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Article

Discrete Integrals Based on Comonotonic Modularity

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Abstract: It is known that several discrete integrals, including the Choquet and Sugeno integrals, as well as some of their generalizations, are comonotonically modular functions. Based on a recent description of the class of comonotonically modular functions, we axiomatically identify more general families of discrete integrals that are comonotonically modular, including signed Choquet integrals and symmetric signed Choquet integrals, as well as natural extensions of Sugeno integrals.

Keywords: aggregation function; discrete Choquet integral; discrete Sugeno integral; functional equation; comonotonic additivity; comonotonic modularity; axiomatization

Classification: MSC Primary 26B35, 39B22, 39B72

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1. Introduction

Aggregation functions arise wherever merging information is needed: Applied and pure mathematics (probability, statistics, decision theory and functional equations), operations research, computer science and many applied fields (economics and finance, pattern recognition and image processing, data fusion, *etc*.). For recent references, see Beliakov *et al*. [1] and Grabisch *et al*. [2].

Discrete Choquet integrals and discrete Sugeno integrals are among the best known functions in aggregation theory, mainly because of their many applications, for instance, in decision-making (see the edited book [3]). More generally, signed Choquet integrals, which need not be nondecreasing in their arguments, and the Lovász extensions of pseudo-Boolean functions, which need not vanish at the origin, are natural extensions of the Choquet integrals and have been thoroughly investigated in aggregation theory. For recent references, see, e.g., [4,5].

The class of n-variable Choquet integrals has been axiomatized independently by means of two noteworthy aggregation properties, namely comonotonic additivity (see, e.g., [6]) and horizontal min-additivity (originally called "horizontal additivity", see [7]). Function classes characterized by these properties have been recently described by the authors [5]. Quasi-Lovász extensions, which generalize signed Choquet integrals and Lovász extensions by transforming the arguments by a one-variable function, have also been recently investigated by the authors [8] through natural aggregation properties.

Lattice polynomial functions and quasi-Sugeno integrals generalize the notion of Sugeno integrals [9–13]: The former by removing the idempotency requirement and the latter also by transforming arguments by a one-variable function. Likewise, these functions have been axiomatized by means of well-known properties, such as comonotonic maxitivity and comonotonic minitivity.

All of these classes share the feature that its members are comonotonically modular. These facts motivated a recent study that led to a description of comonotonically modular functions [8]. In this paper, we survey these and other results and present a somewhat typological study of the vast class of comonotonically modular functions, where we identify several families of discrete integrals within this class using variants of homogeneity as distinguishing feature.

The paper is organized as follows. In Section 2, we recall basic notions and terminology related to the concept of signed Choquet integrals and present some preliminary characterization results. In Section 3, we survey several results that culminate in a description of comonotonic modularity and establish connections to other well studied properties of aggregation functions. These results are then used in Section 4 to provide characterizations of the various classes of functions considered in the previous sections, as well as of classes of functions extending Sugeno integrals.

We employ the following notation throughout the paper. The set of permutations on $X = \{1,\dots,n\}$ is denoted by $\mathfrak{S}_X$. For every $\sigma \in \mathfrak{S}_X$, we define:

$$\mathbb{R}^n_\sigma = \{\mathbf{x} = (x_1,\dots,x_n) \in \mathbb{R}^n : x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)}\}$$

Let $\mathbb{R}_+ = [0,+\infty[$ and $\mathbb{R}_- =]-\infty,0]$. We let I denote a nontrivial (*i.e.*, of positive Lebesgue measure) real interval, possibly unbounded. We also introduce the notation, $I_+ = I \cap \mathbb{R}_+$, $I_- = I \cap \mathbb{R}_-$ and $I^n_\sigma = I^n \cap \mathbb{R}^n_\sigma$. For every $S \subseteq X$, the symbol, $\mathbf{1}_S$, denotes the n-tuple whose ith component is one, if $i \in S$, and zero, otherwise. Let also $\mathbf{1} = \mathbf{1}_X$ and $\mathbf{0} = \mathbf{1}_\varnothing$. The symbols, $\wedge$ and $\vee$, denote the minimum and maximum functions, respectively. For every $\mathbf{x} \in \mathbb{R}^n$, let $\mathbf{x}^+$ be the n-tuple, whose ith component is $x_i \vee 0$, and let $\mathbf{x}^- = (-\mathbf{x})^+$. For every permutation, $\sigma \in \mathfrak{S}_X$, and every $i \in X$, we set $S^\uparrow_\sigma(i) = \{\sigma(i),\dots,\sigma(n)\}$, $S^\downarrow_\sigma(i) = \{\sigma(1),\dots,\sigma(i)\}$ and $S^\uparrow_\sigma(n+1) = S^\downarrow_\sigma(0) = \varnothing$.

2. Signed Choquet Integrals

In this section, we recall the concepts of Choquet integrals, signed Choquet integrals, and symmetric signed Choquet integrals. We also recall some axiomatizations of these function classes. For general background, see [4,5,8].

A capacity on $X = \{1, \ldots, n\}$ is a set function, $\mu: 2^X \to \mathbb{R}$, such that $\mu(\varnothing) = 0$ and $\mu(S) \leqslant \mu(T)$ whenever: $S \subseteq T$.

Definition 1. The Choquet integral with respect to a capacity, μ on X, is the function $C_\mu: \mathbb{R}^n_+ \to \mathbb{R}$ defined as:

$$C_\mu(\mathbf{x}) = \sum_{i=1}^{n} x_{\sigma(i)} \big(\mu(S^\uparrow_\sigma(i)) - \mu(S^\uparrow_\sigma(i+1))\big) \qquad \mathbf{x} \in (\mathbb{R}^n_+)_\sigma,\ \sigma \in \mathfrak{S}_X$$

The concept of a Choquet integral can be formally extended to a more general set of functions and n-tuples of $\mathbb{R}^n$ as follows. A signed capacity on X is a set function, $v: 2^X \to \mathbb{R}$, such that $v(\varnothing) = 0$.

Definition 2. The signed Choquet integral with respect to a signed capacity, v, on X is the function, $C_v: \mathbb{R}^n \to \mathbb{R}$ defined as:

$$C_v(\mathbf{x}) = \sum_{i=1}^{n} x_{\sigma(i)} \big(v(S^\uparrow_\sigma(i)) - v(S^\uparrow_\sigma(i+1))\big) \qquad \mathbf{x} \in \mathbb{R}^n_\sigma,\ \sigma \in \mathfrak{S}_X \tag{1}$$

From (1), it follows that $C_v(\mathbf{1}_S) = v(S)$ for every $S \subseteq X$. Thus, Equation (1) can be rewritten as:

$$C_v(\mathbf{x}) = \sum_{i=1}^{n} x_{\sigma(i)} \big(C_v(\mathbf{1}_{S^\uparrow_\sigma(i)}) - C_v(\mathbf{1}_{S^\uparrow_\sigma(i+1)})\big) \qquad \mathbf{x} \in \mathbb{R}^n_\sigma,\ \sigma \in \mathfrak{S}_X \tag{2}$$

Thus defined, the signed Choquet integral with respect to a signed capacity, v, on X is the continuous function, C_v, whose restriction to each region, $\mathbb{R}^n_\sigma$ ($\sigma \in \mathfrak{S}_X$), is the unique linear function that coincides with v (or equivalently, the corresponding pseudo-Boolean function, $v: \{0,1\}^n \to \mathbb{R}$) at the $n+1$ vertices of the standard simplex, $[0,1]^n \cap \mathbb{R}^n_\sigma$, of the unit cube, $[0,1]^n$. As such, C_v is called the Lovász extension of v.

From this observation, we immediately derive the following axiomatization of the class of n-variable signed Choquet integrals over a real interval, I. A function, $f: I^n \to \mathbb{R}$, is said to be a signed Choquet integral if it is the restriction to I^n of a signed Choquet integral.

Theorem 3 ([4]). *Assume that $0 \in I$. A function, $f: I^n \to \mathbb{R}$, satisfying $f(\mathbf{0}) = 0$ is a signed Choquet integral if and only if:*

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{x}') = \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{x}') \qquad \lambda \in [0,1],\ \mathbf{x}, \mathbf{x}' \in I^n_\sigma,\ \sigma \in \mathfrak{S}_X$$

The next theorem provides an axiomatization of the class of n-variable signed Choquet integrals based on comonotonic additivity, horizontal min-additivity and horizontal max-additivity. Recall that two n-tuples, $\mathbf{x}, \mathbf{x}' \in I^n$, are said to be comonotonic if there exists $\sigma \in \mathfrak{S}_X$, such that $\mathbf{x}, \mathbf{x}' \in I^n_\sigma$. A function, $f: I^n \to \mathbb{R}$, is said to be comonotonically additive if, for every comonotonic n-tuples, $\mathbf{x}, \mathbf{x}' \in I^n$, such that $\mathbf{x} + \mathbf{x}' \in I^n$, we have:

$$f(\mathbf{x} + \mathbf{x}') = f(\mathbf{x}) + f(\mathbf{x}')$$

A function, $f: I^n \to \mathbb{R}$, is said to be horizontally min-additive (respectively (resp.) horizontally max-additive) if, for every $\mathbf{x} \in I^n$ and every $c \in I$, such that $\mathbf{x} - \mathbf{x} \wedge c \in I^n$ (respectively $\mathbf{x} - \mathbf{x} \vee c \in I^n$), we have:

$$f(\mathbf{x}) = f(\mathbf{x} \wedge c) + f(\mathbf{x} - \mathbf{x} \wedge c) \qquad \big(\text{resp. } f(\mathbf{x}) = f(\mathbf{x} \vee c) + f(\mathbf{x} - \mathbf{x} \vee c)\big)$$

Theorem 4 ([5]). *Assume* $[0,1] \subseteq I \subseteq \mathbb{R}_+$ *or* $I = \mathbb{R}$*. Then, a function,* $f: I^n \to \mathbb{R}$*, is a signed Choquet integral if and only if the following conditions hold:*

(i) f *is comonotonically additive or horizontally min-additive (or horizontally max-additive if* $I = \mathbb{R}$*).*

(ii) $f(c\,x\mathbf{1}_S) = c\,f(x\mathbf{1}_S)$ *for all* $x \in I$ *and* $c > 0$*, such that* $c\,x \in I$ *and all* $S \subseteq X$*.*

Remark 1. It is easy to see that condition (ii) of Theorem 4 is equivalent to the following simpler condition: $f(x\mathbf{1}_S) = \operatorname{sign}(x)\,x\,f(\operatorname{sign}(x)\,\mathbf{1}_S)$ for all $x \in I$ and $S \subseteq X$.

We now recall the concept of the symmetric signed Choquet integral. Here, "symmetric" does not refer to invariance under a permutation of variables, but rather to the role of the origin of $\mathbb{R}^n$ as a symmetry center with respect to the function values.

Definition 5. Let v be a signed capacity on X. The symmetric signed Choquet integral with respect to v is the function, $\check{C}_v: \mathbb{R}^n \to \mathbb{R}$, defined as:

$$\check{C}_v(\mathbf{x}) = C_v(\mathbf{x}^+) - C_v(\mathbf{x}^-) \qquad \mathbf{x} \in \mathbb{R}^n \tag{3}$$

Thus defined, a symmetric signed Choquet integral is an odd function in the sense that $\check{C}_v(-\mathbf{x}) = -\check{C}_v(\mathbf{x})$. It is then not difficult to show that the restriction of $\check{C}_v$ to $\mathbb{R}^n_\sigma$ is the function:

$$\begin{aligned}\check{C}_v(\mathbf{x}) &= \sum_{i=1}^{p} x_{\sigma(i)}\big(C_v(\mathbf{1}_{S^{\downarrow}_\sigma(i)}) - C_v(\mathbf{1}_{S^{\downarrow}_\sigma(i-1)})\big)\\ &\quad + \sum_{i=p+1}^{n} x_{\sigma(i)}\big(C_v(\mathbf{1}_{S^{\uparrow}_\sigma(i)}) - C_v(\mathbf{1}_{S^{\uparrow}_\sigma(i+1)})\big) \qquad \mathbf{x} \in \mathbb{R}^n_\sigma\end{aligned} \tag{4}$$

where the integer, $p \in \{0, \ldots, n\}$, is given by the condition, $x_{\sigma(p)} < 0 \leqslant x_{\sigma(p+1)}$, with the convention that $x_{\sigma(0)} = -\infty$ and $x_{\sigma(n+1)} = +\infty$.

The following theorem provides an axiomatization of the class of n-variable symmetric signed Choquet integrals based on horizontal median-additive additivity. Assuming that I is centered at zero, recall that a function, $f: I^n \to \mathbb{R}$, is said to be horizontally median-additive if, for every $\mathbf{x} \in I^n$ and every $c \in I_+$, we have:

$$f(\mathbf{x}) = f\big(\operatorname{med}(-c, \mathbf{x}, c)\big) + f\big(\mathbf{x} - \mathbf{x} \wedge c\big) + f\big(\mathbf{x} - \mathbf{x} \vee (-c)\big) \tag{5}$$

where $\operatorname{med}(-c, \mathbf{x}, c)$ is the n-tuple, whose ith component is the middle value of $\{-c, x_i, c\}$. Equivalently, a function, $f: I^n \to \mathbb{R}$, is horizontally median-additive if and only if its restrictions to I^n_+ and I^n_- are comonotonically additive and:

$$f(\mathbf{x}) = f(\mathbf{x}^+) + f(-\mathbf{x}^-) \qquad \mathbf{x} \in I^n$$

A function, $f: I^n \to \mathbb{R}$, is said to be a symmetric signed Choquet integral if it is the restriction to I^n of a symmetric signed Choquet integral.

Theorem 6 ([5]). *Assume that* I *is centered at zero, with* $[-1,1] \subseteq I$*. Then, a function,* $f: I^n \to \mathbb{R}$*, is a symmetric signed Choquet integral if and only if the following conditions hold:*

(*i*) *f is horizontally median-additive.*

(*ii*) *$f(c\,x\mathbf{1}_S) = c\,f(x\mathbf{1}_S)$ for all $c, x \in I$, such that $c\,x \in I$ and all $S \subseteq X$.*

Remark 2. It is easy to see that condition (ii) of Theorem 6 is equivalent to the following simpler condition: $f(x\mathbf{1}_S) = x\,f(\mathbf{1}_S)$ for all $x \in I$ and $S \subseteq X$.

We end this section by recalling the following important formula. For every signed capacity, v, on X, we have:

$$C_v(\mathbf{x}) \;=\; C_v(\mathbf{x}^+) - C_{v^d}(\mathbf{x}^-) \qquad \mathbf{x} \in \mathbb{R}^n \tag{6}$$

where v^d is the capacity on X, called the dual capacity of v, defined as $v^d(S) = v(X) - v(X \smallsetminus S)$.

3. Comonotonic Modularity

Recall that a function, $f\colon I^n \to \mathbb{R}$, is said to be modular (or a valuation) if:

$$f(\mathbf{x}) + f(\mathbf{x}') = f(\mathbf{x} \wedge \mathbf{x}') + f(\mathbf{x} \vee \mathbf{x}') \tag{7}$$

for every $\mathbf{x}, \mathbf{x}' \in I^n$, where $\wedge$ and $\vee$ are considered componentwise. It was proven [14] that a function, $f\colon I^n \to \mathbb{R}$, is modular if and only if it is separable, that is, there exist n functions, $f_i\colon I \to \mathbb{R}$ $(i = 1, \ldots, n)$, such that $f = \sum_{i=1}^n f_i$. In particular, any one-variable function $f\colon I \to \mathbb{R}$ is modular.

More generally, a function, $f\colon I^n \to \mathbb{R}$, is said to be comonotonically modular (or a comonotonic valuation) if (7) holds for every comonotonic n-tuples, $\mathbf{x}, \mathbf{x}' \in I^n$; see [8,15]. It was shown [8] that a function, $f\colon I^n \to \mathbb{R}$, is comonotonically modular if and only if it is comonotonically separable, that is, for every $\sigma \in \mathfrak{S}_X$, there exist functions, $f_i^\sigma\colon I \to \mathbb{R}$ $(i = 1, \ldots, n)$, such that:

$$f(\mathbf{x}) \;=\; \sum_{i=1}^n f_i^\sigma(x_{\sigma(i)}) \;=\; \sum_{i=1}^n f_{\sigma^{-1}(i)}^\sigma(x_i) \qquad \mathbf{x} \in I_\sigma^n$$

We also have the following important definitions. For every $\mathbf{x} \in \mathbb{R}^n$ and every $c \in \mathbb{R}_+$ (resp. $c \in \mathbb{R}_-$), we denote by $[\mathbf{x}]_c$ (resp. $[\mathbf{x}]^c$) the n-tuple, whose ith component is zero, if $x_i \leqslant c$ (resp. $x_i \geqslant c$), and x_i, otherwise. Recall that a function, $f\colon I^n \to \mathbb{R}$, where $0 \in I \subseteq \mathbb{R}_+$, is invariant under horizontal min-differences if, for every $\mathbf{x} \in I^n$ and every $c \in I$, we have:

$$f(\mathbf{x}) - f(\mathbf{x} \wedge c) = f([\mathbf{x}]_c) - f([\mathbf{x}]_c \wedge c) \tag{8}$$

Dually, a function, $f\colon I^n \to \mathbb{R}$, where $0 \in I \subseteq \mathbb{R}_-$ is invariant under horizontal max-differences if, for every $\mathbf{x} \in I^n$ and every $c \in I$, we have:

$$f(\mathbf{x}) - f(\mathbf{x} \vee c) = f([\mathbf{x}]^c) - f([\mathbf{x}]^c \vee c) \tag{9}$$

The following theorem provides a description of the class of functions that are comonotonically modular.

Theorem 7 ([8])**.** *Assume that $I \ni 0$. For any function, $f\colon I^n \to \mathbb{R}$, the following assertions are equivalent:*

(i) *f is comonotonically modular.*

(ii) *$f|_{I_+^n}$ is comonotonically modular (or invariant under horizontal min-differences); $f|_{I_-^n}$ is comonotonically modular (or invariant under horizontal max-differences); and we have $f(\mathbf{x}) + f(\mathbf{0}) = f(\mathbf{x}^+) + f(-\mathbf{x}^-)$ for every $\mathbf{x} \in I^n$.*

(iii) *There exist, $g\colon I_+^n \to \mathbb{R}$ and $h\colon I_-^n \to \mathbb{R}$, such that, for every $\sigma \in \mathfrak{S}_X$ and every $\mathbf{x} \in I_\sigma^n$:*

$$f(\mathbf{x}) = f(\mathbf{0}) + \sum_{i=1}^{p} \Big(h(x_{\sigma(i)}\mathbf{1}_{S_\sigma^\downarrow(i)}) - h(x_{\sigma(i)}\mathbf{1}_{S_\sigma^\downarrow(i-1)})\Big) + \sum_{i=p+1}^{n} \Big(g(x_{\sigma(i)}\mathbf{1}_{S_\sigma^\uparrow(i)}) - g(x_{\sigma(i)}\mathbf{1}_{S_\sigma^\uparrow(i+1)})\Big)$$

where $p \in \{0, \ldots, n\}$ is such that $x_{\sigma(p)} < 0 \leqslant x_{\sigma(p+1)}$, with the convention that $x_{\sigma(0)} = -\infty$ and $x_{\sigma(n+1)} = +\infty$. In this case, we can choose $g = f|_{I_+^n}$ and: $h = f|_{I_-^n}$.

We finish this section with remarks on some properties subsumed by comonotonic modularity, namely, the following relaxations of maxitivity and minitivity properties.

Recall that a function, $f\colon I^n \to \mathbb{R}$, is said to be maxitive if:

$$f(\mathbf{x} \vee \mathbf{x}') = f(\mathbf{x}) \vee f(\mathbf{x}') \qquad \mathbf{x}, \mathbf{x}' \in I^n \tag{10}$$

and it is said to be minitive if:

$$f(\mathbf{x} \wedge \mathbf{x}') = f(\mathbf{x}) \wedge f(\mathbf{x}') \qquad \mathbf{x}, \mathbf{x}' \in I^n \tag{11}$$

As in the case of modularity, maxitivity and minitivity give rise to noteworthy decompositions of functions into maxima and minima, respectively, of one-variable functions.

In the context of Sugeno integrals (see Section 4), de Campos *et al.* [6] proposed the following comonotonic variants of these properties. A function, $f\colon I^n \to \mathbb{R}$, is said to be comonotonic maxitive (resp. comonotonic minitive) if (10) (resp. (11)) holds for any two comonotonic n-tuples $\mathbf{x}, \mathbf{x}' \in I^n$. It was shown in [12] that any of these properties implies nondecreasing monotonicity, and it is not difficult to observe that comonotonic maxitivity together with comonotonic minitivity imply comonotonic modularity; the converse is not true (e.g., the arithmetic mean).

Explicit descriptions of each one of these properties was given in [9] for functions over bounded chains. For the sake of self-containment, we present these descriptions here. To this end, we now assume that $I = [a, b] \subseteq \mathbb{R}$, and for each $S \subseteq X$, we denote by $\mathbf{e}_S$, the n-tuple in $\{a, b\}^n$, whose i-th component is b, if $i \in S$, and a, otherwise.

Theorem 8 ([9])**.** *Assume $I = [a, b] \subseteq \mathbb{R}$. A function, $f\colon I^n \to \mathbb{R}$, is comonotonic maxitive (resp. comonotonic minitive) if and only if there exists a nondecreasing function, $g\colon I^n \to \mathbb{R}$, such that:*

$$f(\mathbf{x}) = \bigvee_{S \subseteq X} g\Big(\mathbf{e}_S \wedge \bigwedge_{i \in S} x_i\Big) \qquad \Big(\text{resp. } f(\mathbf{x}) = \bigwedge_{S \subseteq X} g\Big(\mathbf{e}_{X \setminus S} \vee \bigvee_{i \in S} x_i\Big)\Big)$$

In this case, we can choose $g = f$.

These descriptions are further refined in the following corollary.

Corollary 9. *Assume $I = [a,b] \subseteq \mathbb{R}$. For any function, $f\colon I^n \to \mathbb{R}$, the following assertions are equivalent:*

(i) *f is comonotonic maxitive (resp. comonotonic minitive).*

(ii) *There are unary nondecreasing functions, $\varphi_S\colon I \to \mathbb{R}$ ($S \subseteq X$), such that:*

$$f(\mathbf{x}) = \bigvee_{S\subseteq X} \varphi_S\Big(\bigwedge_{i\in S} x_i\Big) \qquad \text{(resp. } f(\mathbf{x}) = \bigwedge_{S\subseteq X} \varphi_S\Big(\bigvee_{i\in S} x_i\Big)\text{)}$$

In this case, we can choose $\varphi_S(x) = f(\mathbf{e}_S \wedge x)$ (resp. $\varphi_S(x) = f(\mathbf{e}_{X\setminus S} \vee x)$) for every $S \subseteq X$.

(iii) *For every $\sigma \in \mathfrak{S}_X$, there are nondecreasing functions, $f_i^\sigma\colon I \to \mathbb{R}$ ($i = 1,\ldots,n$), such that for every $\mathbf{x} \in I_\sigma^n$:*

$$f(\mathbf{x}) = \bigvee_{i\in X} f_i^\sigma(x_{\sigma(i)}) \qquad \text{(resp. } f(\mathbf{x}) = \bigwedge_{i\in X} f_i^\sigma(x_{\sigma(i)})\text{)}$$

In this case, we can choose $f_i^\sigma(x) = f(\mathbf{e}_{S_\sigma^\uparrow(i)} \wedge x)$ (resp. $f_i^\sigma(x) = f(\mathbf{e}_{S_\sigma^\downarrow(i-1)} \vee x)$).

Remark 3. (i) Note that the expressions provided in Theorem 8 and Corollary 9 greatly differ from the additive form given in Theorem 7.

(ii) An alternative description of comonotonic maxitive (resp. comonotonic minitive) functions was obtained in Grabisch *et al.* ([2], Chart 2).

4. Classes of Comonotonically Modular Integrals

In this section, we present axiomatizations of classes of functions that naturally generalize Choquet integrals (e.g., signed Choquet integrals and symmetric signed Choquet integrals) by means of comonotonic modularity and variants of homogeneity. From the analysis of the more stringent properties of comonotonic minitivity and comonotonic maxitivity, we also present axiomatizations of classes of functions generalizing Sugeno integrals.

4.1. Comonotonically Modular Integrals Generalizing Choquet Integrals

The following theorem provides an axiomatization of the class of n-variable signed Choquet integrals.

Theorem 10. *Assume $[0,1] \subseteq I \subseteq \mathbb{R}_+$ or $[-1,1] \subseteq I$. Then, a function, $f\colon I^n \to \mathbb{R}$, is a signed Choquet integral if and only if the following conditions hold:*

(i) *f is comonotonically modular.*

(ii) *$f(\mathbf{0}) = 0$ and $f(x\mathbf{1}_S) = \mathrm{sign}(x)\, x\, f(\mathrm{sign}(x)\,\mathbf{1}_S)$ for all $x \in I$ and $S \subseteq X$.*

(iii) *If $[-1,1] \subseteq I$, then $f(\mathbf{1}_{X\setminus S}) = f(\mathbf{1}) + f(-\mathbf{1}_S)$ for all $S \subseteq X$.*

Proof. (Necessity) Assume that f is a signed Choquet integral, $f = C_v$. Then, condition (ii) is satisfied in view of Theorem 4 and Remark 1. If $[-1,1] \subseteq I$, then by (6), we have:

$$C_v(-\mathbf{1}_S) \;=\; -C_{v^d}(\mathbf{1}_S) \;=\; C_v(\mathbf{1}_{X\smallsetminus S}) - C_v(\mathbf{1})$$

which shows that condition (iii) is satisfied. Let us now show that condition (i) is also satisfied. For every $\sigma \in \mathfrak{S}_X$ and every $\mathbf{x} \in \mathbb{R}^n_\sigma$, setting $p \in \{0,\ldots,n\}$, such that $x_{\sigma(p)} < 0 \leqslant x_{\sigma(p+1)}$, by (2) and conditions (iii) and (ii), we have:

$$\begin{aligned} C_v(\mathbf{x}) &= \sum_{i=1}^{n} x_{\sigma(i)} \big(C_v(\mathbf{1}_{S_\sigma^\uparrow(i)}) - C_v(\mathbf{1}_{S_\sigma^\uparrow(i+1)}) \big) \\ &= \sum_{i=1}^{p} x_{\sigma(i)} \big(C_v(-\mathbf{1}_{S_\sigma^\downarrow(i-1)}) - C_v(-\mathbf{1}_{S_\sigma^\downarrow(i)}) \big) + \sum_{i=p+1}^{n} x_{\sigma(i)} \big(C_v(\mathbf{1}_{S_\sigma^\uparrow(i)}) - C_v(\mathbf{1}_{S_\sigma^\uparrow(i+1)}) \big) \\ &= \sum_{i=1}^{p} \big(C_v(x_{\sigma(i)}\,\mathbf{1}_{S_\sigma^\downarrow(i)}) - C_v(x_{\sigma(i)}\,\mathbf{1}_{S_\sigma^\downarrow(i-1)}) \big) + \sum_{i=p+1}^{n} \big(C_v(x_{\sigma(i)}\,\mathbf{1}_{S_\sigma^\uparrow(i)}) - C_v(x_{\sigma(i)}\,\mathbf{1}_{S_\sigma^\uparrow(i+1)}) \big) \end{aligned}$$

which shows that condition (iii) of Theorem 7 is satisfied. Hence, C_v is comonotonically modular.

(Sufficiency) Assume that f satisfies conditions (i)–(iii). By condition (iii) of Theorem 7 and conditions (ii) and (iii), for every $\sigma \in \mathfrak{S}_X$ and every $\mathbf{x} \in \mathbb{R}^n_\sigma$, we have:

$$\begin{aligned} f(\mathbf{x}) &= \sum_{i=1}^{p} \big(f(x_{\sigma(i)}\mathbf{1}_{S_\sigma^\downarrow(i)}) - f(x_{\sigma(i)}\mathbf{1}_{S_\sigma^\downarrow(i-1)}) \big) + \sum_{i=p+1}^{n} \big(f(x_{\sigma(i)}\mathbf{1}_{S_\sigma^\uparrow(i)}) - f(x_{\sigma(i)}\mathbf{1}_{S_\sigma^\uparrow(i+1)}) \big) \\ &= \sum_{i=1}^{p} x_{\sigma(i)} \big(f(-\mathbf{1}_{S_\sigma^\downarrow(i-1)}) - f(-\mathbf{1}_{S_\sigma^\downarrow(i)}) \big) + \sum_{i=p+1}^{n} x_{\sigma(i)} \big(f(\mathbf{1}_{S_\sigma^\uparrow(i)}) - f(\mathbf{1}_{S_\sigma^\uparrow(i+1)}) \big) \\ &= \sum_{i=1}^{n} x_{\sigma(i)} \big(f(\mathbf{1}_{S_\sigma^\uparrow(i)}) - f(\mathbf{1}_{S_\sigma^\uparrow(i+1)}) \big) \end{aligned}$$

which, combined with (2), shows that f is a signed Choquet integral. □

Remark 4. Condition (iii) of Theorem 10 is necessary. Indeed, the function, $f(\mathbf{x}) = C_v(\mathbf{x}^+)$, satisfies conditions (i) and (ii), but fails to satisfy condition (iii).

Theorem 11. *Assume I is centered at zero, with $[-1,1] \subseteq I$. Then, a function, $f\colon I^n \to \mathbb{R}$, is a symmetric signed Choquet integral if and only if the following conditions hold:*

(i) f is comonotonically modular.

(ii) $f(x\mathbf{1}_S) = x\, f(\mathbf{1}_S)$ for all $x \in I$ and $S \subseteq X$.

Proof. (Necessity) Assume that f is a symmetric signed Choquet integral, $f = \check{C}_v$. Then, condition (ii) is satisfied in view of Theorem 6 and Remark 2. Let us now show that condition (i) is also satisfied. For every $\sigma \in \mathfrak{S}_X$ and every $\mathbf{x} \in \mathbb{R}^n_\sigma$, setting $p \in \{0,\ldots,n\}$, such that $x_{\sigma(p)} < 0 \leqslant x_{\sigma(p+1)}$, by (4) and condition (ii), we have:

$$\begin{aligned} C_v(\mathbf{x}) &= \sum_{i=1}^{p} x_{\sigma(i)} \big(C_v(\mathbf{1}_{S_\sigma^\downarrow(i)}) - C_v(\mathbf{1}_{S_\sigma^\downarrow(i-1)}) \big) + \sum_{i=p+1}^{n} x_{\sigma(i)} \big(C_v(\mathbf{1}_{S_\sigma^\uparrow(i)}) - C_v(\mathbf{1}_{S_\sigma^\uparrow(i+1)}) \big) \\ &= \sum_{i=1}^{p} \big(C_v(x_{\sigma(i)}\,\mathbf{1}_{S_\sigma^\downarrow(i)}) - C_v(x_{\sigma(i)}\,\mathbf{1}_{S_\sigma^\downarrow(i-1)}) \big) + \sum_{i=p+1}^{n} \big(C_v(x_{\sigma(i)}\,\mathbf{1}_{S_\sigma^\uparrow(i)}) - C_v(x_{\sigma(i)}\,\mathbf{1}_{S_\sigma^\uparrow(i+1)}) \big) \end{aligned}$$

which shows that condition (iii) of Theorem 7 is satisfied. Hence, C_v is comonotonically modular.

(Sufficiency) Assume that f satisfies conditions (i) and (ii). By condition (iii) of Theorem 7 and condition (ii), for every $\sigma \in \mathfrak{S}_X$ and every $\mathbf{x} \in \mathbb{R}^n_\sigma$, we have:

$$\begin{aligned} f(\mathbf{x}) &= \sum_{i=1}^{p}\big(f(x_{\sigma(i)}\mathbf{1}_{S^{\downarrow}_{\sigma}(i)}) - f(x_{\sigma(i)}\mathbf{1}_{S^{\downarrow}_{\sigma}(i-1)})\big) + \sum_{i=p+1}^{n}\big(f(x_{\sigma(i)}\mathbf{1}_{S^{\uparrow}_{\sigma}(i)}) - f(x_{\sigma(i)}\mathbf{1}_{S^{\uparrow}_{\sigma}(i+1)})\big) \\ &= \sum_{i=1}^{p} x_{\sigma(i)}\big(f(\mathbf{1}_{S^{\downarrow}_{\sigma}(i)}) - f(\mathbf{1}_{S^{\downarrow}_{\sigma}(i-1)})\big) + \sum_{i=p+1}^{n} x_{\sigma(i)}\big(f(\mathbf{1}_{S^{\uparrow}_{\sigma}(i)}) - f(\mathbf{1}_{S^{\uparrow}_{\sigma}(i+1)})\big) \end{aligned}$$

which, combined with (4), shows that f is a symmetric signed Choquet integral. □

The authors [8] showed that comonotonically modular functions also include the class of signed quasi-Choquet integrals on intervals of the forms I_+ and I_- and the class of symmetric signed quasi-Choquet integrals on intervals, I, centered at the origin.

Definition 12. Assume $I \ni 0$ and let v be a signed capacity on X. A signed quasi-Choquet integral with respect to v is a function, $f\colon I^n \to \mathbb{R}$, defined as $f(\mathbf{x}) = C_v(\varphi(x_1),\ldots,\varphi(x_n))$, where $\varphi\colon I \to \mathbb{R}$ is a nondecreasing function satisfying $\varphi(0) = 0$.

We now recall axiomatizations of the class of n-variable signed quasi-Choquet integrals on I_+ and I_- by means of comonotonic modularity and variants of homogeneity.

Theorem 13 ([8])**.** *Assume $[0,1] \subseteq I \subseteq \mathbb{R}_+$ (resp. $[-1,0] \subseteq I \subseteq \mathbb{R}_-$), and let $f\colon I^n \to \mathbb{R}$ be a nonconstant function, such that $f(\mathbf{0}) = 0$. Then, the following assertions are equivalent:*

- (i) *f is a signed quasi-Choquet integral, and there exists $S \subseteq X$, such that $f(\mathbf{1}_S) \neq 0$ (resp. $f(-\mathbf{1}_S) \neq 0$).*
- (ii) *$f|_{I^n_+}$ is comonotonically modular (or invariant under horizontal min-differences); $f|_{I^n_-}$ is comonotonically modular (or invariant under horizontal max-differences); and there exists a nondecreasing function, $\varphi\colon I \to \mathbb{R}$ satisfying $\varphi(0) = 0$, such that $f(x\mathbf{1}_S) = \mathrm{sign}(x)\,\varphi(x)\,f(\mathrm{sign}(x)\,\mathbf{1}_S)$ for every $x \in I$ and every $S \subseteq X$.*

Remark 5. If $I = [0,1]$ (resp. $I = [-1,0]$), then the "nonconstant" assumption and the second condition in assertion (i) of Theorem 13 can be dropped off.

The extension of Theorem 13 to functions on intervals, I, centered at zero and containing $[-1,1]$ remains an interesting open problem.

We now recall the axiomatization obtained by the authors of the class of n-variable symmetric signed quasi-Choquet integrals.

Definition 14. Assume I is centered at zero and let v be a signed capacity on X. A symmetric signed quasi-Choquet integral with respect to v is a function, $f\colon I^n \to \mathbb{R}$, defined as $f(\mathbf{x}) = \check{C}_v(\varphi(x_1),\ldots,\varphi(x_n))$, where $\varphi\colon I \to \mathbb{R}$ is a nondecreasing odd function.

Theorem 15 ([8]). *Assume that I is centered at zero, with $[-1,1] \subseteq I$, and let $f\colon I^n \to \mathbb{R}$ be a function, such that $f|_{I_+^n}$ or $f|_{I_-^n}$ is nonconstant and $f(\mathbf{0}) = 0$. Then, the following assertions are equivalent:*

(i) *f is a symmetric signed quasi-Choquet integral, and there exists $S \subseteq X$, such that $f(\mathbf{1}_S) \neq 0$.*

(ii) *f is comonotonically modular, and there exists a nondecreasing odd function, $\varphi\colon I \to \mathbb{R}$, such that $f(x\mathbf{1}_S) = \varphi(x)\, f(\mathbf{1}_S)$ for every $x \in I$ and every $S \subseteq X$.*

Remark 6. If $I = [-1,1]$, then the "nonconstant" assumption and the second condition in assertion (i) of Theorem 15 can be dropped off.

4.2. Comonotonically Modular Integrals Generalizing Sugeno Integrals

In this subsection, we consider natural extensions of the n-variable Sugeno integrals on a bounded real interval, $I = [a,b]$. By an I-valued capacity on X, we mean an order preserving mapping, $\mu\colon 2^X \to I$, such that $\mu(\varnothing) = a$ and $\mu(X) = b$.

Definition 16. Assume that $I = [a,b]$. The Sugeno integral with respect to an I-valued capacity, μ, on X is the function, $\mathcal{S}_\mu\colon I^n \to I$, defined as:

$$\mathcal{S}_\mu(\mathbf{x}) \;=\; \bigvee_{i\in X} x_{\sigma(i)} \wedge \mu(S_\sigma^\uparrow(i)) \qquad \mathbf{x} \in I_\sigma^n,\ \sigma \in \mathfrak{S}_X$$

As the following proposition suggests, Sugeno integrals can be viewed as idempotent "lattice polynomial functions" (see [16]).

Proposition 17. *Assume that $I = [a,b]$. A function, $f\colon I^n \to I$, is a Sugeno integral if and only if $f(\mathbf{e}_\varnothing) = a$, $f(\mathbf{e}_X) = b$ and for every $\mathbf{x} \in I^n$*

$$f(\mathbf{x}) \;=\; \bigvee_{S\subseteq X} f(\mathbf{e}_S) \wedge \bigwedge_{i\in S} x_i$$

As mentioned, the properties of comonotonic maxitivity and comonotonic minitivity were introduced by de Campos *et al*. in [6] to axiomatize the class of Sugeno integrals. However, without further assumptions, they define a wider class of functions that we now define.

Definition 18. Assume that $I = [a,b]$ and $J = [c,d]$ are real intervals, and let μ be an I-valued capacity on X. A quasi-Sugeno integral with respect to μ is a function, $f\colon J^n \to I$, defined by $f(\mathbf{x}) = \mathcal{S}_\mu(\varphi(x_1),\ldots,\varphi(x_n))$, where $\varphi\colon J \to I$ is a nondecreasing function.

Using Proposition 11 and Corollary 17 in [10], we obtain the following axiomatization of the class of quasi-Sugeno integrals.

Theorem 19. *Let $I = [a,b]$ and $J = [c,d]$ be real intervals and consider a function, $f\colon J^n \to I$. The following assertions are equivalent:*

(i) *f is a quasi-Sugeno integral.*

(ii) *f is comonotonically maxitive and comonotonically minitive.*

(iii) *f is nondecreasing, and there exists a nondecreasing function, $\varphi\colon J \to I$, such that for every $\mathbf{x} \in J^n$ and $r \in J$, we have:*

$$f(r \vee \mathbf{x}) = \varphi(r) \vee f(\mathbf{x}) \qquad \text{and} \qquad f(r \wedge \mathbf{x}) = \varphi(r) \wedge f(\mathbf{x}) \tag{12}$$

where $r \vee \mathbf{x}$ (resp. $r \wedge \mathbf{x}$) is the n-tuple, whose ith component is $r \vee x_i$ (resp. $r \wedge x_i$). In this case, φ can be chosen as $\varphi(x) = f(x, \ldots, x)$.

Remark 7. The two conditions given in (12) are referred to in [10] as quasi-max homogeneity and quasi-min homogeneity, respectively.

As observed at the end of the previous section, condition (ii) (and, hence, (i) or (iii)) of Theorem 19 implies comonotonic modularity. As the following result shows, the converse is true whenever f is nondecreasing and verifies any of the following weaker variants of quasi-max homogeneity and quasi-min homogeneity:

$$f(x \vee \mathbf{e}_S) = f(x, \ldots, x) \vee f(\mathbf{e}_S) \qquad x \in J,\ S \subseteq X \tag{13}$$

$$f(x \wedge \mathbf{e}_S) = f(x, \ldots, x) \wedge f(\mathbf{e}_S) \qquad x \in J,\ S \subseteq X \tag{14}$$

Theorem 20. *Let $I = [a, b]$ and $J = [c, d]$ be real intervals and consider a function, $f\colon J^n \to I$. The following conditions are equivalent:*

(i) *f is a quasi-Sugeno integral, $f(\mathbf{x}) = \mathcal{S}_\mu(\varphi(x_1), \ldots, \varphi(x_n))$, where $\varphi(x) = f(x, \ldots, x)$.*

(ii) *f is a quasi-Sugeno integral.*

(iii) *f is comonotonically modular, nondecreasing and satisfies (13) or (14).*

(iv) *f is nondecreasing and satisfies (13) and (14).*

Proof. $(i) \Rightarrow (ii)$ Trivial.

$(ii) \Rightarrow (iii)$ Follows from Theorem 19.

$(iii) \Rightarrow (iv)$ Suppose that f is comonotonically modular and satisfies (13). Then:

$$\begin{aligned} f(x \wedge \mathbf{e}_S) &= f(x, \ldots, x) + f(\mathbf{e}_S) - f(x \vee \mathbf{e}_S) \\ &= f(x, \ldots, x) + f(\mathbf{e}_S) - f(x, \ldots, x) \vee f(\mathbf{e}_S) = f(x, \ldots, x) \wedge f(\mathbf{e}_S) \end{aligned}$$

Hence, f satisfies (14). The other case can be dealt with dually.

$(iv) \Rightarrow (i)$ Define $\varphi(x) = f(x, \ldots, x)$. By nondecreasing monotonicity and (14), for every $S \subseteq X$, we have:

$$f(\mathbf{x}) \geqslant f\Big(\mathbf{e}_S \wedge \bigwedge_{i \in S} x_i\Big) = f(\mathbf{e}_S) \wedge \varphi\Big(\bigwedge_{i \in S} x_i\Big) = f(\mathbf{e}_S) \wedge \bigwedge_{i \in S} \varphi(x_i)$$

and thus, $f(\mathbf{x}) \geqslant \bigvee_{S\subseteq X} f(\mathbf{e}_S) \wedge \bigwedge_{i\in S} \varphi(x_i)$. To complete the proof, it is enough to establish the converse inequality. Let $S^* \subseteq X$ be such that $f(\mathbf{e}_{S^*}) \wedge \bigwedge_{i\in S^*} \varphi(x_i)$ is maximum. Define

$$T = \Big\{ j \in X : \varphi(x_j) \leqslant f(\mathbf{e}_{S^*}) \wedge \bigwedge_{i\in S^*} \varphi(x_i) \Big\}$$

We claim that $T \neq \varnothing$. Suppose this is not true, that is, $\varphi(x_j) > f(\mathbf{e}_{S^*}) \wedge \bigwedge_{i\in S^*} \varphi(x_i)$ for every $j \in X$. Then, by nondecreasing monotonicity, we have, $f(\mathbf{e}_X) \geqslant f(\mathbf{e}_{S^*})$, and since $f(\mathbf{e}_X) \geqslant \bigwedge_{i\in X} \varphi(x_i)$,

$$f(\mathbf{e}_X) \wedge \bigwedge_{i\in X} \varphi(x_i) \;>\; f(\mathbf{e}_{S^*}) \wedge \bigwedge_{i\in S^*} \varphi(x_i)$$

which contradicts the definition of S^*. Thus $T \neq \varnothing$.

Now, by nondecreasing monotonicity and (13), we have:

$$f(\mathbf{x}) \;\leqslant\; f\Big(\mathbf{e}_{X\setminus T} \vee \bigvee_{j\in T} x_j\Big) \;=\; f(\mathbf{e}_{X\setminus T}) \vee \varphi\Big(\bigvee_{j\in T} x_j\Big) \;=\; f(\mathbf{e}_{X\setminus T}) \vee \bigvee_{j\in T} \varphi(x_j) \;=\; f(\mathbf{e}_{X\setminus T})$$

Indeed, we have $\varphi(x_j) \leqslant f(\mathbf{x})$ for every $j \in T$ and $\mathbf{x} \leqslant \mathbf{e}_{X\setminus T} \vee \bigvee_{j\in T} x_j$.

Note that $f(\mathbf{e}_{X\setminus T}) \leqslant f(\mathbf{e}_{S^*}) \wedge \bigwedge_{i\in S^*} \varphi(x_i)$, since, otherwise, by definition of T, we would have:

$$f(\mathbf{e}_{X\setminus T}) \wedge \bigwedge_{i\in X\setminus T} \varphi(x_i) \;>\; f(\mathbf{e}_{S^*}) \wedge \bigwedge_{i\in S^*} \varphi(x_i)$$

again, contradicting the definition of S^*. Finally:

$$f(\mathbf{x}) \;\leqslant\; f(\mathbf{e}_{S^*}) \wedge \bigwedge_{i\in S^*} \varphi(x_i) \;=\; \bigvee_{S\subseteq X} f(\mathbf{e}_S) \wedge \bigwedge_{i\in S} \varphi(x_i)$$

and the proof is thus complete. □

Remark 8. An axiomatization of the class of Sugeno integrals based on comonotonic modularity can be obtained from Theorems 19 and 20 by adding the idempotency property.

5. Conclusions

In this paper, we analyzed comonotonic modularity as a feature common to many well-known discrete integrals. In doing so, we established its relation to many other noteworthy aggregation properties, such as comonotonic relaxations of additivity, maxitivity and minitivity. In fact, the latter become equivalent in the presence of comonotonic modularity. As a by-product, we immediately see that, e.g., the so-called discrete Shilkret integral lies outside the class of comonotonic modular functions, since this integral is comonotonically maxitive, but not comonotonically minitive.

Albeit, such an example, the class of comonotonically modular functions, is rather vast and includes several important extensions of the Choquet and Sugeno integrals. The results presented in Section 4 seem to indicate that suitable variants of homogeneity suffice to distinguish and fully describe these extensions. This naturally asks for an exhaustive study of homogeneity-like properties, which may lead to a complete classification of all subclasses of comonotonically modular functions.

Another question that still eludes us is the relation between the additive forms given by comonotonic modularity and the max-min forms. As shown in Theorem 20, the latter are particular instances of the former; in fact, proof of Theorem 20 provides a procedure to construct max-min representations of comonotonically modular functions, whenever they exist. However, we were not able to present a direct translation between the two. This remains as a relevant open question, since its answer will inevitably provide a better understanding of the synergy between these intrinsically different normal forms.

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Conflict of Interest

The authors declare no conflict of interest.

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