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Article

Orthogonality and Dimensionality

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Abstract: In this article, we present what we believe to be a simple way to motivate the use of Hilbert spaces in quantum mechanics. To achieve this, we study the way the notion of dimension can, at a very primitive level, be defined as the cardinality of a maximal collection of mutually orthogonal elements (which, for instance, can be seen as spatial directions). Following this idea, we develop a formalism based on two basic ingredients, namely an orthogonality relation and matroids which are a very generic algebraic structure permitting to define a notion of dimension. Having obtained what we call *orthomatroids*, we then show that, in high enough dimension, the basic constituants of orthomatroids (more precisely the simple and irreducible ones) are isomorphic to generalized Hilbert lattices, so that their presence is a direct consequence of an orthogonality-based characterization of dimension.

Keywords: quantum logic; Piron's representation theorem; foundations of quantum mechanics

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1. Introduction

One of the most striking peculiarities of the mathematical formulation of quantum mechanics is its heavy reliance on the use of complex Hilbert spaces. In its core formulation, the state of a quantum system is represented by a normalized vector of a Hilbert space and the probabilistic nature of quantum mechanics is formalized by interpreting, using the Born rule, the quantity $\langle\psi|P_i|\psi\rangle$ as the probability of obtaining eigenvalue λ_i (with P_i denoting the orthogonal projection on the associated eigenspace) when measuring a quantum system in state $|\psi\rangle$, thus emphasizing the importance of the inner product.

However, fundamental as it is, it seems that there still exists no consensus regarding any satisfactory justification of the use of Hilbert spaces, so that one might wonder how far we stand from Mackey which had to state, in a rather *ad hoc* manner, its seventh axiom [1] as "*the partially ordered set (or poset) of* *all questions in quantum mechanics is isomorphic to the poset of all closed subspaces of a [...] Hilbert space.*" This requirement can usually be split into two parts, and many efforts have tried to address both of them: first that the poset of all questions forms an orthomodular lattice, and that this orthomodular lattice is indeed isomorphic to the lattice of all closed subspaces of a Hilbert space.

Regarding orthomodularity, described by Beltrametti and Cassinelli [2] as "*the survival [...] of a notion of the logical conditional, which takes the place of the classical implication associated with Boolean algebra*", some attempts to justify this as a property verified by the poset of all questions include works by Grinbaum [3] based on the notion of "*relevant information*" and by the author of the present article [4] where it is assumed that the considered lattice ensures the definition of sufficiently many "*points of view*."

As for the second requirement, a central result, Piron's celebrated Representation Theorem, provides conditions for an orthomodular lattice to be isomorphic to the lattice of all closed subspaces of a Hilbert space (or, more precisely, to a *generalized* Hilbert space, that is where it is not required that the used field is a "classical" one) [5–7]. However, one of those conditions, namely the *covering law*, *"presents a [...] delicate problem. [...] It is probably safe to say that no simple and entirely compelling argument has been given for assuming its general validity"* [8].

In the present article, we introduce some simple geometric conditions which are sufficient to ensure that the hypotheses of Piron's Representation Theorem are verified. Our approach is based on the notion of dimension and, more precisely, on the fact that this very notion can indeed be defined. Actually, this fact is taken for granted to the point that this requirement, if not implicit, often appears at the very beginning of axiomatic formulations of physics and, more specifically, of quantum mechanics. For instance, in [9], Hardy assumes that one can define a *number of degrees of freedom* K before presenting the eponymous *five reasonable axioms* of the article. In Rovelli's Relational Quantum Mechanics [10], Postulate 1 states that "*there is a maximum amount of relevant information that can be extracted from a system,*" and directly leads to considerations about the existence of the dimension of a system.

Thus, rather than questioning the role and meaning of concepts like dimension, degree of freedom amount of information in physical sciences, we propose instead to study the way the presence of dimension can shape the underlying mathematical structure of axiomatic formulations of physical theories.

Typically, such formulations involve the use of euclidean or Hilbert spaces, where the dimension is defined as the common cardinality of every bases of the considered space. This definition can even be refined in the presence of an inner product by restricting it to orthogonal bases. In other words, the dimension of a euclidean or Hilbert space is the common cardinality of every maximal set of mutually orthogonal vector rays (which we will call a maximal *orthoindependent* set of vector rays), so that dimension can be defined using orthogonality alone. However, it is also clear that orthogonality can be defined without relying upon the sophisticated machinery of linear and bilinear algebra. Indeed, one can easily imagine a Greek philosopher who would, during the antiquity, justify that the space surrounding him is 3-dimensional by exhibiting three orthogonal rods and arguing that he cannot add another rod orthogonal to the first three.

In that respect, we believe that orthogonality should be considered as a primitive notion and we will study the way it can lead to a reasonable definition of concepts such as dimension or bases.

Our primary tool will be the theory of matroids [11–14] which provides an algebraic framework for dealing with those concepts. This theory relies upon an abstract notion of independence which generalizes that of linear independence in linear algebra, so that the notion of linearly independent family of vectors is generalized to the notion of independent subset. An central feature of this theory is that it provides conditions regarding independence so that if a matroid admits a finite maximal (w.r.t. inclusion) independent subset (which is called a *basis* of the matroid), then every independent subset can be extended to a basis, and every bases of the matroid have the same cardinality, which is called the *rank* of the matroid and corresponds to the usual idea of dimension. It should however be noted that this result is, in general, only true in finite dimension (or finite rank), but it also emphasizes the role played by bases.

In our approach based on the notion of orthogonality, we will define a independence criterion based solely on an orthogonality relation, and such that any orthoindependent subset is independent, and we will demand that the obtained structure is indeed a matroid.

Moreover, a direct application of Zorn's Lemma shows that any orthoindependent subset can be extended to a maximal one w.r.t. inclusion, which differs greatly from the situation of basis in matroids in general. A natural requirement is then that every maximal orthoindependent subset should be a basis of the matroid.

The study of this requirements will constitute the next section. We will then show that the obtained algebraic structure, which we call *orthomatroids*, is closely related to Piron's propositional systems, and finally we will present an adaptation of Piron's Representation Theorem to orthomatroids.

2. Orthomatroids

Let us start by defining the general notion of orthogonality on a set.

Definition 1 *Given a set E, a binary relation $\perp$ on E is an* orthogonality relation *if, and only if it verifies*

$$\forall\, a, b \in E,\, a \perp b \iff b \perp a \qquad \textit{Symmetry}$$
$$\forall\, a, b \in E,\, a \perp b \implies a \neq b \qquad \textit{Anti-reflexivity}$$

In the following, such a pair $(E, \perp)$ will be called an orthoset*.*

The basic intuition behind an orthoset $(E, \perp)$ is to think of its elements as spatial directions. Obviously, the study of such structures is neither new nor original [15–17]. However, we present some basic results in order to introduce the important ideas before considering orthosets in the light of matroids.

Definition 2 *Given an orthoset $(E, \perp)$ and a subset $F \in \mathcal{P}(E)$ (where $\mathcal{P}(E)$ denotes the* power set *of E), we define its* orthogonal complement *$F^\perp$ as*

$$F^\perp = \{x \in E \mid \forall\, y \in F,\, x \perp y\}$$

Proposition 1 *Given an orthoset $(E, \perp)$, we have*

$$\forall\, F \in \mathcal{P}(E),\, F \cap F^\perp = \emptyset$$
$$\forall\, F, G \in \mathcal{P}(E),\, G \subseteq F^\perp \iff F \subseteq G^\perp \qquad \text{(C)}$$

Proof For $F \in \mathcal{P}(E)$ and $x \in F$, having $x \in F^\perp$ would imply that $\forall\, y \in F$, $x \perp y$ and, in particular, $x \perp x$ which is not possible. As a consequence, $F \cap F^\perp = \emptyset$. Now, contraposition (C) trivially follows from the symmetry of our orthogonality relation, since

$$G \subseteq F^\perp \iff \forall\, x \in F, \forall\, y \in G, x \perp y \iff F \subseteq G^\perp$$

□

This proposition leads directly to the definition of a *closure operator* [18–20] on E. We recall that a closure operator on a set E is a function $\mathrm{Cl} : \mathcal{P}(E) \to \mathcal{P}(E)$ which verifies the following three properties:

$$\begin{array}{cr} \forall\, F \in \mathcal{P}(E),\, F \subseteq \mathrm{Cl}(F) & \text{Extensivity} \\ \forall\, F, G \in \mathcal{P}(E),\, F \subseteq G \implies \mathrm{Cl}(F) \subseteq \mathrm{Cl}(G) & \text{Monotony} \\ \forall\, F \in \mathcal{P}(E),\ \mathrm{Cl}\big(\mathrm{Cl}(F)\big) = \mathrm{Cl}(F) & \text{Idempotence} \end{array}$$

Proposition 2 *The operation* $\cdot^{\perp\!\perp} : F \mapsto \left(F^\perp\right)^\perp$ *is a closure operator on* E*.*

Proof Extensivity trivially follows from (C), since

$$F^\perp \subseteq F^\perp \iff F \subseteq F^{\perp\!\perp}$$

Now, if $F \subseteq G$, then $F \subseteq G^{\perp\!\perp}$ and hence $G^\perp \subseteq F^\perp$. Repeating this step, we obtain

$$F \subseteq G \implies G^\perp \subseteq F^\perp \implies F^{\perp\!\perp} \subseteq G^{\perp\!\perp}$$

Finally, for idempotence, we only need to prove that $\left(F^{\perp\!\perp}\right)^{\perp\!\perp} \subseteq F^{\perp\!\perp}$ which is equivalent, because of (C), to $F^\perp \subseteq \left((F^{\perp\!\perp})^{\perp\!\perp}\right)^\perp = \left((F^\perp)^{\perp\!\perp}\right)^{\perp\!\perp}$. □

Definition 3 *Given a subset* $F \in \mathcal{P}(E)$, $F^{\perp\!\perp}$ *is the* closure *of* F*, and* F *is said to be* closed *if and only if it is its own closure,* i.e., $F = F^{\perp\!\perp}$.

Following our initial discussion, we now want to turn an orthoset $(E, \perp)$ into a matroid in order to have a suitable definition of dimension. These exists many equivalent definitions of matroids [12–14], including the following one, based on closure operators:

Definition 4 *A set* S *equipped with a closure operator* Cl *forms a matroid if and only if it verifies the* MacLane-Steinitz Exchange Property*:*

$$\forall\, F \in \mathcal{P}(S), \forall\, x, y \in S, x \in \mathrm{Cl}(F + x) \setminus \mathrm{Cl}(F) \implies x \in \mathrm{Cl}(F + y)$$

where $F + x$ *denotes the set* $F \cup \{x\}$*. In that case, a subset* F *of* S *is said to be* independent *if and only if it verifies*

$$\forall\, x \in F, x \notin \mathrm{Cl}(F - x)$$

with $F - x = F \setminus \{x\}$*, and a* basis *of the matroid is a maximal independent subset.*

Having defined a closure operation on E using our orthogonality relation, it is natural to base our definition of a matroid on it. As a consequence, E is a matroid with closure operator $\cdot^{\perp\!\perp}$ if and only if it verifies

$$\forall F \in \mathcal{P}(E), \forall x, y \in E, x \in (F+y)^{\perp\!\perp} \setminus F^{\perp\!\perp} \implies y \in (F+x)^{\perp\!\perp} \tag{EP}$$

With this definition, a subset $F \in \mathcal{P}(E)$ is independent if, and only if

$$\forall x \in F, x \notin (F-x)^{\perp\!\perp}$$

A matroid is said of *finite rank* if it admits a finite basis. In that case, it is a basic result of matroid theory that any two bases have the same cardinality (the *rank* of the matroid) and that any independent subset can be extended to a basis. However, in general, it is possible that a matroid does not have any bases.

The next step is to study the possibility of having bases made of mutually orthogonal elements.

Definition 5 *A subset F of E is said to be* orthoindependent *if and only if*

$$\forall x, y \in F, x \neq y \implies x \perp y$$

Obviously, every orthoindependent subset is independent. Moreover, orthoindependent subsets verify the nice following property:

Proposition 3 *If $\{F_i\}$ is a chain of orthoindependent subsets, then $\bigcup F_i$ is also orthoindependent.*

By direct application of Zorn's Lemma, the previous proposition implies that there exists maximal orthoindependent subsets and that every orthoindependent subset is included in a maximal one.

As explained in the introduction, it is natural, considering the previous proposition, to demand that every maximal orthoindependent subset B of any closed subset $F^{\perp\!\perp}$ of E is indeed a basis of $F^{\perp\!\perp}$. This suggests the following definition:

Definition 6 *For $F \in \mathcal{P}(E)$, an* orthobasis *of $F^{\perp\!\perp}$ is a orthoindependent subset B of E such that $B^{\perp\!\perp} = F^{\perp\!\perp}$*

Clearly, any orthobasis of E is also a basis of E. We will demand that:

> " (OB) Given a subset $F \in \mathcal{P}(E)$, every maximal orthoindependent subset B of $F^{\perp\!\perp}$ is an orthobasis of $F^{\perp\!\perp}$. "

This requirement is equivalent to the fact that for all $F \in \mathcal{P}(E)$, any orthoindependent subset I of $F^{\perp\!\perp}$ can be completed to an orthobasis of $F^{\perp\!\perp}$. It also implies, as we will show later in Proposition 6, that any independent subset can be extended to a basis, even in infinite rank.

The next proposition provides an equivalent and convenient way to state this requirement.

Proposition 4 *If $(E, \perp)$ verifies the MacLane-Steinitz Exchange Property, then axiom (OB) is equivalent to the* Straightening Property*, which we define as*

$$\forall F \in \mathcal{P}(E), \forall x \in E, x \notin F^{\perp\!\perp} \implies \exists y \in F^{\perp} : x \in (F+y)^{\perp\!\perp} \tag{SP}$$

Proof Suppose first that (OB) holds, and let $F \in \mathcal{P}(E)$ and $x \in E$ be such that $x \notin F^{\perp\perp}$. Moreover, let B be an orthobasis of $F^{\perp\perp}$ which we extend to an orthobasis B' of $(F+x)^{\perp\perp}$, using (OB). For $y \in B' \setminus B$, it is clear from the orthoindependence of B' that $y \in F^{\perp}$. This means that $y \in (F+x)^{\perp\perp} \setminus F^{\perp\perp}$ from which, using the Exchange Property, we deduce that $x \in (F+y)^{\perp\perp}$.

Conversely, suppose that the Straightening Property is verified. It can be remarked that, because of the Exchange Property, it can be equivalently stated as

$$\forall F \in \mathcal{P}(E), \forall x \in E, x \notin F^{\perp\perp} \implies \exists y \in F^{\perp} : (F+x)^{\perp\perp} = (F+y)^{\perp\perp}$$

Now, given an orthoindependent subset I of $F^{\perp\perp}$, let J be a maximal orthoindependent subset of $F^{\perp\perp}$ such that $I \subseteq J$. If $J^{\perp\perp} \neq F^{\perp\perp}$, then there exists an element $x \in F^{\perp\perp} \setminus J^{\perp\perp}$ and, following from the Straightening Property, there exists an element $y \in J^{\perp}$ such that $(J+x)^{\perp\perp} = (J+y)^{\perp\perp}$. This implies in particular that $y \in F^{\perp\perp}$ and hence $J+y$ is also an orthoindependent subset of $F^{\perp\perp}$ which is absurd since J was supposed maximal. As a consequence, we have $J^{\perp\perp} = F^{\perp\perp}$, that is J is an orthobasis of $F^{\perp\perp}$. □

We now summarize all these properties into the definition of what we call an *orthomatroid*:

Definition 7 (Orthomatroid) *An* orthomatroid *is an orthoset* $(E, \perp)$ *which verifies the two following properties:*

1. *Exchange Property*

$$\forall F \in \mathcal{P}(E), \forall x, y \in E, x \in (F+y)^{\perp\perp} \setminus F^{\perp\perp} \implies y \in (F+x)^{\perp\perp}$$

2. *Straightening Property*

$$\forall F \in \mathcal{P}(E), \forall x \in E, x \notin F^{\perp\perp} \implies \exists y \in F^{\perp} : x \in (F+y)^{\perp\perp}$$

Moreover, in order to talk about orthomatroids *up to isomorphism*, we will say that two orthomatroids $(E_1, \perp_1)$ and $(E_2, \perp_2)$ are *orthoisomorphic* if there exists a bijection $\varphi : E_1 \to E_2$ such that

$$\forall x, y \in E_1, x \perp_1 y \iff \varphi(x) \perp_2 \varphi(y)$$

We believe that orthomatroids provide a reasonable answer to the initial objective of formalizing a notion of orthogonality-based dimension. Here, the Exchange Property ensures an orthomatroid is indeed a matroid and, to that respect, that a correct notion of dimension can be defined through basis (at least in finite rank): every independent subset can be extended to a basis, any two bases of the same orthomatroid have the same cardinality. Moreover, considering orthoindependent subsets, the Straightening Property ensures that every orthoindependent subset can be extended to an orthobasis.

In the next section, we will study some properties of the lattices associated to orthomatroids, and show their relationship with propositional systems.

3. The Lattice Associated to an Orthomatroid

Definition 8 *Given an orthomatroid $M = (E, \perp)$, we define the lattice $\mathcal{L}(M)$ associated to M as the set $\{F^{\perp\!\!\perp} \mid F \in \mathcal{P}(E)\}$ of its closed subsets ordered by inclusion.*

Since $\mathcal{L}(M)$ is defined as the set of closed elements of a closure operator, it is a complete lattice [18,19,21], with operations

$$P \wedge Q = P \cap Q \qquad P \vee Q = (P \cup Q)^{\perp\!\!\perp} = (P^{\perp} \cap Q^{\perp})^{\perp}$$

It is also clearly atomistic (meaning that every element is the join of its atoms), and it is an ortholattice with $\cdot^{\perp}$ as orthocomplementation.

We now present two important properties that are verified by $\mathcal{L}(M)$.

Proposition 5 (Orthomodularity) *The ortholattice $\mathcal{L}(M)$ is orthomodular,* i.e.*, it verifies*

$$\forall P, Q \in \mathcal{L}(M), P \leq Q \implies P = Q \wedge (P \vee Q^{\perp})$$

Proof Let P and Q be in $\mathcal{L}(M)$ such that $P \leq Q$. Clearly, $P \leq Q \wedge (P \vee Q^{\perp})$. Conversely, let x be in $Q \wedge (P \vee Q^{\perp})$ and suppose that $x \notin P$. One can then define $y \in P^{\perp}$ such that $(P + y)^{\perp\!\!\perp} = (P + x)^{\perp\!\!\perp}$. In particular, since $P \leq Q$, $x \in Q$ and $y \in (P + x)^{\perp\!\!\perp}$, one has $y \in Q$, hence $y \in Q \wedge P^{\perp}$ and $y \in (Q \wedge P^{\perp}) \vee Q^{\perp} = \big(Q \wedge (P \vee Q^{\perp})\big)^{\perp}$. Since $x \in Q \wedge (P \vee Q^{\perp})$, this implies that $y \perp x$. But then, from $y \in P^{\perp}$ and $y \perp x$, one can deduce that $y \in (P + x)^{\perp}$ which is absurd since $y \in (P + x)^{\perp\!\!\perp}$. As a consequence, we have shown that $\forall x \in E, x \in Q \wedge (P \vee Q^{\perp}) \implies x \in P$ and finally that $P = Q \wedge (P \vee Q^{\perp})$. $\square$

In terms of set inclusion, this can be written equivalently as:

$$\forall P, Q \in \mathcal{P}(E), P^{\perp\!\!\perp} \subseteq Q^{\perp\!\!\perp} \implies P^{\perp\!\!\perp} = Q^{\perp\!\!\perp} \cap \big(P^{\perp\!\!\perp} \cup Q^{\perp}\big)^{\perp\!\!\perp} = Q^{\perp\!\!\perp} \cap \big(P^{\perp} \cap Q^{\perp\!\!\perp}\big)^{\perp}$$

As a consequence of orthomodularity, we can now easily prove the previously announced statement that every independent subset of an orthomatroid can be extended to a basis:

Proposition 6 *Given an independent subset I of E and an orthobasis B of $I^{\perp}$, $B' = I \cup B$ is a basis of E.*

Proof Let us first show that B' is independent. If $x \in B$, this is clear since

$$\forall y \in B' - x, x \perp y$$

so that $x \in (B' - x)^{\perp}$ and hence $x \notin (B' - x)^{\perp\!\!\perp}$. If $x \in I$, then one has

$$\begin{aligned}
(B' - x)^{\perp} &= \{y \in E \mid \forall z \in (I - x) \cup B, y \perp z\} \\
&= \{y \in B^{\perp} \mid \forall z \in I - x, y \perp z\} \\
&= \{y \in I^{\perp\!\!\perp} \mid \forall z \in I - x, y \perp z\} && \text{since } B^{\perp} = I^{\perp\!\!\perp} \\
&= I^{\perp\!\!\perp} \cap (I - x)^{\perp}
\end{aligned}$$

Moreover, by orthomodularity, it follows from $(I-x)^{\perp\!\!\perp} \subseteq I^{\perp\!\!\perp}$ that

$$(I-x)^{\perp\!\!\perp} = I^{\perp\!\!\perp} \cap \big(I^{\perp\!\!\perp} \cap (I-x)^{\perp}\big)^{\perp} = I^{\perp\!\!\perp} \cap (B'-x)^{\perp\!\!\perp}$$

As a consequence, if $x \in (B'-x)^{\perp\!\!\perp}$, then $x \in I^{\perp\!\!\perp} \cap (B'-x)^{\perp\!\!\perp}$ or, equivalently, $x \in (I-x)^{\perp\!\!\perp}$ which is not possible, as it would contradict the independence of I. Thus, we have shown that B' is independent. Finally, it is a basis since $B'^{\perp} = (I \cup B)^{\perp} = I^{\perp} \cap B^{\perp} = I^{\perp} \cap I^{\perp\!\!\perp} = \emptyset$ so that $B'^{\perp\!\!\perp} = \emptyset^{\perp} = E$ and hence for all $x \in E$, $B'+x$ is dependent. □

We now present a second property verified by $\mathcal{L}(M)$:

Proposition 7 (Atom-covering) *The ortholattice $\mathcal{L}(M)$ verifies the atom-covering property,* i.e.*, for all $F \in \mathcal{P}(E)$ and $x \in E$, if $x \notin F^{\perp\!\!\perp}$, then $(F+x)^{\perp\!\!\perp}$ covers $F^{\perp\!\!\perp}$:*

$$\forall\, G \in \mathcal{P}(E),\, F^{\perp\!\!\perp} < G^{\perp\!\!\perp} \leq (F+x)^{\perp\!\!\perp} \implies G^{\perp\!\!\perp} = (F+x)^{\perp\!\!\perp}$$

Proof Let $F \in \mathcal{P}(E)$, $x \in E \setminus F^{\perp\!\!\perp}$, and $G \in \mathcal{P}(E)$ be such that $F^{\perp\!\!\perp} < G^{\perp\!\!\perp} \leq (F+x)^{\perp\!\!\perp}$. We have to show that $G^{\perp\!\!\perp} = (F+x)^{\perp\!\!\perp}$. But since $F^{\perp\!\!\perp} < G^{\perp\!\!\perp}$, one can define $y \in G^{\perp\!\!\perp} \setminus F^{\perp\!\!\perp}$. Thus, we have

$$(F+y)^{\perp\!\!\perp} \leq G^{\perp\!\!\perp} \leq (F+x)^{\perp\!\!\perp}$$

But then, $y \in (F+x)^{\perp\!\!\perp} \setminus F^{\perp\!\!\perp}$ which implies that $(F+x)^{\perp\!\!\perp} = (F+y)^{\perp\!\!\perp}$ and finally that $G^{\perp\!\!\perp} = (F+x)^{\perp\!\!\perp}$. □

This shows that if M is an orthomatroid, then its associated lattice $\mathcal{L}(M)$ is a complete atomistic orthomodular lattice that satisfies the covering law or, following Piron's terminology [5,6], $\mathcal{L}(M)$ is a *propositional system*:

Theorem 1 *The lattice $\mathcal{L}(M)$ associated to any orthomatroid M is a propositional system.*

Conversely, given a propositional system S, we define the associated orthoset $\mathcal{O}(S)$ made of the atoms of S with orthogonality relation $p \perp q \iff p \leq q^{\perp}$. Moreover, for $F \in S$, let $\mathrm{At}(F)$ denote the set of atoms contained in F:

$$\mathrm{At}(F) = \{a \in \mathcal{O}(S) \mid a \leq F\}$$

Proposition 8 *If S is a propositional system, then $\mathcal{O}(S)$ is an orthomatroid.*

Proof Let us first consider the exchange property, and let $F \in S$, and x and y be two atoms of S such that $x \in \mathrm{At}(F \vee y) \setminus \mathrm{At}(F)$ (or, written in a lattice way, $x \leq F \vee y$ and $x \not\leq F$). Since y is an atom not contained in F, $F \vee y$ covers F. But then $F < F \vee x \leq F \vee y$ so that $F \vee x = F \vee y$.

Now, regarding the straightening property, if $x \notin \mathrm{At}(F)$, then considering orthomodularity, one has

$$F \vee x = F \vee \big((F \vee x) \wedge F^{\perp}\big)$$

A first consequence of this equality is that $(F \vee x) \wedge F^{\perp}$ contains at least one atom. Let y be such an atom (so that $y \in F^{\perp}$). Then, since $F \vee x$ covers F, it follows from the exchange property that $F \vee x = F \vee y$ or, equivalently that $x \in F \vee y$. □

Proposition 9 *If S is a propositional system, then $\mathcal{L}(\mathcal{O}(S))$ is orthoisomorphic (as a propositional system) to S.*

Proof For the sake of clarity, let $\cdot^{\perp_S}$ denote the orthogonal of S and $\cdot^{\perp_M}$ the orthogonal of orthomatroid $\mathcal{O}(S)$. For every subset F of $\mathcal{O}(S)$ (or, equivalently, for every set of atoms of S), one has

$$F^{\perp_M} = \left\{a \in \mathcal{O}(S) \middle| \forall\, b \in F,\, a \perp b\right\} = \left\{a \in \mathcal{O}(S) \mid a \perp \bigvee F\right\} = \mathrm{At}\Big(\big(\bigvee F\big)^{\perp_S}\Big)$$

so that $F^{\perp\!\!\perp_M} = \mathrm{At}\Big(\big(\bigvee F\big)^{\perp\!\!\perp_S}\Big) = \mathrm{At}\big(\bigvee F\big)$ and

$$\begin{aligned} \mathcal{L}(\mathcal{O}(S)) &= \left\{F^{\perp\!\!\perp_M} \mid F \subseteq \mathcal{O}(S)\right\} \\ &= \left\{\mathrm{At}\big(\bigvee F\big) \,\middle|\, F \subseteq \mathcal{O}(S)\right\} \\ &\subseteq \{\mathrm{At}(F) \mid F \in S\} \end{aligned}$$

Finally, the atomisticity of S implies that

$$\forall\, F \in S,\, F = \bigvee \mathrm{At}(F)$$

so that finally

$$\mathcal{L}(\mathcal{O}(S)) = \{\mathrm{At}(F) \mid F \in S\}$$

The rest of proof follows directly from this equality. □

From these results, it is clear that the lattice associated to any orthomatroid is a propositional system, and also that any propositional system S is the lattice associated to an orthomatroid, namely $\mathcal{O}(S)$. This illustrates the fact that orthomatroids do exactly capture the structure of the set of atoms of a propositional systems or, stated the other way, that propositional systems are exactly the lattices associated to orthomatroids.

Moreover, any orthomatroid of the form $\mathcal{O}(S)$ for S a propositional system is *simple*: for every atom $x \in \mathcal{O}(S)$, the closure of $\{x\}$ (that is $\{x\}^{\perp\!\!\perp}$) equals $\{x\}$. This means that there is a bijection between $\mathcal{O}(S)$ and $\mathrm{At}(S)$. It is a well known fact in matroid theory that every matroid can be *simplified*, *i.e.*, transformed into a simple matroid having the same associated lattice. Here, the simplification of an orthomatroid M is, up to orthoisomorphism, $\mathcal{O}(\mathcal{L}(M))$. As a consequence, in the following, we will restrict ourselves, without loss of generality, to simple orthomatroids.

4. A Representation Theorem for Orthomatroids

Having studied the close relationship between orthomatroids and propositional systems, we will now adapt Piron's Representation Theorem [5–7] to orthomatroids. This theorem can be stated as:

Theorem 2 (Pirons Representation Theorem) *Every irreducible propositional system of rank at least 4 is orthoisomorphic to the lattice of (biorthogonally) closed subspaces of a generalized Hilbert space.*

Let us recall that a generalized Hilbert space $(\mathcal{H}, \mathbf{K}, \cdot^\star, \langle\cdot|\cdot\rangle)$ consists in a vector space $\mathcal{H}$ over a field $\mathbf{K}$ with an involution anti-automorphism $\cdot^\star : \alpha \in \mathbf{K} \mapsto \alpha^\star$ and an orthomodular Hermitian form $\langle\cdot|\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbf{K}$ satisfying

$$\begin{gathered}\forall\, x, y, z \in \mathcal{H}, \forall\, \lambda \in \mathbf{K}, \langle \lambda x + y|z\rangle = \lambda\langle x|z\rangle + \langle y|z\rangle \\ \forall\, x, y \in \mathcal{H}, \langle x|y\rangle = \langle y|x\rangle^\star \\ \forall\, S \in \mathcal{P}(\mathcal{H}), S^\perp \oplus S^{\perp\!\perp} = \mathcal{H}\end{gathered}$$

where $S^\perp = \big\{x \in \mathcal{H} \,\big|\, \forall\, y \in S, \langle x|y\rangle = 0\big\}$. We invite the reader to consult [7] for more informations.

It is well known that the set of bi-orthogonally closed subsets of a generalized Hilbert space forms a propositional system. In terms of orthomatroids, this corresponds to the fact that the set $A(\mathcal{H})$ of vector rays of a generalized Hilbert space forms a (simple) orthomatroid with orthogonality relation

$$\mathbf{K}x \perp_{\mathcal{H}} \mathbf{K}y \iff \langle x|y\rangle = 0$$

In order to express Piron's Representation Theorem in terms of orthomatroids, we need to define the notion of irreducibility. Following [7] again, given a propositional system S, the binary relation $\sim$ defined on $\mathrm{At}(S)$ by

$$\forall\, x, y \in \mathrm{At}(S), x \sim y \iff \big(x \neq y \implies \exists\, z \in \mathrm{At}(S) \setminus \{x, y\} : z \leq x \vee y\big)$$

is an equivalence relation on $\mathrm{At}(S)$. The equivalence classes of $\mathrm{At}(S)$ are then called irreducible components of $\mathrm{At}(S)$, and S is said to be *irreducible* if it has a single irreducible component.

Given a simple orthomatroid $(E, \perp)$, the previous equivalence relation can be re-expressed as

$$\forall\, x, y \in E, x \sim y \iff \mathrm{Card}\{x, y\}^{\perp\!\perp} \neq 2$$

If $\big\{(E_i, \perp_i)\big\}_{i\in\mathcal{I}}$ denotes the irreducible components of E (with $\perp_i$ being the restriction of $\perp$ to E_i), then $(E, \perp)$ can be seen as the disjoint union of its irreducible components (up to orthoisomorphism):

$$\begin{gathered} E = \biguplus_{i\in\mathcal{I}} E_i = \bigcup_{i\in\mathcal{I}} \big\{(i, x) \,\big|\, x \in E_i\big\} \\ (i, x) \perp (j, y) \iff \big(i \neq j \text{ or } (i = j \text{ and } x \perp_i y)\big)\end{gathered}$$

We are finally able to express Piron's Representation Theorem in terms of orthomatroids, and we finally obtain the following representation theorem:

Theorem 3 (Representation of Orthomatroids) *Every simple and irreducible orthomatroid $(E, \perp)$ of rank at least 4 is orthoisomorphic to the orthomatroid $(A(\mathcal{H}), \perp_{\mathcal{H}})$ associated to a generalized Hilbert space $(\mathcal{H}, \mathbf{K}, \cdot^\star, \langle\cdot|\cdot\rangle)$.*

5. Conclusions

In this article, we have presented a formalism based on the idea that dimension can (at least in the finite case) be defined, at a very primitive level, as the common cardinality of a maximal collections of mutually orthogonal elements. Using a generic definition of orthogonality, together with the theory of matroids which provides a algebraic structure permitting to define a convenient notion of dimension, and a last requirement regarding the existence of orthobases, we have obtained what we call *orthomatroids* which provide a general framework for dealing with dimension in an orthogonality-based context. We then have shown that orthomatroids do actually exactly capture the structure of propositional systems (or, more precisely, of the set of atoms $\mathrm{At}(S)$ of a propositional system S, which is sufficient for reconstructing S) with the consequence that, in high enough dimension, irreducible matroids can be represented by generalized Hilbert spaces.

We insist on the fact that in this approach, the use of generalized Hilbert lattices (or, at least, of propositional systems) has entirely been derived from simple and generic assumptions. As a result, this suggests that instead of seeing the use of generalized Hilbert spaces (or, again, of propositional systems) in quantum physics as puzzling, it should in the contrary be seen as the most general (if not natural) way to model situations where dimension can be defined in terms of orthogonality, such as in quantum mechanics.

By way of conclusion, we would like to present a few directions for further study. First, even though the study of infinite matroids has long been problematic due to the lack of a unified approach capturing all the important aspects of finite matroid theory, it appears that recently, a satisfactory approach has been obtained [22], where an additional axiom (M) is introduced, stating that for every subset X of a matroid E and every independent subset $I \subseteq X$, the set of independent subsets J such that $I \subseteq J \subseteq X$ has a maximal element. The obtained structure is that of B-matroids [23] which are known to have equicardinal bases [24]. Since orthomatroids do not, in general, verify (M), it would be interesting to study the addition of (M) to the definition of orthomatroids.

Another direction is the relationship between orthomatroids and regular Hilbert spaces, as Piron's Representation Theorem provides no information about the field $\mathbf{K}$ appearing in the statement of the theorem. Is it possible to find some extra conditions which would ensure that $\mathbf{K}$ is a number field, *i.e.*, either $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$? Such a condition is provided by Solèr's theorem [25,26] in infinite dimension (namely, the existence of an infinite orthonormal sequence), but a condition for the general case remains to find. In that respect, we suggest another approach. In [27], the author has defined a general topology on orthomodular lattices which, in the case of a Hilbert lattice, is equivalent to the usual metrics-induced one. It is easy to adapt this topology to orthomatroids and the question is, then, whether it is possible to express some topological properties for orthomatroids which would, in turn, impose that $\mathbf{K}$ is a number field when applying Piron's Representation Theorem.

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Conflicts of Interest

The authors declare no conflict of interest.

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