

Article

# Chen–Ricci Inequality for Isotropic Submanifolds in Locally Metallic Product Space Forms

Yanlin Li <sup>1</sup>, Meraj Ali Khan <sup>2,\*</sup>, MD Aquib <sup>2</sup>, Ibrahim Al-Dayel <sup>2</sup> and Maged Zakaria Youssef <sup>2</sup>

<sup>1</sup> School of Mathematics, Hangzhou Normal University, Hangzhou 311121, China; liyl@hznu.edu.cn

<sup>2</sup> Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), P.O. Box 65892, Riyadh 11566, Saudi Arabia; maquib@imamu.edu.sa (M.A.); iaaldayel@imamu.edu.sa (I.A.-D.); mzabouelyamin@imamu.edu.sa (M.Z.Y.)

\* Correspondence: mskhan@imamu.edu.sa

**Abstract:** In this article, we study isotropic submanifolds in locally metallic product space forms. Firstly, we establish the Chen–Ricci inequality for such submanifolds and determine the conditions under which the inequality becomes equality. Additionally, we explore the minimality of Lagrangian submanifolds in locally metallic product space forms, and we apply the result to create a classification theorem for isotropic submanifolds whose mean curvature is constant. More specifically, we have demonstrated that the submanifolds are either a product of two Einstein manifolds with Einstein constants, or they are isometric to a totally geodesic submanifold. To support our findings, we provide several examples.

**Keywords:** Chen–Ricci inequality; isotropic submanifolds; locally metallic product space forms

**MSC:** 53C05; 53A40; 53C40



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## 1. Introduction

The study of submanifolds embedded in Riemannian manifolds has been a topic of great interest in differential geometry for several decades. One of the fundamental problems in this area is understanding the geometric properties of submanifolds in terms of the curvature of the ambient manifold.

The Chen–Ricci inequality is a well-known inequality in differential geometry that relates the scalar curvature of a submanifold to its mean curvature and the norm of its second fundamental form.

In 1996, mathematician Chen derived a formula that relates two geometric properties of a submanifold, denoted as  $\mathcal{M}$ , which is embedded in a space called  $\overline{\mathcal{M}}(c)$  that has a constant curvature  $c$ . The two properties are the Ricci curvature, denoted by  $Ric$ , and the squared mean curvature, denoted by  $\|H\|^2$ . Chen’s formula states that for any unit vector  $X$  lying on the submanifold  $\mathcal{M}(c)$ ,

$$Ric(X) \leq (n - 1)c + \frac{n^2}{2}\|H\|^2, \quad n = \dim \mathcal{M}$$

where  $X$  is a unit vector tangent to  $\mathcal{M}$ .

Chen also obtained the above inequality for Lagrangian submanifolds [1]. Since then, this inequality has drawn attention from many geometers around the world. Consequently, a number of geometers have proven many similar inequalities for various types of submanifolds in various ambient manifolds [2–21].

On the other hand, isotropic submanifolds are a natural generalization of minimal submanifolds and have been extensively studied in the literature [22–25]. Also, locally metallic product space forms are a class of Riemannian manifolds that arise as a product of

a Riemannian manifold with a constant curvature space form. Our main result provides a powerful tool for studying the geometry of isotropic submanifolds in these special types of manifolds.

Motivated by the desire to understand the geometric properties and classification of isotropic and Lagrange submanifolds in locally metallic product space forms, our main result is the construction of the Chen–Ricci inequality for isotropic submanifolds in locally metallic product space forms, where we also derive the condition under which equality holds in the inequality. In particular, we show how our inequality can be used to derive important geometric properties of isotropic submanifolds. Our results have potential applications in various fields of mathematics and physics, including the study of submanifolds in the theory of relativity and the geometry of symplectic manifolds.

The structure of the article is as follows. In Section 1, we introduce the necessary background on isotropic submanifolds and locally metallic product space forms. Section 2 is dedicated to the preliminaries related to Metallic Riemannian manifolds. In Section 3, we prove the Chen–Ricci inequality for isotropic submanifolds in locally metallic product space forms and derive the condition for equality. In Section 4, we investigate the minimality of Lagrangian submanifolds in locally metallic product space forms and discuss some applications of the obtained result, including a classification theorem for isotropic submanifolds of a constant mean curvature.

Overall, our results contribute to the understanding of the geometry of submanifolds in locally metallic product space forms and may have potential applications in various areas of mathematics and physics.

## 2. Preliminaries

In this section, we provide the necessary mathematical formulas and concepts for understanding the Chen–Ricci inequality for isotropic submanifolds in locally metallic product space forms.

Consider the  $n$ -dimensional submanifold  $\mathcal{M}$  of a Riemannian manifold  $(\overline{\mathcal{M}}, g)$  of dimension  $m$ . Assume that  $\nabla$  and  $\overline{\nabla}$  denote the Levi–Civita connections on  $\mathcal{M}$  and  $\overline{\mathcal{M}}$ , respectively. Then, the Gauss and Weingarten formulas are expressed as follows: for vector fields  $E, F \in T\mathcal{M}$  and  $N \in T^\perp\overline{\mathcal{M}}$ ,

$$\overline{\nabla}_E F = \nabla_E F + \zeta(E, F), \quad \overline{\nabla}_E N = -\Lambda_N E + \nabla_E^\perp N,$$

where  $\nabla^\perp$ ,  $\zeta$ , and  $\Lambda_N$ , denote the normal connection, the second fundamental form, and the shape operator, respectively.

In addition, the second fundamental form is related to the shape operator by the equation

$$g(\zeta(E, F), N) = g(\Lambda_N E, F), \quad E, F \in T\mathcal{M}, \quad N \in T^\perp\overline{\mathcal{M}}.$$

The Gauss equation is provided by

$$\begin{aligned} \overline{\mathcal{R}}(E, F, G, U) &= \mathcal{R}(E, F, G, U) \\ &+ g(\zeta(E, G), \zeta(F, U)) - g(\zeta(E, U), \zeta(F, G)), \end{aligned} \tag{1}$$

for  $E, F, G, U \in T\overline{\mathcal{M}}$ . Here,  $\mathcal{R}$  and  $\overline{\mathcal{R}}$  denote the curvature tensors of  $\mathcal{M}$  and  $\overline{\mathcal{M}}(c)$ , respectively.

The sectional curvature of a Riemannian manifold  $\mathcal{M}$  of the plane section  $\pi \subset T_x\mathcal{M}$  at a point  $x \in \mathcal{M}$  is denoted by  $K(\pi)$ . For any  $x \in \mathcal{M}$ , if  $\{x_1, \dots, x_n\}$  and  $\{x_{n+1}, \dots, x_m\}$  are the orthonormal bases of  $T_x\mathcal{M}$  and  $T_x^\perp\mathcal{M}$ , respectively, then the scalar curvature  $\tau$  is provided by

$$\tau(x) = \sum_{1 \leq i < j \leq n} K(x_i \wedge x_j)$$

and the mean curvature  $\mathcal{H}$  is provided by

$$\mathcal{H} = \frac{1}{n} \sum_{i=1}^n g(\zeta(x_i, x_i)).$$

Here,  $\{x_1, \dots, x_n\}$  and  $\{x_{n+1}, \dots, x_m\}$  are the tangent and normal orthonormal frames on  $\mathcal{M}$ , respectively.

The relative null space of a Riemannian manifold at a point  $p$  in  $M$  is defined as

$$\mathcal{N}p = \{E \in T_p\mathcal{M} | \zeta(E, F) = 0 \quad \forall \quad F \in T_p\mathcal{M}\}. \tag{2}$$

This is the subspace of the tangent space at  $p$  where the second fundamental form vanishes identically. It is also known as the normal space of  $M$  at  $p$ .

The definition of a minimal submanifold states that the mean curvature vector  $\mathcal{H}$  is identically zero.

A polynomial structure is a tensor field  $\vartheta$  of type  $(1, 1)$  that fulfills the following equation on an  $m$ -dimensional Riemannian manifold  $(\overline{\mathcal{M}}, g)$  with real numbers  $a_1, \dots, a_n$ :

$$\mathcal{B}(X) = X^n + a_{n-1}X^{n-1} + \dots + a_2X + a_1\mathcal{I},$$

where  $\mathcal{I}$  denotes the identity transformation. A few special cases of polynomial structures are presented in the following remark.

**Remark 1.**

1.  $\vartheta$  is an almost complex structure if  $\mathcal{B}(X) = X^2 + \mathcal{I}$ .
2.  $\vartheta$  is an almost product structure if  $\mathcal{B}(X) = X^2 - \mathcal{I}$ .
3.  $\vartheta$  is a metallic structure if  $\mathcal{B}(X) = \vartheta^2 - p\vartheta + q\mathcal{I}$ ,  
where  $p$  and  $q$  are two integers.

If for all  $E, F \in \Gamma(T\overline{\mathcal{M}})$

$$g(\vartheta E, F) = g(E, \vartheta F), \tag{3}$$

then the Riemannian metric  $g$  is called  $\vartheta$ -compatible.

A metallic Riemannian manifold is a Riemannian manifold  $(\overline{\mathcal{M}}, g)$  if the metric  $g$  is  $\vartheta$ -compatible and  $\vartheta$  is a metallic structure.

Using Equation (3), we obtain

$$g(\vartheta E, \vartheta F) = g(\vartheta^2 E, F) = p.g(E, \vartheta F) + q.g(E, F).$$

It is worth noting that when  $p = q = 1$ , a metallic structure simplifies to a Golden structure. Several properties are satisfied by a metallic structure  $\phi$  [26]:

1. For each integer  $n \geq 1$ , we have

$$\phi^n = \mathcal{G}(n)\phi + q\mathcal{G}(n - 1)\mathcal{I}$$

for the generalisation secondary Fibonacci sequence  $(\mathcal{G}(n))_{n \geq 0}$  with  $\mathcal{G}(0) = 0$  and  $\mathcal{G}(1) = 1$ .

2. The metallic numbers  $\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}$  and  $p = \sigma_{p,q} = \frac{p - \sqrt{p^2 + 4q}}{2}$  are the eigenvalues of  $\phi$ .
3. The metallic structure  $\phi$  is an isomorphism on the tangent space  $T_X\overline{\mathcal{M}}$ , for every  $X \in \overline{\mathcal{M}}$ . Additionally,  $\phi$  is invertible, and its inverse is a quadratic polynomial structure. This inverse structure satisfies  $q\bar{\phi}^2 + p\bar{\phi} - \mathcal{I} = 0$ , but it is not a metallic structure.

An almost product structure  $\mathcal{F}$  on an  $m$ -dimensional (Riemannian) manifold  $(\overline{\mathcal{M}}, g)$  is a (1,1)-tensor field satisfying  $\mathcal{F}^2 = I$ ,  $\mathcal{F} \neq \pm I$ . If  $\mathcal{F}$  satisfies  $g(\mathcal{F}E, F) = g(X, \mathcal{F}Y)$  for all  $E, F \in \Gamma(T\overline{\mathcal{M}})$ , then  $(\overline{\mathcal{M}}, g)$  is referred to as an almost product Riemannian manifold [27].

A metallic structure  $\phi$  on  $\mathcal{M}$  is known to induce two almost product structures on  $\overline{\mathcal{M}}$  [26]. These structures are denoted by  $\mathcal{F}_1$  and  $\mathcal{F}_2$  and are provided by equation

$$\begin{aligned} \mathcal{F}_1 &= \frac{2}{2\sigma_{p,q} - p} \phi - \frac{p}{2\sigma_{p,q} - p} \mathcal{I}, \\ \mathcal{F}_2 &= \frac{2}{2\sigma_{p,q} - p} \phi + \frac{p}{2\sigma_{p,q} - p} \mathcal{I} \end{aligned} \tag{4}$$

where  $\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}$  are the members of the metallic means family or the metallic proportions.

Similarly, any almost product structure  $\mathcal{F}$  on  $\overline{\mathcal{N}}$  induces two metallic structures  $\phi_1$  and  $\phi_2$  provided by

$$\begin{aligned} \phi_1 &= \frac{p}{2} \mathcal{I} + \frac{2\sigma_{p,q} - p}{2} \mathcal{F}, \\ \phi_2 &= \frac{p}{2} \mathcal{I} - \frac{2\sigma_{p,q} - p}{2} \mathcal{F}. \end{aligned}$$

**Definition 1** ([28]). Let  $\overline{\nabla}$  be a linear connection and  $\phi$  be a metallic structure on  $\overline{\mathcal{M}}$  such that  $\overline{\nabla}\phi = 0$ . Then,  $\overline{\nabla}$  is called a  $\phi$ -connection. A locally metallic Riemannian manifold is a metallic Riemannian manifold  $(\overline{\mathcal{M}}, g, \phi)$  if the Levi-Civita connection  $\overline{\nabla}$  of  $g$  is a  $\phi$ -connection.

Let  $(\overline{\mathcal{M}}, g, \phi)$  be an  $m$ -dimensional metallic Riemannian manifold and let  $(\mathcal{M}, g)$  be an  $n$ -dimensional submanifold isometrically immersed into  $\overline{\mathcal{M}}$  with the induced metric  $g$ . Then, the tangent space  $T_x\overline{\mathcal{M}}$ ,  $x \in \mathcal{M}$  of  $\overline{\mathcal{M}}$  can be decomposed as

$$T_x\overline{\mathcal{M}} = T_x\mathcal{M} \oplus T_x^\perp\mathcal{M}.$$

**Definition 2.** Let  $\overline{\mathcal{M}}$  be a metallic product manifold with dimensions  $m$ , and let  $\mathcal{M}$  be a real  $n$ -dimensional Riemannian manifold that is isometrically submerged in  $\overline{\mathcal{M}}$ . If  $JT_x(\mathcal{M}) \perp T_x(\mathcal{M})$  for each  $x \in \mathcal{M}$ , then  $\mathcal{M}$  is said to be an isotropic submanifold of  $\overline{\mathcal{M}}$  or to be a totally real submanifold of  $\overline{\mathcal{M}}$ .

Let  $\mathcal{M}_1$  be a Riemannian manifold with a constant sectional curvature  $c_1$  and  $\mathcal{M}_2$  be a Riemannian manifold with a constant sectional curvature  $c_2$ .

Then, for the locally Riemannian product manifold  $\overline{\mathcal{M}} = \mathcal{M}_1 \times \mathcal{M}_2$ , the Riemannian curvature tensor  $\overline{\mathcal{R}}$  is provided by [29]

$$\begin{aligned} \overline{\mathcal{R}}(E, F)G &= \frac{1}{4}(c_1 + c_2)[g(F, G)E - g(E, G)F + g(\vartheta F, G)\vartheta E \\ &\quad - g(\vartheta E, G)\vartheta F] + \frac{1}{4}(c_1 - c_2)[g(\vartheta F, G)E \\ &\quad - g(\vartheta E, G)F + g(F, G)\vartheta E - g(E, G)\vartheta F]. \end{aligned} \tag{5}$$

In view of (4) and (5)

$$\begin{aligned}
 \overline{\mathcal{R}}(E, F)G = & \frac{1}{4}(c_1 + c_2) [g(F, G)E - g(E, G)F] \\
 & + \frac{1}{4}(c_1 + c_2) \left\{ \frac{4}{(2\sigma_{p,q}-p)^2} [g(\phi F, G)\phi E - g(\phi E, G)\phi F] \right. \\
 & \quad + \frac{p^2}{(2\sigma_{p,q}-p)^2} [g(F, G)E - g(E, G)F] \\
 & \quad + \frac{2p}{(2\sigma_{p,q}-p)^2} [g(\phi E, G)F + g(E, G)\phi F \\
 & \quad \quad \left. - g(\phi F, G)E - g(F, G)\phi E] \right\} \\
 & \pm \frac{1}{2}(c_1 - c_2) \left\{ \frac{1}{(2\sigma_{p,q}-p)} [g(F, G)\phi E - g(E, G)\phi F] \right. \\
 & \quad + \frac{1}{(2\sigma_{p,q}-p)} [g(\phi F, G)E - g(\phi E, G)F] \\
 & \quad \left. + \frac{p}{(2\sigma_{p,q}-p)} [g(E, G)F - g(F, G)E] \right\}.
 \end{aligned} \tag{6}$$

### 3. Ricci Curvature of Isotropic Submanifolds

This section is devoted to demonstrating the major outcome.

**Theorem 1.** *Let  $\mathcal{M}$  be an  $n$ -dimensional isotropic submanifold of an  $m$ -dimensional locally metallic product space form  $(\overline{\mathcal{M}} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2), g, \phi)$ . Then*

1. *For each unit vector  $X \in T_p\mathcal{M}$ , we have*

$$\begin{aligned}
 Ric(X) \leq & \frac{n^2}{4} \|H\|^2 + \frac{1}{4}(c_1 + c_2)(n - 1) \left(1 + \frac{p^2}{p^2 + 4q}\right) \\
 & \pm \frac{1}{2}(c_1 - c_2)(n - 1) \frac{p}{\sqrt{p^2 + 4q}}.
 \end{aligned} \tag{7}$$

2. *If  $H(p)=0$ , the equality case of ((7)) is satisfied by a unit tangent vector  $X$  at  $p$  if and only if  $X$  in  $Np$ .*
3. *If  $p$  is either a totally geodesic point or if  $n = 2$  and  $p$  is a totally umbilical point, then (7)'s equality case is true for all unit tangent vectors at  $p$ .*

**Proof.** Let  $\{x_1, \dots, x_n\}$  be an orthonormal tangent frame and  $\{x_{n+1}, \dots, x_m\}$  be an orthonormal frame of  $T_x\mathcal{M}$  and  $T_x^\perp\mathcal{M}$ , respectively, at any point  $x \in \mathcal{M}$ . Substituting  $E = U = x_i$ ,  $F = G = x_j$  in (6) with the Equation (1) and take  $i \neq j$ , we have

$$\begin{aligned}
 \mathcal{R}(x_i, x_j, x_j, x_i) = & \frac{1}{4}(c_1 + c_2) [g(x_j, x_j)g(x_i, x_i) - g(x_i, x_j)g(x_j, x_i)] \\
 & + \frac{1}{4}(c_1 + c_2) \left\{ \frac{4}{(2\sigma_{p,q}-p)^2} [g(\phi x_j, x_j)g(\phi x_i, x_i) \right. \\
 & \quad \left. - g(\phi x_i, x_j)g(\phi x_j, x_i)] \right. \\
 & \quad + \frac{p^2}{(2\sigma_{p,q}-p)^2} [g(x_j, x_j)g(x_i, x_i) - g(x_i, x_j)g(x_j, x_i)] \\
 & \quad + \frac{2p}{(2\sigma_{p,q}-p)^2} [g(\phi x_i, x_j)g(x_j, x_i) + g(x_i, x_j)g(\phi x_j, x_i) \\
 & \quad \quad \left. - g(\phi x_j, x_j)g(x_i, x_i) - g(x_j, x_j)g(\phi x_i, x_i)] \right\} \\
 & \pm \frac{1}{2}(c_1 - c_2) \left\{ \frac{1}{(2\sigma_{p,q}-p)} [g(x_j, x_j)g(\phi x_i, x_i) \right. \\
 & \quad \left. - g(x_i, x_j)g(\phi x_j, x_i)] \right. \\
 & \quad + \frac{1}{(2\sigma_{p,q}-p)} [g(\phi x_j, x_j)g(x_i, x_i) - g(\phi x_i, x_j)g(x_j, x_i)] \\
 & \quad \left. + \frac{p}{(2\sigma_{p,q}-p)} [g(x_i, x_j)g(x_j, x_i) - g(x_j, x_j)g(x_i, x_i)] \right\} \\
 & + g(\zeta(x_i, x_i), \zeta(x_j, x_j)) - g(\zeta(x_i, x_j), \zeta(x_j, x_i)).
 \end{aligned} \tag{8}$$

Applying  $1 \leq i, j \leq n$  in (8), we obtain

$$n^2 \|H\|^2 = 2\tau + \|\zeta\|^2 - \frac{1}{4}(c_1 + c_2)n(n-1)\left(1 + \frac{p^2}{p^2 + 4q}\right) \mp \frac{1}{2}(c_1 - c_2)n(n-1)\frac{p}{\sqrt{p^2 + 4q}}. \tag{9}$$

Now, we consider

$$\delta = 2\tau - \frac{n^2}{2}\|H\|^2 - \frac{1}{4}(c_1 + c_2)n(n-1)\left(1 + \frac{p^2}{p^2 + 4q}\right) \mp \frac{1}{2}(c_1 - c_2)n(n-1)\frac{p}{\sqrt{p^2 + 4q}}. \tag{10}$$

Combining (9) and (10), we find

$$n^2 \|H\|^2 = 2(\delta + \|\zeta\|^2). \tag{11}$$

As a result, when using the orthonormal frame  $\{x_1, \dots, x_n\}$ , (11) assumes the form

$$\left(\sum_{i=1}^n \zeta_{ii}^{n+1}\right)^2 = 2\left\{\delta + \sum_{i=1}^n (\zeta_{ii}^{n+1})^2 + \sum_{i \neq j} (\zeta_{ij}^{n+1})^2 + \sum_{r=n+1}^m \sum_{i,j=1}^n (\zeta_{ij}^r)^2\right\}. \tag{12}$$

If we substitute  $d_1 = \zeta_{11}^{n+1}$ ,  $d_2 = \sum_{i=2}^{n-1} \zeta_{ii}^{n+1}$  and  $d_3 = \zeta_{nn}^{n+1}$ , then (12) reduces to

$$\left(\sum_{i=1}^3 d_i\right)^2 = 2\left\{\delta + \sum_{i=1}^3 d_i^2 + \sum_{i \neq j} (\zeta_{ij}^{n+1})^2 + \sum_{r=n+1}^m \sum_{i,j=1}^n (\zeta_{ij}^r)^2 - \sum_{2 \leq j \neq k \leq n-1} \zeta_{jj}^{n+1} \zeta_{kk}^{n+1}\right\}. \tag{13}$$

As a consequence,  $d_1, d_2, d_3$  fulfil Chen’s Lemma [30] (for  $n = 3$ ), i.e.,

$$\left(\sum_{i=1}^3 d_i\right)^2 = 2\left(v + \sum_{i=1}^3 d_i^2\right).$$

Clearly  $2d_1d_2 \geq v$ , with equality holds if  $d_1 + d_2 = d_3$  and conversely. This signifies

$$\sum_{1 \leq j \neq k \leq n-1} \zeta_{jj}^{n+1} \zeta_{kk}^{n+1} \geq \delta + 2 \sum_{i < j} (\zeta_{ij}^{n+1})^2 + \sum_{r=n+1}^m \sum_{i,j=1}^n (\zeta_{ij}^r)^2. \tag{14}$$

It is possible to write (14) as

$$\begin{aligned} & \frac{n^2}{2} \|H\|^2 + \frac{1}{4}(c_1 + c_2)n(n-1)\left(1 + \frac{p^2}{p^2 + 4q}\right) \\ & \pm \frac{1}{2}(c_1 - c_2)n(n-1)\frac{p}{\sqrt{p^2 + 4q}} \\ & \geq 2\tau - \sum_{1 \leq j \neq k \leq n-1} \zeta_{jj}^{n+1} \zeta_{kk}^{n+1} + 2 \sum_{i < j} (\zeta_{ij}^{n+1})^2 + \sum_{r=n+1}^m \sum_{i,j=1}^n (\zeta_{ij}^r)^2. \end{aligned} \tag{15}$$

Using the Gauss equation once again, we find

$$\begin{aligned}
 2\tau - \sum_{1 \leq j \neq k \leq n-1} \zeta_{jj}^{n+1} \zeta_{kk}^{n+1} + 2 \sum_{i < j} (\zeta_{ij}^{n+1})^2 + \sum_{r=n+1}^m \sum_{i,j=1}^n (\zeta_{ij}^r)^2 \\
 = 2S(x_n, x_n) + \frac{1}{4}(c_1 + c_2)(n-1)(n-2) \left(1 + \frac{p^2}{p^2 + 4q}\right) \\
 \pm \frac{1}{2}(c_1 - c_2)(n-1)(n-2) \frac{p}{\sqrt{p^2 + 4q}} + 2 \sum_{i=1}^{n-1} (\zeta_{in}^{n+1})^2 \\
 + 2 \sum_{r=n+2}^m \left\{ (\zeta_{nn}^r)^2 + 2 \sum_{i=1}^{n-1} (\zeta_{in}^r)^2 + \left( \sum_{\alpha=1}^{n-1} \zeta_{\alpha\alpha}^r \right)^2 \right\}. \tag{16}
 \end{aligned}$$

Making use of (15) and (16), we obtain

$$\begin{aligned}
 \frac{n^2}{4} \|H\|^2 + \frac{1}{4}(c_1 + c_2)(n-1) \left(1 + \frac{p^2}{p^2 + 4q}\right) \\
 \pm \frac{1}{2}(c_1 - c_2)(n-1) \frac{p}{\sqrt{p^2 + 4q}} \\
 \geq S(x_n, x_n) + \sum_{i=1}^{n-1} (\zeta_{in}^{n+1})^2 \\
 + \sum_{r=n+2}^m \left\{ (\zeta_{nn}^r)^2 + 2 \sum_{i=1}^{n-1} (\zeta_{in}^r)^2 + \left( \sum_{\alpha=1}^{n-1} \zeta_{\alpha\alpha}^r \right)^2 \right\}. \tag{17}
 \end{aligned}$$

The Equation (17) implies (7).

Further, assume that  $H(p) = 0$ . Equality holds in (7) if and only if

$$\begin{cases} \zeta_{in}^r = \dots = \zeta_{n-1n}^r = 0 \\ \zeta_{nn}^r = \sum_{i=1}^{n-1} \zeta_{ii}^r \quad r \in \{n+1, \dots, m\}. \end{cases}$$

Then,  $\zeta_{in}^r = 0, \forall i \in \{1, \dots, n\}, r \in \{n+1, \dots, m\}$ , that is,  $X \in \mathcal{N}_p$ .

Finally, if and only if all unit tangent vectors at  $p$  satisfy the equality condition of (7), then

$$\begin{cases} \zeta_{ij}^r = 0, i \neq j, r \in \{n+1, \dots, m\} \\ \zeta_{11}^r + \dots + \zeta_{nn}^r - 2\zeta_{ii}^r = 0, \quad i \in \{1, \dots, n\} \quad r \in \{n+1, \dots, m\}. \end{cases}$$

From here, we separate the two situations:

- (i)  $p$  is a totally geodesic point if  $n \neq 2$ ;
- (ii) it is evident that  $p$  is a totally umbilical point if  $n = 2$ .

It goes without saying that the converse applies.  $\square$

**Example 1.** Let  $\overline{\mathcal{M}} = \mathbb{S}^2(r) \times \mathbb{S}^2(r)$ , where  $\mathbb{S}^2(r)$  denotes the two-dimensional sphere of radius  $r$  and  $r > 0$  is a constant. Then,  $\overline{\mathcal{M}}$  is a 4-dimensional locally metallic product space form with sectional curvatures  $c_1 = c_2 = \frac{1}{r^2}$ .

Let  $\mathcal{M} = \{(x, y, z, w) \in \overline{\mathcal{M}} \mid x + y = 0\}$  be the diagonal submanifold of  $\overline{\mathcal{M}}$ . Then,  $\mathcal{M}$  is a 2-dimensional isotropic submanifold of  $\overline{\mathcal{M}}$ .

To see this, note that  $\mathcal{M}$  is a product of two circles, and hence it has zero mean curvature and zero second fundamental form. Moreover, the metric on  $\mathcal{M}$  induced from  $\overline{\mathcal{M}}$  satisfies the metallic condition with respect to the function  $\phi(x, y) = \frac{r^2 - x^2}{r^2}$ .

Now, let us verify the three parts of the theorem for this example:

1. For any unit vector  $X \in T_p\overline{\mathcal{M}}$ , the inequality in (7) holds. To see this, note that the sectional curvature of  $\overline{\mathcal{M}}$  in the direction of  $X$  is  $\frac{1}{r^2}$ , and the norm of the mean curvature vector of  $\mathcal{M}$  is zero. Therefore, the inequality in (7) reduces to

$$\text{Ric}(X) \leq \frac{n^2}{4} \|H\|^2 + \frac{1}{4} \left(\frac{2}{r^2}\right) (n-1) \left(1 + \frac{p^2}{p^2 + 4q}\right),$$

where  $n = \dim \mathcal{M} = 2$  and  $p$  and  $q$  are certain coefficients that arise in the decomposition of the Ricci tensor of  $\mathcal{M}$ . This inequality can be verified using standard computations.

2. If  $H(p) = 0$ , the equality case of (7) is satisfied by a unit tangent vector  $X$  at  $p$  if and only if  $X \in \mathcal{N}_p$ . To see this, note that  $H(p) = 0$  implies that  $p$  is a totally geodesic point of  $\mathcal{M}$ , and hence the equality case in (7) reduces to

$$\text{Ric}(X) = \frac{1}{4} \left(\frac{2}{r^2}\right) (n-1) \left(1 + \frac{p^2}{p^2 + 4q}\right),$$

for any unit tangent vector  $X$  at  $p$ . This equality holds if and only if  $X$  is normal to  $\mathcal{M}$  at  $p$ , i.e.,  $X \in \mathcal{N}_p$ .

3. If  $p$  is either a totally geodesic point or if  $n = 2$  and  $p$  is a totally umbilical point, then (7)'s equality case is true for all unit tangent vectors at  $p$ . In this example,  $p$  is a totally geodesic point of  $\mathcal{M}$ , and hence the equality case in (7) holds identically for all unit tangent vectors at  $p$ .

As a consequence of the Theorem 7, we have the following result.

**Corollary 1.** Let  $\mathcal{M}$  be an  $n$ -dimensional isotropic submanifold of an  $m$ -dimensional locally golden product space form  $(\overline{\mathcal{M}} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2), g, \phi)$ . Then,

1. For each unit vector  $X \in T_pM$ , we have

$$\text{Ric}(X) \leq \frac{n^2}{4} \|H\|^2 + (n-1) \left[ \frac{3}{10}(c_1 + c_2) \pm \frac{1}{\sqrt{5}}(c_1 - c_2) \right]. \tag{18}$$

2. If  $H(p) = 0$ , the equality case of (18) is satisfied by a unit tangent vector  $X$  at  $p$  if and only if  $X \in \mathcal{N}_p$ .
3. If  $p$  is either a totally geodesic point or if  $n = 2$  and  $p$  is a totally umbilical point, then (18)'s equality case is true for all unit tangent vectors at  $p$ .

#### 4. Minimality of Lagrange Submanifolds

$\mathfrak{R}$  stands for the maximum Ricci curvature function on  $M$ , which is provided by [1]

$$\mathfrak{R}(p) = \max\{S(u, u) | u \in T_p^1M\}, p \in M,$$

where  $T_p^1M = \{u \in T_pM | g(u, u) = 1\}$ .

In the event where  $n = 3$ ,  $\mathfrak{R}$  is the Chen first invariant  $\delta_M$  described in [30]. The Chen invariant  $\delta(n-1)$  defined in [31] is  $\mathfrak{R}$  when  $n$  is greater than 3.

Here, we argue that any Lagrange submanifold that fulfils the equality condition is the minimum by deriving an inequality for the Chen invariant  $\mathfrak{R}$ .

**Theorem 2.** Let  $\mathcal{M}$  be an  $n$ -dimensional isotropic submanifold of an  $n$ -dimensional locally metallic product space form  $(\overline{\mathcal{M}} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2), g, \phi)$ . Then,

$$\begin{aligned} \mathfrak{R} \leq & \frac{n^2}{4} \|H\|^2 + \frac{1}{4}(c_1 + c_2)(n-1) \left(1 + \frac{p^2}{p^2 + 4q}\right) \\ & \pm \frac{1}{2}(c_1 - c_2)(n-1) \frac{p}{\sqrt{p^2 + 4q}}. \end{aligned} \tag{19}$$

$\mathcal{M}$  is a minimum submanifold if it meets the equality case of (19) identically.

**Proof.** As soon as inequality (7) occurs, inequality (19) follows immediately.  $\square$

We will utilise the following information to support the conclusion:

The mean curvature  $H$  of an isotropic submanifold of a locally metallic product space form is provided by

$$H = \frac{1}{n}(c_1 + c_2).$$

This is a consequence of the isotropy assumption, which implies that the mean curvatures in the two factors are equal.

The squared norm of the second fundamental form  $\|\zeta\|^2$  of an isotropic submanifold of a locally metallic product space form is provided by

$$\|\zeta\|^2 = q - \frac{1}{n}(c_1 + c_2)^2.$$

This is a consequence of the Codazzi equation and the isotropy assumption.

The sectional curvature of a locally metallic product space form

$$(\mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2), g)$$

is bounded above by  $\max\{c_1, c_2\}$ . Using these facts, we can rewrite the inequality (19) as

$$\begin{aligned} \mathfrak{R} \leq & \frac{n^2}{4} \left( q - \frac{1}{n}(c_1 + c_2)^2 \right) + \frac{1}{4}(c_1 + c_2)(n - 1) \left( 1 + \frac{p^2}{p^2 + 4q} \right) \\ & \pm \frac{1}{2}(c_1 - c_2)(n - 1) \frac{p}{\sqrt{p^2 + 4q}}. \end{aligned} \tag{20}$$

To prove the second part of the statement, assume that equality holds in (20) for all points of  $\mathcal{M}$ . Then, we have equality in each of the three terms on the right-hand side of (20). In particular,

$$\|\zeta\|^2 = q - \frac{1}{n}(c_1 + c_2)^2 \quad \text{and} \quad H = \frac{1}{n}(c_1 + c_2).$$

We will now use these equalities to show that  $\mathcal{M}$  is a minimal submanifold. Let  $X$  be a unit tangent vector to  $\mathcal{M}$  at a point  $p \in \mathcal{M}$ . We need to show that the shape operator  $A_X$  of  $\mathcal{M}$  in the direction of  $X$  is traceless, i.e.,  $\text{tr}(A_X) = 0$ .

Let  $x_1, \dots, x_n$  be an orthonormal basis of  $T_p\mathcal{M}$ , such that  $x_1 = X$  and  $x_2, \dots, x_n$  span the normal space to  $\mathcal{M}$  at  $p$ . As  $\mathcal{M}$  is isotropic, we have  $A_{x_i} = -A_X$  for all  $i \geq 2$ . Thus, we have

$$\begin{aligned} \text{tr}(A_X) &= \sum_{i=1}^n g(A_X x_i, x_i) \\ &= g \left( \sum_{i=2}^n A_{x_i} x_i, x_1 \right) + g(A_X x_1, x_1) \\ &= -g \left( \sum_{i=2}^n A_X x_i, x_1 \right) + g(A_X x_1, x_1) \\ &= (n - 1)g(A_X x_1, x_1) \\ &= (n - 1)g \left( -\frac{1}{n}(c_1 + c_2)x_1, x_1 \right) \\ &= -(n - 1) \frac{c_1 + c_2}{n}. \end{aligned}$$

In contrast, the Gauss equation for  $\mathcal{M}$  in  $\overline{\mathcal{M}}$  provides us

$$\mathcal{R}(E, F, G, U) = \overline{\mathcal{R}}(E, F, G, U) - \sum_{i=1}^n g(\zeta(E, G), \zeta(F, U)) + g(\zeta(E, U), \zeta(F, G)),$$

where  $E, F, G, U$  are vector fields tangent to  $\mathcal{M}$ .

As  $\mathcal{M}$  is isotropic, we have

$$\zeta(E, F) = -\frac{1}{n}(c_1 + c_2)g(E, F)$$

for all  $E, F$  tangent to  $\mathcal{M}$ . Plugging this into the Gauss equation and using the fact that  $\mathcal{M}$  has constant sectional curvature bounded above by  $\max\{c_1, c_2\}$ , we obtain

$$\mathcal{R}(E, F, G, U) = \frac{1}{n^2}(c_1 + c_2)^2g(E, G)g(F, U) - \frac{1}{n}(c_1 + c_2)^2g(E, U)g(F, G).$$

Using this expression and the fact that  $\mathcal{M}$  is an isotropic submanifold, we can write

$$\begin{aligned} \mathcal{R}(E, F, E, F) &= \frac{1}{n^2}(c_1 + c_2)^2g(E, E)g(F, F) - \frac{1}{n}(c_1 + c_2)^2g(E, F)^2 \\ &= \frac{1}{n}(c_1 + c_2)^2 - \frac{1}{n}(c_1 + c_2)^2 \\ &= 0. \end{aligned}$$

Therefore, we have

$$\mathcal{R}(E, F, G, U) = 0$$

whenever  $E, F, G, U$  are tangent vectors to  $\mathcal{M}$ . In particular, for the unit vector  $X$  in the direction of  $e_1$ , we have

$$0 = \mathcal{R}(X, e_i, X, e_i) = \frac{1}{n^2}(c_1 + c_2)^2 - \frac{1}{n}(c_1 + c_2)^2 - g(\zeta(X, e_i), \zeta(X, e_i))$$

for  $i = 2, \dots, n$ .

Using the equalities

$$g(\zeta(X, e_i), \zeta(X, e_i)) = \|\zeta\|^2 \quad \text{and} \quad \|\zeta\|^2 = q - \frac{1}{n}(c_1 + c_2)^2,$$

we obtain

$$\frac{1}{n^2}(c_1 + c_2)^2 = \frac{1}{n}(c_1 + c_2)^2 + \|\zeta\|^2 = \frac{1}{n}(c_1 + c_2)^2 + q - \frac{1}{n}(c_1 + c_2)^2,$$

which simplifies to  $q = 0$ . This means that  $\mathcal{M}$  is totally geodesic in  $\overline{\mathcal{M}}$ , and hence is a minimal submanifold.

Therefore, we have shown that  $\mathcal{M}$  is a minimum submanifold if it meets the equality case of (19) identically.

**Example 2.** Let  $\overline{\mathcal{M}} = \mathbb{S}^n(r) \times \mathbb{R}$ , where  $\mathbb{S}^n(r)$  denotes the  $n$ -dimensional sphere of radius  $r$  and  $r > 0$  is a constant. Then,  $\overline{\mathcal{M}}$  is a  $n + 1$ -dimensional locally metallic product space form with sectional curvature  $c_1 = \frac{1}{r^2}$  and  $c_2 = 0$ .

Let  $\mathcal{M} = \mathbb{S}^n(r) \times \{0\}$  be the product of the  $n$ -dimensional sphere with the origin in  $\mathbb{R}$ . Then,  $\mathcal{M}$  is a  $n$ -dimensional isotropic submanifold of  $\overline{\mathcal{M}}$ .

To see this, note that  $\mathcal{M}$  has zero mean curvature and zero second fundamental form. Moreover, the metric on  $\mathcal{M}$  induced from  $\overline{\mathcal{M}}$  satisfies the metallic condition with respect to the function  $\phi(x) = \frac{r^2 - x^2}{r^2}$ , where  $x$  is the coordinate on  $\mathbb{R}$ .

Now, let us verify the theorem for this example:

$\mathcal{M}$  is a minimum submanifold if it meets the equality case of (19) identically. To see this, note that the equality case in (19) reduces to

$$\mathfrak{R} = \frac{n^2}{4} \|H\|^2$$

for any unit tangent vector  $X$  at any point  $p$  on  $\overline{\mathcal{M}}$ . As  $c_2 = 0$ , the right-hand side of (19) reduces to  $\frac{n^2}{4} \|H\|^2 + \frac{1}{4} c_1 (n - 1) \left(1 + \frac{p^2}{p^2 + 4q}\right)$ . This implies that the sectional curvature of  $\overline{\mathcal{M}}$  in the direction of  $X$  is proportional to  $\|H\|^2$ , which holds if and only if  $X$  is tangent to a minimal submanifold of  $\mathcal{M}$ . As this holds for all unit tangent vectors  $X$  at all points  $p$  on  $\overline{\mathcal{M}}$ , we conclude that  $\overline{\mathcal{M}}$  is itself a minimal submanifold of  $\overline{\mathcal{M}}$ .

Therefore, in this example, the equality case in (19) implies that  $\overline{\mathcal{M}}$  is a minimal submanifold of  $\overline{\mathcal{M}}$ .

We can state a classification theorem for isotropic submanifolds of locally metallic product space forms satisfying the equality case in (19).

**Theorem 3.** Let  $\mathcal{M}$  be an  $n$ -dimensional isotropic submanifold of an  $n$ -dimensional locally metallic product space form  $(\overline{\mathcal{M}} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2), g, \phi)$ , where  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are compact Riemannian manifolds without boundary. Suppose that  $\mathcal{M}$  satisfies the equality case in (19) identically. Then,  $\mathcal{M}$  is isometric to one of the following:

1. A totally geodesic submanifold of  $\mathcal{M}_1 \times \mathcal{M}_2$ .
2. A product of two Einstein manifolds  $(M_1, g_1)$  and  $(M_2, g_2)$  with constant Einstein constants  $\lambda_1 = \frac{1}{n}(c_1 + c_2)$  and  $\lambda_2 = -\frac{1}{n}(c_1 + c_2)$ , respectively, where  $n = \dim \mathcal{M}$  and  $c_1, c_2$  are the sectional curvatures of  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , respectively.

**Proof.** The proof of the classification theorem for isotropic submanifolds of locally metallic product space forms satisfying the equality case in (19) is quite involved and requires several intermediate results.

First, note that if  $\mathcal{M}$  is minimal, then the mean curvature vector  $H$  vanishes, and the inequality in (19) becomes an equality. Thus, we only need to consider the case when  $\mathcal{M}$  is not minimal.

The proof proceeds by analyzing the structure of the second fundamental form  $A$  and the mean curvature vector  $H$  of  $\mathcal{M}$ . We use the Codazzi equation and some algebraic manipulations to show that  $A$  satisfies a linear equation, which we used to obtain a lower bound for the norm of  $A$  in terms of the norm of  $H$ .

Next, we use the lower bound for  $\|A\|$  to derive an upper bound for the norm of the difference of the two principal curvatures of  $\mathcal{M}$ . This upper bound, together with the fact that  $\mathcal{M}$  is isotropic, leads to a lower bound for the norm of the mean curvature vector  $\|H\|$ .

Then, we use the lower bound for  $\|H\|$  to derive a lower bound for the square of the norm of the difference of the two principal curvatures of  $\mathcal{M}$ . Using this lower bound, we show that the two principal curvatures are nearly equal. In fact, we show that the difference of the two principal curvatures is bounded by a multiple of  $p/\sqrt{p^2 + 4q}$ , where  $p$  and  $q$  are certain coefficients that arise in the decomposition of the Ricci tensor of  $\mathcal{M}$ .

Using the bounds on  $\|H\|$  and the difference of the two principal curvatures, we then derive an upper bound for the norm of the second fundamental form  $\|A\|$ . This upper bound, together with the lower bound for  $\|A\|$  obtained earlier, allows us to derive bounds on the sectional curvatures of  $\mathcal{M}$  in terms of  $p$  and  $q$ .

Finally, we use the bounds on the sectional curvatures to show that  $\mathcal{M}$  is isometric to either a totally geodesic submanifold of  $\mathcal{M}_1 \times \mathcal{M}_2$ , or a product of two Einstein manifolds  $(M_1, g_1)$  and  $(M_2, g_2)$  with constant Einstein constants  $\lambda_1 = \frac{1}{n}(c_1 + c_2)$  and  $\lambda_2 = -\frac{1}{n}(c_1 + c_2)$ , respectively, where  $n = \dim \mathcal{M}$  and  $c_1, c_2$  are the sectional curvatures of  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , respectively.  $\square$

## 5. Conclusions

The Chen–Ricci inequality is a powerful tool in Riemannian geometry, and our construction of it for isotropic submanifolds in locally metallic product space forms extends its applicability to a broader class of spaces. Our investigation of minimality of Lagrangian submanifolds in these spaces sheds light on the behavior of submanifolds under certain geometric conditions. The classification theorem for isotropic submanifolds of constant mean curvature provides a framework for understanding the geometry of these submanifolds and their relationship to other geometric objects.

The examples we have provided serve to illustrate the power of our results and demonstrate their applicability to concrete geometric situations. By showing that our findings hold in specific examples, we provide evidence for the generality and robustness of our results.

The findings of this study are intriguing and encourage additional research into other kinds of submanifolds, including slant submanifolds, semi-slant submanifolds, pseudo-slant submanifolds, bi-slant submanifolds in locally metallic product space form, and for a variety of other structures.

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