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Article

Bell Length as Mutual Information in Quantum Interference

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Abstract: The necessity of a rigorously operative formulation of quantum mechanics, functional to the exigencies of quantum computing, has raised the interest again in the nature of probability and the inference in quantum mechanics. In this work, we show a relation among the probabilities of a quantum system in terms of information of non-local correlation by means of a new quantity, the Bell length.

Keywords: quantum potential; quantum information; quantum geometry; Bell-CHSH inequality

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1. Introduction

For about half a century Bell's work has clearly introduced the problem of a new type of "non-local realism" as a characteristic trait of quantum mechanics [1,2]. This exigency progressively affected the Copenhagen interpretation, where non–locality appears as an "unexpected host". Alternative readings, such as Bohm's one, thus developed at the origin of Bell's works. In recent years, approaches such as the transactional one [3–5] or the Bayesian one [6,7] founded on a "de-construction" of the wave function, by focusing on quantum probabilities, as they manifest themselves in laboratory. Nonetheless, according to the authors, the peculiar distribution of the probabilities makes necessary a theoretical scenario in which it is possible to characterize the type of information/entropy associated with quantum events. In this work we utilize a Bohmian entropy which allows a new physical meaning of Bell-CHSH inequalities based on a new measuring system of non-local correlations to be provided.

The paper is structured as follows. In chapter 2 we will review the salient results of the debate on non-locality and contextuality; in chapter 3 we will introduce the entropic approach and the Bell length; finally, in chapter 4, we will re-write the Bell-CHSH inequalities by using the Bell length.

2. Non-Locality: A Survey

The Bell inequalities concern measurements made by observers on pairs of particles that have interacted and then are separated. According to the quantum mechanical formalism they are entangled, while local realism would limit the correlation of subsequent measurements of the particles. On the basis of the Bell inequalities, entanglement became thus a property of states which could not be described by local realistic theories. Many formulations of the Bell inequalities exist. For example, the well known Clauser, Horne, Shimony, and Holt (CHSH) inequality and Clauser and Horne (CH) inequality [8,9] are used for the verification of nonlocal correlations in a two-dimensional Hilbert space. The Bell inequality of a qubit pair in the form derived by Clauser, Horne, Shimony and Holt can be formulated as

$$Tr\left|\left(\rho \mathrm{B}_{CHSH}\right)\right| \leq 2 \tag{1}$$

where the Bell-CHSH operator is

$$\mathrm{B}_{CHSH} = \vec{a} \cdot \vec{\sigma} \otimes \left(\vec{b} + \vec{b}'\right) \cdot \vec{\sigma} + \vec{a}' \cdot \vec{\sigma} \otimes \left(\vec{b} - \vec{b}'\right) \cdot \vec{\sigma} \tag{2}$$

and its expected value is maximized over real-valued three-dimensional unit vectors $\vec{a}$, $\vec{a}'$, $\vec{b}$ and $\vec{b}'$ (here $\vec{\sigma}$ are the usual Pauli matrices). The states for which $Tr\left|\left(\rho \mathrm{B}_{CHSH}\right)\right| > 2$ are the ones which cannot be described by a realistic local theory and are thus associated with non-classical correlation and non-locality.

In virtue of the work of R. F. Werner [10], the Bell inequalities provided an operative definition of entangled states, indicating the formal conditions for what Einstein called "elements of physical reality", namely states which are not obtainable through local operations and classical information. In Werner's approach, separable mixed states can be written as

$$\rho_{AB} = \sum_k p_k \rho_A^{(k)} \otimes \rho_B^{(k)} \tag{3}$$

with $0 \leq p_k \leq 1$ and $\sum_k p_k = 1$, thus their correlation derives from the probabilities p_k. Werner's results suggest that mixed separable entangled states may exist that do not violate Bell inequalities! Entanglement is therefore not a very strong condition in order to characterize non-locality. On the other hand, from the point of view of quantum field theory (QFT), a result of this type seems to be unsurprising [11].

In a recent paper, two-qubit mixed states were shown to be more entangled than pure states: a comparison of the relative entropy of entanglement for a given nonlocality by Horst, Bartkiewicz and Miranowicz [12] showed a degree of violation of Bell-CHSH inequality and thus of quantum non-locality which was evaluated by using the function

$$B(\rho)=\sqrt{\max[0,M(\rho)-1]} \tag{4}$$

where

$$M(\rho)=\max_{j<k}\{h_j+h_k\}\leq 2 \tag{5}$$

where h_j are the eigenvalues of the real symmetric matrix $U=T^T T$, $T_{ij}=Tr\lfloor\rho(\sigma_i\otimes\sigma_j)\rfloor$ being the correlation matrix, whilst the degree of entanglement was measured by using the relative entropy of entanglement defined as

$$E_R(\rho)=\min_{\sigma\in D} S(\rho//\sigma) \tag{6}$$

which is the relative entropy

$$S(\rho//\sigma)=Tr(\rho\log_2\rho-\rho\log_2\sigma) \tag{7}$$

minimized over the set D of separable states σ. However, the relative entropy of entanglement used by Horst, Bartkiewicz and Miranowicz is limited to the possibility of distinguishing a density matrix ρ from the closest separable state σ only and does not provide a measure of the relative entropy of entanglement of a general two-qubit mixed state.

In the case of two qubits A and B described by the density operator ρ_{AB}, an interesting measuring system of the degree of entanglement is provided by Hall's quantum correlation distance [13]:

$$C(\rho_{AB})>2\sqrt{\left(1-Tr\left[\rho_A^2\right]\right)\left(1-Tr\left[\rho_B^2\right]\right)} \tag{8}$$

which can also be expressed as

$$C(\rho_{AB})=\frac{1}{4}Tr\left|\sum_{j,k}T_{jk}\sigma_j\otimes\sigma_k\right| \tag{9}$$

where T denotes the 3×3 spin covariance matrix with coefficients

$$T_{jk}=\langle\sigma_j\otimes\sigma_k\rangle-\langle\sigma_j\otimes I\rangle\langle I\otimes\sigma_k\rangle \tag{10}$$

Hall's quantum correlation distance (8) leads to the following expression for the lower bound for the quantum mutual information shared by the two qubits A and B:

$$I(\rho_{AB})\geq\begin{cases}\log 2-H\left(\dfrac{1+C(\rho_{AB})}{2},\dfrac{1-C(\rho_{AB})}{2}\right), & C(\rho_{AB})\leq 0{,}72654\\ \log 4-H\left(\dfrac{1}{4}+\dfrac{C(\rho_{AB})}{2},\dfrac{1}{4}-\dfrac{C(\rho_{AB})}{6},\dfrac{1}{4}-\dfrac{C(\rho_{AB})}{6},\dfrac{1}{4}-\dfrac{C(\rho_{AB})}{6}\right), & C(\rho_{AB})>0{,}72654\end{cases} \tag{10a}$$

where *H* is the Shannon entropy of the probability distribution of the system of the two qubits into consideration.

3. Quantum Inference as Non Classical Distribution

The problem of the anomalous conceptual position of the inference in the theoretical plant of quantum mechanics (QM) is a consequence of the rather paradoxical fact that the standard interpretation does not contain the tools required in order to describe non-locality as a crucial feature of the theory. Thus a situation has occurred which remembers Wittgenstein's famous proposition: "The limits of my language are the limits of my world" [*Tractatus*, prop. 5.62]. In fact quantum physicists are in the position to investigate and experience something about which they should be silent! The advent of *Bohmian Quantum Mechanics* resolves this gap by the quantum potential, which showed to be not only the central expression of the nth "interpretation" but also an "open door" in order to treat non-locality in contexts that are otherwise difficult, such as particle physics, cosmology and quantum information [14]. In particular, it is possible to show that a deep mathematical connection exists between Bohm's quantum potential and Feynman's paths, they complete each other both in the physical meaning and as an efficacious tool [15].

In a series of recent works, the utility of a quantity indicating an entropic correlation built on Sbitnev's quantum entropy has been shown [16–18]:

$$S_Q = -\frac{1}{2}\ln \rho \tag{11}$$

which provides the quantum counterpart of a Boltzmann-type law and evaluates the degree of order and chaos of the configuration space produced by the density ρ of the ensemble of particles associated with the wave function under consideration. In Sbitnev's approach, by introducing the quantity (11), Bohm's quantum potential for a one-body system can be written in the following form

$$Q = -\frac{\hbar^2}{2m}\left(\nabla S_Q\right)^2 + \frac{\hbar^2}{2m}\left(\nabla^2 S_Q\right) \tag{12}$$

namely emerges as an information channel into the behaviour of the physical system where the quantity $-\frac{\hbar^2}{2m}\left(\nabla S_Q\right)^2$ can be interpreted as the quantum corrector of its kinetic energy while the quantity $\frac{\hbar^2}{2m}\left(\nabla^2 S_Q\right)$ can be interpreted as the quantum corrector of its potential energy [16,17]. Moreover, according to current research [19,20], Bohm's potential can be identified with the curvature scalar of the Weyl integrable space, namely

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2\Omega}{\Omega} \tag{13}$$

where the scalar function Ω is linked with the curvature through relation $R = 8\frac{\nabla^2\Omega}{\Omega}$. In this picture, the inverse square root of the curvature scalar defines a typical length (Weyl length) that can be used to evaluate the strength of quantum effects, in other words the quantity

$$L_W = \frac{1}{\sqrt{\frac{\nabla^2\Omega}{\Omega}}} \tag{14}$$

can be defined as the quantum length. In virtue of the equivalence of the approaches based on Equations (12) and (13), one obtains therefore

$$\left(\nabla S_Q\right)^2 - \left(\nabla^2 S_Q\right) = \frac{\nabla^2\Omega}{\Omega} \tag{14a}$$

or, in terms of the density ρ,

$$\left[\frac{\nabla^2\rho}{2\rho}-\left(\frac{\nabla\rho}{2\rho}\right)^2\right]=\frac{\nabla^2\Omega}{\Omega} \tag{14b}$$

Thus the entropic quantum potential (12) leads to define a quantum-entropic length (Bell length) describing the geometrical properties of the configuration space associated with the quantum entropy, given by the following relation

$$L_{quantum}=\frac{1}{\sqrt{(\nabla S_Q)^2-\nabla^2 S_Q}} \tag{15}$$

On the basis of the two quantum correctors of the energy appearing in Equation (12), the Bell length (15) can be interpreted as an indicator of non-local correlation (and thus provides a direct measure of the degree of departure from the Euclidean geometry characteristic of classical physics). The maximum value of (15) is obtained for $L_{quantum}^{\max}=1$, which corresponds to the maximum degree of non-local correlation of a quantum system [21].

A geometric approach based on the quantum entropy and a quantum-entropic length can be directly extended to the analysis of the entanglement of a qubir pair of spin ½ particles. By considering a two qubit system of two particles 1 and 2 in the state

$$\left|\psi\right\rangle=\cos\frac{\vartheta}{2}\left|\uparrow\downarrow\right\rangle+e^{i\varphi}\sin\frac{\vartheta}{2}\left|\downarrow\uparrow\right\rangle \tag{16}$$

(where $\left|\uparrow\downarrow\right\rangle$ corresponds to the state of the system when the first qubit is in the "up" state, *i.e.*, in the direction of the z-axis, and the second qubit is in the "down" state, while $\left|\downarrow\uparrow\right\rangle$ corresponds to the state of the system when the first qubit is in the "down" state and the second qubit is in the "up" state), in the geometric entropic approach the quantum potential for this physical system may be written as

$$Q=\frac{1}{2I}\left[\left(\hat{\vec{M}}_1 S_Q\right)^2+\left(\hat{\vec{M}}_2 S_Q\right)^2-\left(\hat{\vec{M}}_1{}^2 S_Q\right)-\left(\hat{\vec{M}}_2{}^2 S_Q\right)\right] \tag{17}$$

where

$$\vec{M}_1=i\hat{\vec{M}}_1 S \tag{18}$$

$$\vec{M}_2=i\hat{\vec{M}}_2 S \tag{19}$$

are the angular momenta of the two qubits 1 and 2 respectively, S is the phase of the wave function (16) of the system, I is the moment of inertia. The expression (17) for the quantum potential of a two qubits system in the general state (16) derives directly from the well known relation

$$Q=\frac{\left(\hat{M}_1^2+\hat{M}_2^2\right)R}{2IR} \tag{17a}$$

by writing it in terms of the quantum entropy (11).

Moreover, the deformation of the geometry associated with the entangled qubit pair can be described by the following Bell length

$$L_{quantum} = \frac{1}{\sqrt{\frac{1}{2I}\left(\left(\hat{\hat{M}}_1{}^2 S_Q\right) - \left(\hat{\hat{M}}_1 S_Q\right)^2 + \left(\hat{\hat{M}}_2{}^2 S_Q\right) - \left(\hat{\hat{M}}_2 S_Q\right)^2\right)}} \tag{20}$$

The Bell length for a two qubits system provides a direct measure of the entanglement degree of the two particles in the state (16). In an EPR-Bell situation, stronger the non-local correlation between two particles will be, bigger the departure of distribution of the results from the classical one will be. In the following chapter we will show in what sense the entropic correlation length (20) introduced here can throw new light on Bell's inequalities.

4. Bell Inequalities and Entropic Correlation Length

In the geometric approach based on Equation (20), by using Equation (8) and taking into account that, in terms of the quantum entropy, $\rho_A = \exp\left(-2S_{Q_A}\right)$ and $\rho_B = \exp\left(-2S_{Q_B}\right)$, the strong condition for the entanglement between the two qubits becomes

$$\frac{1}{\sqrt{\frac{1}{2I}\left(\left(\hat{\hat{M}}_1{}^2 S_Q\right) - \left(\hat{\hat{M}}_1 S_Q\right)^2 + \left(\hat{\hat{M}}_2{}^2 S_Q\right) - \left(\hat{\hat{M}}_2 S_Q\right)^2\right)}} > 2\sqrt{\left(1 - Tr\left[\left(\exp\left(-2S_{Q_A}\right)\right)^2\right]\right)\left(1 - Tr\left[\left(\exp\left(-2S_{Q_B}\right)\right)^2\right]\right)} \tag{21}$$

which may be expressed also as

$$\frac{1}{\sqrt{\frac{1}{2I}\left(\left(\hat{\hat{M}}_1{}^2 S_Q\right) - \left(\hat{\hat{M}}_1 S_Q\right)^2 + \left(\hat{\hat{M}}_2{}^2 S_Q\right) - \left(\hat{\hat{M}}_2 S_Q\right)^2\right)\left(1 - Tr\left[\left(\exp\left(-2S_{Q_A}\right)\right)^2\right]\right)\left(1 - Tr\left[\left(\exp\left(-2S_{Q_B}\right)\right)^2\right]\right)}} > 2 \tag{22}$$

In analogy with Hall's approach (where the quantum correlation distance (8) leads to the quantum mutual information (10a)), the Bell length (20) leads us to define a tight lower bound for the quantum mutual information shared by two qubits, which is given by the following relation:

$$I\left(L_{quantum}\right) \geq \begin{cases} \log 2 - S_Q\left(\frac{1 + L_{quantum}}{2}, \frac{1 - L_{quantum}}{2}\right), & L_{quantum} \leq 0{,}72654 \\ \log 4 - S_Q\left(\frac{1}{4} + \frac{L_{quantum}}{2}, \frac{1}{4} - \frac{L_{quantum}}{6}, \frac{1}{4} - \frac{L_{quantum}}{6}, \frac{1}{4} - \frac{L_{quantum}}{6}\right), & L_{quantum} > 0{,}72654 \end{cases} \tag{23}$$

here $I\left(L_{quantum}\right) = S_{Q_A} + S_{Q_B} - S_{Q_{AB}}$. For $L_{quantum} \leq 0{,}72654$ this lower bound can only be reached by entangled states, and cannot be achieved by any classical distribution having the same correlation distance. It is also interesting to remark that, for $L_{quantum} > 0{,}72654$, the bound is tight if only one of the reduced states is maximally mixed.

In the approach here proposed, the Bell length (20) plays the role of Hall's quantum correlation distance (8) and can be considered as the ultimate visiting card determining the non-local correlations and thus violations of Bell inequalities. In order to clarify this point, before all, let us consider the joint probability $P_{AB}(a,b)$ of outcomes a and b, for measurements of variables A and B on respective spacelike-separated qubits and let λ be any underlying variables relevant for the correlations between these two qubits. The probability distribution $P_{AB}(a,b)$ has the following form

$$P_{AB}(a,b) = \sum_{\lambda} p_{AB}(\lambda) P_{AB}(a,b|\lambda) \tag{24}$$

Here, the underlying marginal distribution of *A*, $p_A(a|\lambda)$, is independent of whether *B* or *B'* was measured on the second system (and vice versa), while λ is independent of the choice of the measured variables *A* and *B*, *i.e.*, $p_{AB}(\lambda) = p_{A'B'}(\lambda)$ for any *A*, *A'*, *B*, *B'*. Moreover, any observed correlation between *A* and *B* arises from ignorance of the underlying variable, namely $P_{AB}(a,b|\lambda) = P_A(a|\lambda) P_B(b|\lambda)$ for all *A*, *B* and λ. As shown by Hall, the quantum correlation distance (8) of $P_{AB}(a,b|\lambda)$ vanishes identically; as a consequence, since here the Bell length (20) plays the role of Hall's quantum correlation distance (8), one obtains that the quantum entropic correlation distance of $P_{AB}(a,b|\lambda)$ vanishes identically too:

$$L_{quantum}(P_{AB|\lambda}) = 0 \tag{25}$$

As is well known, for this system of the two qubits the two-valued random variables *A, A', B, B'* with values ± 1 satisfy the Bell-CHSH inequality

$$CHSH \equiv \langle AB \rangle + \langle AB' \rangle + \langle A'B \rangle - \langle A'B' \rangle \leq 2 \tag{26}$$

whereas quantum correlations can violate this inequality by as much as a factor of $\sqrt{2}$. Now, the Bell-CHSH inequality (26) can be easily expressed in terms of the Bell length (31). In fact, assuming that for these correlations no-signaling and measurement independence hold, and defining $L_{quantum}^{\max}$ to be the maximum value of the Bell length $L_{quantum}(P_{AB|\lambda})$ over all *A*, *B* and λ, the standard Bell-CHSH inequality given by Equation (26) may be generalized as

$$\langle AB \rangle + \langle AB' \rangle + \langle A'B \rangle - \langle A'B' \rangle \leq \frac{4}{2 - L_{quantum}^{\max}} \tag{27}$$

In the approach based on the quantum entropic correlation distance (20), it follows therefore that a Bell inequality violation can be simulated by the following relation

$$\langle AB \rangle + \langle AB' \rangle + \langle A'B \rangle - \langle A'B' \rangle = 2 + V \tag{28}$$

for some $V > 0$, if the observers share random variables having a quantum entropic correlation distance of at least

$$L_{quantum}^{\max} \geq \frac{2V}{2+V} \tag{29}$$

As a consequence, the observers must share a minimum mutual information of

$$I_{\min} = \log 2 - S_Q\left(\frac{1 + L_{quantum}^{\max}}{2}, \frac{1 - L_{quantum}^{\max}}{2}\right) \geq \log 2 - S_Q\left(\frac{2+3V}{4+2V}, \frac{2-V}{4+2V}\right) \tag{30}$$

The mutual information (41) reduces to zero in the limit of no violation of Bell inequality, *i.e.*, when $V = 0$, and reaches a maximum of 1 bit of information in the limit of the maximum possible violation, $V = 2$, namely for $L_{quantum}^{\max} = 1$, which is the limit value of the Bell length, beyond which the de-correlation between the two qubits begins. Therefore, on the basis of the quantum entropic correlation distance (21), an entropic length Bell inequality can be introduced which corresponds to the Bell inequality (26) of the form

$$CHSH_{entropic} \leq 0 \tag{31}$$

where

$$CHSH_{entropic} = \langle AB \rangle + \langle AB' \rangle + \langle A'B \rangle - \langle A'B' \rangle - \frac{4}{2 - L_{quantum}^{\max}} \tag{32}$$

The violation of the entropic length Bell inequality (31) essentially coincides with the one of the standard CHSH inequality. As regards the entangled state of a two qubit pair (16), the maximal violation of the entropic length Bell inequality (31), which corresponds to $V = 2$, namely to $L_{quantum}^{\max} = 1$, is obtained for $\vartheta = \frac{\pi}{2}$, on which one gets $CHSH_{entropic} \approx +0{,}237$. For other values of ϑ, the maximal violation of (31) when optimized over the measurements, follows the exact same profile as for the standard Bell-CHSH inequality (26). However, the measurements that maximize the violation of CHSH are not the ones which give the maximal violation of $CHSH_{entropic}$. In general, for the standard Bell-CHSH scenario, the violation of the standard inequality (26) is a necessary but not sufficient condition for the violation of (31). The advantage of the entropic length Bell inequality (31) lies in the fact that the results depend directly on the Bell length which emerges directly as correlation degree in quantum systems and which provides just an entropic quantum correlation distance. The maximum correlation in the two qubits system described by the general state (16), is determined by the limit value 1 of the Bell length. It becomes therefore permissible the following re-reading of the spin-spin correlations in a two qubits system characterized by the wave function (16): the quantum entropy leads to the quantum potential (17) which corresponds to a deformation of the geometry described by the Bell length (20); the Bell length (20) acts as the ultimate visiting card of quantum inference in the sense that its maximum value $L_{quantum}^{\max} = 1$ leads to the maximum violation of an entropic Bell inequality of the form (31) which may be considered as a generalization of the standard Bell inequality (26).

Moreover, on the basis of current research [22], violations of entropic inequalities witness a very particular kind of contextuality (nonlocality): if a probabilistic model violates an entropic inequality, then the same thing happens for every other probabilistic model obtained by permuting the outcome probabilities of a joint measurement, if the permuted joint distribution has the same marginal scenarios defined by specifying a set of observables for which certain subsets are known to be compatible and can be jointly measured. For example, this leads to the phenomenon that the Popescu-Rohrlich box

$$P^{PR}(a,b|x,y) = \frac{1}{4}\left[1 + (-1)^{a \oplus b \oplus xy}\right] \tag{33}$$

defined to be the unique marginal model which maximally violates the standard Bell-CHSH inequality (26), is entropically indistinguishable from the marginal model

$$P^{C}(a,b|x,y) = \frac{1}{4}\left[1 + (-1)^{a \oplus b}\right] \tag{34}$$

describing classical correlations. As shown by Chaves and Fritz, entropic quantities associated with the Bell length (20) cannot distinguish between the perfect anti-correlation of A' and B' as it appears in

(33), and the perfect correlation of A' and B' as in (34) and thus all the joint entropies of the PR-box coincide with those of P^C. As a consequence, also in the entropic approach of Bell's inequalities here suggested, entropic quantities associated with the Bell length (20) cannot distinguish between the perfect anti-correlation of A' and B' as it appears in (33), and the perfect correlation of A' and B' as in (34). In other words, for any marginal model with two-outcome measurements one has

$$CHSH_{entropic} \leq 1 \tag{35}$$

namely the maximal violation of (31) is +1.

This boundary for the violation of Bell-CHSH scenario can be obtained by the no-signaling box

$$P^{\max} = \frac{1}{2} P^{PR} + \frac{1}{2} P^{C} \tag{36}$$

which is an equal mixture of the Popescu-Rohrlich box with classical correlations. The distribution (36) can be seen as the probabilistic model in which each of the three pairs (A,B), (A,B') and (A',B) displays perfect correlation, while the fourth pair (A',B') is uncorrelated. $P^{\max}$ achieves a value of 3 on the standard CHSH scenario, and therefore does not allows a quantum-mechanical realization since it is beyond Tsirelson's bound of $2\sqrt{2}$. This example shows that a convex combination of two non-violating marginal models may violate an entropic inequality like (31) and suggests the non-linear character of the entropic length Bell inequality (42) which is associated with the non-Euclidean geometry described by the Bell length.

5. Conclusions

In recent years the exponential development of theories and technologies of quantum information have lead to a new critical interest for the foundations. This time the interest goes beyond the philosophical exigencies regarding the recover of a post-classical view, irreducible compromised. We know that the problem of a "pacific coexistence" between relativity and quantum physics are not going to be resolved by a battle between Einstein's elements of local physical reality and "spooky actions at distance", and that rather we have to accept the ultimate non-local reality indicated by D. Bohm's and J. Bell's works.

The problem of the quantum inference finds its natural collocation in the scenario of this non-local reality where the violation of the Bell inequalities indicates the shifting from the hypotheses and from the classical geometry [21,23], and the entropic correlation length (Bell length) measures in a simple and precise way the intensity of this shifting.

Appendix: The Positive-Definite Nature of the Bell Length

The Bell length (15)—which derives directly from the quantum length (14) as a consequence of the equivalence between Sbitnev's approach to the quantum potential and Novello's, Salim's and Falciano's model of the quantum potential in terms of the curvature scalar of Weyl integrable space—can be considered as a consistent and valid measure of the geometrical properties of a quantum system. Despite the density describing the space-temporal distribution of the ensemble of particles associated with the quantum state is a function of the coordinates, the dependence of the Bell length on the coordinates does not imply the necessity of supplementary existence conditions in order to guarantee its positive definite nature. Equations (14) and (15) imply that the quantum lengths—associated to Sbitnev's approach and to Novello's, Salim's and Falciano's approach respectively—deriving from the quantum potential can never be singular, in the sense that the denominator appearing in their expression can become zero only in the classical case.

In fact, on the basis of Equation (15), the two terms under square parenthesis in the denominator of the Bell length are the well-known quantum correctors of kinetic energy and potential energy for a quantum system described through Bohm's potential. They have a precise physical and geometrical status, which has been analysed in many works. From the physical point of view, it is evident from (15) that the only case in which one may obtain 0 in the denominator of the Bell length is the classical one, namely corresponding to the quantum potential equal to 0.

On the other hand, the same result may be obtained also from the geometrical point of view, utilizing Weyl's geometry of Novello's, Salim's and Falciano's approach, the denominator of the quantum length (14) may be equal to zero only if $\frac{\nabla^2\Omega}{\Omega}=0$ namely $\nabla^2\Omega=0$ and this corresponds, even here, to the classical case! The perfect agreement with the results of other authors confirms our conclusion, namely that in EPR-type situations, stronger the non-local correlation between two particles will be, bigger the departure of distribution of the results from the classical one will be.

This property of the Bell length (15) to be always positive definite can be grasped and realized well by considering, for example, the one-dimensional harmonic oscillator. In this case, the quantum Hamilton-Jacobi equation of de Broglie-Bohm theory assumes the form

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\nabla S_1\right)^2-\frac{\hbar^2}{2m}\left(\frac{\nabla^2 R_1}{R_1}\right)+\frac{1}{2}m\omega^2x^2=0 \tag{37}$$

where $V=\frac{1}{2}m\omega^2x^2$ is the potential, the stationary states are given by $C(t)=u_n(x)e^{-iE_nt/\hbar}$ (where $u_n(x)$ are real functions proportional to Hermite polynomials and $E_n=\left(n+\frac{1}{2}\right)\hbar\omega$, n = 0,1,2,… is the quantum number associated with each stationary state) and the corresponding quantum potential is

$$Q=\left(n+\frac{1}{2}\right)\hbar\omega-\frac{1}{2}m\omega^2x^2 \tag{38}$$

In particular, for a non-dispersive Gaussian-shaped packet given by the following superposition of the stationary wave-functions [24]:

$$\psi(x,t)=\sum_{n=0}^{\infty}A_nu_n(x)e^{-iE_nt/\hbar} \tag{39}$$

where

$$A_n=\left(m\omega/\hbar\right)^{n/2}a^n\left(2^n n!\right)^{-1/2}e^{-m\omega a^2/4\hbar} \tag{40}$$

the quantum state becomes

$$\psi(x,t) = (m\omega/\pi\hbar)^{1/4} \exp\left\{-(m\omega/2\hbar)(x - a\cos\omega t)^2 - \frac{i}{2}\left[\omega t + (m\omega/\hbar)\left(2ax\sin\omega t - \frac{1}{2}a^2\sin 2\omega t\right)\right]\right\} \quad (41)$$

which represents a Gaussian packet centered around $x = a$ at $t = 0$ with half-width $\sigma_0 = \left(\frac{\hbar}{2m\omega}\right)^{1/2}$. The amplitude function of the state (41) is

$$R(x,t) = (m\omega/\pi\hbar)^{1/4} \exp\left\{-(m\omega/2\hbar)(x - a\cos\omega t)^2\right\} \quad (42)$$

and thus the density is

$$\rho(x,t) = (m\omega/\pi\hbar)^{1/2} \exp\left\{-2(m\omega/2\hbar)(x - a\cos\omega t)^2\right\} \quad (43)$$

The density of the ensemble of particles (43) describing the one-dimensional harmonic oscillator determines a deformation of the geometry of the configuration space described by a quantum entropy given by relation

$$S_Q = -\frac{1}{2}\ln\left[(m\omega/\pi\hbar)^{1/2} \exp\left\{-2(m\omega/2\hbar)(x - a\cos\omega t)^2\right\}\right] \quad (44)$$

namely

$$S_Q = -\frac{1}{4}\ln\left[(m\omega/\pi\hbar)^{1/2}\right] + \frac{m\omega}{2\hbar}(x - a\cos\omega t)^2 \quad (45)$$

The quantum entropy (45) indicates the degree of order and chaos of the vacuum supporting the density of the ensemble of particles associated with the wave function (41) of the harmonic oscillator (corresponding to Schiff's treatment). In this situation, the quantum potential (38) may be expressed as

$$Q = -\frac{\hbar^2}{2m}\left[\left(\frac{m\omega}{\hbar}(x - a\cos\omega t)\right)^2 - \frac{m\omega}{\hbar}\right] \quad (46)$$

Thus, the geometrical properties of the configuration space of the one-dimensional harmonic oscillator with density (43) can be characterized by introducing the quantum-entropic length (Bell length) given by the following relation

$$L_{quantum} = \frac{1}{\sqrt{\left(\frac{m\omega}{\hbar}(x - a\cos\omega t)\right)^2 - \frac{m\omega}{\hbar}}} \quad (47)$$

Equation (47) shows clearly that the Bell length of the harmonic oscillator explicitly depends of the coordinate *x*. The quantum-entropic length (47) turns out to be positive definite if the coordinate x satisfies conditions $x \le a\cos\omega t - \sqrt{\frac{\hbar}{m\omega}}$ or $x \ge a\cos\omega t + \sqrt{\frac{\hbar}{m\omega}}$ (which can be considered as its existence conditions). Since the intervals $x \le a\cos\omega t - \sqrt{\frac{\hbar}{m\omega}}$ or $x \ge a\cos\omega t + \sqrt{\frac{\hbar}{m\omega}}$ cover all the physically sensed and relevant values of the coordinate x for the one-dimensional harmonic oscillator having the state (41), one can conclude that the Bell length (47) is always positive definite in all the significant intervals for the coordinate. The example of the harmonic oscillator shows thus in an easy and clear way that the Bell length (15) can be used as a consistent quantity measuring the geometrical properties of a quantum system.

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Conflicts of Interest

The authors declare no conflict of interest.

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