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Article

Continuous Stieltjes-Wigert Limiting Behaviour of a Family of Confluent q-Chu-Vandermonde Distributions

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Abstract: From Kemp [1], we have a family of confluent q-Chu- Vandermonde distributions, consisted by three members I, II and III, interpreted as a family of q-steady-state distributions from Markov chains. In this article, we provide the moments of the distributions of this family and we establish a continuous limiting behavior for the members I and II, in the sense of pointwise convergence, by applying a q-analogue of the usual Stirling asymptotic formula for the factorial number of order n. Specifically, we initially give the q-factorial moments and the usual moments for the family of confluent q-Chu- Vandermonde distributions and then we designate as a main theorem the conditions under which the confluent q-Chu-Vandermonde distributions I and II converge to a continuous Stieltjes-Wigert distribution. For the member III we give a continuous analogue. Moreover, as applications of this study we present a modified q-Bessel distribution, a generalized q-negative Binomial distribution and a generalized over/underdispersed (O/U) distribution. Note that in this article we prove the convergence of a family of discrete distributions to a continuous distribution which is not of a Gaussian type.

Keywords: stirling asymptotic formula; q-factorial number of order n; confluent q-Chu-Vandermonde distributions; q-factorial moments; modified q-Bessel distribution; generalized q-negative Binomial distribution; Over/Underdispersed (O/U) distribution; pointwise convergence; continuous Stieltjes-Wigert distribution

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1. Introduction and Preliminaries

From Kemp [1], we have that the confluent q-Chu-Vandermonde hypergeometric sum,

$$ {}_1\phi_1(b;c;q,c/b) = \sum_{n=0}^{\infty} \frac{(b;q)_n}{(c;q)_n(q;q)_n}(-c/b)^n q^{\binom{n}{2}} = \frac{(c/b;q)_\infty}{(c;q)_\infty} \tag{1} $$

where $0 < q < 1$ and $(a;q)_x = \prod_{j=1}^{x}(1-aq^{j-1}), x = 0,1,2,\ldots$, gives rise to a family of q-Chu-Vandermonde distributions for suitable values of c and b, interpreted as a family q-steady-state distributions from Markov chains, with probability generating function (p.g.f.)

$$ G(z) = \frac{{}_1\phi_1(b;c;q,c/bz)}{{}_1\phi_1(b;c;q,c/b)},\ z \in \mathcal{R} \tag{2} $$

and with probability function (p.f.)

$$ p(x) = P[X=x] = f_X(x) = \frac{\frac{(b;q)_x}{(c;q)_x(q;q)_x} q^{\binom{x}{2}} (-c/b)^x}{\frac{(c/b;q)_\infty}{(c;q)_\infty}},\ x = 0,1,\ldots. \tag{3} $$

Note that Equation (1) is a generalization of the q-binomial theorem and gives rise to two q-confluent distributions with infinite support and one with finite support.

The members of the above family of q-Chu-Vandermonde hypergeometric series discrete distributions are listed in Table 1.

Table 1. Confluent q-Chu-Vandermonde Distributions.

Confluent q-Chu-Vandermonde Distributions	**Symbol of $G(z)$**	**Symbol of $p(x)$**	**Parameters b and c**	**Support**
q-CCV-I	$G_{qCCVI}(z)$	$p_{qCCVI}(x)$	$b=-h,\ h>0,\ 0<c<1$	$x=0,1,2,\ldots$
q-CCV-II	$G_{qCCVII}(z)$	$p_{qCCVII}(x)$	$0<b<1,\ c=-\eta,\ \eta>0$	$x=0,1,2,\ldots$
q-CCV-III	$G_{qCCVIII}(z)$	$p_{qCCVIII}(x)$	$b=q^{-n},\ n=0,1,\ldots,\ 0<c<1$	$x=0,1,\ldots,n$

The distributions of the above table have finite mean and variance when $n \to \infty$ and we cannot conclude the asymptotic normality in the sense of the DeMoivre-Laplace classical limit theorem, as in the case of ordinary hypergeometric series discrete distributions. Also, we cannot apply asymptotic methods –central or/and local limit theorems– as in Bender [2], Canfield [3], Flajolet and Soria [4], Odlyzko [5] *et al.*

Thus an important question is arisen about the asymptotic behaviour for $n \to \infty$ of this family of q-Chu-Vandermonde hypergeometric series discrete distributions.

Recently, the authors investigated the asymptotic behaviour of another member of q-hypergeometric series discrete distributions, having also finite mean and variance, that of a q-Binomial one [6]. Specifically it has been established a pointwise convergence to a continuous Stieltjes-Wigert distribution.

In this article, we provide a continuous limiting behaviour of the above family of confluent q- Chu- Vandermonde discrete distributions, for $0 < q < 1$, in the sense of pointwise convergence.

Specifically, we initially give the q-factorial moments and the usual moments of this family and then we designate as a main theorem the conditions under which the confluent q-Chu-Vandermonde distributions I and II converge to a continuous Stieltjes-Wigert distribution. For the member III we give a continuous analogue. Moreover, as applications of this study we present a modified q-Bessel distribution, a generalized q-negative Binomial distribution and a generalized over / underdispersed (O/U) distribution. Note that, the main contribution of this article is that a family of discrete distributions converges to a continuous distribution which is not of a Gaussian type.

To establish the proof of our main theorem we apply a q-analogue of the well known Stirling asymptotic formula for the n factorial ($n!$) established by Kyriakoussis and Vamvakari [6]. The authors have derived an asymptotic expansion for $n \to \infty$ of the q-factorial number of order n ,

$$[n]_q! = [1]_q[2]_q \dots [n]_q = \prod_{k=1}^{n} \frac{1-q^k}{(1-q)^n} = \frac{(q;q)_n}{(1-q)^n} \tag{4}$$

where $0 < q < 1$ and $[t]_q = \frac{1-q^t}{1-q}$, the q-number t. Analytically we have

$$[n]_q! \;=\; \frac{(2\pi(1-q))^{1/2}}{(q\log q^{-1})^{1/2}} \frac{q^{\binom{n}{2}} q^{-n/2} [n]_{1/q}^{n+1/2}}{\prod_{j=1}^{\infty}(1+q(q^{-n}-1)q^{j-1})} \left(1+O(n^{-1})\right) \tag{5}$$

For answering the main question of this study we apply our above asymptotic formula for the q-factorial number of order n to provide pointwise convergence of the family of confluent q-Chu-Vandermonde distributions to a continuous Stieltjes-Wigert distribution with probability density function

$$v_q^{SW}(x) = \frac{q^{1/8}}{\sqrt{2\pi \log q^{-1}}\, x} e^{\frac{(\log x)^2}{2\log q}}, \; x > 0 \tag{6}$$

with mean value $\mu^{SW} = q^{-1}$ and standard deviation $\sigma^{SW} = q^{-3/2}(1-q)^{1/2}$

Remark 1. We note that the corresponding to the probability measure Equation (3) orthogonal polynomials are the q-Meixner ones (see [7]). Also, we have that the q-Meixner orthogonal polynomials converge to the Stieltjes-Wigert ones, both members of the q-Askey scheme (see [7,8]). But, from the convergence of the orthogonal polynomials one cannot conclude the convergence of the corresponding probability measures (see [9,10]). So, in this paper the method of pointwise convergence is followed.

2. On Factorial Moments of the Confluent q-Chu-Vandermonde Distributions

In this section, we first transfer from the random variable X of the family of confluent q-Chu-Vandermonde distributions Equation (3) to the equal-distributed deformed random variable $Y = [X]_{1/q}$, and we then compute the mean value and variance of the random variable Y, say μ_q and σ_q^2 respectively. We also derive all the descending factorial k-th order moments of the random variable X through the computation of all the r-th orders factorials of the random variable Y, named q-factorial moments of the r.v. X.

Proposition 1. *The q-mean and q-variance of the family of confluent q-Chu-Vandermonde distributions are given respectively by*

$$\mu_q = -\frac{c}{b}\frac{1-b}{1-q} \text{ and } \sigma_q^2 = \left(-\frac{c}{b}\right)^2 \frac{1-b}{q(1-q)} - \frac{c}{b}\frac{1-b}{1-q}$$

Proof. The q-mean of the family of confluent q-Chu-Vandermonde distributions is given by

$$\mu_q = E(Y) = E([X]_{1/q}) = \sum_{x=0}^{n} [x]_{1/q} f_X(x) = \frac{(c;q)_\infty}{(c/b;q)_\infty} \sum_{x=0}^{\infty} [x]_{1/q} \frac{(b;q)_x}{(c;q)_x (q;q)_x} q^{\binom{x}{2}} \left(-\frac{c}{b}\right)^x \tag{7}$$

and since

$$[x]_{1/q} = q^{-x+1}[x]_q, \;\; q^{-x+1} q^{\binom{x}{2}} = q^{\binom{x-1}{2}}, \;\; \frac{[x]_q}{(q;q)_x} = \frac{1}{(1-q)(q;q)_{x-1}}$$

and

$$(b;q)_x = (1-b)(bq;q)_{x-1}, \;\; (c;q)_x = (1-c)(cq;q)_{x-1}$$

it is written as

$$\mu_q = -\frac{c}{b} \frac{1-b}{(1-c)(1-q)} \frac{(c;q)_\infty}{(c/b;q)_\infty} \sum_{x=1}^{\infty} \frac{(bq;q)_{x-1}}{(cq;q)_{x-1}(q;q)_{x-1}} q^{\binom{x-1}{2}} \left(-\frac{c}{b}\right)^{x-1} \tag{8}$$

Using the confluent q-Chu-Vandermonde hypergeometric sum Equation (1) we obtain the formula of the q-mean in Equation (7).

For the evaluation of the q-variance we need to find the second order moment of the r.v. $Y = [X]_{1/q}$ which is given by

$$E[Y^2] = E[[X]_{1/q}^2] = \sum_{x=0}^{\infty} [x]_{1/q}^2 f_X(x) = \frac{(c;q)_\infty}{(c/b;q)_\infty} \sum_{x=0}^{\infty} [x]_{1/q}^2 \frac{(b;q)_x}{(c;q)_x (q;q)_x} q^{\binom{x}{2}} \left(-\frac{c}{b}\right)^x \tag{9}$$

Since

$$[x]_q = [x-1]_q + q^{x-1}, \;\; q^{-2x+2} q^{\binom{x}{2}} = q^{-1} q^{\binom{x-2}{2}}$$

and

$$(b;q)_x = (1-b)(1-bq)(bq^2;q)_{x-2}, \;\; (c;q)_x = (1-c)(1-cq)(cq^2;q)_{x-2}$$

Equation (9) becomes

$$\begin{aligned} E[Y^2] &= \left(-\frac{c}{b}\right)^2 \frac{(1-b)(1-bq)}{q(1-q)^2(1-c)(1-cq)} \frac{(c;q)_\infty}{(c/b;q)_\infty} \sum_{x=2}^{\infty} \frac{(bq^2;q)_{x-2}}{(cq^2;q)_{x-2}(q;q)_{x-2}} q^{\binom{x-2}{2}} \left(-\frac{c}{b}\right)^{x-2} \\ &= \left(-\frac{c}{b}\right)^2 \frac{(1-b)(1-bq)}{q(1-q)^2(1-c)(1-cq)} \frac{(c;q)_\infty}{(cq^2;q)_\infty} = \left(-\frac{c}{b}\right)^2 \frac{(1-b)(1-bq)}{q(1-q)^2} \end{aligned}$$

So,

$$\sigma_q^2 = V(Y) = V([X]_{1/q}) = \left(-\frac{c}{b}\right)^2 \frac{(1-b)(1-bq)}{q(1-q)^2} - \frac{c}{b}\frac{1-b}{1-q} - \left(-\frac{c}{b}\right)^2 \frac{(1-b)^2}{(1-q)^2} \tag{10}$$

from which we obtain the formula of the q-variance given in Equation (7).

Proposition 2. *The r-th order q-factorial moments of the family of confluent q-Chu-Vandermonde distributions are given by*

$$E([X]_{r,1/q}) = \frac{(b;q)_r}{(1-q)^r} \left(-\frac{c}{b}\right)^r, \; r = 1, 2, \ldots. \tag{11}$$

Proof. The r-th order q-factorial moments of the family of confluent q-Chu-Vandermonde distributions is

$$\begin{aligned} E([X]_{r,1/q}) &= \sum_{x=r}^{\infty}[x]_{r,1/q}f_X(x) \\ &= \frac{(c;q)_\infty}{(c/b;q)_\infty}\sum_{x=r}^{\infty}[x]_{1/q}[x-1]_{1/q}\cdots[x-r+1]_{1/q}\frac{(b;q)_x}{(c;q)_x(q;q)_x}q^{\binom{x}{2}}(-c/b)^x \end{aligned} \tag{12}$$

Since

$$[x]_{1/q}=q^{-x+1}[x]_q,\ \binom{x}{2}=\binom{x-r}{2}+\binom{r}{2}+r(x-r),\ \frac{[x]_{r,q}}{(q;q)_x}=\frac{1}{(1-q)^r(q;q)_{x-r}}$$

$$(b;q)_x=(b;q)_r(bq^r;q)_{x-r}\ \text{ and }\ \frac{(c;q)_\infty}{(c;q)_x}=\frac{(cq^r;q)_\infty}{(cq^r;q)_{x-r}}$$

the sum Equation (12) becomes

$$E([X]_{r,1/q})=\frac{(cq^r;q)_\infty(b;q)_r}{(c/b;q)_\infty(1-q)^r}(-c/b)^r\sum_{x=r}^{\infty}\frac{(bq^r;q)_{x-r}}{(cq^r;q)_{x-r}(q;q)_{x-r}}q^{\binom{x-r}{2}}(-c/b)^{x-r} \tag{13}$$

By the confluent q-Vandermonde sum the r-th order q-factorial moments of the family of confluent q-Chu-Vandermonde distributions, reduces to Equation (11).

Proposition 3. *The descending factorial k-th order moments of the r.v. X of the family of q-Chu-Vandermonde distributions are given by*

$$E((X)_k)=\frac{k!}{(c/b;q)_\infty}\sum_{r=k}^{\infty}\frac{s_q(r,k)(q-1)^{r-k}(cq^r;q)_\infty}{[r]_{1/q}!}E([X]_{r,1/q})\,{}_1\phi_1(bq^r;cq^r;q,cq^r/b) \tag{14}$$

Proof. The relation of the factorial descending moments with the q-factorial descending moments through the q-Stirling numbers of the first kind is given by the sum

$$E((X)_k)=k!\sum_{r=k}^{\infty}\frac{s_q(r,k)(q-1)^{r-k}}{[r]_q!}E([X]_{r,q}) \tag{15}$$

where $s_q(r,k)$ the q-Stirling numbers of the first kind (see Charalambides [11]).
Since

$$\binom{x}{r}_q=q^{r(x-r)}\binom{x}{r}_{1/q}$$

the sum Equation (15) is written as

$$\begin{aligned} E((X)_k) &= k!\sum_{r=k}^{\infty}\frac{s_q(r,k)(q-1)^{r-k}}{[r]_{1/q}!}E(q^{r(x-r)}[X]_{r,1/q}) \\ &= k!\sum_{r=k}^{\infty}\frac{s_q(r,k)(q-1)^{r-k}}{[r]_{1/q}!}\frac{(cq^r;q)_\infty(b;q)_r(-c/b)^r}{(c/b;q)_\infty(1-q)^r}\sum_{x=r}^{\infty}\frac{(bq^r;q)_{x-r}}{(cq^r;q)_{x-r}(q;q)_{x-r}}q^{\binom{x-r}{2}}(-cq^r/b)^{x-r} \end{aligned} \tag{16}$$

By Equation (11) of the previous proposition 2 and the definition of the q-hypergeometric function Equation (1), Equation (16) reduces to Equation (14).

3. Pointwise Convergence of A Family of Confluent q-Chu-Vandermonde Distributions to the Stieltjes-Wigert Distribution

In this section, we transfer from the random variable X of the family of confluent q-Chu-Vandermonde distributions Equation (3) to the equal-distributed deformed random variable $Y = [X]_{1/q}$, and using the q-analogue Stirling asymptotic formula (5), we establish the convergence to a deformed standardized continuous Stieltjes-Wigert distribution of the members I and II of the family of q-Chu-Vandermonde distributions.

Theorem 1. *Let the p.f. of the family of confluent q-Chu-Vandermonde distributions be of the form*

$$f_X(x) = \frac{\frac{(b;q)_x}{(c;q)_x(q;q)_x} q^{\binom{x}{2}} (-c/b)^x}{\frac{(c/b;q)_\infty}{(c;q)_\infty}}, \quad x = 0, 1, \ldots \tag{17}$$

where $b = b_n,\ c = c_n,\ n = 0, 1, 2, \ldots,$ such that $b_n = o(1)$ and $-c_n/b_n \to \infty$, as $n \to \infty$. Then, for $n \to \infty$, the p.f. $f_X(x),\ x = 0, 1, 2, \ldots$ is approximated by a deformed standardized continuous Stieltjes-Wigert distribution as follows

$$\begin{aligned} f_X(x) \quad &\cong \quad \frac{q^{1/8}(\log q^{-1})^{1/2}}{(2\pi)^{1/2}} \left(q^{-3/2}(1-q)^{1/2} \frac{[x]_{1/q} - \mu_q}{\sigma_q} + q^{-1} \right)^{1/2} \\ &\quad \cdot \exp\left(\frac{1}{2\log q} \log^2 \left(q^{-3/2}(1-q)^{1/2} \frac{[x]_{1/q} - \mu_q}{\sigma_q} + q^{-1} \right) \right), \quad x \geq 0 \end{aligned} \tag{18}$$

Proof. Since the product $(b; q)_x = \prod_{j=1}^{x}(1 - bq^{j-1}) = (1 - b)(1 - bq) \cdots (1 - bq^{x-1})$ for $b = b_n$ with $b_n \to 0$ as $n \to \infty$ is approximated by $(b_n; q)_x \cong 1$ the p.f. of the family of confluent q-Chu-Vandermonde distributions is discretely approximated as

$$f_X(x) \cong \frac{\frac{q^{\binom{x}{2}}(-c_n/b_n)^x}{(c_n;q)_x(q;q)_x}}{\frac{(c_n/b_n;q)_\infty}{(c_n;q)_\infty}}, \quad x = 0, 1, \ldots. \tag{19}$$

By using the q-Stirling asymptotic formula (5) we get the following approximation for the p.f. $f_X(x)$ with $b = b_n,\ c = c_n$ such that $b_n = o(1)$ and $-c_n/b_n \to \infty$, as $n \to \infty$,

$$f_X(x) \quad \cong \quad \frac{(q\log q^{-1})^{1/2}}{(2\pi(1-q))^{1/2}} \frac{(-c_n/b_n)^x}{(1-q)^x} \frac{\prod_{j=1}^{\infty}(1 + q(q^{-x} - 1)q^{j-1})(c_n; q)_\infty}{q^{-x/2}[x]_{1/q}^{x+1/2}(c_n; q)_x(c_n/b_n; q)_\infty} \tag{20}$$

From the standardized r.v. $Z = \frac{[X]_{1/q} - \mu_q}{\sigma_q}$ with μ_q and σ_q given in Equation (7), we get

$$\begin{aligned} [x]_{1/q} \quad &= \quad \sigma_q z + \mu_q = \left[\left(-\frac{c_n}{b_n} \right)^2 \frac{1-q}{q(1-q)} - \frac{c_n}{b_n} \frac{1-b_n}{1-q} \right]^{1/2} z - \frac{c_n}{b_n} \frac{1-b_n}{1-q} \\ &= \quad -\frac{c_n}{b_n} \frac{1-b_n}{1-q} \left[\left(\frac{1-q}{q(1-b_n)} - \frac{b_n}{c_n} \frac{1-q}{1-b_n} \right)^{1/2} z + 1 \right] \end{aligned} \tag{21}$$

Using the assumptions $b_n = o(1)$ and $-c_n/b_n \to \infty$ as $n \to \infty$, we have

$$[x]_{1/q} \cong -\frac{c_n}{b_n} \frac{q}{1-q} (q^{-3/2}(1-q)^{1/2} z + q^{-1}) \tag{22}$$

Also, by the previous two equations we get

$$q^{-x} = -\frac{c_n}{b_n}\frac{1-b_n}{q}\left[\left(\frac{1-q}{q(1-b_n)} - \frac{b_n}{c_n}\frac{1-q}{1-b_n}\right)^{1/2} z+1\right]+1 \tag{23}$$

and

$$q^{-x} \cong -\frac{c_n}{b_n}\left(q^{-3/2}(1-q)z+q^{-1}\right) \tag{24}$$

Moreover, by the Equation (23) we find

$$x = \frac{1}{\log q^{-1}}\log\left(-\frac{c_n}{b_n}\frac{1-b_n}{q}\left[\left(\frac{1-q}{q(1-b_n)} - \frac{b_n}{c_n}\frac{1-q}{1-b_n}\right)^{1/2} z+1\right]+1\right) \tag{25}$$

and

$$x \cong \frac{1}{\log q^{-1}}\log\left(-\frac{c_n}{b_n}\left(q^{-3/2}(1-q)z+q^{-1}\right)\right) \tag{26}$$

Finally, by the Equation (21) we get

$$\begin{aligned}
[x]_{1/q}^x &= \left(-\frac{c_n}{b_n}\right)^x\left(\frac{1-b_n}{1-q}\right)^x\left[\left(\frac{1-q}{q(1-b_n)} - \frac{b_n}{c_n}\frac{1-q}{1-b_n}\right)^{1/2} z+1\right]^x \\
&= \left(-\frac{c_n}{b_n}\right)^x\left(\frac{1-b_n}{1-q}\right)^x \exp\left(x\log\left[\left(\frac{1-q}{q(1-b_n)} - \frac{b_n}{c_n}\frac{1-q}{1-b_n}\right)^{1/2} z+1\right]\right)
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
[x]_{1/q}^x &\cong \left(-\frac{c_n}{b_n}\right)^x\left(\frac{1}{1-q}\right)^x \\
&\cdot \exp\left(\frac{1}{\log q^{-1}}\log\left(-\frac{c_n}{b_n}\left(q^{-3/2}(1-q)z+q^{-1}\right)\right)\log\left(q\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)\right)\right)
\end{aligned} \tag{28}$$

with $z = \frac{[x]_{1/q}-\mu_q}{\sigma_q}$

Substituting all the previous approximations Equations (22),(24),(26),(28) to the p.f. $f_X(x)$ we get the approximation

$$\begin{aligned}
f_X(x) &\cong \frac{(q(1-q)\log q^{-1})^{1/2}}{(2\pi q(1-q))^{1/2}} \frac{\prod_{j=1}^{\infty}\left(1-\frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)q^{j-1}\right)(c_n;q)_\infty}{(c_n;q)_x(c_n/b_n;q)_\infty} \\
&\cdot \exp\left(\frac{1}{\log q}\log\left(-\frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)\right)\log\left(q\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)\right)\right) \\
&\cdot \left(-\frac{c_n}{b_n}\frac{q^{1/2}}{(1-q)^{1/2}}\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)\right)^{-1}, \quad z = \frac{[x]_{1/q}-\mu_q}{\sigma_q}
\end{aligned} \tag{29}$$

As a last step, we need to estimate the products $\prod_{j=1}^{\infty}\left(1-\frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)q^{j-1}\right)$, $(c_n/b_n;q)_x = \prod_{j=1}^{\infty}(1-c_n/b_nq^{j-1})$ and $(c_n;q)_\infty/(c_n;q)_x = (c_nq^x;q)_\infty = \prod_{j=1}^{\infty}(1-c_nq^xq^{j-1})$ by integrals. Since the first product is written as

$$\prod_{j=1}^{\infty}\left(1-\frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)q^{j-1}\right) = \exp\left(\sum_{j=1}^{\infty}\log\left(1-\frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)q^{j-1}\right)\right) \tag{30}$$

and the function

$$h(x) = \log\left(1 - \frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z + q^{-1}\right)q^{x-1}\right)$$

has all orders continuous derivatives in $[1, \infty)$, we can apply the Euler-Maclaurin summation formula (see [5], p. 1090) in the sum of the Equation (30).

So,

$$\sum_{j=1}^{\infty}\log\left(1 - \frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z + q^{-1}\right)q^{j-1}\right) = \int\limits_1^{\infty}\log\left(1 - \frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z + q^{-1}\right)q^{u-1}\right)du$$
$$+\frac{1}{2}\log\left(1 - \frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z + q^{-1}\right)\right) + \sum_{k=1}^{m}\frac{\beta_{2k}}{(2k)!}h^{(2k-1)}\left(1 - \frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z + q^{-1}\right)\right) + R_k \tag{31}$$

where

$$|R_k| \le \frac{|\beta_{2k}|}{(2k)!}\int\limits_1^{\infty}|g^{(2k)}\left(1 - \frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z + q^{-1}\right)q^{u-1}\right)|du \tag{32}$$

with β_k the Bernoulli numbers.

Now, expressing the integral appearing in Equation (31) through the dilogarithm function we get

$$\int\limits_1^{\infty}\log\left(1 - \frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z + q^{-1}\right)q^{u-1}\right)du = \frac{1}{\log q}\mathrm{Li}_2\left(-\left(-\frac{c_n}{b_n}\right)\left(q^{-3/2}(1-q)^{1/2}z + q^{-1}\right)\right) \tag{33}$$

where $\mathrm{Li}_2(y) = \sum_{k\ge 1}\frac{y^k}{k^2}$ the dilogarithm function. The dilogarithm satisfies the Landen's identity

$$\mathrm{Li}_2(-y) = \mathrm{Li}_2\left(\frac{y}{y+1}\right) - \frac{1}{2}\log^2(1+y) \tag{34}$$

Applying the Landen's identity to Equation (33) we obtain

$$\begin{aligned}\int\limits_1^{\infty}\log\left(1 - \frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z + q^{-1}\right)q^{u-1}\right)du &= \frac{1}{2\log q^{-1}}\log^2\left(-\frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z + q^{-1}\right)\right)\\ &+ \mathrm{Li}_2\left(\frac{-\frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z + q^{-1}\right)}{-\frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z + q^{-1}\right)+1}\right)\end{aligned} \tag{35}$$

Next, we estimate the sum and the quantity R_k appearing in Equation (31)

$$\begin{aligned}&\sum_{k=1}^{m}\frac{\beta_{2k}}{(2k)!}h^{(2k-1)}\left(1 - \frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z + q^{-1}\right)\right) + R_k\\ &= \frac{\beta_2}{2}h'\left(1 - \frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z + q^{-1}\right)\right) + R_1 + O\left(\left(-\frac{c_n}{b_n}\right)^{-2}\right)\\ &= \frac{\beta_2\log q}{2}\frac{\left(-\frac{c_n}{b_n}\right)\left(q^{-3/2}(1-q)^{1/2}z + q^{-1}\right)}{1 - \frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z + q^{-1}\right)} + O\left(-\frac{b_n}{c_n}\right)\end{aligned} \tag{36}$$

So, by applying Equations (35) and (36) to Equation (31) we obtain

$$\sum_{j=1}^{\infty}\log\left(1-\frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)q^{j-1}\right)=\frac{1}{2\log q^{-1}}\log^2\left(-\frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)\right)$$
$$+\text{Li}_2\left(\frac{-\frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)}{-\frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)+1}\right)+\frac{1}{2}\log\left(1-\frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)\right)$$
$$+\frac{\beta_2\log q}{2}\frac{\left(-\frac{c_n}{b_n}\right)\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)}{1-\frac{c_n}{b_n}\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)}+O\left(-\frac{b_n}{c_n}\right) \tag{37}$$

Working similarly for the sum appearing in the product

$$\prod_{j=1}^{\infty}\left(1-\frac{c_n}{b_n}q^{j-1}\right)=\exp\left(\sum_{j=1}^{\infty}\log\left(1-\frac{c_n}{b_n}q^{j-1}\right)\right) \tag{38}$$

we obtain

$$\begin{aligned}\sum_{j=1}^{\infty}\log\left(1-\frac{c_n}{b_n}q^{j-1}\right) &= \frac{1}{2\log q^{-1}}\log^2\left(-\frac{c}{b_n}\right)+\text{Li}_2\left(\frac{-\frac{c}{b_n}}{-\frac{c}{b_n}+1}\right)\\ &+ \frac{1}{2}\log\left(1-\frac{c}{b_n}\right)+\frac{\beta_2\log q}{2}\frac{\left(-\frac{c}{b_n}\right)}{1-\frac{c}{b_n}}+O\left(-\frac{b_n}{c}\right)\end{aligned} \tag{39}$$

We need now to estimate the sum appearing in the last product

$$\begin{aligned}(c_n;q)_\infty/(c_n;q)_x &= (c_nq^x;q)_\infty=\prod_{j=1}^{\infty}(1-c_nq^xq^{j-1})\\ &= \exp\left(\sum_{j=1}^{\infty}\log\left(1+b_n\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)^{-1}q^{j-1}\right)\right)\end{aligned} \tag{40}$$

and working analogously as previous we get

$$\sum_{j=1}^{\infty}\log\left(1+b_n\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)^{-1}q^{j-1}\right)=\frac{1}{2\log q^{-1}}\log^2\left(b_n\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)^{-1}\right)$$
$$+\text{Li}_2\left(\frac{b_n\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)^{-1}}{b_n\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)+1}^{-1}\right)+\frac{1}{2}\log\left(1+b_n\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)^{-1}\right)$$
$$+\frac{\beta_2\log q}{2}\frac{b_n\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)^{-1}}{1+b_n\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)^{-1}}+O\left(b_n\right) \tag{41}$$

Applying the estimations Equations (37),(39),(41) to the approximation Equation (29), carrying out all the necessary manipulations and by the assumptions $b_n=o(1)$ and $-c_n/b_n\to\infty$, as $n\to\infty$, we derive

$$f_X(x) \cong \frac{q^{1/8}(\log q^{-1})^{1/2}}{(2\pi)^{1/2}} \left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)^{1/2} \exp\left(\frac{1}{2\log q}\log^2\left(q^{-3/2}(1-q)^{1/2}z+q^{-1}\right)\right)$$
$$z = \frac{[x]_{1/q}-\mu_q}{\sigma_q}, \quad x \geq 0 \tag{42}$$

that is Equation (18).

Moreover, by standardizing the continuous Stieltjes-Wigert distribution Equation (6) and then deforming this by the random variable $\frac{[X]_{1/q}-\mu_q}{\sigma_q}$, we obtain Equation (18) and our proof is completed.

Remark 1. Under the assumptions of the theorem 1 the probability functions I and II of the Table 1 have the asymptotic approximation (18). Note that we do not have the same conclusion for the p.f. III of the table 1 since the assumption $b_n = o(1)$ does not hold.

Remark 2. From the proof of the theorem 1 we have that the p.f. of the family of confluent q- Chu -Vandermonde discrete distributions is discretely approximated by

$$f_X(x) \cong \frac{\frac{q^{\binom{x}{2}}(-c_n/b_n)^x}{(c_n;q)_x(q;q)_x}}{\frac{(c_n/b_n;q)_\infty}{(c_n;q)_\infty}}, \quad x = 0, 1, \dots .$$

From the q-CCVI of the table 1, for $b_n = -h_n = o(1)$, $h_n > 0$ and $c_n = c$ constant with $0 < c < 1$, $n = 0, 1, 2, \dots$, we get

$$p_{qCCVI}(x) \cong \frac{\frac{q^{\binom{x}{2}}(c/h_n)^x}{(c;q)_x(q;q)_x}}{\frac{(-c/h_n;q)_\infty}{(c;q)_\infty}}, \quad x = 0, 1, \dots . \tag{43}$$

Consequently, $p_{qCCVI}(x)$ is discretely approximated by a *modified q-Bessel* distribution.

Applications

1. A modified q-Bessel distribution: From the remarks 1 and 2 we get an asymptotic expression as in Equation (18) for the modified q-Bessel distribution with p.f.

$$p_{MB}(x) = \frac{\frac{q^{\binom{x}{2}}(c/h_n)^x}{(c;q)_x(q;q)_x}}{\frac{(-c/h_n;q)_\infty}{(c;q)_\infty}}, \quad x = 0, 1, \dots, \tag{44}$$

and with

$$\mu_q = \frac{c}{1-q} h_n^{-1} \ \ and \ \ \sigma_q^2 = \frac{c^2}{q(1-q)} h_n^{-2} - \frac{c}{1-q} h_n^{-1}$$

where $0 < c < 1$, $h_n > 0$, $n = 0, 1, 2, \dots$ with $h_n = o(1)$.

2. A Generalized q-Negative Binomial Distribution: The q-CCVII for $b = q^n$, $n = 1, 2, \dots$ and $\eta = q$ becomes a generalized q-negative binomial distribution with p.f.

$$p(x) = P(X = x) = \frac{\frac{\binom{n+x-1}{x}_q}{(-q;q)_x} q^{\binom{x}{2}} q^{-nx+x}}{\frac{(-q^{-n+1};q)_\infty}{(-q;q)_\infty}}, \ x = 0, 1, \dots \text{ with } \binom{n}{x}_q = \frac{[n]_q!}{[x]_q![n-x]_q!} \tag{45}$$

From the proposition 3 the mean value and the variance of the r.v. X of this q-negative binomial distribution are given respectively by

$$E(X) = \sum_{r=1}^{\infty} a_1^{II}(r)\,{}_1\phi_1(q^{n+1}; -q^{r+1}; q, -q^{r+1-n}) \tag{46}$$

where

$$a_1^{II}(r) = \frac{(-q^{r+1};q)_\infty}{(-q^{1-n};q)_\infty} \frac{|s_q(r,1)|(q^n;q)_r q^r}{(1-q)q^{nr}[r]_{1/q}!} \tag{47}$$

and

$$V(X) = E(X(X-1)) + E(X) - E(X)^2 \tag{48}$$

with

$$E(X(X-1)) = \sum_{r=2}^{\infty} a_2^{II}(r)\,{}_1\phi_1(q^{n+1}; -q^{r+1}; q, -q^{r+1-n}) \tag{49}$$

where

$$a_2^{II}(r) = \frac{(-q^{r+1};q)_\infty}{(-q^{1-n};q)_\infty} \frac{2|s_q(r,2)|(q^n;q)_r q^r}{(1-q)^2 q^{nr}[r]_{1/q}!} \tag{50}$$

By remark 1 the above q-negative binomial distribution has the Stieltjes-Wigert asymptotic behavior for $n \to \infty$ as in Equation (18) with μ_q and σ_q^2 given by proposition 1

$$\mu_q = q^{-n+1}\frac{1-q^n}{1-q} \text{ and } \sigma_q^2 = q^{-2n+1}\frac{1-q^n}{q(1-q)} + q^{-n+1}\frac{1-q^n}{1-q}$$

3. The Generalized Over / Underdispersed (O/U) Distribution: The q-CCVII for $b = q^n$ and $\eta = \lambda q^n$ becomes a generalized O/U distribution with p.f.

$$p(x) = P(X = x) = \frac{(-\lambda q^n;q)_\infty}{(-\lambda;q)_\infty} \frac{(q^n;q)_x q^{\binom{x}{2}} \lambda^x}{(-\lambda q;q)_x (q;q)_x},\ x = 0, 1, \ldots \text{ (see Kemp [1])} \tag{51}$$

From the proposition 3 the mean value and the variance of the r.v. X of the generalized O/U distribution are given respectively by

$$E(X) = \sum_{r=1}^{\infty} a_1^{II}(r)\,{}_1\phi_1(q^{r+n}; -\lambda q^{r+n}; q, -\lambda q^r) \tag{52}$$

where

$$a_1^{II}(r) = \frac{(-\lambda q^{r+n};q)_\infty}{(-\lambda;q)_\infty} \frac{|s_q(r,1)|(q^n;q)_r \lambda^r}{(1-q)[r]_{1/q}!} \tag{53}$$

and

$$V(X) = E(X(X-1)) + E(X) - E(X)^2 \tag{54}$$

with

$$E(X(X-1)) = \sum_{r=2}^{\infty} a_2^{II}(r)\,{}_1\phi_1(q^{r+n}; -\lambda q^{r+n}; q, -\lambda q^r) \tag{55}$$

where

$$a_2^{II}(r) = \frac{(-\lambda q^{r+n};q)_\infty}{(-\lambda;q)_\infty} \frac{2|s_q(r,2)|(q^n;q)_r \lambda^r}{(1-q)^2[r]_{1/q}!} \tag{56}$$

By remark 1 the generalized O/U distribution for $\lambda = \lambda_n \to \infty$, has the Stieltjes-Wigert asymptotic behavior for $n \to \infty$ as in Equation (18) with $\mu_q = \lambda_n/(1-q)$ and $\sigma_q^2 = \frac{\lambda_n^2}{q(1-q)} + \frac{\lambda_n}{1-q}$ given by proposition 1.

Remark 3. As it was noted in remark 1, theorem 1 is not sufficed for the confluent q-Chu-Vandermonde hypergeometric series discrete distribution III with p.f.

$$p_{qCCVI}(x) = P[X = x] = \frac{\frac{(q^{-n};q)_x}{(c;q)_x(q;q)_x} q^{\binom{x}{2}} (-cq^n)^x}{\frac{(cq^n;q)_\infty}{(c;q)_\infty}} = \frac{(c;q)_\infty}{(cq^n;q)_\infty} \frac{\binom{n}{x}_q q^{2\binom{x}{2}} c^x}{(c;q)_x(q;q)_x},\ x = 0, 1, \ldots, n \quad (57)$$

where c constant. However, the discrete approximation of the above q-CCV-III distribution for $n \to \infty$, is given by

$$p_{qCCVIII}(x) \cong (c;q)_\infty \frac{q^{2\binom{x}{2}} c^x}{(c;q)_x(q;q)_x},\ x = 0, 1, \ldots. \quad (58)$$

Berg and Valent [12], have proved that for $q < a < 1/q$, the above discrete probability measure Equation (58) has a continuous analogue counterpart family of absolutely continuous probability measures on $(0, \infty)$ defined by

$$v^{SC}(dx) = \frac{p}{\pi}\{(\frac{a}{a-1}\frac{(x/a;q)_\infty^2}{(q/a;q)_\infty})^2 + p^2(\frac{(x;q)_\infty^2}{(q;q)_\infty(qa;q)_\infty})^2\}^{-1}dx \quad (59)$$

where the parameter $p > 0$ is given by $p = \gamma/(t^2 + \gamma^2)$ with $\gamma^2 = -t(1/\psi(a) + t)$, where t belongs to the interval with endpoints 0 and $-1/\psi(a)$ and is given by $\psi(a) = (q;q)_\infty \sum_{j=0}^{\infty} \frac{q^j}{(a-q^j)(q;q)_j}$ with $\psi(q^+) = \infty$.

Remark 4. Gould and Srivastava [13] have presented a unification of some combinatorial identities associated with ordinary Gauss's summation theorem and their basic (or $q-$) extension associated with the $q-$ analogue Gauss' s theorem. They have also shown a generalization of their unification for the ordinary case involving a bilateral series and have posed as an open problem the q-extension of their bilateral result. In our work it is considered a family of confluent q-Chu-Vnadermonde distributions which can be associated with the q-analogue of Gauss's theorem and it would be an interesting closlely-related open problem to study a bilateral family of the considered distributions.

4. Concluding Remarks

In this article, we have provided a continuous limiting behavior of a family of confluent q -Chu- Vandermonde distributions, for $0 < q < 1$, in the sense of pointwise convergence, by applying a q-analogue of the usual Stirling asymptotic formula for the factorial number of order n. Specifically, we have designated as a main theorem the conditions under which the confluent q-Chu-Vandermonde discrete distributions q-CCVI and II converge to a continuous Stieltjes-Wigert distribution. Moreover, as applications for this study we present a modified q-Bessel distribution, a generalized q- negative Binomial distribution and a generalized over/underdispersed O/U distribution, converging to a continuous Stieltjes-Wigert distribution. Note that the main contribution of this article is that a discrete distribution congerges to a continuous one, which is not of a Gaussian type distribution.

Conflicts of Interest

The authors declare no conflict of interest.

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