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${}_3F_4$ Hypergeometric Functions as a Sum of a Product of ${}_2F_3$ Functions

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Abstract: This paper shows that certain ${}_3F_4$ hypergeometric functions can be expanded in sums of pair products of ${}_2F_3$ functions, which reduce in special cases to ${}_2F_3$ functions expanded in sums of pair products of ${}_1F_2$ functions. This expands the class of hypergeometric functions having summation theorems beyond those expressible as pair-products of generalized Whittaker functions, ${}_2F_1$ functions, and ${}_3F_2$ functions into the realm of ${}_pF_q$ functions where $p < q$ for both the summand and terms in the series. In addition to its intrinsic value, this result has a specific application in calculating the response of the atoms to laser stimulation in the Strong Field Approximation.

Keywords: ${}_3F_4$ hypergeometric functions; ${}_2F_3$ hypergeometric functions; ${}_1F_2$ hypergeometric functions; Strong Field Approximation; laser stimulation; summation theorem

MSC: 33C10; 42C10; 41A10; 33F10; 65D20; 68W30; 33D50; 33C05

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1. Introduction

In 1942, Jackson [1,2] extended the concept of *summation theorems* over pair-products of functions such as Bessel functions [3] (p. 992 No. 8.53) to sums of pair-products of the broader class of Generalized Hypergeometric functions. Though his focus was on hypergeometric functions of two variables (x, y), as a special case, he replaced $y = xq^b$, resulting in ${}_2F_1$ functions expanded as pairs of ${}_2F_1$ functions (his Equation (I.55)) and ${}_1F_1$ functions (Whittaker functions) expanded as pairs of ${}_1F_1$ functions (his Equation (II.69)). In 1962, Ragab [4] found six such expressions involving Slater's [5] generalization of Whittaker functions to ${}_pF_p$ functions having $p \geq 1$, all but one of which have x^2 rather than x as the argument in the sum, such as

$$ {}_2F_3\left(\frac{b}{2}+\frac{1}{2},\frac{b}{2};a+\frac{1}{2},\frac{c}{2}+\frac{1}{2},\frac{c}{2};\frac{x^2}{16}\right) = \sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}(a)_r(b)_r(c-b)_r}{r!(2a)_r(c)_{2r}(c+r-1)_r} \times\ {}_2F_2(a+r,b+r;2a+r,c+2r;x)\,{}_1F_1\left(b+r;c+2r;-\frac{x}{2}\right). \quad (1) $$

In addition to the sums that were ${}_2F_3$ hypergeometric functions for many of his results, he expressed a ${}_4F_5$ function as a sum of products of ${}_2F_2$ functions. Verma [6] rederived Jackson's and some of Ragab's results in 1964 and added expansions of ${}_3F_2$ functions as a product of a ${}_2F_1$ function with another ${}_3F_2$ function. He also expressed a ${}_5F_5$ generalized Whittaker function as a sum of products of ${}_2F_2$ functions.

If one excludes *pair-products* of generalized hypergeometric functions in the summands, one finds a very active modern field of study of summations theorems. To cite just a few of many, Choi, Milovanovi and Rathie [7] express Kampé de Fériet functions as certain finite sums, as do Wang and Chen [8]. Wang [9] expresses Kampé de Fériet functions and various related functions as infinite sums, as does Yakubovich [10] for generalized hypergeometric functions. Liu and Wang [11] reduce Kampé de Fériet functions to Appell series and generalized hypergeometric functions via generalizations of classical summation theorems due to Kummer, Gauss, and Bailey, with extensions by Choi and Rathie [12].

Awad et al. [13] give an excellent summary of those classical summation theorems for generalized hypergeometric functions and provide extensions.

The present paper, in contrast, derives infinite summation theorems for ${}_3F_4$ hypergeometric functions in terms of *pair-products* of ${}_2F_3$ functions. In special cases these reduce to ${}_2F_3$ functions expanded in sums of pair products of ${}_1F_2$ functions. While interesting in itself, this result has a specific application in calculating the response of the atoms to laser stimulation in the Strong Field Approximation (SFA) [14–19]. Whereas perturbation expansions will not converge if the applied laser field is sufficiently large, the Strong Field Approximation (SFA) is an analytical approximation that is non-perturbative. Keating [20] applied it specifically to the production of the positive antihydrogen ion.

As a sketch of the difference between perturbation expansions and the SFA, consider the exact transition amplitude for a one-electron system,

$$T_{fi} = \left\langle \Psi_f | H_{int} | \phi_i \right\rangle \equiv -\frac{i}{\hbar} \int \int \Psi_f^*(x,\, t) H_{int} \phi_i^{(0)}(x,\, t) d^3x dt\,, \tag{2}$$

where

$$H_0 = -\frac{\hbar^2}{2m}\nabla^2 - \frac{Ze^2}{(4\pi\epsilon_0)r} \tag{3}$$

is the time-independent Hamiltonian describing the hydrogenic atom or ion in the absence of the radiation field (or Coulomb interaction for particle scattering problems), and

$$H_{int}(t) \quad = \quad -i\hbar\tfrac{e}{m}\mathbf{A}\cdot\boldsymbol{\nabla} + \tfrac{e^2}{2m}\mathbf{A}^2 = \tfrac{e}{m}\mathbf{A}\cdot\mathbf{p} + \tfrac{e^2}{2m}\mathbf{A}^2 \tag{4}$$

is the perturbing Hamiltonian describing the interaction of the hydrogenic atom with the radiation field. Since the unperturbed wave functions $\phi_i^{(0)}(x,\, t)$ form a complete set, one may expand the exact final state $\Psi_f(x,\, t)$ in them and produce a perturbation series,

$$T_{fi} \quad = \quad \left\langle \phi_f | H_{int} | \phi_i \right\rangle + \left\langle \phi_f | H_{int} \frac{1}{E_i - H_0 + i\epsilon} H_{int} | \phi_i \right\rangle + \cdots\,, \tag{5}$$

where in the second-order term one would normally insert a sum over the bound states n, and integral over the continuum states k, of a complete set of intermediate states $|\phi_n\rangle\langle\phi_n|$. For strong laser fields, the series (5) will not converge.

In the Strong Field Approximation, one says that the photodetached state $\Psi_f(t)$ is fully detached: it is so far from the atomic nucleus (and any other bound electrons) so as to make negligible the effects of the inter-particle interactions compared to the effects of a strong laser field. One may therefore approximate $\Psi_f(t)$ by the Volkov solution [21] for a “free” particle in a field,

$$\Psi_f(\mathbf{x},t) \approx \chi(\mathbf{x},t) = exp(-i\mathbf{k}\cdot\mathbf{x}) exp\left[\frac{i}{\hbar}\left(E_k t + \int H_{int}(t')dt'\right)\right] \tag{6}$$

so that the T-matrix is

$$T_{fi} \simeq -\frac{i}{\hbar} \int \int \chi^*(\mathbf{x},t) H_{int} \Psi_i^{(0)}(x,\, t) d^3x\, dt \quad . \tag{7}$$

This is the (zeroth order) Strong Field Approximation.

As one solves for this T-matrix, the Volkov solution generates terms

$$exp\left[\frac{1}{\hbar}\int H_{int}(t')dt'\right] = exp[\mathbf{k}\cdot\boldsymbol{\alpha}_0\, sin(\omega t) + k\beta_0 sin(2\omega t) + 2k\beta_0\omega t], \tag{8}$$

where $\boldsymbol{\alpha}_0$ is called the “quiver” (motion) of the electron in the laser field (not the fine structure constant) and β_0 is called the “shiver”. One uses the generalized Bessel function $J_n(x,y)$ in what follows because its integral representation

$$J_n(x,y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} exp[i(x\,sin(\theta) + y\,sin(2\theta) - n\theta)]d\theta \tag{9}$$

is a convenient form in which to represent the exponential in (8).

The next section maps Keating's terminology onto the present problem. The third section lays out how one would perform angular integrals over these generalized Bessel functions using a series expansion, with a subsection that shows how to perform these angular integrals without the need for a series expansion. Comparison of these two approaches gives the desired summation theorem. The fourth section shows how to obtain a summation theorem for a closely related problem.

2. The Transition Amplitude

The SFA transition amplitudes involve integrals over differential angles, in Keating's notation,

$$\Xi_n\left(k\alpha_0, -\frac{z}{2}\right) = \int J_n^2\left(\mathbf{k}\cdot\boldsymbol{\alpha}_0, -\frac{z}{2}\right)d\Omega \tag{10}$$

of the generalized Bessel function, introduced above, which also has a series expansion

$$J_n(x,y) = \sum_{h=-\infty}^{\infty} J_{n-2h}(x)J_h(y)\,. \tag{11}$$

We extend Keating's function somewhat with an additional cosine-squared factor

$$\Xi_n^p\left(k\alpha_0, -\frac{z}{2}\right) = \int \cos^{2p}(\theta)J_n^2\left(\mathbf{k}\cdot\boldsymbol{\alpha}_0, -\frac{z}{2}\right)d\Omega \tag{12}$$

whose power *2p* can be set to 0 to reproduce Keating's result or retained with higher integers.

This integral can be solved analytically by expanding $\cos^p(\theta)J_n\left(\mathbf{k}\cdot\boldsymbol{\alpha}_0, -\frac{z}{2}\right)$ in a Laplace series, that is in terms of spherical harmonics [22]

$$\cos^p(\theta)J_n\left(\mathbf{k}\cdot\boldsymbol{\alpha}_0, -\frac{z}{2}\right) = \sum_{lm} F_{lm}^{pn}(k\alpha_0, -\frac{z}{2})Y_{lm}(\theta,\phi)\,, \tag{13}$$

where, in a parallel notation to Keating's,

$$F_{lm}^{pn}(k\alpha_0, -\frac{z}{2}) = \int \cos^p(\theta)J_n\left(\mathbf{k}\cdot\boldsymbol{\alpha}_0, -\frac{z}{2}\right)Y_{lm}^*(\theta,\phi)d\Omega\,. \tag{14}$$

With these definitions,

$$\begin{array}{rcc} \Xi_n^p(k\alpha_0, -\frac{z}{2}) & = & \int \cos^{2p}(\theta)J_n^2(\mathbf{k}\cdot\boldsymbol{\alpha}_0, -\frac{z}{2})d\Omega \\ & = & \int \sum_{lm} F_{lm}^{pn}(k\alpha_0, -\frac{z}{2})Y_{lm}(\theta,\phi)\sum_{l'm'} F_{l'm'}^{pn}(k\alpha_0, -\frac{z}{2})Y_{l'm'}^*(\theta,\phi)d\Omega \\ & = & \sum_{lm} F_{lm}^{pn}(k\alpha_0, -\frac{z}{2}))\sum_{l'm'} F_{l'm'}^{pn}(k\alpha_0, -\frac{z}{2})\delta_{l'l}\delta_{m'm} \\ & = & \sum_{lm} F_{lm}^{pn}(k\alpha_0, -\frac{z}{2}))F_{lm}^{pn}(k\alpha_0, -\frac{z}{2})\,. \end{array} \tag{15}$$

Using the definition of the Generalized Bessel function, we find

$$F_{lm}^{pn}(k\alpha_0, -\frac{z}{2}) = \sum_{h=-\infty}^{\infty}\int \cos^p(\theta)J_{n-2h}(\mathbf{k}\cdot\boldsymbol{\alpha}_0)Y_{lm}^*(\theta,\phi)d\Omega\,J_h(-\frac{z}{2})\,. \tag{16}$$

In the derivation leading to the ${}_2F_3$ functions that are the focus of this work, we wish to avoid the infinities that come from negative indices of the Bessel functions that we will integrate over. So, using $\delta = \left\{\begin{array}{c}1\\0\end{array}\right\}$ for $n = \left\{\begin{array}{c}odd\\even\end{array}\right\}$ and [3] (p. 979 No. 8.472.5)

$$J_{-N}(z) = (-1)^N J_N(z)\ \text{[N is a natural number]}, \tag{17}$$

this becomes

$$
\begin{aligned}
F_{lm}^{pn}\left(k\alpha_0,-\tfrac{z}{2}\right) &= \left(\sum_{h=-\infty}^{(n-\delta)/2} + \sum_{h=(n-\delta)/2+1}^{\infty}\right) \int \cos^p(\theta) J_{n-2h}(\mathbf{k}\cdot\boldsymbol{\alpha}_0) Y_{lm}^*(\theta,\phi) d\Omega\, J_h\left(-\tfrac{z}{2}\right) \\
&= \sum_{j=(n-\delta)/2}^{\infty} \int \cos^p(\theta) J_{n+2j}(\mathbf{k}\cdot\boldsymbol{\alpha}_0) Y_{lm}^*(\theta,\phi) d\Omega\, J_{-j}\left(-\tfrac{z}{2}\right) \\
&+ \sum_{j=(n-\delta)/2+1}^{\infty} \int \cos^p(\theta) J_{n-2j}(\mathbf{k}\cdot\boldsymbol{\alpha}_0) Y_{lm}^*(\theta,\phi) d\Omega\, J_j\left(-\tfrac{z}{2}\right) \\
&= \sum_{j=(n-\delta)/2}^{\infty} \int \cos^p(\theta) J_{2j+n}(\mathbf{k}\cdot\boldsymbol{\alpha}_0) Y_{lm}^*(\theta,\phi) d\Omega\, J_{-j}\left(-\tfrac{z}{2}\right) \\
&+ (-1)^{n-2j} \sum_{j=(n-\delta)/2+1}^{\infty} \int \cos^p(\theta) J_{2j-n}(\mathbf{k}\cdot\boldsymbol{\alpha}_0) Y_{lm}^*(\theta,\phi) d\Omega\, J_j\left(-\tfrac{z}{2}\right) .
\end{aligned} \tag{18}
$$

Then

$$
\begin{aligned}
F_{lm}^{pn}(k\alpha_0,-\tfrac{z}{2}) &= \sum_{h=-\infty}^{\infty} \mathbb{J}_{lm}^{p,n-2h}(k\alpha_0) J_h(-\tfrac{z}{2}) \\
&= \sum_{j=(n-\delta)/2}^{\infty} \mathbb{J}_{lm}^{p,2j+n}(k\alpha_0) J_{-j}\left(-\tfrac{z}{2}\right) \\
&+ (-1)^n \sum_{j=(n-\delta)/2+1}^{\infty} \mathbb{J}_{lm}^{p,2j-n}(k\alpha_0)\, J_j\left(-\tfrac{z}{2}\right) ,
\end{aligned} \tag{19}
$$

where we have extended Keating's definition to

$$
\mathbb{J}_{lm}^{p,2j\pm n}(k\alpha_0) = \int \cos^p(\theta) J_{2j\pm n}(\mathbf{k}\cdot\boldsymbol{\alpha}_0) Y_{lm}^*(\theta,\phi) d\Omega . \tag{20}
$$

We pause to note that if $p = 0$, this is simply the Laplace series of a conventional Bessel function with integer indices $J_{2j\pm n}(\mathbf{k}\cdot\boldsymbol{\alpha}_0) \equiv J_{2j\pm n}(k\alpha_0\cos\theta)$, whose derivation we have not seen in the literature prior to Keating's.

Since this function is independent of the azimuthal angle, we can reduce the Laplace series to a sum over Legendre polynomials. Using the definition of the spherical harmonics

$$
Y_{lm}^*(\theta,\phi) = (-1)^l \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l(\cos\theta) e^{-im\phi} , \tag{21}
$$

we obtain

$$
\mathbb{J}_{lm}^{p,2j\pm n}(k\alpha_0)=(-1)^l \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} \int \cos^p(\theta) J_{2j\pm n}(k\alpha_0\cos\theta) P_l(\cos\theta) d(\cos\theta) d\theta \int e^{-im\phi} d\phi \tag{22}
$$

or

$$
\mathbb{J}_{lm}^{p,2j\pm n}(k\alpha_0) = (-1)^l \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} \mathcal{J}_l^{p,2j\pm n}(k\alpha_0)\delta_{m0} , \tag{23}
$$

where we again extend Keating's defined function to

$$
\mathcal{J}_l^{p,2j\pm n}(k\alpha_0) = 2\pi \int_{-\pi}^{\pi} \cos^p(\theta) J_{2j\pm n}(k\alpha_0\cos\theta) P_l(\cos\theta) d(\cos\theta) . \tag{24}
$$

Let $x = \cos(\theta)$, then

$$
\mathcal{J}_l^{p,2j\pm n}(k\alpha_0) = 2\pi \int_{-1}^{1} x^p J_{2j\pm n}(k\alpha_0 x) P_l(x) dx . \tag{25}
$$

3. The Fourier–Legendre Series of a Bessel Function of the First Kind

In a prior paper [23], we dropped the complicated indices to cleanly express the Bessel function in a series of Legendre polynomials, on the assumption [24] that the series

$$J_N(kx) = \sum_{L=0}^{\infty} a_{LN}(k) P_L(x) \tag{26}$$

converges uniformly. (Let D be a region in which the above series converges for each value of x. Then the series can be said to converge uniformly in D if, for every $\varepsilon > 0$, there exists a number $N'(\varepsilon)$ such that, for $n > N'$ it follows that

$$\left| J_N(kx) - \sum_{L=0}^{n} a_{LN}(k) P_L(x) \right| = \left| \sum_{L=n+1}^{\infty} a_{LN}(k) P_L(x) \right| < \varepsilon \tag{27}$$

for all x in D). The coefficients are given by the orthogonality of the Legendre polynomials,

$$a_{LN}(k) = \frac{2L+1}{2} \int_{-1}^{1} J_N(kx) P_L(x) dx \,. \tag{28}$$

Following Keating's lead, but without the complicated indices, we showed that

$$\begin{aligned} a_{LN}(k) &= \sqrt{\pi}(2L+1)2^{-L-1} i^{L-\mathrm{N}} \textstyle\sum_{M=0}^{\infty} \frac{\left(\left(-\frac{1}{4}\right)^{M} k^{L+2M}\right)}{2^{L+2M+1}\left(M!\Gamma\left(L+M+\frac{3}{2}\right)\right)} \\ &\times \left(1+(-1)^{L+2M+\mathrm{N}}\right)\binom{L+2M}{\frac{1}{2}(L+2M-\mathrm{N})} \\ &= \frac{\sqrt{\pi}2^{-2L-2}(2L+1)k^L i^{L-N}}{\Gamma\left(\frac{1}{2}(2L+3)\right)}\left(1+(-1)^{L+N}\right)\binom{L}{\frac{L-N}{2}} \\ &\times {}_2F_3\left(\tfrac{L}{2}+\tfrac{1}{2}, \tfrac{L}{2}+1; L+\tfrac{3}{2}, \tfrac{L}{2}-\tfrac{N}{2}+1, \tfrac{L}{2}+\tfrac{N}{2}+1; -\tfrac{k^2}{4}\right) \\ &= \sqrt{\pi}\, 2^{-2L-2}(2L+1)k^L i^{L-N}\left(1+(-1)^{L+N}\right)\Gamma(L+1) \\ &\times {}_2\tilde{F}_3\left(\tfrac{L}{2}+\tfrac{1}{2}, \tfrac{L}{2}+1; L+\tfrac{3}{2}, \tfrac{L}{2}-\tfrac{N}{2}+1, \tfrac{L}{2}+\tfrac{N}{2}+1; -\tfrac{k^2}{4}\right), \end{aligned} \tag{29}$$

where the final two steps are new with the prior work [23]. Whenever $N > 1$ is an integer larger than L, and of the same parity, the conventional form of the hypergeometric function in the second expression—with its prefactors—gives indeterminacies (ratios of infinities) in computation. For this reason, we have included the final form involving regularized hypergeometric functions [25]

$${}_2F_3(a_1, a_2; b_1, b_2, b_3; z) = \Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\, {}_2\tilde{F}_3(a_1, a_2; b_1, b_2, b_3; z) \tag{30}$$

and cancelled the $\Gamma(b_i)$ with gamma functions in the denominators of the prefactors that would otherwise each give infinities in this case.

For the special cases of $N = 0$ and 1 the order of the hypergeometric functions is reduced since the parameters $a_2 = b_3$ and $a_1 = b_2$, resp., giving

$$\begin{aligned} a_{L0}(k) &= \frac{\sqrt{\pi} i^L 2^{-2L-2}(2L+1)k^L}{\Gamma\left(\frac{1}{2}(2L+3)\right)}\left(1+(-1)^L\right)\binom{L}{\frac{L}{2}} \\ &\times {}_1F_2\left(\tfrac{L}{2}+\tfrac{1}{2}; \tfrac{L}{2}+1, L+\tfrac{3}{2}; -\tfrac{k^2}{4}\right) \\ &= \sqrt{\pi} i^L 2^{-2L-2}(2L+1)k^L \Gamma\left(\tfrac{L}{2}+1\right)\left(1+(-1)^L\right)\binom{L}{\frac{L}{2}} \\ &\times {}_1\tilde{F}_2\left(\tfrac{L}{2}+\tfrac{1}{2}; \tfrac{L}{2}+1, L+\tfrac{3}{2}; -\tfrac{k^2}{4}\right), \end{aligned} \tag{31}$$

and

$$
\begin{aligned}
a_{L1}(k) &= \frac{\sqrt{\pi} i^{L-1} 2^{-2L-2}(2L+1)k^L}{\Gamma\left(\frac{1}{2}(2L+3)\right)} \left(1+(-1)^{L+1}\right)\binom{L}{\frac{L-1}{2}} \\
&\times {}_1F_2\left(\tfrac{L}{2}+1; \tfrac{L}{2}+\tfrac{3}{2}, L+\tfrac{3}{2}; -\tfrac{k^2}{4}\right) \\
&= i^{L-1}2^{-L-2}(2L+1)k^L\Gamma\left(\tfrac{L}{2}+1\right)\left(1+(-1)^{L+1}\right) \\
&\times {}_1\tilde{F}_2\left(\tfrac{L}{2}+1; \tfrac{L}{2}+\tfrac{3}{2}, L+\tfrac{3}{2}; -\tfrac{k^2}{4}\right).
\end{aligned} \tag{32}
$$

In each special case, the first form involving a hypergeometric function has no indeterminacies, but we include the regularized hypergeometric function version for completeness.

We now have everything in place to prove the following theorem.

Theorem 1. *There exists a summation theorem for* ${}_3F_4$ *hypergeometric functions of the following form:*

$$
\begin{aligned}
&{}_3F_4\left(\tfrac{a}{2}+\tfrac{1}{2}, \tfrac{a}{2}, \tfrac{a}{2}; \tfrac{a}{2}+1, a, a-b+1, b; z\right) = a\Gamma(b)\Gamma(a-b+1) \\
&\times \sum_{L=0}^{\infty} \frac{\pi^2(2L+1)i^{-2a+4L+2}2^{a-8L-7}\left((-1)^{b+L-1}+1\right)\left((-1)^{a-b+L}+1\right)}{\Gamma\left(L+\frac{3}{2}\right)^2\Gamma\left(\frac{1}{2}(2L+3)\right)^2\Gamma\left(-\frac{b}{2}+\frac{L}{2}+\frac{3}{2}\right)\Gamma\left(\frac{b}{2}+\frac{L}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}-\frac{b}{2}+\frac{L}{2}+1\right)} \\
&\times \frac{\Gamma(2L+2)^2}{\Gamma\left(-\frac{a}{2}+\frac{b}{2}+\frac{L}{2}+1\right)} z^{\frac{1-a}{2}+L} {}_2F_3\left(\tfrac{L}{2}+\tfrac{1}{2}, \tfrac{L}{2}+1; -\tfrac{b}{2}+\tfrac{L}{2}+\tfrac{3}{2}, \tfrac{b}{2}+\tfrac{L}{2}+\tfrac{1}{2}, L+\tfrac{3}{2}; \tfrac{z}{4}\right) \\
&\times {}_2F_3\left(\tfrac{L}{2}+\tfrac{1}{2}, \tfrac{L}{2}+1; \tfrac{a}{2}-\tfrac{b}{2}+\tfrac{L}{2}+1, -\tfrac{a}{2}+\tfrac{b}{2}+\tfrac{L}{2}+1, L+\tfrac{3}{2}; \tfrac{z}{4}\right),
\end{aligned} \tag{33}
$$

where $a = 1, 3, 5, 7\cdots$ *and* $b = 1, 2, 3, \cdots, a$.

Proof of Theorem 1. Stepping outward from (29) in the string of definitions, with $k \to k\alpha_0$,

$$
\begin{aligned}
\mathbb{J}_{lm}^{0,2j\pm n}(k\alpha_0) &= (-1)^l\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\mathcal{J}_l^{0,2j\pm n}(k\alpha_0)\delta_{m0}, \\
&= (-1)^l\sqrt{\frac{2l+1}{4\pi}}\frac{\pi^{3/2}2^{-2l}(k\alpha_0)^l i^{(l\mp n)-2j}}{\Gamma\left(\frac{1}{2}(2l+3)\right)}\left((-1)^{2j+(l\pm n)}+1\right)\binom{l}{\frac{1}{2}((l\mp n)-2j)} \\
&\times {}_2F_3\left(\tfrac{l}{2}+\tfrac{1}{2}, \tfrac{l}{2}+1; l+\tfrac{3}{2}, -j+\left(\tfrac{l}{2}\mp\tfrac{n}{2}\right)+1, j+\left(\tfrac{l}{2}\pm\tfrac{n}{2}\right)+1; -\tfrac{1}{4}k^2\alpha_0^2\right)
\end{aligned} \tag{34}
$$

so that

$$
\begin{aligned}
F_{lm}^{0n}\left(k\alpha_0, -\tfrac{z}{2}\right) &= \sum_{j=(n-\delta)/2}^{\infty}\mathbb{J}_{lm}^{0,2j+n}(k\alpha_0)J_{-j}\left(-\tfrac{z}{2}\right) + (-1)^n\sum_{j=(n-\delta)/2+1}^{\infty}\mathbb{J}_{lm}^{0,2j-n}(k\alpha_0)J_j\left(-\tfrac{z}{2}\right) \\
&= \sum_{j=(n-\delta)/2}^{\infty}(-1)^l\sqrt{\frac{2l+1}{4\pi}}\frac{\pi^{3/2}2^{-2l}(k\alpha_0)^l i^{(l-n)-2j}}{\Gamma\left(\frac{1}{2}(2l+3)\right)}\left((-1)^{2j+(l+n)}+1\right)\binom{l}{\frac{1}{2}((l-n)-2j)} \\
&\times J_{-j}\left(-\tfrac{z}{2}\right){}_2F_3\left(\tfrac{l}{2}+\tfrac{1}{2}, \tfrac{l}{2}+1; l+\tfrac{3}{2}, -j+\left(\tfrac{l}{2}-\tfrac{n}{2}\right)+1, j+\left(\tfrac{l}{2}+\tfrac{n}{2}\right)+1; -\tfrac{1}{4}k^2\alpha_0^2\right) \\
&+ (-1)^n\sum_{j=(n-\delta)/2+1}^{\infty}(-1)^l\sqrt{\frac{2l+1}{4\pi}}\frac{\pi^{3/2}2^{-2l}(k\alpha_0)^l i^{(l+n)-2j}}{\Gamma\left(\frac{1}{2}(2l+3)\right)}\left((-1)^{2j+(l-n)}+1\right)\binom{l}{\frac{1}{2}((l+n)-2j)} \\
&\times J_j\left(-\tfrac{z}{2}\right){}_2F_3\left(\tfrac{l}{2}+\tfrac{1}{2}, \tfrac{l}{2}+1; l+\tfrac{3}{2}, -j+\left(\tfrac{l}{2}+\tfrac{n}{2}\right)+1, j+\left(\tfrac{l}{2}-\tfrac{n}{2}\right)+1; -\tfrac{1}{4}k^2\alpha_0^2\right),
\end{aligned} \tag{35}
$$

and finally

$$
\begin{aligned}
\Xi_n^0\left(k\alpha_0, -\tfrac{z}{2}\right) &= \textstyle\sum_{l0} F_{l0}^{0n}\left(k\alpha_0, -\tfrac{z}{2}\right)F_{l0}^{0n}\left(k\alpha_0, -\tfrac{z}{2}\right) \\
&= \textstyle\sum_{l0}\frac{2l+1}{4\pi}\left(\displaystyle\sum_{j=-(n-\delta)/2}^{\infty}\sum_{M=-(n-\delta)/2}^{\infty}J_{-j}\left(-\tfrac{z}{2}\right)J_{-M}\left(-\tfrac{z}{2}\right)\right. \\
&\times \frac{\pi^{3/2}2^{-2l}(k\alpha_0)^l i^{(l-n)-2j}}{\Gamma\left(\frac{1}{2}(2l+3)\right)}\left((-1)^{2j+(l+n)}+1\right)\binom{l}{\frac{1}{2}((l-n)-2j)} \\
&\times \frac{\pi^{3/2}2^{-2l}(k\alpha_0)^l i^{l-2M-n}}{\Gamma\left(\frac{1}{2}(2l+3)\right)}\left((-1)^{l+2M+n}+1\right)\binom{l}{\frac{1}{2}(l-2M-n)} \\
&\times {}_2F_3\left(\tfrac{l}{2}+\tfrac{1}{2}, \tfrac{l}{2}+1; l+\tfrac{3}{2}, -j+\left(\tfrac{l}{2}-\tfrac{n}{2}\right)+1, j+\left(\tfrac{l}{2}+\tfrac{n}{2}\right)+1; -\tfrac{1}{4}k^2\alpha_0^2\right) \\
&\times \left.{}_2F_3\left(\tfrac{l}{2}+\tfrac{1}{2}, \tfrac{l}{2}+1; l+\tfrac{3}{2}, \tfrac{l}{2}-M-\tfrac{n}{2}+1, \tfrac{l}{2}+M+\tfrac{n}{2}+1; -\tfrac{1}{4}k^2\alpha_0^2\right)\right) \\
&+ \quad \cdots.
\end{aligned} \tag{36}
$$

A More Direct Approach

On the other hand, we can transform from Bessel to hypergeometric functions using [26] (p. 220 No. 2.21.2.11), [27] (p. 212 No. 6.2.7.1), or [28]

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu {}_0F_1\left(;\nu+1;-\frac{z^2}{4}\right)}{\Gamma(\nu+1)} \tag{37}$$

and combine pairs using [29,30] or [27] (p. 228 No. 6.4.1.26)

$${}_0F_1(;b;z)\,{}_0F_1(;c;z) = {}_2F_3\left(\frac{b}{2}+\frac{c}{2}-\frac{1}{2},\frac{b}{2}+\frac{c}{2};b,c,b+c-1;4z\right) \tag{38}$$

so that [27] (p. 216 No. 6.2.7.39)

$$J_\mu(z)J_\nu(z) = \frac{2^{-\mu-\nu}z^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)}\,{}_2F_3\left(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2},\frac{\mu}{2}+\frac{\nu}{2}+1;\mu+1,\nu+1,\mu+\nu+1;-z^2\right). \tag{39}$$

Then

$$\begin{aligned}
J_n^2(k\alpha_0\cos(\theta),-\tfrac{z}{2}) &= \sum_{j=-(n-\delta)/2}^{\infty}\ \sum_{M=-(n-\delta)/2}^{\infty} \frac{2^{-2j-2M-2n}(k\alpha_0\cos(\theta))^{2j+2M+2n}}{\Gamma(2j+n+1)\Gamma(2M+n+1)} J_{-j}(-\tfrac{z}{2})J_{-M}(-\tfrac{z}{2})\\
&\times\ {}_2F_3\left(j+M+n+\tfrac{1}{2},j+M+n+1;2j+n+1,2M+n+1,2j+2M+2n+1;-k^2{\alpha_0}^2\cos^2(\theta)\right)\\
&+\ (-1)^n \sum_{j=(n-\delta)/2+1}^{\infty}\ \sum_{M=-(n-\delta)/2}^{\infty} \frac{2^{-2j-2M}(k\alpha_0\cos(\theta))^{2j+2M}}{\Gamma(2j-n+1)\Gamma(2M+n+1)} J_j(-\tfrac{z}{2})J_{-M}(-\tfrac{z}{2})\\
&\times\ {}_2F_3\left(j+M+\tfrac{1}{2},j+M+1;2j+2M+1,2j-n+1,2M+n+1;-k^2\cos^2(\theta)\alpha_0^2\right)\\
&+\ (-1)^n \sum_{j=-(n-\delta)/2}^{\infty}\ \sum_{M=(n-\delta)/2+1}^{\infty} \frac{2^{-2j-2M}(k\alpha_0\cos(\theta))^{2j+2M}}{\Gamma(2j+n+1)\Gamma(2M-n+1)} J_{-j}(-\tfrac{z}{2})J_M(-\tfrac{z}{2})\\
&\times\ {}_2F_3\left(j+M+\tfrac{1}{2},j+M+1;2j+2M+1,2M-n+1,2j+n+1;-k^2\cos^2(\theta)\alpha_0^2\right)\\
&+\ \sum_{j=(n-\delta)/2+1}^{\infty}\ \sum_{M=(n-\delta)/2+1}^{\infty} \frac{2^{-2j-2M+2n}(k\alpha_0\cos(\theta))^{2j+2M-2n}}{\Gamma(2j-n+1)\Gamma(2M-n+1)} J_j(-\tfrac{z}{2})J_M(-\tfrac{z}{2})\\
&\times\ {}_2F_3\left(j+M-n+\tfrac{1}{2},j+M-n+1;2j+2M-2n+1,2j-n+1,2M-n+1;-k^2\cos^2(\theta)\alpha_0^2\right).
\end{aligned} \tag{40}$$

The integral we wish to perform is

$$\begin{aligned}
\Xi_n^p\left(k\alpha_0,-\frac{z}{2}\right) &= \int \cos^{2p}(\theta) J_n^2\left(\mathbf{k}\cdot\boldsymbol{\alpha}_0,-\frac{z}{2}\right)d\Omega\\
&= 2\pi\int_{-\pi}^{\pi}\cos^{2p}(\theta)J_n^2\left(k\alpha_0\cos(\theta),-\frac{z}{2}\right)d(\cos(\theta))\\
&= 2\pi\int_{-1}^{1} f\left(u^2\right)du = 4\pi\int_0^1 f\left(u^2\right)du = 2\pi\int_0^1 f(y)y^{-1/2}dy\,.
\end{aligned} \tag{41}$$

We can then use [26] (p. 334 No. 2.22.2.1)

$$\begin{aligned}
&\int_0^a y^{\alpha-1}(a-y)^{\beta-1}\,{}_pF_q(a_1,\ldots,a_p,b_1,\ldots,b_q;-\omega y)\,dy\\
&\quad= \frac{\Gamma(\alpha)\Gamma(\beta)a^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\,{}_{p+1}F_{q+1}(a_1,\ldots,a_p,\alpha;b_1,\ldots,b_q,\alpha+\beta;-a\omega)\\
&\qquad[\Re(\alpha)>0\wedge\Re(\beta)>0\wedge a>0]
\end{aligned} \tag{42}$$

with $\alpha = \{j+M+n+1,\ j+M+1,\ j+M+1,\ j+M-n+1\}+p$ for the four terms, respectively, and with $a=1$ and $\beta=1$ for each of the four.

Then

$$
\begin{aligned}
\Xi_n\left(k\alpha_0,-\frac{z}{2}\right) &= \int J_n^2\left(\mathbf{k}\cdot\boldsymbol{\alpha}_0,-\frac{z}{2}\right)d\Omega \\
&= 2\pi\sum_{j=-(n-\delta)/2}^{\infty}\sum_{M=-(n-\delta)/2}^{\infty}\frac{2^{-2j-2M-2n+1}(k\alpha_0)^{2j+2M+2n}}{(2j+2M+2n+1)\Gamma(2j+n+1)\Gamma(2M+n+1)}J_{-j}\left(-\frac{z}{2}\right)J_{-M}\left(-\frac{z}{2}\right) \\
&\times {}_3F_4\Big(j+M+n+\frac{1}{2},j+M+n+\frac{1}{2},j+M+n+1; \\
&\qquad 2j+n+1,j+M+n+\frac{3}{2},2M+n+1,2j+2M+2n+1;-k^2\alpha_0^2\Big) \\
&+ 2\pi(-1)^n\sum_{j=(n-\delta)/2+1}^{\infty}\sum_{M=-(n-\delta)/2}^{\infty}\frac{2^{-2j-2M+1}(k\alpha_0)^{2j+2M}}{(2j+2M+1)\Gamma(2j-n+1)\Gamma(2M+n+1)}J_{j}\left(-\frac{z}{2}\right)J_{-M}\left(-\frac{z}{2}\right) \\
&\times {}_3F_4\left(j+M+\frac{1}{2},j+M+\frac{1}{2},j+M+1;j+M+\frac{3}{2},2j+2M+1,2j-n+1,2M+n+1;-k^2\alpha_0^2\right) \\
&+ 2\pi(-1)^n\sum_{j=-(n-\delta)/2}^{\infty}\sum_{M=(n-\delta)/2+1}^{\infty}\frac{2^{-2j-2M+1}(k\alpha_0)^{2j+2M}}{(2j+2M+1)\Gamma(2j+n+1)\Gamma(2M-n+1)}J_{-j}\left(-\frac{z}{2}\right)J_{M}\left(-\frac{z}{2}\right) \\
&\times {}_3F_4\left(j+M+\frac{1}{2},j+M+\frac{1}{2},j+M+1;j+M+\frac{3}{2},2j+2M+1,2M-n+1,2j+n+1;-k^2\alpha_0^2\right) \\
&+ 2\pi\sum_{j=(n-\delta)/2+1}^{\infty}\sum_{M=(n-\delta)/2+1}^{\infty}\frac{2^{-2j-2M+2n+1}(k\alpha_0)^{2j+2M-2n}}{(2j+2M-2n+1)\Gamma(2j-n+1)\Gamma(2M-n+1)}J_{j}\left(-\frac{z}{2}\right)J_{M}\left(-\frac{z}{2}\right) \\
&\times {}_3F_4\Big(j+M-n+\frac{1}{2},j+M-n+\frac{1}{2},j+M-n+1; \\
&\qquad 2j+2M-2n+1,2j-n+1,j+M-n+\frac{3}{2},2M-n+1;-k^2\alpha_0^2\Big).
\end{aligned} \tag{43}
$$

Examination of Equations (36) and (43) shows that

$$
\begin{aligned}
&{}_3F_4\left(\tfrac{\mu}{2}+\tfrac{\nu}{2}+\tfrac{1}{2},\tfrac{\mu}{2}+\tfrac{\nu}{2}+\tfrac{1}{2},\tfrac{\mu}{2}+\tfrac{\nu}{2}+1;\mu+1,\tfrac{\mu}{2}+\tfrac{\nu}{2}+\tfrac{3}{2},\nu+1,\mu+\nu+1;z\right) = (\mu+\nu+1)\Gamma(\mu+1)\Gamma(\nu+1) \\
&\times \sum_{L=0}^{\infty}\frac{\pi^2(2L+1)\left((-1)^{L+\mu}+1\right)\left((-1)^{L+\nu}+1\right)\Gamma(2L+2)^2 i^{4L-2\mu-2\nu}2^{-8L+\mu+\nu-6}}{\Gamma\left(L+\frac{3}{2}\right)^2\Gamma\left(\frac{1}{2}(2L+3)\right)^2\Gamma\left(\frac{L}{2}-\frac{\mu}{2}+1\right)\Gamma\left(\frac{L}{2}+\frac{\mu}{2}+1\right)\Gamma\left(\frac{L}{2}-\frac{\nu}{2}+1\right)\Gamma\left(\frac{L}{2}+\frac{\nu}{2}+1\right)} \\
&\times z^{L-(\mu+\nu)/2}\,{}_2F_3\left(\tfrac{L}{2}+\tfrac{1}{2},\tfrac{L}{2}+1;L+\tfrac{3}{2},\tfrac{L}{2}-\tfrac{\mu}{2}+1,\tfrac{L}{2}+\tfrac{\mu}{2}+1;\tfrac{z}{4}\right) \\
&\times {}_2F_3\left(\tfrac{L}{2}+\tfrac{1}{2},\tfrac{L}{2}+1;L+\tfrac{3}{2},\tfrac{L}{2}-\tfrac{\nu}{2}+1,\tfrac{L}{2}+\tfrac{\nu}{2}+1;\tfrac{z}{4}\right) \\
&= (\mu+\nu+1)\Gamma(\mu+1)\Gamma(\nu+1) \\
&\times \sum_{L=0}^{\infty}\frac{\pi^2(2L+1)\left((-1)^{L+\mu}+1\right)\left((-1)^{L+\nu}+1\right)\Gamma(2L+2)^2 i^{4L-2\mu-2\nu}2^{-8L+\mu+\nu-6}}{\Gamma\left(\frac{1}{2}(2L+3)\right)^2} \\
&\times z^{L-(\mu+\nu)/2}\,{}_2\tilde{F}_3\left(\tfrac{L}{2}+\tfrac{1}{2},\tfrac{L}{2}+1;L+\tfrac{3}{2},\tfrac{L}{2}-\tfrac{\mu}{2}+1,\tfrac{L}{2}+\tfrac{\mu}{2}+1;\tfrac{z}{4}\right) \\
&\times {}_2\tilde{F}_3\left(\tfrac{L}{2}+\tfrac{1}{2},\tfrac{L}{2}+1;L+\tfrac{3}{2},\tfrac{L}{2}-\tfrac{\nu}{2}+1,\tfrac{L}{2}+\tfrac{\nu}{2}+1;\tfrac{z}{4}\right).
\end{aligned} \tag{44}
$$

Numerical checks show that the right-hand side requires as few as three nonzero terms—$L = 0, 2, 4$ or $L = 1, 3, 5$, depending on the parity of μ—in the sum to obtain seven-digit accuracy for $\mu+\nu < 10$ when the variable is set to the arbitrary value $z = 0.17$, as seen in Table 1.

Table 1. The left and right sides of (44) when the variable is set to the arbitrary value $z = 0.17$ and we include only three nonzero terms in the sum, shown through the digit with which the two sides disagree. When $z = 0.0017$ the accuracy increases, as seen in the last line, while for $z = 17$ the accuracy decreases; the penultimate line. The results are symmetrical with respect to μ and ν.

Left-Hand Side of (44)	Right-Hand Side of (44)	μ	ν	z
1.028881345119003	1.028881345119001	0	0	0.17
1.0344878191148	1.0344878191146	0	2	0.17
1.0369001971	1.0369001970	0	4	0.17
1.020434382759	1.020434382749	2	2	0.17
1.01777403	1.01777403	2	4	0.17
1.0140011	1.0140009	4	4	0.17
1.0258250454427744	1.0258250454427744	1	1	0.17
1.0230034607369	1.0230034607370	1	3	0.17
1.022243424630	1.022243424628	1	5	0.17
1.016657722535	1.016657722534	3	3	0.17
1.014587307	1.014587305	3	5	0.17
1.01205576	1.01205571	5	5	0.17
23.049	23.044	0	0	17.0
1.00013910008	1.00013910005	4	4	0.0017

Parameters μ and ν must be of the same parity or the sum is zero. They also must be non-negative to avoid infinities. The conventional hypergeometric functions on the right-hand side—with their prefactors—give indeterminacies in computation (infinities divided by infinities) unless μ and ν are both zero or one, so we relied on the second version, using regularized hypergeometric functions—having cancelled the $\Gamma(b_i)$ with gamma functions in the denominators of the prefactors that also give infinities in this case—for numerical checks.

The derivation of the Fourier–Legendre series for $J_\mu(z)$ in the prior paper [23], upon which the present work relies, was restricted to integer μ (and, hence, ν in the present work) at two places, Equations (3) and (10) of that paper, with the latter reproduced below in (58). Equation (3) of that paper is an integer-restricted version (the first term) of the more general integral representation due to Heine of the Bessel function [31]

$$J_\mu(kx\cos(\theta)) = \frac{e^{i\mu\pi/2}}{\pi}\left[\int_0^\pi dt\, e^{-ikx\cos(\theta)\cos(t)}\cos(\mu t) - \sin(\mu\pi)\int_0^\infty dt\, e^{ikx\cos(\theta)\cosh(t)-\mu t}\right] \tag{45}$$

and, hence, is not the blockage to generalization. The essential blockage is that we found no non-integer version of (58); however, it is possible that one could be found. If found, one would have to investigate whether the second term of (45) can be integrated over both $\cos(\theta)$ and t if one wanted to remove the integer restriction on μ and ν.

When $\mu = \nu = 0$ the order of the hypergeometric functions is reduced, since the parameters $a_3 = b_4$ on the left-hand side and $a_2 = b_3$ on the right-hand side. This results in the following special case:

$$\begin{aligned} {}_2F_3\left(\tfrac{1}{2},\tfrac{1}{2};1,1,\tfrac{3}{2};z\right) &= \sum_{L=0}^\infty \frac{\pi^2 i^{4L} 2^{-8L-6}\left((-1)^L+1\right)^2(2L+1)\Gamma(2L+2)^2}{\Gamma\left(\frac{L}{2}+1\right)^4\Gamma\left(L+\frac{3}{2}\right)^2\Gamma\left(\frac{1}{2}(2L+3)\right)^2} z^L\, {}_1F_2\left(\tfrac{L}{2}+\tfrac{1}{2};\tfrac{L}{2}+1,L+\tfrac{3}{2};\tfrac{z}{4}\right)^2 \\ &= \sum_{L=0}^\infty \frac{\pi^2 i^{4L} 2^{-8L-6}\left((-1)^L+1\right)^2(2L+1)\Gamma(2L+2)^2}{\Gamma\left(\frac{L}{2}+1\right)^2\Gamma\left(\frac{1}{2}(2L+3)\right)^2} z^L\, {}_1\tilde{F}_2\left(\tfrac{L}{2}+\tfrac{1}{2};L+\tfrac{3}{2},\tfrac{L}{2}+1;\tfrac{z}{4}\right)^2. \end{aligned} \tag{46}$$

When $\mu = \nu = 1$, the order of the hypergeometric functions is also reduced, since the parameters $a_3 = b_3$ on the left-hand side and $a_2 = b_2$ on the right-hand side. This results following the special case:

$$
\begin{aligned}
{}_2F_3\left(\tfrac{3}{2},\tfrac{3}{2};2,\tfrac{5}{2},3;z\right) &= 3\sum_{L=0}^{\infty}\frac{\pi^2 i^{4L-4}2^{-8L-4}\left((-1)^{L+1}+1\right)^2(2L+1)\Gamma(2L+2)^2}{\Gamma\left(\frac{L}{2}+\frac{1}{2}\right)^2\Gamma\left(\frac{L}{2}+\frac{3}{2}\right)^2\Gamma\left(L+\frac{3}{2}\right)^2\Gamma\left(\frac{1}{2}(2L+3)\right)^2} z^{L-1}\,{}_1F_2\left(\frac{L}{2}+1;\frac{L}{2}+\frac{3}{2},L+\frac{3}{2};\frac{z}{4}\right)^2 \\
&= 3\sum_{L=0}^{\infty}\frac{\pi^2 i^{4L-4}2^{-8L-4}\left((-1)^{L+1}+1\right)^2(2L+1)\Gamma(2L+2)^2}{\Gamma\left(\frac{L}{2}+\frac{1}{2}\right)^2\Gamma\left(\frac{1}{2}(2L+3)\right)^2} z^{L-1}\,{}_1\tilde{F}_2\left(\frac{L}{2}+1;L+\frac{3}{2},\frac{L}{2}+\frac{3}{2};\frac{z}{4}\right)^2.
\end{aligned}
\tag{47}
$$

Since the ${}_3F_4$ hypergeometric function in (44) contains $\frac{\mu}{2}+\frac{\nu}{2}$ in most of its parameters, we can simplify the parameters on the left-hand side somewhat by letting $\mu \to a-\nu-1$ and $\nu \to b-1$ so that

$$
\begin{aligned}
&{}_3F_4\left(\frac{a}{2}+\frac{1}{2},\frac{a}{2},\frac{a}{2};\frac{a}{2}+1,a,a-b+1,b;z\right) = a\Gamma(b)\Gamma(a-b+1) \\
&\times \sum_{L=0}^{\infty}\frac{\pi^2(2L+1)i^{-2a+4L+2}2^{a-8L-7}\left((-1)^{b+L-1}+1\right)\left((-1)^{a-b+L}+1\right)}{\Gamma\left(L+\frac{3}{2}\right)^2\Gamma\left(\frac{1}{2}(2L+3)\right)^2\Gamma\left(-\frac{b}{2}+\frac{L}{2}+\frac{3}{2}\right)\Gamma\left(\frac{b}{2}+\frac{L}{2}+\frac{1}{2}\right)\Gamma\left(\frac{a}{2}-\frac{b}{2}+\frac{L}{2}+1\right)} \\
&\times \frac{\Gamma(2L+2)^2}{\Gamma\left(-\frac{a}{2}+\frac{b}{2}+\frac{L}{2}+1\right)} z^{\frac{1-a}{2}+L}\,{}_2F_3\left(\frac{L}{2}+\frac{1}{2},\frac{L}{2}+1;-\frac{b}{2}+\frac{L}{2}+\frac{3}{2},\frac{b}{2}+\frac{L}{2}+\frac{1}{2},L+\frac{3}{2};\frac{z}{4}\right) \\
&\times {}_2F_3\left(\frac{L}{2}+\frac{1}{2},\frac{L}{2}+1;\frac{a}{2}-\frac{b}{2}+\frac{L}{2}+1,-\frac{a}{2}+\frac{b}{2}+\frac{L}{2}+1,L+\frac{3}{2};\frac{z}{4}\right) \\
&= a\Gamma(b)\Gamma(a-b+1) \\
&\times \sum_{L=0}^{\infty}\frac{\pi^2(2L+1)i^{-2a+4L+2}2^{a-8L-7}\left((-1)^{b+L-1}+1\right)\Gamma(2L+2)^2\left((-1)^{a-b+L}+1\right)}{\Gamma\left(\frac{1}{2}(2L+3)\right)^2} \\
&\times z^{\frac{1-a}{2}+L}\,{}_2\tilde{F}_3\left(\frac{L}{2}+\frac{1}{2},\frac{L}{2}+1;L+\frac{3}{2},-\frac{b}{2}+\frac{L}{2}+\frac{3}{2},\frac{b}{2}+\frac{L}{2}+\frac{1}{2};\frac{z}{4}\right) \\
&\times {}_2\tilde{F}_3\left(\frac{L}{2}+\frac{1}{2},\frac{L}{2}+1;L+\frac{3}{2},-\frac{a}{2}+\frac{b}{2}+\frac{L}{2}+1,\frac{a}{2}-\frac{b}{2}+\frac{L}{2}+1;\frac{z}{4}\right).
\end{aligned}
\tag{48}
$$

In this form, we must have $a = 1, 3, 5, 7\cdots$ and $b = 1, 2, 3, \cdots, a$. □

The reduction in order (46) occurs when $a = b = 1$, and $a = 3$, $b = 2$ gives (47).

4. For p = 1

Consider, now, an additional power of the cosine function (which equals x in what follows) that will give a different Fourier–Legendre series than does a pure Bessel function,

$$
xJ_N(kx) = \sum_{L=0}^{\infty} a^1_{LN}(k)P_L(x)\,. \tag{49}
$$

Theorem 2. *The Fourier–Legendre series coefficients of $xJ_N(kx)$ are*

$$
\begin{aligned}
a^1_{LN}(k) &= \frac{\sqrt{\pi}}{2}i^{-N}\Bigg[\frac{i^{L-1}2^{1-2L}L\,k^{L-1}}{\Gamma\left(\frac{1}{2}(2L+1)\right)}\left((-1)^{L+N+1}+1\right)\binom{L-1}{\frac{1}{2}(L-N-1)} \\
&\times {}_2F_3\left(\frac{L}{2}+\frac{1}{2},\frac{L}{2};L+\frac{1}{2},\frac{L}{2}-\frac{N}{2}+\frac{1}{2},\frac{L}{2}+\frac{N}{2}+\frac{1}{2};-\frac{k^2}{4}\right) \\
&+ \frac{i^{L+1}2^{-2L-3}(L+1)k^{L+1}}{\Gamma\left(\frac{1}{2}(2L+5)\right)}\left(1+(-1)^{L+N-1}\right)\binom{L+1}{\frac{1}{2}(L-N+1)} \\
&\times {}_2F_3\left(\frac{L}{2}+1,\frac{L}{2}+\frac{3}{2};L+\frac{5}{2},\frac{L}{2}-\frac{N}{2}+\frac{3}{2},\frac{L}{2}+\frac{N}{2}+\frac{3}{2};-\frac{k^2}{4}\right)\Bigg].
\end{aligned}
\tag{50}
$$

Proof of Theorem 2. The key step in modifying the prior derivation [23], generalizes the expansion coefficient at one point to

$$
a^p_{LN}(k) = \frac{2L+1}{2}\frac{i^{-N}}{\pi}\int_0^{\pi}\left[\int_{-1}^{1} e^{ikx\cos\theta}x^p P_L(x)dx\right]\cos(N\theta)d\theta\,, \tag{51}
$$

in which we use [26] (p. 987 No. 8.511.4) or [32] (p. 671 Eq. (B.46))

$$
e^{ikx\cos\theta} = \sum_{l'}(2l'+1)i^{l'}j_{l'}(k\cos\theta)P_{l'}(x)\,. \tag{52}
$$

Then, using the recurrence relation for Legendre functions [32] (p. 666 eq. (B.4)) , one finds

$$\begin{aligned} a^{p}_{LN}(k) &= \tfrac{2L+1}{2}\tfrac{i^{-N}}{\pi}\int_0^{\pi}\left[\int_{-1}^{1}\left(\sum_{l'=0}^{\infty}(2l'+1)i^{l'}j_{l'}(k\cos\theta)P_{l'}(x)\right)x^{p}P_L(x)dx\right]\cos(N\theta)d\theta \\ a^{1}_{LN}(k) &= \tfrac{2L+1}{2}\tfrac{i^{-N}}{\pi}\int_0^{\pi}\left[\int_{-1}^{1}\left(\sum_{l'=0}^{\infty}(2l'+1)i^{l'}j_{l'}(k\cos\theta)\right)\left\{\tfrac{l'P_{l'-1}(x)}{2l'+1}+\tfrac{(l'+1)P_{l'+1}(x)}{2l'+1}\right\}P_L(x)dx\right]\cos(N\theta)d\theta\,, \end{aligned} \tag{53}$$

which complicates the use of Legendre function orthogonality [32] (p. 666 eq. (B.5)) to truncate the series. For $L=0$ only the left-hand term in the curly brackets will contribute with the $l'=1$ term in the sum. When $L=1$, the left-hand term will again contribute with the $l'=2$ term in the sum, as will the right-hand term with the $l'=0$ term in the sum. This pattern continues with the first term nonzero only when $l'=L+1$, and the second nonzero only when $l'=L-1$. Thus,

$$\begin{aligned} a^{1}_{LN}\;(k) &= \tfrac{2L+1}{2}\tfrac{i^{-N}}{\pi}\int_0^{\pi}\left[\int_{-1}^{1}\left(\sum_{l'=0}^{\infty}i^{l'}j_{l'}(k\cos\theta)\right)\left\{\tfrac{2l'+1}{2l'+1}\right\}\{l'P_{l'-1}(x)+(l'+1)P_{l'+1}(x)\}P_L(x)dx\right]\cos(N\theta)d\theta \\ &= \tfrac{2L+1}{2}\tfrac{i^{-N}}{\pi}\int_0^{\pi}\left[\int_{-1}^{1}\left(\sum_{l'=0}^{\infty}i^{l'}j_{l'}(k\cos\theta)\right)\left\{\tfrac{2l'}{2l'-1}\delta_{l'-1,L}+\tfrac{2(l'+1)}{2l'+3}\delta_{l'+1,L}\right\}\right]\cos(N\theta)d\theta \\ &= \tfrac{2L+1}{2}\tfrac{i^{-N}}{\pi}\int_0^{\pi}\left[\sum_{\ell=0}^{\infty}i^{\ell+1}j_{\ell+1}(k\cos\theta)\frac{2(\ell+1)}{2\ell+1}\delta_{\ell L}+\sum_{\ell=1}^{\infty}i^{\ell-1}j_{\ell-1}(k\cos\theta)\frac{2\ell}{2\ell+1}\delta_{\ell L}\right]\cos(N\theta)d\theta \\ &= \tfrac{i^{-N}}{\pi}\int_0^{\pi}\left[i^{L+1}j_{L+1}(k\cos\theta)(L+1)+i^{L-1}j_{L-1}(k\cos\theta)L\right]\cos(N\theta)d\theta \end{aligned} \tag{54}$$

where the factor of L takes the second term to zero when $L=0$ so we need not worry about spherical Bessel functions with an index less than zero. (This is the same reason why the first term of the l' sum in the second line really starts at 1 so that the ℓ sum starts at 0 for the first sum in the third line). Note that [33] (p. 23, No. 171.4) gives only the second of these two terms, but one can interchange the meaning of m and n in this to give the first, the central line in

$$\int_{-1}^{1}xP_n(x)P_m(x)\,dx=\begin{cases}\frac{2n+2}{(2n+1)(2n+3)} & m=n+1\\ \frac{2n}{(2n-1)(2n+1)} & m=n-1\,.\\ 0 & otherwise\end{cases} \tag{55}$$

Using the series expansion [34]

$$j_{L\pm1}(x)=\sqrt{\pi}\,2^{-(L\pm1)-1}x^{L\pm1}\sum_{M=0}^{\infty}\frac{\left(-\frac{1}{4}\right)^{M}x^{2M}}{M!\Gamma\left(M+(L\pm1)+\frac{3}{2}\right)}, \tag{56}$$

we find

$$\begin{aligned} a^{1}_{LN}(k) &= \tfrac{i^{-N}}{\pi}\int_0^{\pi}\left[i^{L+1}j_{L+1}(k\alpha_0\cos\theta)(L+1)+i^{L-1}j_{L-1}(k\alpha_0\cos\theta)L\right]\cos(N\theta)d\theta \\ &= \tfrac{1}{2\sqrt{\pi}}i^{-N}\int_0^{\pi}\left[\left(\tfrac{i}{2}\right)^{L+1}(L+1)\sum_{M=0}^{\infty}\frac{(-1)^{M}}{4^{M}M!\Gamma\left(L+M+\frac{5}{2}\right)}(k\cos(\theta))^{2M+L+1}\right. \\ &+ \left.\left(\tfrac{i}{2}\right)^{L-1}L\sum_{M=0}^{\infty}\frac{(-1)^{M}}{4^{M}M!\Gamma\left(L+M+\frac{1}{2}\right)}(k\cos(\theta))^{2M+L-1}\right]\cos(N\theta)d\theta\,. \end{aligned} \tag{57}$$

This integral can be performed using an integral—that has three branches over the interval $\left[0,\frac{\pi}{2}\right]$—that Gröbner and Hofreiter [33] (p. 110, No. 332.14a) extended to the interval $[0,\pi]$ with a prefactor $(1+(-1)^{m+n})$, which renders the central one of the three possibilities nonzero only for even values for $m+n$:

$$\int_0^{\pi}cos^{m}\theta\cos(n\theta)d\theta=\left(1+(-1)^{m+n}\right)\frac{\pi}{2^{m+1}}\binom{m}{\frac{m-n}{2}}\quad[m\geq n>-1,\;m-n=2K]\,. \tag{58}$$

The other two branches, for odd $m+n$ on $\left[0,\frac{\pi}{2}\right]$, are zero on $[0,\pi]$ when this prefactor is included. (The lower limit on m $[m\geq n>-1]$ was a finding of the prior paper). Then

$$
\begin{aligned}
a^1_{LN}(k) &= \tfrac{1}{2\sqrt{\pi}} i^{-N} \Bigg[\left(\tfrac{i}{2}\right)^{L+1} (L+1) \sum_{M=0}^{\infty} \frac{(-1)^M (k)^{2M+L+1}}{4^M M! \Gamma\left(L+M+\frac{5}{2}\right)} \left(1+(-1)^{2M+L+1+N}\right) \\
&\times \tfrac{\pi}{2^{2M+L+2}} \binom{2M+L+1}{\frac{1}{2}(2M+L+1-N)} \\
&+ \left(\tfrac{i}{2}\right)^{L-1} L \sum_{M=0}^{\infty} \frac{(-1)^M}{4^M M! \Gamma\left(L+M+\frac{1}{2}\right)} (k)^{2M+L-1} \left(1+(-1)^{2M+L-1+N}\right) \\
&\times \tfrac{\pi}{2^{2M+L}} \binom{2M+L-1}{\frac{1}{2}(2M+L-1-N)} \Bigg] \\
&= \tfrac{\sqrt{\pi}}{2} i^{-N} \Bigg[\tfrac{i^{L-1} 2^{1-2L} L\, k^{L-1}}{\Gamma\left(\frac{1}{2}(2L+1)\right)} \left((-1)^{L+N+1}+1\right) \binom{L-1}{\frac{1}{2}(L-N-1)} \\
&\times {}_2F_3\left(\tfrac{L}{2}+\tfrac{1}{2}, \tfrac{L}{2}; L+\tfrac{1}{2}, \tfrac{L}{2}-\tfrac{N}{2}+\tfrac{1}{2}, \tfrac{L}{2}+\tfrac{N}{2}+\tfrac{1}{2}; -\tfrac{k^2}{4}\right) \\
&+ \tfrac{i^{L+1} 2^{-2L-3} (L+1) k^{L+1}}{\Gamma\left(\frac{1}{2}(2L+5)\right)} \left(1+(-1)^{L+N-1}\right) \binom{L+1}{\frac{1}{2}(L-N+1)} \\
&\times {}_2F_3\left(\tfrac{L}{2}+1, \tfrac{L}{2}+\tfrac{3}{2}; L+\tfrac{5}{2}, \tfrac{L}{2}-\tfrac{N}{2}+\tfrac{3}{2}, \tfrac{L}{2}+\tfrac{N}{2}+\tfrac{3}{2}; -\tfrac{k^2}{4}\right) \Bigg] \\
&= \tfrac{\sqrt{\pi}}{2} i^{-N} \big[i^{L-1} 2^{1-2L} L\, k^{L-1} \left(1+(-1)^{L+N+1}\right) \Gamma(L) \\
&\times {}_2\tilde{F}_3\left(\tfrac{L}{2}+\tfrac{1}{2}, \tfrac{L}{2}; L+\tfrac{1}{2}, \tfrac{L}{2}-\tfrac{N}{2}+\tfrac{1}{2}, \tfrac{L}{2}+\tfrac{N}{2}+\tfrac{1}{2}; -\tfrac{k^2}{4}\right) \\
&+ i^{L+1} 2^{-2L-3} (L+1) k^{L+1} \left(1+(-1)^{L+N-1}\right) \Gamma(L+2) \\
&\times {}_2\tilde{F}_3\left(\tfrac{L}{2}+1, \tfrac{L}{2}+\tfrac{3}{2}; L+\tfrac{5}{2}, \tfrac{L}{2}-\tfrac{N}{2}+\tfrac{3}{2}, \tfrac{L}{2}+\tfrac{N}{2}+\tfrac{3}{2}; -\tfrac{k^2}{4}\right) \Bigg].
\end{aligned} \tag{59}
$$

□

For the special cases $N = \{0,1\}$, these reduce somewhat [35] to

$$
\begin{aligned}
a^1_{L0}(k) &= \tfrac{1}{2}\sqrt{\pi} \Bigg(\tfrac{i^{L-1} 2^{1-2L}}{\Gamma\left(\frac{1}{2}(2L+1)\right)} \left(1+(-1)^{L+1}\right) L \binom{L-1}{\frac{L-1}{2}} k^{L-1}\, {}_1F_2\left(\tfrac{L}{2}; \tfrac{L}{2}+\tfrac{1}{2}, L+\tfrac{1}{2}; -\tfrac{k^2}{4}\right) \\
&+ \tfrac{i^{L+1} 2^{-2L-3}}{\Gamma\left(\frac{1}{2}(2L+5)\right)} \left(1+(-1)^{L-1}\right) (L+1) \binom{L+1}{\frac{L+1}{2}} k^{L+1}\, {}_1F_2\left(\tfrac{L}{2}+1; \tfrac{L}{2}+\tfrac{3}{2}, L+\tfrac{5}{2}; -\tfrac{k^2}{4}\right) \Bigg) \\
&= \tfrac{\sqrt{\pi}}{2} \Bigg(\tfrac{i^{L-1} 2^{1-2L}}{\Gamma\left(\frac{L}{2}+\frac{1}{2}\right)} \left(1+(-1)^{L+1}\right) L k^{L-1} \Gamma(L)\, {}_1\tilde{F}_2\left(\tfrac{L}{2}; L+\tfrac{1}{2}, \tfrac{L}{2}+\tfrac{1}{2}; -\tfrac{k^2}{4}\right). \\
&+ \tfrac{i^{L+1} 2^{-2L-3}}{\Gamma\left(\frac{L}{2}+\frac{3}{2}\right)} \left(1+(-1)^{L-1}\right) (L+1) k^{L+1} \Gamma(L+2)\, {}_1\tilde{F}_2\left(\tfrac{L}{2}+1; L+\tfrac{5}{2}, \tfrac{L}{2}+\tfrac{3}{2}; -\tfrac{k^2}{4}\right) \Bigg)
\end{aligned} \tag{60}
$$

and

$$
\begin{aligned}
a^1_{L1}(k) &= -\tfrac{1}{2} i\sqrt{\pi} \Bigg(\tfrac{i^{L-1} 2^{1-2L}}{\Gamma\left(\frac{1}{2}(2L+1)\right)} \left(1+(-1)^{L+2}\right) L \binom{L-1}{\frac{L-2}{2}} k^{L-1}\, {}_1F_2\left(\tfrac{L}{2}+\tfrac{1}{2}; \tfrac{L}{2}+1, L+\tfrac{1}{2}; -\tfrac{k^2}{4}\right) \\
&+ \tfrac{i^{L+1} 2^{-2L-3}}{\Gamma\left(\frac{1}{2}(2L+5)\right)} \left(1+(-1)^{L}\right) (L+1) \binom{L+1}{\frac{L}{2}} k^{L+1}\, {}_1F_2\left(\tfrac{L}{2}+\tfrac{3}{2}; \tfrac{L}{2}+2, L+\tfrac{5}{2}; -\tfrac{k^2}{4}\right) \Bigg) \\
&= \tfrac{\sqrt{\pi}}{2} (-i) \Bigg[\tfrac{i^{L-1} 2^{1-2L}}{\Gamma\left(\frac{L}{2}\right)} \left(1+(-1)^{L+2}\right) L k^{L-1} \Gamma(L)\, {}_1\tilde{F}_2\left(\tfrac{L}{2}+\tfrac{1}{2}; L+\tfrac{1}{2}, \tfrac{L}{2}+1; -\tfrac{k^2}{4}\right) \\
&+ \tfrac{i^{L+1} 2^{-2L-3}}{\Gamma\left(\frac{L}{2}+1\right)} \left(1+(-1)^{L}\right) (L+1) k^{L+1} \Gamma(L+2)\, {}_1\tilde{F}_2\left(\tfrac{L}{2}+\tfrac{3}{2}; L+\tfrac{5}{2}, \tfrac{L}{2}+2; -\tfrac{k^2}{4}\right) \Bigg].
\end{aligned} \tag{61}
$$

We now have the necessary elements in place to prove the following:

Theorem 3. *A second summation theorem for ${}_3F_4$ hypergeometric functions:*

$$
\begin{aligned}
&{}_3F_4\left(\tfrac{a}{2}+\tfrac{1}{2},\tfrac{a}{2},\tfrac{a}{2}+p;a,a-b+1,b,\tfrac{a}{2}+p+1;z\right)_{p\to 1} \\
&= \pi i^{1-a}2^{a-3}z^{\frac{1-a}{2}}\Gamma(b)\Gamma(a-b+1)(a+2p)_{p\to 1} \\
&\times \sum_{L=0}^{\infty}\frac{i^{2L-2}2^{2-4L}\left(1+(-1)^{b+L}\right)(-1)^{\frac{b-a}{2}+\frac{1-b}{2}+L-1}\left(1+(-1)^{a-b+L+1}\right)}{2L+1}z^{L-1} \\
&\times \left(\tfrac{L}{\Gamma\left(\frac{1}{2}(2L+1)\right)}\tbinom{L-1}{\frac{L-b}{2}}\,{}_2F_3\left(\tfrac{L}{2}+\tfrac{1}{2},\tfrac{L}{2};-\tfrac{b}{2}+\tfrac{L}{2}+1,\tfrac{b}{2}+\tfrac{L}{2},L+\tfrac{1}{2};\tfrac{z}{4}\right)\right. \\
&+ \left.\tfrac{(L+1)}{16\Gamma\left(\frac{1}{2}(2L+5)\right)}\tbinom{L+1}{\frac{1}{2}(-b+L+2)}\,z\,{}_2F_3\left(\tfrac{L}{2}+1,\tfrac{L}{2}+\tfrac{3}{2};-\tfrac{b}{2}+\tfrac{L}{2}+2,\tfrac{b}{2}+\tfrac{L}{2}+1,L+\tfrac{5}{2};\tfrac{z}{4}\right)\right) \\
&\times \left(\tfrac{L}{\Gamma\left(\frac{1}{2}(2L+1)\right)}\tbinom{L-1}{\frac{1}{2}(-a+b+L-1)}\,{}_2F_3\left(\tfrac{L}{2}+\tfrac{1}{2},\tfrac{L}{2};\tfrac{a}{2}-\tfrac{b}{2}+\tfrac{L}{2}+\tfrac{1}{2},-\tfrac{a}{2}+\tfrac{b}{2}+\tfrac{L}{2}+\tfrac{1}{2},L+\tfrac{1}{2};\tfrac{z}{4}\right)\right. \\
&+ \left.\tfrac{(L+1)}{16\Gamma\left(\frac{1}{2}(2L+5)\right)}\tbinom{L+1}{\frac{1}{2}(-a+b+L+1)}\,z\,{}_2F_3\left(\tfrac{L}{2}+1,\tfrac{L}{2}+\tfrac{3}{2};\tfrac{a}{2}-\tfrac{b}{2}+\tfrac{L}{2}+\tfrac{3}{2},-\tfrac{a}{2}+\tfrac{b}{2}+\tfrac{L}{2}+\tfrac{3}{2},L+\tfrac{5}{2};\tfrac{z}{4}\right)\right),
\end{aligned}
\tag{62}
$$

where $a = 1, 3, 5, 7\cdots$ *and* $b = 1, 2, 3, \cdots, a$.

Proof of Theorem 3. We will not display intermediate results as we step outward from (59) since they mirror those for $p = 0$ but with twice as many terms. With $x = k\alpha_0$, the final form is

$$
\begin{aligned}
&\Xi_n^1(k\alpha_0,-\tfrac{z}{2}) = \textstyle\sum_{l0} F_{l0}^{1n}(k\alpha_0,-\tfrac{z}{2})F_{l0}^{1n}(k\alpha_0,-\tfrac{z}{2}) \\
&= \textstyle\sum_{l0}\frac{2l+1}{4\pi}\left[\displaystyle\sum_{j=-(n-\delta)/2}^{\infty}\ \sum_{M=-(n-\delta)/2}^{\infty} J_{-j}(-\tfrac{z}{2})J_{-M}(-\tfrac{z}{2})\right. \\
&\times \frac{\pi^3 2^{4-4l}x^{2l-2}}{(2l+1)^2}\left((-1)^{2j+l+n+1}+1\right)\left((-1)^{l+2M+n+1}+1\right)(-1)^{-j+l-M-n-1} \\
&\times \left(\tfrac{l}{\Gamma\left(\frac{1}{2}(2l+1)\right)}\tbinom{l-1}{\frac{1}{2}(-2j+l-n-1)}\,{}_2F_3\left(\tfrac{l}{2}+\tfrac{1}{2},\tfrac{l}{2};l+\tfrac{1}{2},-j+\tfrac{l}{2}-\tfrac{n}{2}+\tfrac{1}{2},j+\tfrac{l}{2}+\tfrac{n}{2}+\tfrac{1}{2};-\tfrac{x^2}{4}\right)\right. \\
&- \left.\tfrac{(l+1)x^2}{16\Gamma\left(\frac{1}{2}(2l+5)\right)}\tbinom{l+1}{\frac{1}{2}(-2j+l-n+1)}\,{}_2F_3\left(\tfrac{l}{2}+1,\tfrac{l}{2}+\tfrac{3}{2};l+\tfrac{5}{2},-j+\tfrac{l}{2}-\tfrac{n}{2}+\tfrac{3}{2},j+\tfrac{l}{2}+\tfrac{n}{2}+\tfrac{3}{2};-\tfrac{x^2}{4}\right)\right) \\
&\times \left(\tfrac{l}{\Gamma\left(\frac{1}{2}(2l+1)\right)}\tbinom{l-1}{\frac{1}{2}(l-2M-n-1)}\,{}_2F_3\left(\tfrac{l}{2}+\tfrac{1}{2},\tfrac{l}{2};l+\tfrac{1}{2},\tfrac{l}{2}-M-\tfrac{n}{2}+\tfrac{1}{2},\tfrac{l}{2}+M+\tfrac{n}{2}+\tfrac{1}{2};-\tfrac{x^2}{4}\right)\right. \\
&- \left.\tfrac{(l+1)x^2}{16\Gamma\left(\frac{1}{2}(2l+5)\right)}\tbinom{l+1}{\frac{1}{2}(l-2M-n+1)}\,{}_2F_3\left(\tfrac{l}{2}+1,\tfrac{l}{2}+\tfrac{3}{2};l+\tfrac{5}{2},\tfrac{l}{2}-M-\tfrac{n}{2}+\tfrac{3}{2},\tfrac{l}{2}+M+\tfrac{n}{2}+\tfrac{3}{2};-\tfrac{x^2}{4}\right)\right) \\
&+ \left.\ldots\right],
\end{aligned}
\tag{63}
$$

where we have displayed just the first of four sums over j and M since they all lead to the same summation theorem.

On the other hand, for $p = 1$ in the direct integration method, the only thing that changes is the power of the $\cos^2(\theta) = u^2 = y$ factor in

$$
\begin{aligned}
\Xi_n^1\left(k\alpha_0,-\frac{z}{2}\right) &= \int \cos^2(\theta)J_n^2\left(\mathbf{k}\cdot\boldsymbol{\alpha}_0,-\frac{z}{2}\right)d\Omega \\
&= 2\pi\int_{-\pi}^{\pi}\cos^2(\theta)J_n^2\left(k\alpha_0\cos(\theta),-\frac{z}{2}\right)d(\cos(\theta)) \\
&= 2\pi\int_{-1}^{1}f\left(u^2\right)du = 4\pi\int_0^1 f\left(u^2\right)du = 2\pi\int_0^1 f(y)y^{-1/2}dy\,.
\end{aligned}
\tag{64}
$$

and in the four versions of $\alpha = \{j+M+n+1,\ j+M+1,\ j+M+1,\ j+M-n+1\}+p$ (only the first of which is needed for the one of three terms we displayed in (63)), in the following

$$
\begin{aligned}
&\int_0^a y^{\alpha-1}(a-y)^{\beta-1}\,{}_pF_q(a_1,\ldots,a_p,b_1,\ldots,b_q;-\omega y)\,dy\\
&\quad=\frac{\Gamma(\alpha)\Gamma(\beta)a^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\,{}_{p+1}F_{q+1}(a_1,\ldots,a_p,\alpha;b_1,\ldots,b_q,\alpha+\beta;-a\omega)\\
&\quad[\Re(\alpha)>0\wedge\Re(\beta)>0\wedge a>0]
\end{aligned}
\tag{65}
$$

that simply raises α by one for each of the four terms; with $a=1$ and $\beta=1$ unchanged.

Then the above and (63) give

$$
\begin{aligned}
&{}_3F_4\left(\tfrac{\mu}{2}+\tfrac{\nu}{2}+\tfrac12,\tfrac{\mu}{2}+\tfrac{\nu}{2}+1,\tfrac{\mu}{2}+\tfrac{\nu}{2}+\tfrac12+p;\mu+1,\nu+1,\mu+\nu+1,\tfrac{\mu}{2}+\tfrac{\nu}{2}+\tfrac32+p;z\right)_{p\to1}\\
=\;&\pi i^{-\mu-\nu}2^{\mu+\nu-2}\Gamma(\mu+1)\Gamma(\nu+1)(2p+\mu+\nu+1)z^{\frac12(-\mu-\nu)}\\
\times\;&\sum_{L=0}^{\infty}\frac{i^{2L-2}2^{2-4L}\left(1+(-1)^{L+\mu+1}\right)\left(1+(-1)^{L+\nu+1}\right)(-1)^{L-\frac{\mu}{2}-\frac{\nu}{2}-1}}{2L+1}z^{L-1}\\
\times\;&\left(\tfrac{L}{\Gamma\left(\frac12(2L+1)\right)}\tbinom{L-1}{\frac12(L-\mu-1)}\,{}_2F_3\left(\tfrac L2+\tfrac12,\tfrac L2;L+\tfrac12,\tfrac L2-\tfrac{\mu}{2}+\tfrac12,\tfrac L2+\tfrac{\mu}{2}+\tfrac12;\tfrac z4\right)\right.\\
+\;&\left.\tfrac{(L+1)}{16\Gamma\left(\frac12(2L+5)\right)}z\tbinom{L+1}{\frac12(L-\mu+1)}\,{}_2F_3\left(\tfrac L2+1,\tfrac L2+\tfrac32;L+\tfrac52,\tfrac L2-\tfrac{\mu}{2}+\tfrac32,\tfrac L2+\tfrac{\mu}{2}+\tfrac32;\tfrac z4\right)\right)\\
\times\;&\left(\tfrac{L}{\Gamma\left(\frac12(2L+1)\right)}\tbinom{L-1}{\frac12(L-\nu-1)}\,{}_2F_3\left(\tfrac L2+\tfrac12,\tfrac L2;L+\tfrac12,\tfrac L2-\tfrac{\nu}{2}+\tfrac12,\tfrac L2+\tfrac{\nu}{2}+\tfrac12;\tfrac z4\right)\right.\\
+\;&\left.\tfrac{(L+1)}{16\Gamma\left(\frac12(2L+5)\right)}z\tbinom{L+1}{\frac12(L-\nu+1)}\,{}_2F_3\left(\tfrac L2+1,\tfrac L2+\tfrac32;L+\tfrac52,\tfrac L2-\tfrac{\nu}{2}+\tfrac32,\tfrac L2+\tfrac{\nu}{2}+\tfrac32;\tfrac z4\right)\right)\\
=\;&\pi i^{-\mu-\nu}2^{\mu+\nu-2}\Gamma(\mu+1)\Gamma(\nu+1)(2p+\mu+\nu+1)z^{\frac12(-\mu-\nu)}\\
\times\;&\sum_{L=0}^{\infty}\frac{i^{2L-2}2^{2-4L}\left(1+(-1)^{L+\mu+1}\right)\left(1+(-1)^{L+\nu+1}\right)(-1)^{L-\frac{\mu}{2}-\frac{\nu}{2}-1}}{2L+1}z^{L-1}\\
\times\;&\left(\Gamma(L+1)\,{}_2\tilde F_3\left(\tfrac L2+\tfrac12,\tfrac L2;L+\tfrac12,\tfrac L2-\tfrac{\mu}{2}+\tfrac12,\tfrac L2+\tfrac{\mu}{2}+\tfrac12;\tfrac z4\right)\right.\\
+\;&\left.\tfrac{1}{16}(L+1)\Gamma(L+2)\,z\,{}_2\tilde F_3\left(\tfrac L2+1,\tfrac L2+\tfrac32;L+\tfrac52,\tfrac L2-\tfrac{\mu}{2}+\tfrac32,\tfrac L2+\tfrac{\mu}{2}+\tfrac32;\tfrac z4\right)\right)\\
\times\;&\left(\Gamma(L+1)\,{}_2\tilde F_3\left(\tfrac L2+\tfrac12,\tfrac L2;L+\tfrac12,\tfrac L2-\tfrac{\nu}{2}+\tfrac12,\tfrac L2+\tfrac{\nu}{2}+\tfrac12;\tfrac z4\right)\right.\\
+\;&\left.\tfrac{1}{16}(L+1)\Gamma(L+2)\,z\,{}_2\tilde F_3\left(\tfrac L2+1,\tfrac L2+\tfrac32;L+\tfrac52,\tfrac L2-\tfrac{\nu}{2}+\tfrac32,\tfrac L2+\tfrac{\nu}{2}+\tfrac32;\tfrac z4\right)\right).
\end{aligned}
\tag{66}
$$

For $p=1$, and the variable set to the arbitrary value $z=0.17$, the right-hand side requires four nonzero terms in the sum to get seven-digit accuracy for $\mu+\nu<10$, as seen in Table 2.

Table 2. The left and right sides of (66) when the variable is set to the arbitrary value $z=0.17$ and we include four nonzero terms in the sum, shown through the digit with which the two sides disagree.

Left-Hand Side of (66)	**Right-Hand Side of (66)**	μ	ν	**z**
1.052175485266236	1.052175485266234	0	0	0.17
1.01448216	1.01448208	4	4	0.17
1.0307786670736816	1.03077866707368164	1	1	0.17
1.01234929	1.01234927	5	5	0.17

When $\mu=\nu=0$, the order of the hypergeometric functions is reduced since the parameters a_2 and b_1 are equal on the left-hand side. On the right-hand side, $a_1=b_2$ in the first hypergeometric function in the summed pair within the products and $a_2=b_2$ in the second. This results in the following special case:

$$
\begin{aligned}
&{}_2F_3\Big(\tfrac{1}{2}, p+\tfrac{1}{2}; 1, 1, p+\tfrac{3}{2}; z\Big)_{p\to 1} \\
&= \tfrac{1}{4}\pi(2p+1)_{p\to 1}\sum_{L=0}^{\infty}\frac{(-1)^{L-1}i^{2L-2}2^{2-4L}\big(1+(-1)^{L+1}\big)^2}{2L+1}z^{L-1} \\
&\times \Bigg(\frac{(L+1)}{16\Gamma\big(\frac{1}{2}(2L+5)\big)}\binom{L+1}{\frac{L+1}{2}}z\,{}_1F_2\Big(\tfrac{L}{2}+1; \tfrac{L}{2}+\tfrac{3}{2}, L+\tfrac{5}{2}; \tfrac{z}{4}\Big) \\
&+ \frac{L}{\Gamma\big(\frac{1}{2}(2L+1)\big)}\binom{L-1}{\frac{L-1}{2}}{}_1F_2\Big(\tfrac{L}{2}; \tfrac{L}{2}+\tfrac{1}{2}, L+\tfrac{1}{2}; \tfrac{z}{4}\Big)\Bigg)^2 \\
&= \tfrac{1}{4}\pi(2p+1)_{p\to 1}\sum_{L=0}^{\infty}\frac{(-1)^{L-1}i^{2L-2}2^{2-4L}\big(1+(-1)^{L+1}\big)^2}{2L+1}z^{L-1} \\
&\times \Bigg(\frac{(L+1)\Gamma(L+2)}{16\Gamma\big(\frac{L}{2}+\frac{3}{2}\big)}z\,{}_1\tilde{F}_2\Big(\tfrac{L}{2}+1; L+\tfrac{5}{2}, \tfrac{L}{2}+\tfrac{3}{2}; \tfrac{z}{4}\Big) \\
&+ \frac{\Gamma(L+1)}{\Gamma\big(\frac{L}{2}+\frac{1}{2}\big)}{}_1\tilde{F}_2\Big(\tfrac{L}{2}; L+\tfrac{1}{2}, \tfrac{L}{2}+\tfrac{1}{2}; \tfrac{z}{4}\Big)\Bigg)^2.
\end{aligned} \tag{67}
$$

When $\mu = \nu = 1$, the order of the hypergeometric functions is also reduced since the parameters a_2 and b_1 are equal on the left-hand side. On the right-hand side, $a_2 = b_2$ in the first hypergeometric function in the summed pair within the products and $a_1 = b_2$ in the second, giving the special case,

$$
\begin{aligned}
&{}_2F_3\big(\tfrac{3}{2}, p+\tfrac{3}{2}; 2, 3, p+\tfrac{5}{2}; z\big)_{p\to 1} \\
&= -\tfrac{\pi}{z}(2p+3)_{p\to 1}\sum_{L=0}^{\infty}\frac{(-1)^{L-2}i^{2L-2}2^{2-4L}\big(1+(-1)^{L+2}\big)^2}{2L+1}z^{L-1} \\
&\times \Bigg(\frac{L}{\Gamma\big(\frac{1}{2}(2L+1)\big)}\binom{L-1}{\frac{L-2}{2}}{}_1F_2\Big(\tfrac{L}{2}+\tfrac{1}{2}; \tfrac{L}{2}+1, L+\tfrac{1}{2}; \tfrac{z}{4}\Big) \\
&+ \frac{(L+1)}{16\Gamma\big(\frac{1}{2}(2L+5)\big)}z\binom{L+1}{\frac{L}{2}}{}_1F_2\Big(\tfrac{L}{2}+\tfrac{3}{2}; \tfrac{L}{2}+2, L+\tfrac{5}{2}; \tfrac{z}{4}\Big)\Bigg)^2 \\
&= -\tfrac{\pi}{z}(2p+3)_{p\to 1}\sum_{L=0}^{\infty}\frac{(-1)^{L-2}i^{2L-2}2^{2-4L}\big(1(-1)^{L+2}\big)^2}{2L+1}z^{L-1} \\
&\times \Bigg(\frac{\Gamma(L+1)}{\Gamma\big(\frac{L}{2}\big)}{}_1\tilde{F}_2\Big(\tfrac{L}{2}+\tfrac{1}{2}; L+\tfrac{1}{2}, \tfrac{L}{2}+1; \tfrac{z}{4}\Big) \\
&+ \frac{(L+1)\Gamma(L+2)}{16\Gamma\big(\frac{L}{2}+1\big)}z\,{}_1\tilde{F}_2\Big(\tfrac{L}{2}+\tfrac{3}{2}; L+\tfrac{5}{2}, \tfrac{L}{2}+2; \tfrac{z}{4}\Big)\Bigg)^2.
\end{aligned} \tag{68}
$$

As with $p = 0$, we can simplify the left-hand side of (66) somewhat by letting $\mu \to a - \nu - 1$ and $\nu \to b - 1$ so that

$$\begin{aligned}
&{}_3F_4\left(\tfrac{a}{2}+\tfrac{1}{2},\tfrac{a}{2},\tfrac{a}{2}+p;a,a-b+1,b,\tfrac{a}{2}+p+1;z\right)_{p\to 1}\\
=\;& \pi i^{1-a}2^{a-3}z^{\frac{1-a}{2}}\Gamma(b)\Gamma(a-b+1)(a+2p)_{p\to 1}\\
\times\;& \sum_{L=0}^{\infty}\frac{i^{2L-2}2^{2-4L}\left(1+(-1)^{b+L}\right)(-1)^{\frac{b-a}{2}+\frac{1-b}{2}+L-1}\left(1+(-1)^{a-b+L+1}\right)}{2L+1}z^{L-1}\\
\times\;& \left(\frac{L}{\Gamma\left(\frac{1}{2}(2L+1)\right)}\binom{L-1}{\frac{L-b}{2}}\,{}_2F_3\left(\tfrac{L}{2}+\tfrac{1}{2},\tfrac{L}{2};-\tfrac{b}{2}+\tfrac{L}{2}+1,\tfrac{b}{2}+\tfrac{L}{2},L+\tfrac{1}{2};\tfrac{z}{4}\right)\right.\\
+\;& \left.\frac{(L+1)}{16\Gamma\left(\frac{1}{2}(2L+5)\right)}\binom{L+1}{\frac{1}{2}(-b+L+2)}z\,{}_2F_3\left(\tfrac{L}{2}+1,\tfrac{L}{2}+\tfrac{3}{2};-\tfrac{b}{2}+\tfrac{L}{2}+2,\tfrac{b}{2}+\tfrac{L}{2}+1,L+\tfrac{5}{2};\tfrac{z}{4}\right)\right)\\
\times\;& \left(\frac{L}{\Gamma\left(\frac{1}{2}(2L+1)\right)}\binom{L-1}{\frac{1}{2}(-a+b+L-1)}\,{}_2F_3\left(\tfrac{L}{2}+\tfrac{1}{2},\tfrac{L}{2};\tfrac{a}{2}-\tfrac{b}{2}+\tfrac{L}{2}+\tfrac{1}{2},-\tfrac{a}{2}+\tfrac{b}{2}+\tfrac{L}{2}+\tfrac{1}{2},L+\tfrac{1}{2};\tfrac{z}{4}\right)\right.\\
+\;& \left.\frac{(L+1)}{16\Gamma\left(\frac{1}{2}(2L+5)\right)}\binom{L+1}{\frac{1}{2}(-a+b+L+1)}z\,{}_2F_3\left(\tfrac{L}{2}+1,\tfrac{L}{2}+\tfrac{3}{2};\tfrac{a}{2}-\tfrac{b}{2}+\tfrac{L}{2}+\tfrac{3}{2},-\tfrac{a}{2}+\tfrac{b}{2}+\tfrac{L}{2}+\tfrac{3}{2},L+\tfrac{5}{2};\tfrac{z}{4}\right)\right)\\
=\;& \pi i^{1-a}2^{a-3}z^{\frac{1-a}{2}}\Gamma(b)\Gamma(a-b+1)(a+2p)_{p\to 1}\\
\times\;& \sum_{L=0}^{\infty}\frac{i^{2L-2}2^{2-4L}\left(1+(-1)^{b+L}\right)(-1)^{\frac{b-a}{2}+\frac{1-b}{2}+L-1}\left(1+(-1)^{a-b+L+1}\right)}{2L+1}z^{L-1}\\
\times\;& \left(\Gamma(L+1)\,{}_2\tilde{F}_3\left(\tfrac{L}{2}+\tfrac{1}{2},\tfrac{L}{2};L+\tfrac{1}{2},\tfrac{1-b}{2}+\tfrac{L}{2}+\tfrac{1}{2},\tfrac{b-1}{2}+\tfrac{L}{2}+\tfrac{1}{2};\tfrac{z}{4}\right)\right.\\
+\;& \left.\tfrac{1}{16}(L+1)z\Gamma(L+2)\,{}_2\tilde{F}_3\left(\tfrac{L}{2}+1,\tfrac{L}{2}+\tfrac{3}{2};L+\tfrac{5}{2},\tfrac{1-b}{2}+\tfrac{L}{2}+\tfrac{3}{2},\tfrac{b-1}{2}+\tfrac{L}{2}+\tfrac{3}{2};\tfrac{z}{4}\right)\right)\\
\times\;& \left(\Gamma(L+1)\,{}_2\tilde{F}_3\left(\tfrac{L}{2}+\tfrac{1}{2},\tfrac{L}{2};L+\tfrac{1}{2},\tfrac{b-a}{2}+\tfrac{L}{2}+\tfrac{1}{2},\tfrac{a-b}{2}+\tfrac{L}{2}+\tfrac{1}{2};\tfrac{z}{4}\right)\right.\\
+\;& \left.\tfrac{1}{16}(L+1)z\Gamma(L+2)\,{}_2\tilde{F}_3\left(\tfrac{L}{2}+1,\tfrac{L}{2}+\tfrac{3}{2};L+\tfrac{5}{2},\tfrac{b-a}{2}+\tfrac{L}{2}+\tfrac{3}{2},\tfrac{a-b}{2}+\tfrac{L}{2}+\tfrac{3}{2};\tfrac{z}{4}\right)\right).
\end{aligned} \tag{69}$$

In this form, we must have $a = 1, 3, 5, 7\cdots$ and $b = 1, 2, 3, \cdots, a$. $\square$

One could continue on in this fashion for $p = 2$ and beyond. However, we would have a product of triplets for $p = 2$ since

$$\begin{aligned}
&x^2P_{l'}(x)P_l(x)\\
=\;& \left(\frac{l'\left(2(l')^2+l'-3\right)P_{l'-2}(x)+\left(2(l')^3+5(l')^2+l'-2\right)P_{l'+2}(x)+\left(4(l')^3+6(l')^2-1\right)P_{l'}(x)}{(2l'+1)\left(4(l')^2+4l'-3\right)}\right)P_l(x)
\end{aligned} \tag{70}$$

and the number of terms increases with p. The process of finding the ${}_2F_3$ products of sums would, thus, be straight-forward though increasingly complicated as p increases.

5. Conclusions

We have found that the angular integration of transition amplitudes for radiative attachment, arising from the Strong Field Approximation (SFA), provides a means to express certain ${}_3F_4$ hypergeometric functions as infinite sums over pair products of ${}_2F_3$ functions. For special values of the parameters, the order is reduced to ${}_2F_3$ functions expressed as infinite sums over pair products of ${}_1F_2$ functions.

The SFA transition amplitudes include products of generalized Bessel functions that each comprise an infinite sum of products of two conventional Bessel functions, one of which contains angular dependence. That one of each pair is expanded in a series of Spherical Harmonics times ${}_2F_3$ functions, whose angular integral reduces the product to a single infinite series. On the other hand, one can express the product of these two conventional Bessel functions as a (different) ${}_2F_3$ function, whose angular integral is a ${}_3F_4$ function. Equating the results of these two methods gives the the desired relationship.

We have also stepped somewhat away from the physical application of this relationship by including a multiplicative factor of $\cos^2\theta$ to generate a second sort of ${}_3F_4$ hypergeometric function expressed as an infinite sum over pair products of a sum of two ${}_2F_3$ functions. If one were to entirely divorce this procedure from transition amplitudes and replace the

Bessel functions with other conventional functions that can each be expanded in a series of Spherical Harmonics, and whose product is some ${}_pF_q$ function, one might be able to generate additional relations of this general type.

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