{12}$$

By (2), $I_1(x) = f(x)$. We now define the function $\tilde{f}(x)$ by:

$$\tilde{f}(x) = I_2(x) = -i \cdot \int_{-R}^{R} \frac{t}{|t|} e^{ixt} d\alpha(t). \tag{13}$$

We then note that the derivatives of $f(x)$ and $\tilde{f}(x)$ are expressed as follows:

$$f'(x) = R \cdot I_3(x) = i \int_{-R}^{R} t \cdot e^{ixt} d\alpha(t), \quad \tilde{f}'(x) = R \cdot I_4(x) = \int_{-R}^{R} |t| e^{ixt} d\alpha(t). \tag{14}$$

We confirm that the exchange of integration and differentiation in each of these relations and (17) given below is allowed, with the aid of the method presented in [6] (Section 4.2).

We define $h_{l,a}(t)$ by $h_{l,a}(t) = h_l(t)e^{iat}$ for $a \in \mathbb{R}$ and $t \in [-R, R]$. They are listed in the second column of Table 1. For $a \in \mathbb{R}$ and $y \in \mathbb{R}$, we define $I_{l,a}(y)$ by:

$$I_{l,a}(y) = \langle h_{l,a}, e^{iyt} \rangle = \int_{-R}^{R} h_{l,a}(t) e^{iyt} d\alpha(t). \tag{15}$$

The following lemma is easily confirmed.

Lemma 3. *Let $a = x - y$. Then $I_l(x) = I_{l,a}(y)$.*

Since $h_{l,a}(t)$ depends on l and a, its Fourier coefficients c_n depend on l and a, and hence we express them by $c_n(l, a)$, and the Fourier series (7) as:

$$h_{l,a}(t) = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n(l, a) e^{in\pi t/R}. \tag{16}$$

In the third column of Table 1, $c_n(l, a)$ satisfying (16) are given for four functions $h_{l,a}(t)$. We note that $h_{l,a}(t) = \frac{1}{R}\frac{\partial}{\partial a} h_{l-2,a}(t)$ for $l = 3, 4$, and hence:

$$c_n(l, a) = \frac{1}{R}\frac{\partial}{\partial a} c_n(l-2, a), \quad l = 3, 4. \tag{17}$$

Because of (9) and (2), when $g(t) = e^{iyt}$, d_n is given by $d_n = f(y + n\pi/R)$.

Now the Parseval formula (8) for (6) gives:

$$I_{l,a}(y) = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n(l, a) f(y + \frac{n\pi}{R}), \tag{18}$$

for (15). By using this in Lemma 3, we obtain:

$$I_l(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n(l, x-y) f(y + \frac{n\pi}{R}), \tag{19}$$

for arbitrary value of $y \in \mathbb{R}$.

Definition 1. *We call the formula (19) the sampling formula (S-formula) for the function $I_l(x)$.*

Lemma 4. *The formula* (5) *for* $f(x)$ *and the corresponding S-formulas for* $\tilde{f}(x)$*,* $f'(x)$ *and* $\tilde{f}'(x)$*, are obtained by using* $c_n(l,a)$ *and* $I_l(x)$*, for* $l = 1$*,* 2*,* 3 *and* 4*, respectively, of Table 1, in* (19)*, where we put* $a = x - y$*.*

Table 1. Fourier coefficients $c_n(l,a)$ of function $h_{l,a}(t) = h_l(t)e^{iat}$ in $[-R, R]$, and $I_l(x)$ satisfying (12).

l	$h_{l,a}(t)$	$c_n(l,a)$	$I_l(x)$
1	e^{iat}	$\frac{\sin(a \cdot R - n\pi)}{a \cdot R - n\pi}$	$f(x)$
2	$-i\frac{t}{\|t\|}e^{iat}$	$\frac{1-\cos(a \cdot R - n\pi)}{a \cdot R - n\pi}$	$\tilde{f}(x)$
3	$i\frac{t}{R}e^{iat}$	$\frac{\cos(a \cdot R - n\pi)}{a \cdot R - n\pi} - \frac{\sin(a \cdot R - n\pi)}{(a \cdot R - n\pi)^2}$	$\frac{1}{R}f'(x)$
4	$\frac{\|t\|}{R}e^{iat}$	$\frac{\sin(a \cdot R - n\pi)}{a \cdot R - n\pi} - \frac{1-\cos(a \cdot R - n\pi)}{(a \cdot R - n\pi)^2}$	$\frac{1}{R}\tilde{f}'(x)$

Proof of Theorem 3 Lemma 4 shows that the formula (5) for $f(x)$ takes the form (19) which is (8) for the present case, and hence is proved by using Lemma 2. ∎

Theorem 5. *Let* $\alpha(t)$ *be continuous at* $t = -R$ *and* $t = R$*. Then the S-formulas for* $\tilde{f}(x)$*,* $f'(x)$ *and* $\tilde{f}'(x)$ *obtained in Lemma 4, are valid. In the case of* $\tilde{f}(x)$*,* $\alpha(t)$ *is required to be continuous also at* $t = 0$*.*

Proof The proof follows to the proof of Theorem 3 given above. ∎

Lemma 5. *Let* $l = 1$ *or* $l = 2$*. Then if the formula* (19) *is valid when* $h_{l,a}(t) = h_l(t)e^{iat}$*, and if* $\alpha(t)$ *satisfies the condition for it in Lemma 2 for* $h_{l,a}(t) = i \cdot t \cdot h_l(t)e^{iat}$*, then we have:*

$$i\int_{-R}^{R} t \cdot h_l(t)e^{iat}e^{iyt}d\alpha(t) = \lim_{N\to\infty}\sum_{n=-N}^{N} f(y + \frac{n\pi}{R})\frac{d}{da}c_n(l,a). \tag{20}$$

Proof We use (17). ∎

Remark 3. *Formula* (20) *is obtained from* (8) *by term-by-term differentiation with respect to* a*. The* $h_{l,a}(t)$*,* $c_n(l,a)$ *and* $I_l(x)$ *given in rows for* $l = 3$ *and* $l = 4$ *in Table 1 are obtained from those in the rows for* $l = 1$ *and* $l = 2$*, respectively, by differentiation with respect to* a *or to* x*, and then dividing by* R*.*

As a consequence of Remark 3, we have the following lemma.

Lemma 6. *The S-formulas for* $f'(x)$ *and* $\tilde{f}'(x)$ *are obtained by term-by-term differentiation of the corresponding S-formulas for* $f(x)$ *and* $\tilde{f}(x)$*, respectively.*

3.1. Derivation of Boas' Formula (3)

In this section, we put $y^* = x + \frac{\pi}{2R}$ and $a^* = -\frac{\pi}{2R}$.

Lemma 7. *Boas' formula* (3) *is derived by putting* $y = y^*$ *in the S-formula for* $f'(x)$ *obtained in Lemma 4.*

Proof When $y = y^*$, $c_n(3, a) = c_n(3, a^*) = \frac{4}{\pi^2} \frac{(-1)^n}{(2n+1)^2}$, as seen from Table 1. ■

Theorem 6. *Boas' formula* (3) *is valid, without the additional assumptions on* $\alpha(t)$ *given in Theorem 3.*

Proof Lemmas 4 and 7 show that (3) takes the form of (19), and hence of the form (8), with $h(t) = h_{3,a^*}(t) = i\frac{t}{R}e^{-i\pi t/(2R)}$. For this $h_{3,a}(t)$, the validity of (3) is concluded by Lemma 1, with the aid of Remark 2 or (21) given below, without invoking Lemma 2. ■

Remark 4. *When* (3) *is proved in the proof of Lemma 7,* (16) *is expressed as follows:*

$$h_{3,a^*}(t) = i\frac{t}{R}e^{-i\pi t/(2R)} = \frac{4}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(2n+1)^2} e^{in\pi t/R}, \quad t \in [-R, R]. \tag{21}$$

By putting $t = R$ *in* (21)*, we obtain the summation formula* (4)*, which was used in the proof of Theorem 2.*

Proof of Proposition 1 This is a consequence of Lemmas 6 and 7. ■

Remark 5. *We can use the steps in Proposition 1 to derive* (3) *from* (5)*. In the course of the steps, it is assumed that* $\alpha(t)$ *is continuous at* $t = R$ *and* $t = -R$*. But in the final form* (3)*, the proof of Theorem 6 shows that Lemma 1 applies, and the additional assumption on* $\alpha(t)$ *is not necessary.*

3.2. Proof of Theorem 4

Theorem 7. *Let* $I_5(x)$*,*$h_{5,a}(t)$ *and* $c_n(5, a)$ *be defined by:*

$$I_5(x) = \frac{1}{R^3} f'''(x), \quad h_{5,a}(t) = -i\frac{t^3}{R^3}e^{iat}, \tag{22}$$

$$c_n(5, a) = \frac{1}{R^3}\frac{d^3}{da^3}\frac{S(a)}{D(a)} = -\frac{C(a)}{D(a)} + 3\frac{S(a)}{D(a)^2} + 6\frac{C(a)}{D(a)^3} - 6\frac{S(a)}{D(a)^4}, \tag{23}$$

where $C(a) = \cos(a \cdot R - n\pi)$*,* $S(a) = \sin(a \cdot R - n\pi)$ *and* $D(a) = a \cdot R - n\pi$*. Then* (19) *and* (16) *for* $l = 5$ *are valid.*

Lemma 8. *We put* $a = a^* := -\frac{\pi}{2R}$ *and* $t = R$*. Then* $D(a^*) = \frac{\pi}{2}(-1-2n)$*,* $C(a^*) = 0$*,* $S(a^*) = -(-1)^n$*,* $h_{5,a}(t) = h_{5,a^*}(t) = -1$*, and*

$$c_n(5, a^*) = -\frac{12}{\pi^2}\frac{(-1)^n}{(2n+1)^2} + \frac{96}{\pi^4}\frac{(-1)^n}{(2n+1)^4}. \tag{24}$$

Now (19) *and* (16) *are:*

$$f'''(x) = R^3 \sum_{n=-\infty}^{\infty} c_n(5, a^*) f(x + \frac{\pi}{2R} + \frac{n\pi}{R}), \tag{25}$$

$$h_{5,a^*}(R) = -1 = -\frac{12}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^2} + \frac{96}{\pi^4} \sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^4}. \tag{26}$$

By comparing (26) with (4), we have the well-known formula:

$$\sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{48}. \tag{27}$$

Comparing (25) with (3), we have:

$$f'''(x) + 3R^2 f'(x) = \frac{96R^3}{\pi^4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(2n+1)^4} f(x + \frac{\pi}{2R} + \frac{n\pi}{R}). \tag{28}$$

We confirm Theorem 4 by using this with (27).

4. Generalized Szegö Theorem and Its Proof

In [3], the Szegö theorem is given as follows.

Theorem 8. *Let $f_T(x)$ be the trigonometric polynomial of order $R_T \in \mathbb{Z}_{>0}$, given by* (1) *in Theorem 1, $M_T \in \mathbb{R}_{>0}$, $|f_T(x)| \leqq M_T$ hold for all $x \in \mathbb{R}$, and let $A_T(x)$ be defined by:*

$$A_T(x) = R_T \cdot \sigma_T(x) + f_T'(x) \cos\theta + \tilde{f}_T'(x) \sin\theta, \tag{29}$$

where θ is any real number, and

$$\sigma_T(x) = \frac{a_0}{2} + \sum_{n=1}^{R_T-1} (1 - \frac{n}{R_T})(a_n \cos nx + b_n \sin nx),$$

$$\tilde{f}_T'(x) = \frac{a_0}{2} + \sum_{n=1}^{R_T} n(a_n \cos nx + b_n \sin nx). \tag{30}$$

Then $|A_T(x)| \leqq M_T R_T$.

In the paper by Boas [4], the generalized Szegö theorem is given without proof. It is as follows.

Theorem 9. *Let $f(x)$, $\tilde{f}'(x)$ and $\tilde{f}(x)$ be defined by* (2), (13) *and* (14)*, respectively, and let $A(x)$ be defined by:*

$$A(x) := R \cdot \sigma(x) + \sigma_2'(x), \tag{31}$$

where

$$\sigma(x) := f(x) - \frac{1}{R}\tilde{f}'(x), \quad \sigma_2(x) := f(x)\cos\theta + \tilde{f}(x)\sin\theta, \tag{32}$$

and θ is any real number. If $|f(x)| \leqq M \in \mathbb{R}_{>0}$ holds for all $x \in \mathbb{R}$, then $|A(x)| \leqq MR$.

A proof of this theorem is given in Section 4.1 below.

Lemma 9. *By using the S-formulas for $f(x)$, $\tilde{f}'(x)$ and $\tilde{f}(x)$ obtained in Lemma 4, in (32), we obtain:*

$$\sigma(x)=\sum_{n=-\infty}^{\infty} f(y+\frac{n\pi}{R})\cdot\frac{1-\cos(a\cdot R-n\pi)}{(a\cdot R-n\pi)^2}, \tag{33}$$

$$\sigma_2(x)=\lim_{N\to\infty}\sum_{n=-N}^{N} f(y+\frac{n\pi}{R})\cdot\frac{\sin\theta+\sin(a\cdot R-n\pi-\theta)}{a\cdot R-n\pi}, \tag{34}$$

where $a=x-y$.

Remark 6. *From the rows for $l=1$ and $l=4$ in Table 1, the formula (19) for $\sigma(x)$ given by (32), becomes (33), and the corresponding Fourier series (16) is given by:*

$$(1-\frac{|t|}{R})e^{iat}=\sum_{n=-\infty}^{\infty}\frac{1-\cos(a\cdot R-n\pi)}{(a\cdot R-n\pi)^2}\,e^{in\pi t/R},\quad t\in[-R,R]. \tag{35}$$

Writing this formula (35) with $a\cdot R=-(\xi+1)\pi/2$ for $t=0$ and for $t=R$, we derive the summation formulas:

$$\sum_{m=-\infty}^{\infty}\frac{1}{(\xi+2m+1)^2}=\frac{\pi^2}{4}\sec^2\frac{\pi\xi}{2},\quad \sum_{m=-\infty}^{\infty}\frac{(-1)^m}{(\xi+2m+1)^2}=-\frac{\pi^2}{4}\sec^2\frac{\pi\xi}{2}\sin\frac{\pi\xi}{2}. \tag{36}$$

4.1. Proof of Theorem 9

In this section, we put $y^*=x-(\theta-\frac{\pi}{2})/R$ and $a^*=(\theta-\frac{\pi}{2})/R$.

We obtain the S-formula for $A(x)$ by using (33) and (34) in (31). We then put $y=y^*$ and $a=a^*$ in the obtained S-formula for $A(x)$. Then we obtain:

$$A(x)=\sum_{n=-\infty}^{\infty} f(y^*+\frac{n\pi}{R})c_n(6,a^*), \tag{37}$$

where

$$c_n(6,a^*)=R\frac{1-\sin\theta-\cos(\theta-\frac{\pi}{2}-n\pi)-\sin(-\frac{\pi}{2}-n\pi)}{(\theta-\frac{\pi}{2}-n\pi)^2}=\begin{cases}\frac{2R(1-\sin\theta)}{(\theta-\frac{\pi}{2}-2m\pi)^2}, & n=2m,\\ 0, & n=2m+1,\end{cases} \tag{38}$$

where $m\in\mathbb{Z}$. When $|f(x)|\leqq M$ for all $x\in\mathbb{R}$, we obtain $|A(x)|\leqq MR$ from (37) with (38), by using the first summation formula in (36).

Before we put $y=y^*$ and $a=a^*$, the S-formula for $A(x)$ is valid when an additional assumption on $\alpha(t)$ given in Theorem 3 is assumed. But the coefficients $c_n(6,a^*)$ in the final form (37) are such that the series (16) converges absolutely and uniformly, and hence Lemma 1 applies. As the result, the additional assumption on $\alpha(t)$ is not required in the validity of (37).

Remark 7. *Here we note that $A(x)$ given by (31) with (33) and (34) is expressed as (15), if we put*

$$I_6(x)=A(x),\quad h_{6,a}(t)=(R-|t|+i\cdot t\cdot\cos\theta+|t|\sin\theta)e^{iat}. \tag{39}$$

When y is chosen to be $y=y^*$, so that $a=a^*$, $h_{6,a}(t)=h_{6,a^*}(t)$ is continuous as a function of t in $[-R,R]$, and satisfies $h_{6,a^*}(-R)=h_{6,a^*}(R)=R$, and hence we obtain (37) by using the Fourier coefficients $c_n(6,a^*)$ of the Fourier series of $h_{6,a^*}(t)$, with the aid only of Lemma 1 and Remark 2.

5. Concluding Remarks

5.1. Concluding Remark 1

In the present paper, the function $f(x)$ is expressed in the form of the Stieltjes integral as (2), and hence the sampling theorem presented here applies to the cases when the spectrum is discrete as well as continuous.

If $f(x)$ is assumed to be expressed in terms of an integrable function $F(t)$ in $(-R, R)$, by

$$f(x) = \int_{-R}^{R} e^{ixt} F(t) dt, \tag{40}$$

then a very simple proof of the sampling theorem was presented by Boas [7] and Pollard and Shisha [8]. In this case, $\frac{1}{2R} f(-\frac{n\pi}{R})$ for $n \in \mathbb{Z}$ are the Fourier coefficients of $F(t)$, and hence

$$F(t) = \frac{1}{2R} \lim_{N\to\infty} \sum_{n=-N}^{N} f(\frac{n\pi}{R}) e^{-in\pi t/R}. \tag{41}$$

Substituting this into the right-hand side of (40) and performing the term-by-term integration, we obtain the Shannon sampling theorem (5) for $y = 0$. The term-by-term integration in this case is justified by the Lebesgue theorem [9] (p. 37).

It is recalled here that Campbell [10] presented the sampling theorem for the case when $F(t)$ is a distribution. In that paper, the author mentioned that the Shannon sampling theorem (5) for $y = 0$ is valid when $F(t)$ is a Dirac's delta function.

For the case when (40) applies, an extensive review of works related on the Shannon sampling theorem is found in the book [11]. In recent papers [12,13], extensions of the sampling theorem to the Hilbert and the Banach space are discussed.

5.2. Concluding Remark 2

In this section, we denote the scalar product of two functions $h(t)$ and $G(t)$ which are integrable in $(-R, R)$, by $(h, G) := \int_{-R}^{R} h(t)\overline{G(t)}dt$. The quantity calculated in (40) is expressed as (h, G) for $h(t) = e^{ixt}$ and $G(t) = \overline{F(t)}$. When the squares of both of these functions are integrable in $(-R, R)$, and the Fourier series of the two functions in the interval $(-R, R)$ are expressed as follows:

$$h(t) = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n \phi_n(t), \quad \overline{G(t)} = \frac{1}{2R} \lim_{N\to\infty} \sum_{n=-N}^{N} d_n \overline{\phi_n(t)}, \tag{42}$$

where $\phi_n(t) = e^{in\pi t/R}$, then (5) for $y = 0$ is the result of the Parseval formula $(h, G) = \sum_{n=-\infty}^{\infty} c_n d_n$.

We now note that the Parseval formula is achieved by using only one of the Fourier series either of $h(t)$ or $G(t)$ and integrating term by term, as

$$(h, G) = \int_{-R}^{R} [\lim_{N\to\infty} \sum_{n=-N}^{N} c_n \phi_n(t)] \overline{G(t)} dt = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n d_n, \tag{43}$$

or as

$$(h, G) = \frac{1}{2R}\int_{-R}^{R} h(t)[\lim_{N\to\infty}\sum_{n=-N}^{N} d_n\overline{\phi_n(t)}]dt = \lim_{N\to\infty}\sum_{n=-N}^{N} c_n d_n, \tag{44}$$

with the aid of the Lebesgue theorem [9] (p. 37). In fact, Boas [7] and Pollard and Shisha [8] proposed to use this calculation using the Fourier series of $\overline{G(t)} = F(t)$ which is assumed to be integrable in $(-R, R)$, without assuming that the square of $F(t)$ is integrable in $(-R, R)$, in deriving the Shannon sampling theorem, as stated above.

In the present paper, $\alpha(t)$ may be expressed as a sum of an absolutely continuous function and a step function, $F(t) = \frac{d\alpha(t)}{dt}$ is a distribution involving Dirac's delta function and hence we have to use the Fourier series of $h(t)$.

5.3. Concluding Remark 3

At the end of Section 1, it was mentioned that $|f(x)| \leqq T$, for the function $f(x)$ expressed as (2), where T is the total variation of $\alpha(t)$. If we use the first equation in (14) for $f'(x)$, we can confirm that $|f'(x)| \leqq TR$. In Remark 7, $A(x)$ defined by (29) is expressed as (15) by using (39). By confirming that $|h_{a^*}(t)| \leqq R$ by (39), we see that $|A(x)| \leqq TR$.

Hence Theorems 2 and 9 are proved in this way, if $M = T$.

5.4. Concluding Remark 4

In recent papers [14,15], a generalized Shannon sampling theorem is applied to non-limited-band signal, on the basis of an inequality given in a Boas's paper [7]. The inequality is given as in the following lemma.

Lemma 10. *Let $g(x) \in \mathcal{L}^1(\mathbb{R})$ and $G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)e^{-itx}dx$. Then*

$$\Big|g(x) - \lim_{N\to\infty}\sum_{n=-N}^{N} g(n)\frac{\sin\pi(x-n)}{\pi(x-n)}\Big| \leq \frac{1}{\pi}\int_{|t|>\pi}|G(t)|dt. \tag{45}$$

We put $g(x) = f(\frac{x\pi}{R})$ in (45), and then we obtain

Lemma 11. *Let $f(x) \in \mathcal{L}^1(\mathbb{R})$ and $F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-itx}dx$. Then*

$$\Big|f(x) - \lim_{N\to\infty}\sum_{n=-N}^{N} f(\frac{n\pi}{R})\frac{\sin(xR-n\pi)}{xR-n\pi}\Big| \leq \epsilon(R) := \frac{1}{\pi}\int_{|t|>R}|F(t)|dt. \tag{46}$$

In confirming this, we note that $G(t) = \frac{R}{\pi}R \cdot F(\frac{R}{\pi}t)$.

We find that this lemma is a part of Theorem 3.12 in the book [11].

When $f(x) = e^{-x^2/(2\sigma^2)}$, $F(t) = \frac{\sigma}{\sqrt{2\pi}}e^{-\sigma^2t^2/2}$ and hence

$$\epsilon(R) = \frac{\sqrt{2}\sigma}{\pi^{3/2}}\int_{R}^{\infty} e^{-\sigma^2t^2/2}dt < \frac{\sqrt{2}}{\pi^{3/2}\sigma R}e^{-\sigma^2R^2/2}. \tag{47}$$

In [15], the case of $f(t) = e^{-\pi t^2}$ is studied. We now recommend to use the inequality (46) in the study.

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Author Contributions

Ken-ichi Sato showed Boas' formula to Morita and asked the possibility of deriving it from the sampling formula which appears in the Shannon sampling theorem. Answering this question, Tohru Morita proposed such a modified sampling formula that its derivative gives Boas' formula, and wrote a manuscript to explain it. Tohru Morita and Ken-ichi Sato kept revising the manuscript to the present form.

Conflicts of Interest

The author declares no conflict of interest.

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