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Communication

Joint Distributions and Quantum Nonlocal Models

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Abstract: A standard result in quantum mechanics is this: if two observables are commuting then they have a classical joint distribution in every state. A converse is demonstrated here: If a classical joint distribution for the pair agrees with standard quantum facts, then the observables must commute. This has consequences for some historical and recent quantum nonlocal models: they are analytically disallowed without the need for experiment, as they imply that all *local* observables must commute among themselves.

Keywords: quantum nonlocal models; testing nonlocal models; quantum conditional probability; commutativity; joint distributions

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1. Introduction

Consider two commuting quantum observables. It is standard that they have a classical joint probability distribution, and this is the well-known result of von Neumann relating to the construction of spectral measures for quantum observables; see [1,2]. Here a converse to the von Neumann result is derived. This result has consequences for versions of quantum nonlocal models that assert or imply the existence of joint distributions for pairs of local observables: Under these models it is shown that all local observables must commute among themselves. Hence these models can be disallowed analytically and without the need for refined experiments or concerns for loophole avoidance.

One example of a nonlocal model that starts from probability assumptions is the familiar Bell-CHSH setup; see [1,2] for example. This family has been studied experimentally and analytically for several decades and with varying, increasingly stronger claims for loophole avoidance. More recent examples of quantum nonlocal models have also been driven by joint probability and distribution assumptions, and these include those proposed in Leggett [3], Eisaman *et al.* [4], Gröblacher *et al.* [5], and Branciard *et al.* [6,7].

In outline, all these nonlocal models introduce, under a no-signaling condition, classical probability distributions for separated observables. Precise statements of these models are given in Section 6. Using just arguments from classical probability, it is then seen that the models necessarily assert classical joint distributions for pairs of local observables, those pairs *within* each lab. Next, the main technical result, given in Section 7, shows that the existence of a classical joint distribution function over any pair of quantum observables—if it agrees with the usual and testable quantum facts—implies that the pair must commute. Hence under the probability distributions assumed by these nonlocal models, since they imply the existence of these joint distributions for local observables, must in turn imply that all local observables must commute among themselves.

Effectively, any carefully designed experimental disallow of these models only implies that some local observables do not commute. In still other words, the initial assumptions of these models are too strongly grounded in classical probability assumptions and only a classical, non-quantum outcome is possible.

The discussion opens by recalling some standard facts about probability, joint distributions, joint measurements, and quantum outcomes. In particular, an essential distinction is noted between quantum joint measurements and classical joint distributions. Next, the technical details of the main argument are presented over several sections, and then finally application is made to recent models for quantum nonlocality.

2. Background on Probability and Joint Distributions

Suppose given two projector observables, *A* and *B*, each taking on the values a, b = 0 or 1, and recall that such observables are defined by the property that $A^2 = A, B^2 = B$. Next, a 2×2 table of nonnegative values over the four pairs of zero/one outcomes will define a joint distribution function if all the values are in the interval [0, 1] and if the assignments to the four cells add up to 1. That's all that is required for the construction of an arbitrary classical joint distribution, $\Pr(a,b) = \Pr(A = a, B = b)$, and there are generally an infinite number of ways of assigning probabilities to the cells.

Next consider how the marginal probabilities, $\Pr(A = a)$ and $\Pr(B = b)$, might fit together consistently with the joint distribution. There always is at least one joint distribution function that agrees with these marginals, namely the product distribution function:

$$\Pr(A = b, B = b) = \Pr(A = a) \times \Pr(B = b) \tag{1}$$

for $a, b = 0$ or 1. But suppose it is asked that this joint distribution survive a minimal test of quantum experimental consistency by requiring that the marginal probabilities for *A* and *B* agree with standard quantum facts. Then, following the usual form of the Born trace-probability rule—for which see [1]—and for any quantum state *D*, it must be the case that for projector observables *A*, *B*:

$$\Pr(A = 1) = tr[DA] \text{ and } \Pr(B = 1) = tr[DB] \tag{2}$$

Note that a model having a joint distribution for outcomes of the product form, as in Equation (1), may or may not agree with other theoretical or experimental properties of the observables, even if the marginal probabilities in Equation (2) are correct.

Next, recall from classical probability—for which again see [1]—that any joint distribution function, $\Pr(A = a, B = b)$, is specified uniquely and exactly by its marginal values, $\Pr(A = a)$, and $\Pr(B = b)$, *together* with its two conditional distributions, $\Pr(A = a \mid B = b)$ and $\Pr(B = b \mid A = a)$. Given these specifications it is then required that, in a simplified but obvious notation:

$$\Pr(A, B) = \Pr(A \mid B) \times \Pr(B) = \Pr(B \mid A) \times \Pr(A) \tag{3}$$

Again, this joint distribution function by itself does not name or require a feasible experimental procedure for joint outcomes and joint measurement; more on this below.

Continuing, given projector observables, if the joint distribution is required to be consistent with standard quantum facts then necessarily the conditional distributions for these projectors must be such that:

$$\begin{aligned} \Pr(A \mid B) &= tr[DBAB] / tr[DB] \\ \Pr(B \mid A) &= tr[DABA] / tr[DA] \end{aligned} \tag{4}$$

As background for Equations (2) and (4) see [1,2].

Finally, if the marginal and the conditional probabilities are *both* required to agree with standard quantum facts then from Equations (2) and (4):

$$\begin{aligned} \Pr(A, B) &= (tr[DBAB] / tr[DB]) \times (tr[DB]) \\ \Pr(A, B) &= (tr[DABA] / tr[DA]) \times (tr[DA]) \end{aligned} \tag{5}$$

By inspection this implies

$$tr[DBAB] = tr[DABA] \tag{6}$$

As shown below, this simple equation, if valid for certain states, eventually leads to this: $AB = BA$.

Observe that no additional mathematical assumptions or model variants have been introduced in order to arrive at Equation (6). It is only necessary that some joint distribution function should exist that is consistent with the usual experimentally testable quantum facts as given in Equations (2) and (4). No experimental scheme or joint measurement process is assumed or required for the pair of observables, and both necessary and sufficient here is the agreement of the marginal and conditional probabilities, as in Equations (2) and (4), with standard quantum outcomes.

3. Joint Measurements or Joint Distributions?

The main result presented below is that requiring the marginals and the conditionals of a joint distribution function to agree with experiment and the usual quantum facts usually leads to a contradiction unless the observables were commutative in the first place. Re-expressed, there are necessary and unavoidable consequences for assigning any joint probability distribution for pairs of observables, if these mathematical objects are to agree with experiment and standard quantum facts.

However, there is an alternative and apparently common outlook on joint probability distributions that appears in the quantum physics literature. It is important to distinguish this view from the classical probability framework considered here. The standard outlook begins this way: If a pair of observables has a shared basis of eigenvectors then they have a joint distribution. This is certainly valid and well known in any discussion of quantum observables considered as Hilbert space operators. See for example ([8]; Theorem 2.2). In words, given a shared eigenbasis the pair can be measured jointly in this sense: applying one operator to the system leaves the state of the system unchanged, and the second operator can then be applied to obtain a "joint measurement." Thus, a pair of outcomes is always unambiguously obtained, given any initial state, and it makes sense and is routine and reasonable to call the paired outcome the joint measurement. And this joint measurement then leads to a classical joint probability distribution function over the pair, exactly as in Section 2, that is quantum consistent for both the marginal and conditional probabilities.

But note that the eigenbasis property of the observables is a Hilbert space property of the pair and is not itself a property derived by consideration of a classical joint distribution that might be defined or constructed in terms of marginal and conditional probabilities. If a shared eigenbasis exists then so does the classical joint probability and the quantum joint measurement. But this is also true: specifying marginal and conditional probabilities is classical probabilistic task not dependent on locating a shared eigenbasis.

The result below offers a precise characterization of joint probability distributions over quantum observables, when presented as a classical statistical problem. There is a substantial history of the problem of joint distributions for quantum outcomes; see for example ([1]; Section 3.3). It is assumed in all that follows that quantum outcomes are observable, and result from measurement and experiment. It is also assume throughout that quantum observables have a finite spectrum; see [1] for details. The result presented here is conclusive, once agreement with simple, testable experimental outcomes is assumed necessary.

4. Quantum Outcomes and Joint Distributions

Consider any model for quantum outcomes that asserts agreement with conventional quantum facts. As discussed above, if the model asserts the existence of joint distributions over a pair of projector observables, it should plausibly return single and conditional distributions, as given by standard quantum facts. Hence two properties seem reasonable for a model that is consistent with quantum outcomes for arbitrary quantum projectors A, B:

(*i*) both marginal probabilities for A and B, those on the left in Equation (2), agree with the Born trace-probability rule values given on the right in Equation (2);

(*ii*) both conditional probabilities, A given B, and B given A, those on the left on Equation (4), agree with the conditional probability rule values given on the right at Equation (4).

Some comments are in order. *First*, by an application of the Lagrange interpolation formula, every projector appearing in a spectral decomposition of an observable can be written as a real polynomial in that observable. It follows that an arbitrary pair of observables has a classical joint distribution if and only if the collection of their spectral decomposition projectors do as well. Hence the assertion of a joint distribution consistent with quantum facts can be evaluated in terms of the corresponding projectors. For example, if a pair of spin observables, A, B, taking on values $\{+1,-1\}$, have a classical joint distribution then so must all four of their associated projectors: $\{A^+, B^+\}$, $\{A^+, B^-\}$, $\{A^-, B^+\}$, $\{A^-, B^-\}$.

Second, quantum conditional probability is most naturally defined in terms of projectors, rather than arbitrary observables; see for example ([1]; Chapter 26).

Third, the conditional probability requirement for projectors as in (*ii*) is just that for the marginal requirement (*i*) when the state is changed in the usual way, since:

$$\Pr(A \mid B) = tr[DBAB] / tr[DB] = tr[(D_B)A] = \Pr_{D_B}(A = 1) \tag{7}$$

for state $D_B = BDB / tr[DB]$. Hence, if Condition (*i*) is valid then Condition (*ii*) is experimentally testable using just marginal outcomes.

Assume the system is in state D and refer to the Conditions (*i*) and (*ii*) above. Then the remarks above motivate these definitions:

Definition. If two projectors have a classical joint distribution satisfying (*i*) and (*ii*) they are said to have a *quantum consistent* joint distribution. An arbitrary pair of observables has a *quantum consistent* joint distribution if every pair of their spectral projectors does so. A model for quantum outcomes is *quantum consistent* if every pair of observables has a quantum consistent joint distribution.

The precise connection between joint distributions and commutativity for quantum observables is now taken up.

5. Joint Distributions and Commutativity

Given projectors A, B, a useful relation between the commutator $C = [A, B] = AB - BA$, and the compound commutator $[AB, BA]$, is: $C^* C = [A, B]^* [A, B] = (A - B)[AB, BA]$. Define the observable

$$G = G(A, B) \equiv [AB, BA] = ABA - BAB \tag{8}$$

Then $(A - B)G = G(A - B) = CC^*$. Therefore there exists a joint spectral resolution of G, $A - B$, and CC^*, such that

$$G = \Sigma \lambda_i P_i, \quad A - B = \Sigma \zeta_i P_i, \quad CC^* = \Sigma \lambda_i \zeta_i P_i \tag{9}$$

for orthogonal projectors P_i, and eigenvalues λ_i, ζ_i. Let $S(CC^*)$ be the set of all those projectors P_i appearing nontrivially in the decomposition for CC^*, so for such i: $\lambda_i \zeta_i > 0$. In particular, if $CC^* = G(A - B) \neq 0$ then there exists at least one projector P_i such that $tr[P_i CC^*] = \lambda_i \zeta_i \neq 0$. For a system in state $D = P_i$ it follows from [9; Proof of *Theorem* 1, Case (*a*) and Case (*b*)] that $AP_i \neq 0$, $BP_i \neq 0$. Hence $tr[AP_i] = tr[(AP_i)^*(AP_i)] \neq 0$, and $tr[BP_i] = tr[(BP_i)^*(BP_i)] \neq 0$. Thus under the assumption that A, B have a joint distribution for the system in state $D = P_i$, the univariate marginals for both A and B are nonvanishing. Therefore, for $D = P_i$:

$$tr[DA] \neq 0, tr[DB] \neq 0 \text{ and } tr[DABA] = tr[DBAB] \tag{10}$$

so that

$$tr[DG] = 0 \tag{11}$$

For convenience, and it is hoped without confusion, call a pair of projectors *quantum consistent* if they have a quantum consistent joint distribution. Similarly for a pair of observables, they are *quantum consistent* if every pair of their spectral projectors has a quantum consistent joint distribution. An important consequence of quantum consistency is this:

Lemma 1. Consider two projectors, *A*, *B*, with the associated decomposition $\{P_i\}$ as in Equation (9). If *A* and *B* are quantum consistent for every state of the form $P = \Sigma\delta_i P_i$, with all $\delta_i > 0$, and P_i in *S*(*CC**), then *A* and *B* commute.

Proof. For *G* as in Equation (8), if $G(A-B)=0$ then $CC^*=0$ and $AB=BA$, so assume $CC^* = G(A-B) \neq 0$. It follows that $tr[PCC^*] = \Sigma\,\lambda_i\zeta_i\delta_i > 0$. Hence

$$tr[PCC^*] = tr[\Sigma\,\delta_i P_i(A-B)G] = \Sigma\,\lambda_i\delta_i tr[P_i G] > 0 \tag{12}$$

Given quantum consistency for *A*, *B*, in every decomposition state P_i, and using (11), it follows that $tr[P_i G] = 0$, for all *i*. But this contradicts Equation (12), so only $G(A-B)=0$ is possible, and the result follows.

The main result on joint distributions and quantum outcomes is this immediate consequence of *Lemma* 1:

Theorem 1. If two observables are quantum consistent for every state then they must commute.

Proof. By definition two observables have a quantum consistent joint distribution for all states if all their spectral projector pairs do so. From *Lemma* 1, it follows that all the pairs commute, and so the observables do as well.

Some remarks are in order concerning quantum consistency for arbitrary models for quantum outcomes. Suppose a model for quantum outcomes induces or implies the existence of a joint distribution for every pair of projectors, *A*, *B*. And suppose a specific pair is commuting. Then for the model to be valid for quantum outcomes it should minimally satisfy Equation (2). Next, suppose the pair is not necessarily commuting, and consider that the model is under experimental control and study. Then the conditional probability rule at Equation (4) can be validated—or not—by experiment on that pair. If the rule is confirmed than the pair is quantum consistent in terms of the stated joint distribution function. In which case *Theorem* 1 applies, and the pair must be commuting after all. But then the conclusion is that the model itself is not quantum consistent. On the other hand, examining a given quantum model by a search over all quantum pairs is prohibitive and also unnecessary, as it is always possible to challenge a model using a single pair of observables. This is taken up next, specifically for the recently devised quantum nonlocal models.

6. The Leggett-Branciard Nonlocal Model

Leggett [3] proposed a nonlocal model of considerable interest. A less constrained version of the Leggett model was given in Branciard *et al.* [7], and this is considered here.

Consider binary outcomes $\alpha, \beta = \pm 1$, for observables A, B, corresponding to spin observables measured in the **a**, **b**, directions in two separated labs. Using the notation in [7] the model assumes that the probability of the event (α, β) is given by

$$P_\lambda(\alpha,\beta\,|\,\mathbf{a},\mathbf{b})=\frac{1}{4}(1+\alpha M_\lambda^A(\mathbf{a},\mathbf{b})+\beta M_\lambda^B(\mathbf{a},\mathbf{b})+\alpha\beta C_\lambda(\mathbf{a},\mathbf{b}))\tag{13}$$

where: (*a*) the model is parameterized by an unobserved nonlocal variable λ; (*b*) the state of the system is initially left unspecified in Equation (13) but for showing how the model might be disallowed it is usually assumed to be an entangled state; (*c*) the two terms $M_\lambda^A(\mathbf{a},\mathbf{b})$, $M_\lambda^B(\mathbf{a},\mathbf{b})$ refer to marginal expectations for the observables *A*, *B*, at each unobserved but fixed value of λ, and where the marginal, within-lab, expectations might be dependent on settings in the other lab; and (*d*) the term $C_\lambda(\mathbf{a},\mathbf{b})$ in Equation (13) is the "correlation" of the pair *A*, *B*, at each fixed value of λ.

Some elementary but clarifying comments are useful. *First*, Equation (13) is simply a parameterized version of a result from classical probability, here re-written to allow for a possible dependence on λ and a possible dependence on settings in the separated labs, that is, a nonlocal interaction.

Second, the expectation functions *M* are not $\lambda-$level *probabilities*, and these expectations can be zero without the underlying probability functions being identically zero.

Third, the function *C* is, at each fixed λ, the expectation of the product of the observables, so *C* is not a correlation function unless the *M* functions, that is the individual expectations, are themselves identically zero.

And *Fourth*, suppose the hidden variable λ is assumed to operate as a classical, though unobserved random variable. Then integrating (summing) over Equation (13) with respect to the appropriate density function for λ necessarily returns a classical joint probability function for the pair *A*, *B*. Quantum consistency of this joint distribution is then an appropriate question for theory and experiment.

Now it is standard in experimental study of a nonlocal model that the system is assumed to be in a singlet state. And then for binary $\{+1,-1\}$ spin observables the expected values for the observables are the ± 1 weighted average of these outcomes, namely zero. As the model assumes that the *M* functions should agree with the required marginal expectations, the function *C* in Equation (13) does correctly return the observed correlation function for the observables in this state.

However, any analytic derivation showing that the *M* functions are zero does not invalidate the model, and indeed only confirms that the model returns the correct singlet state expectations. If the *M* functions were probability functions rather than expectations, then such a demonstration would indeed invalidate the model.

Still more simply, for an observable *A* that takes on values $\{+1,-1\}$ the relation is $E(A)=2\Pr(A)-1$, where $E(A)$ is the expectation of *A*, and $\Pr(A)$ is the probability that the outcome is +1. For *projectors* the equivalent relation is $E(A)=\Pr(A)$. So $E(A)=0$ would imply an analytic disallowal of the model over projectors, but not for the spin observables as in [7]. There it implies $\Pr(A)=1/2$ exactly as it should be for measurements in the standard entangled state.

At this point it is helpful to consider the derivation in [7] showing that the *M* functions must vanish. Under the *no-signaling condition* it is assumed that $M_\lambda^A(\mathbf{a},\mathbf{b})=$ $M_\lambda^A(\mathbf{a},\mathbf{b})=$ $M_\lambda^A(\mathbf{a})$, and $M_\lambda^B(\mathbf{a},\mathbf{b})=M_\lambda^B(\mathbf{b})$, In ([7]; Methods section) it is shown that a simple and important inequality connects expectations over a triple $(\mathbf{a},\mathbf{b},\mathbf{b'})$ of spin measurements, where measurements in the $\mathbf{b},\mathbf{b'}$ directions are made in one lab and $\mathbf{a}$ is made in a separated lab. That is:

$$|C_{\lambda}(\mathbf{a},\mathbf{b}) \pm C_{\lambda}(\mathbf{a},\mathbf{b'})| \leq 2 - |M_{\lambda}^{B}(\mathbf{b}) \mp M_{\lambda}^{B}(\mathbf{b'})| \tag{14}$$

From this, as accurately analytically demonstrated in [7], it follows that any no-signaling model for a system in an entangled state must have vanishing marginals: $M_{\lambda}^{A}(\mathbf{a}) = M_{\lambda}^{B}(\mathbf{b}) = 0$.

It is noteworthy that this conclusion is derived in [7] by integrating over λ with respect to a suitable density function for λ, exactly as suggested in the *Fourth* comment above following Equation (13). The point here is that integrating out the hidden variable should return probabilities and facts that are consistent with standard quantum outcomes. This is exactly the claim made in [7], that the vanishing marginals are a foreshadowing of the experimental conflict with Equation (13) and standard quantum facts. However, as noted above just before Equation (14), the *M* functions are marginal expectations as taken in the singlet state, and they are not marginal probabilities.

To summarize: Given the nonlocal model, as at Equation (13), there are consequences for the existence of joint distributions over triples of measurements, as in Equation (14). Reasoning as above the inequality at Equation (14) correctly shows that expectations in the model must be zero, but not that marginal probabilities are necessarily zero. If the latter were true then Equation (14) would indeed disallow the model. Yet, by using the main result above, the probability model at Equation (13) applied to certain triples of measurements, does indeed lead to a basic model conflict with standard quantum facts. This is now presented.

7. The Leggett-Branciard Model and Joint Distributions

Consider two pairs of observables, (A,B) and (A,B'), where, as above, observables B and B' are in one lab and A is in a separated lab. Under the no-signaling condition the univariate marginal expectation for A in each pair remains the same, independently of settings in the opposite lab. Hence, under the no-signaling condition, the marginal probabilities within each lab are uniquely specified by the *M* expectation functions.

Next, consider another result from classical probability: Given a joint probability distribution for the pair (A,B) and another for the pair (A,B'), such that the marginal probability for A is the same in each, there exists a joint probability distribution function for the triple (A,B,B'). If the marginal probability for A is $f_A(a)$, while the two joint distributions are $f_{A,B}(a,b)$, and $f_{A,B'}(a,b')$, then a valid joint distribution for the triple is given by

$$f_{A,B,B'}(a,b,b') = f_{A,B}(a,b)\, f_{A,B'}(a,b') \,/\, f_A(a) \tag{15}$$

For complete details on the connections between marginal and joint probabilities see ([2], *Theorems* 1 and 2).

Three remarks are in order. *First*, invoking a connection between the observables (B,B') in the same lab is an essential feature of the derivation of Equation (14), as in [7];

Second, while the joint distribution over the triple is possibly dependent on the choice of observable *A*, this plays no part in what follows, where attention is focused on the joint distribution for the pair of local observables (B,B');

And *Third*, the same argument leading to Equation (15) is valid if any of the random variables are themselves vector-valued, for example if *A* is bivariate and $A = (A_1, A_2)$. In this case a joint distribution over (A_1, A_2, B, B') is given by an appropriately notated, right hand side of Equation (15). And then integrating A_1 with respect to a suitable density returns a joint distribution for (A_2, B, B').

Now, using the nonlocal joint probability model at Equation (13) and the classical probability result at Equation (15), the joint distribution for the triple (A, B, B') is immediately given by:

$$P_\lambda(\alpha, \beta, \beta' \mid \mathbf{a}, \mathbf{b}, \mathbf{b}') = \{P_\lambda(\alpha, \beta \mid \mathbf{a}, \mathbf{b}) P_\lambda(\alpha, \beta' \mid \mathbf{a}, \mathbf{b}')\} \,/\, \{P_\lambda(\alpha \mid \mathbf{a})\} \tag{16}$$

Summing over $A = \alpha = \pm 1$, and integrating over λ with respect to a suitable density always returns a joint probability distribution for the pair (B, B'). This step is a version of that given in the *Third* remark just above and is only classical probability, requiring no quantum mechanical side conditions or assumptions.

Three remarks are in order. *First*, invoking a connection between the observables (B, B') in the same lab is an essential feature of the derivation of Equation (14), as in [7]; *Second*, similar to the discussion above, the existence of a joint distribution over the triple (A, B, B') implies a joint distribution over all the projectors in the spectral decomposition of the triple. And if all the local projectors should happen to commute then so must the original spin observables within each lab; *Third*, while the joint distribution over the triple is possibly dependent on the choice of observable *A*, this plays no part in what follows, where attention is focused on the joint distribution for the pair of local observables (B, B'). The existence of a joint probability distribution for the pair (B, B') in the same lab, given any state of the system, has consequences for the model if it is assumed to return valid and testable outcomes.

From these remarks, and using *Theorem* 1, the following is immediate:

Theorem 2. If the model in Equation (16) is quantum consistent, then every pair of local observables within each of the separated labs must commute.

In other words, if the Leggett-Branciard quantum nonlocal model agrees with standard testable properties for quantum outcomes for local observables, then it implies that all local observables commute among themselves. More precisely, the Leggett-Branciard model, with Equation (13) and no-signaling, always returns a joint distribution for local pairs of observables by integrating over *A* and the hidden variable. And as in the discussion following Equation (14) this integration has local consequences. In particular, the model then implies the existence of a joint distribution for each pair with univariate marginal and conditional probabilities, but this joint distribution is not quantum consistent unless the pair commutes.

Two worked examples are useful:

Example (1) Under no-signaling and the assumption that Equation (13) is valid for any state of the system, consider a pure state of the compound system. Given arbitrary projectors, B, B', in the same lab, select any two orthonormal eigenvectors φ_1, φ_2, in a decomposition for $G = G(B, B')$ with eigenvalues λ_1, λ_2. If both are zero then $G = 0$ so B and B' must commute. So suppose $\lambda_1 \neq 0$ and let the system state be $\varphi = \varphi_1 \otimes \varphi_2$. Write $\tilde{A} = A \otimes I$, $\tilde{B} = B \otimes I$, and $\tilde{G} = G(\tilde{B}, \tilde{B}') = G \otimes I$. It is straightforward to show that if (B, B') have a joint distribution then so must $(\tilde{B}, \tilde{B}')$. But then $\tilde{G}\varphi = \lambda_1 \varphi \neq 0$, and this contradicts $tr[D\tilde{G}] = 0$ for state $D = |\varphi\rangle\langle\varphi|$, using Equation (5.4). Hence $\tilde{G} = 0$, and also $G = 0$. Therefore *A* and *B* commute. Since the argument applies to any pair of observables under the model at Equation (13) with no-signaling, it follows that all local observables commute.

Example (2) Select any two orthonormal vectors φ_1, φ_2, and consider the entangled states

$$\varphi^+ = (1/\sqrt{2})(\varphi_1 \otimes \varphi_2 + \varphi_2 \otimes \varphi_1), \quad \varphi^- = (1/\sqrt{2})(\varphi_1 \otimes \varphi_2 - \varphi_2 \otimes \varphi_1) \tag{17}$$

and define projector observables and a subsequent compound function:

$$\begin{aligned} &A = P_{\varphi_1}, B = P_{\varphi_1 + \varphi_2}, \\ &\tilde{G} = \tilde{G}(A, B) = G(A \otimes B, B \otimes A) \end{aligned} \tag{18}$$

Then

$$\begin{aligned} 1/4 &= \left\langle \varphi^+ \middle| (A \otimes B)\varphi^+ \right\rangle = \left\langle \varphi^- \middle| (A \otimes B)\varphi^- \right\rangle \\ &= \left\langle \varphi^+ \middle| (B \otimes A)\varphi^+ \right\rangle = \left\langle \varphi^- \middle| (B \otimes A)\varphi^- \right\rangle \\ \tilde{G}\varphi^+ &= (1/8)\varphi^-, \ \tilde{G}\varphi^- = (1/8)\varphi^+ \end{aligned} \tag{19}$$

Under Equation (13) and no-signaling, the projectors A, B have a joint distribution, and so also must $A \otimes B, B \otimes A$. For the system in state $\varphi = (1/\sqrt{2})(\varphi^+ + \varphi^-) = \varphi_1 \otimes \varphi_2$, it follows that $\left\langle \varphi \middle| \tilde{G}\varphi \right\rangle = 1/8 \neq 0$, and this contradicts Equation (11) above. Hence A, B cannot have a quantum consistent joint distribution as would follow from Equation (13).

Note that *Example* (1) disallows the Leggett-Branciard model using an arbitrary choice of projectors acting on a compound system in a pure state, while *Example* (2) disallows the Leggett-Branciard model using a superposition over two arbitrary entangled states, and two projectors defined by that state. For either *Example* it might be argued that the proposed nonlocal model is assumed to be valid only under some states and not necessarily others, or, only for some observables and not others. That is, it might be asserted to be valid under only a single entangled state, and not any pure state, and not in any superposition of entangled states. However these types of constraints are not broadly evident in any discussions of nonlocality, or on display in the model specifications given in [7].

8. Conclusions

Some nonlocal models proceed by introducing joint probability distributions for quantum observables. One such model is given by the classic Bell-CHSH construction, as discussed in [2]. A more recent model was introduced by Leggett in [3] and a less restricted version of the model was then considered by Branciard *et al.* [6,7], and others [4,5]. These models necessarily imply joint distributions for within-lab, local observables. Independently, a converse to a standard result on pairs of commuting observables has been derived here, so that any experimentally verifiable model leading to the existence of a joint distribution for a pair of quantum observables must also require the pair to commute. Hence these models can be disallowed analytically, and for a given pair of local observables no experimental verification of noncommutativity seems necessary.

In summary, the assertion of joint distributions for noncommuting quantum observables causes nontrivial problems for any quantum probability model, nonlocal or otherwise. More generally, it is not clear whether it is ever necessary to undertake precise and complex experiments to disallow any model, nonlocal or otherwise, that analytically requires or necessarily implies joint distributions over noncommuting observables.

Conflicts of Interest

The authors declare no conflict of interests.

Author Contributions

JDM invented the original idea. AF helped develop it. Both contributed to writing and editing.

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