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The Existence and Averaging Principle for Caputo Fractional Stochastic Delay Differential Systems with Poisson Jumps

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Abstract: In this paper, we obtain the existence and uniqueness theorem for solutions of Caputo-type fractional stochastic delay differential systems(FSDDSs) with Poisson jumps by utilizing the delayed perturbation of the Mittag–Leffler function. Moreover, by using the Burkholder–Davis–Gundy inequality, Doob's martingale inequality, and Hölder inequality, we prove that the solution of the averaged FSDDSs converges to that of the standard FSDDSs in the sense of L^p. Some known results in the literature are extended.

Keywords: stochastic fractional delay differential systems; delayed Mittag–Leffler-type matrix function; existence and uniqueness; averaging principle; L^p convergence

MSC: 34A08; 34F05; 60H10

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1. Introduction

Fractional stochastic delay differential systems (FSDDs) are mathematical models that involve fractional derivatives, stochastic noise, and time delays. The fractional derivatives represent the memory effects and long-range dependence in the system, while the stochastic noise and delays account for the random fluctuations and time delays, respectively. FSDDs find applications in many fields, including physics, biology, finance, and engineering. They can be used to model systems with memory and randomness, such as anomalous diffusion processes, fractional-order control systems with stochastic disturbances, and biological systems with fractional-order kinetics and stochastic effects. They provide a powerful framework for understanding and predicting the behavior of complex systems with memory, randomness, and time delays. See, for example, [1–6] and the references cited therein.

The averaging principle is a mathematical tool used to simplify the analysis of dynamical systems with fast and slow time scales. It provides an approximate description of the system's behavior. In 1968, Khasminskii [7] first used the average principle to prove that the solution of the average equation converges to the solution of the corresponding equation. In [8], the authors presented an averaging method for stochastic differential equations with non-Gaussian Lévy noise. Due to the importance of fractional calculus in theory and application, many works have emerged that apply the averaging principle to fractional stochastic differential equations (FSDEs). In [9], Xu et al. presented an averaging principle for Caputo FSDEs driven by Brown motion. In [10], Luo et al. established an averaging principle for the solution of a class of FSDEs with time delays. In the sense of a mean square, Ahmed and Zhu [11] studied the averaging principle for the Hilfer FSDEs with Poisson jumps. In [12], Ahmed investigated the periodic averaging method for impulsive and conformable FSDEs with Poisson jumps. In [13], Wang and Lin consider the following FSDEs

$$\begin{cases} {}^{C}D_0^{\alpha}[x(t) - h(t, x(t))] = f(t, x(t)) + g(t, x(t))\frac{dB_t}{dt}, & t \in J = [0, T], \\ x(0) = x_0, \end{cases}$$

the main results obtained extend some of the works on the average principle of FSDES [9,10,14] from L^2 convergence to L^p convergence ($p \geq 2$). In [15], Yang, et al. studied the averaging principle for a class of ψ-Caputo FSDDEs with Poisson jumps.

Recently, Li and Wang in [16] investigated the existence, uniqueness, and averaging principle for the following Caputo-type FSDDEs:

$$\begin{cases} ({}^{C}D_0^{\alpha}Y)(t) = AY(t) + BY(t-h) + f(t, Y(t)) + \sigma(t, Y(t))\frac{dW(t)}{dt}, \ t \in J, \\ Y(t) = \Phi(t), \quad -h \leq t \leq 0, \ h > 0. \end{cases}$$

Motivated by [11,13,16], we will study the following Caputo FSDDSs with Poisson jumps

$$\begin{cases} ({}^{C}D_0^{\alpha}x)(t) = Ax(t) + Bx(t-\sigma) + g(t, x(t), x(t-\sigma)) + \kappa(t, x(t), x(t-\sigma))\frac{dW(t)}{dt}, \\ \qquad\qquad + \int_V f(t, x(t), x(t-\sigma), v)\tilde{N}(dt, dv), \quad t \in J, \\ x(t) = \phi(t), \quad -\sigma \leq t \leq 0, \end{cases} \tag{1}$$

where ${}^{C}D_0^{\alpha}$ is the left Caputo fractional derivative with $\frac{1}{2} < \alpha < 1$, $J = [0, T]$, $A, B \in \mathbb{R}^{n\times n}$ are two constant matrices, the state vector $x \in \mathbb{R}^n$ is a stochastic process, $g : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\kappa : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $f : J \times \mathbb{R}^n \times \mathbb{R}^n \times V \to \mathbb{R}^n$ are measurable continuous functions, $W(t)$ is an m-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$. Let $(V, \Phi, \lambda(dv))$ be a σ-finite measurable space. Define $\tilde{N}(t, dv) := N(t, dv) - t\lambda(dv)$, where $N(t, dv)$ is the counting measure of the stationary Poisson point process p_t.

In this paper, we first prove the existence and uniqueness of solutions of Caputo-type FSDDSs (1) using the delayed perturbation of the Mittag–Leffler function and Banach fixed-point theorem; secondly, we prove the averaging principle for Caputo FSDDSs (1) in the sense of L_p (pth moment) with inequality techniques. The main contributions and advantages of this paper are as follows:

(1) The solution of the averaged FSDDSs converges to that of the standard FSDDSs in the sense of L_p, which is a generalization of the existing result ($p = 2$) of the averaging principle for FSDDSs;
(2) Stochastic inequality, fractional calculus, and Hölder inequality are utilized to establish our results very effectively.
(3) Our work in this article is innovative. Our result extends the main results of [16].

The remainder of this paper is arranged as follows. In Section 2, we give some definitions and preliminaries. In Section 3, we prove the existence and uniqueness of solutions for Caputo-type FSDDSs (1) with Poisson jumps. In Section 4, we prove that the solution of the FSDDSs (1) converges to that of the standard one in the L_p sense. In Section 5, two examples are presented to illustrate our theoretical results. Finally, the paper is concluded in Section 6.

2. Preliminaries

Let $\mathrm{Y} = \mathbb{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ denote the space of all $\mathcal{F}(t)$-measurable, p-square integrable functions $x : \Omega \to \mathbb{R}^n$ with $\|x(t)\|_{ps} := \left(\sum_{i=1}^{n} \mathbb{E}(|x_i(t)|^p)\right)^{1/p}$, and $\|x\| = \sum_{i=1}^{n} |x_i|$ and $\|A\| = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$ be the vector norm and matrix norm, respectively. A process $x : [-\sigma, T] \to \mathbb{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ is said to be $\mathcal{F}(t)$-adapted if $x(t) \in \mathrm{Y}$.

Definition 1 ([17])**.** *Let $\alpha > 0$ and f be a real function defined on $[a, b]$. The left Riemann–Liouville fractional integral operator of order α is defined by*

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad t > a. \tag{2}$$

Definition 2 ([17])**.** *Let $n-1 < \alpha < n$ and $f \in C^n([a,b])$. The left Caputo fractional derivative of order α is defined by*

$$ {}_a^C D_t^\alpha f(t) = ({}_aI_t^{n-\alpha} f^{(n)})(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds, \quad t > a, \tag{3}$$

where $n = [\alpha] + 1$.

Definition 3 ([18])**.** *The coefficient matrices $Q_k(s)$, $k = 0, 1, 2, ...,$ satisfy the following multivariate determining matrix equation*

$$Q_0(s) = Q_k(-\tau) = \Theta, \quad Q_1(0) = I, \quad k = 0, 1, 2, \cdots, \quad s = 0, \tau, 2\tau, \cdots,$$

$$Q_{k+1}(s) = AQ_k(s) + BQ_k(s-\tau), \quad k = 0, 1, 2, \cdots, \quad s = 0, \tau, 2\tau, \cdots,$$

where I is an identity matrix and Θ is a zero matrix.

Definition 4 ([18])**.** *Delayed perturbation of two parameter Mittag–Leffler-type matrix function $X^{A,B}_{\sigma,\alpha,\beta}$ generated by A, and$B$ is defined by*

$$X^{A,B}_{\sigma,\alpha,\beta}(t) := \begin{cases} \Theta, \quad t \in [-\sigma, 0), \\ I, \quad t = 0, \\ \sum\limits_{i=0}^{\infty} Q_{i+1}(0) \frac{t^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)} + \sum\limits_{i=1}^{\infty} Q_{i+1}(\sigma) \frac{(t-\sigma)^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)} \\ \quad + \cdots + \sum\limits_{i=0}^{\infty} Q_{i+1}(p\sigma) \frac{(t-p\sigma)^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)}, \quad p\sigma < t \le (p+1)\sigma. \end{cases} \tag{4}$$

From [18], we can easily obtain the following definition.

Definition 5. *A $\mathbb{R}^n$-value stochastic process $\{x(t) : -\sigma \le t \le T\}$ is called a solution of (1) if $x(t)$ satisfies the following form:*

$$x(t) = \begin{cases} X^{A,B}_{\sigma,\alpha,1}(t+\sigma)\phi(-\sigma) + \int_{-\sigma}^0 X^{A,B}_{\sigma,\alpha,\alpha}(t-s)[{}^C D^\alpha_{-\sigma^+}\phi)(s) - A\phi(s)]ds \\ \quad + \int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s) g(s, x(s), x(s-\sigma))ds \\ \quad + \int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s) \kappa(s, x(s), x(s-\sigma))dW(s) \\ \quad + \int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s) \int_V f(s, x(s), x(s-\sigma), v)\bar{N}(ds, dv), \quad t \in J, \\ \phi(t), \quad t \in [-\sigma, 0], \end{cases} \tag{5}$$

where $x(t)$ is $\mathcal{F}(t)$-adapted and $\mathbb{E}(\int_{-\sigma}^T \|x(t)\|^p dt) < \infty$.

Lemma 1 ([19])**.** *For each $t \ge 0$, $0 < \alpha < 1$, $0 < \beta \le 1$ and $\alpha + \beta \ge 1$, one has*

$$\|X^{A,B}_{\sigma,\alpha,\beta}(t)\| \le t^{\beta-1} E_{\alpha,\beta}((\|A\| + \|B\|)t^\alpha), \tag{6}$$

where $E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(k\alpha+\beta)}$, $z \in \mathbb{R}$ is the Mittag–Leffler function.

Lemma 2. *For any $p \geq 2$, $\alpha \in \left(1-\frac{1}{p}, 1\right)$ and $\mu > 0$, we have*

$$\int_0^t (t-s)^{p\alpha-p} E_{p\alpha-p+1,1}(\mu s^{p\alpha-p+1}) ds \leq \frac{\Gamma(p\alpha-p+1)}{\mu} E_{p\alpha-p+1,1}(\mu t^{p\alpha-p+1}), \tag{7}$$

where $\Gamma(\alpha) := \int_0^{+\infty} s^{\alpha-1} e^{-s} ds$ is the Gamma function.

Proof. Let $\mu > 0$ be arbitrary. Consider the corresponding linear Caputo fractional differential equation of the following form

$${}^C D_{0^+}^{p\alpha-p+1} x(t) = \mu x(t). \tag{8}$$

From [20], it is easy to know that the Mittag–Leffler function $E_{p\alpha-p+1,1}(\mu t^{p\alpha-p+1})$ is a solution of (8). So, the following equality holds:

$$E_{p\alpha-p+1,1}(\mu t^{p\alpha-p+1}) = 1 + \frac{\mu}{\Gamma(p\alpha-p+1)} \int_0^t (t-s)^{p\alpha-p} E_{p\alpha-p+1,1}(\mu s^{p\alpha-p+1}) ds,$$

which completes the proof. □

Lemma 3 ([21,22])**.** *Let $\phi : R_+ \times V \to R^n$ and assume that*

$$\int_0^t \int_V |\phi(s,v)|^p \lambda(dv) ds < \infty, \quad p \geq 2.$$

Then, there exists $D_p > 0$ such that

$$\begin{aligned} &\mathbb{E}\left(\sup_{0\leq t\leq u}\left|\int_0^t \int_V \phi(s,v)\tilde{N}(ds,dv)\right|^p\right) \\ &\quad \leq D_p\left\{\mathbb{E}\left(\int_0^u \int_V |\phi(s,v)|^2 \lambda(dv) ds\right)^{\frac{p}{2}} + \mathbb{E}\int_0^u \int_V |\phi(s,v)|^p \lambda(dv) ds\right\}. \end{aligned} \tag{9}$$

Lemma 4 ([23])**.** *Let u, v be two integrable functions and g be continuously defined on the domain $[a,b]$. Suppose that*

(1) u and v are non-negative, and v is non-decreasing;
(2) g is non-negative and non-decreasing.

If

$$u(t) \leq v(t) + g(t) \int_a^t (t-\tau)^{\alpha-1} u(\tau) d\tau,$$

then

$$u(t) \leq v(t) E_\alpha(g(t)\Gamma(\alpha)(t-a)^\alpha), \quad \forall t \in [a,b],$$

where $E_\alpha(\cdot)$ is the Mittag–Leffler function.

To study the problem (1), we impose the following conditions:

(H1) For each $x_1, x_2, y_1, y_2 \in R^n$ and $t \in J$, there exist two constants $C_1, C_2 > 0$ such that

$$\begin{aligned} &\|g(t,x_1,y_1) - g(t,x_2,y_2)\|^p \vee \|\kappa(t,x_1,y_1) - \kappa(t,x_2,y_2)\|^p \\ &\quad \vee \int_V \|f(t,x_1,y_1,v) - f(t,x_2,y_2,v)\|^p \lambda(dv) \leq C_1^p(\|x_1-x_2\|^p + \|y_1-y_2\|^p), \end{aligned}$$

where $\|\cdot\|$ is the norm of $\mathbb{R}^n$, $x \vee y = max\{x,y\}$.

(H2) Let $\kappa(\cdot,0,0)$ and $f(\cdot,0,0,0)$ be essentially bounded, i.e.,

$$\|\kappa(\cdot,0,0)\|_\infty := ess \sup_{t\in[0,\infty)} \|\kappa(t,0,0)\| < +\infty, \quad \|f(\cdot,0,0,0)\|_\infty := ess \sup_{t\in[0,\infty)} \|f(t,0,0,0)\| < +\infty,$$

and $g(\cdot, 0, 0)$ is $\mathbb{L}^p$ integrable, i.e.,

$$\|g\|_{\mathbb{L}^p} = \int_0^T \|g(t,0,0)\|^p dt < +\infty.$$

3. Existence and Uniqueness Result

Let $\mathbb{H}^p([0,T])$ be the space of all the processes x which are measurable, $\mathcal{F}(t)$-adapted, and satisfied that $\|x\|_{\mathbb{H}^p} := \sup_{0\le t\le T} \|x(t)\|_{ps} < \infty$. Obviously, $(\mathbb{H}^p([0,T]), \|\cdot\|_{\mathbb{H}^p})$ is a Banach space. Set $\gamma = \|A\| + \|B\|$. For each $t \in [-\sigma, T]$ and $\phi \in C([-\sigma, 0], \mathbb{R}^n)$, we define an operator $\mathcal{T} : \mathbb{H}^p([0,T]) \to \mathbb{H}^p([0,T])$ as follows :

$$\begin{aligned}
(\mathcal{T}x)(t) = {} & X^{A,B}_{\sigma,\alpha,1}(t+\sigma)\phi(-\sigma) + \int_{-\sigma}^0 X^{A,B}_{\sigma,\alpha,\alpha}(t-s)[{}^C D^{\alpha}_{-\sigma^+}\phi)(s) - A\phi(s)]ds \\
& + \int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)g(s,x(s),x(s-\sigma))ds \\
& + \int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\kappa(s,x(s),x(s-\sigma))dW(s) \\
& + \int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\int_V f(s,x(s),x(s-\sigma),v)\bar{N}(ds,dv).
\end{aligned} \tag{10}$$

Lemma 5. *Let $1 - \frac{1}{p} < \alpha < 1$. Assume that (H1) and (H2) hold. Then, the operator $\mathcal{T}$ is well defined.*

Proof. For any $x \in \mathbb{H}^p([0,T])$, by (10) and the following elementary inequality,

$$\left\|\sum_{i=1}^m a_i\right\|^p \le m^{p-1}\sum_{i=1}^m \|a_i\|^p, \quad a_i \in \mathbb{R}^n,\ i = 1,2,...,m. \tag{11}$$

we have

$$\begin{aligned}
\|(\mathcal{T}x)(t)\|^p_{ps} \le {} & 5^{p-1}\mathbb{E}(\|X^{A,B}_{\sigma,\alpha,1}(t+\sigma)\phi(-\sigma)\|^p) \\
& + 5^{p-1}\mathbb{E}\left(\left\|\int_{-\sigma}^0 X^{A,B}_{\sigma,\alpha,\alpha}(t-s)[({}^C D^{\alpha}_{-\sigma^+}\phi)(s) - A\phi(s)]ds\right\|^p\right) \\
& + 5^{p-1}\mathbb{E}\left(\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)g(s,x(s),x(s-\sigma))ds\right\|^p\right) \\
& + 5^{p-1}\mathbb{E}\left(\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\kappa(s,x(s),x(s-\sigma))dW(s)\right\|^p\right) \\
& + 5^{p-1}\mathbb{E}\left(\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\int_V f(s,x(s),x(s-\sigma),v)\bar{N}(ds,dv)\right\|^p\right) \\
:= {} & I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{12}$$

For I_1, from Lemma 1, one has

$$\begin{aligned}
I_1 &= 5^{p-1}\mathbb{E}(\|X^{A,B}_{\sigma,\alpha,1}(t+\sigma)\phi(-\sigma)\|^p) \le 5^{p-1}\mathbb{E}(\|X^{A,B}_{\sigma,\alpha,1}(t+\sigma)\|^p\|\phi(-\sigma)\|^p) \\
&\le 5^{p-1}\|\phi(-\sigma)\|^p(E_{\alpha,1}(\gamma(T+\sigma)^{\alpha}))^p.
\end{aligned} \tag{13}$$

For I_2, by Lemma 1, Hölder inequality, and $\alpha > 1 - \frac{1}{p}$, we obtain

$$\begin{aligned}
I_2 &= 5^{p-1}\mathbb{E}\left(\left\|\int_{-\sigma}^{0} X^{A,B}_{\sigma,\alpha,\alpha}(t-s)[({}^C D^{\alpha}_{-\sigma^+}\phi)(s) - A\phi(s)]ds\right\|^p\right) \\
&\leq 5^{p-1}\int_{-\sigma}^{0} \|X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\|^p ds \cdot \mathbb{E}\left(\int_{-\sigma}^{0} \|({}^C D^{\alpha}_{-\sigma^+}\phi)(s) - A\phi(s)\|^q ds\right)^{p-1} \\
&\leq 5^{p-1}\Xi\frac{(T+\sigma)^{p\alpha-p+1}}{p\alpha-p+1}(E_{\alpha,\alpha}(\gamma(T+\sigma)^{\alpha}))^p,
\end{aligned} \tag{14}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\Xi = \left(\int_{-\sigma}^{0} \|({}^C D^{\alpha}_{-\sigma^+}\phi)(s) - A\phi(s)\|^q ds\right)^{p-1} < \infty$.

For I_3, applying (H1), (H2), Hölder inequality, Lemma 1 and Jensen inequality, one has

$$\begin{aligned}
I_3 &= 5^{p-1}\mathbb{E}\left(\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)g(s,x(s),x(s-\sigma))ds\right\|^p\right) \\
&\leq 5^{p-1}\left(\int_0^t \|X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\|^q ds\right)^{\frac{p}{q}} \cdot \mathbb{E}\left(\int_0^t \|g(s,x(s),x(s-\sigma)) - g(s,0,0) + g(s,0,0)\|^p ds\right) \\
&\leq 5^{p-1}\left(\int_0^t t^{q(\alpha-1)}E_{\alpha,\alpha}(\mu(t-s)^{\alpha})^q ds\right)^{\frac{p}{q}} \\
&\quad \cdot 2^{p-1}\mathbb{E}\left(\int_0^t \|g(s,x(s),x(s-\sigma)) - g(s,0,0)\|^p ds + \int_0^t \|g(s,0,0)\|^p ds\right) \\
&\leq 10^{p-1}E_{\alpha,\alpha}(\gamma T^{\alpha})^p\left(\frac{T^{q\alpha-q+1}}{q\alpha-q+1}\right)^{\frac{p}{q}}\mathbb{E}\left(\int_0^t C_1^p(\|x(s)\|^p + \|x(s-\sigma)\|^p)ds + \int_0^t \|g(s,0,0)\|^p ds\right) \\
&\leq 10^{p-1}E_{\alpha,\alpha}(\gamma T^{\alpha})^p\left(\frac{T^{q\alpha-q+1}}{q\alpha-q+1}\right)^{\frac{p}{q}}\left(TC_1^p(2\|x\|^p_{\mathbb{H}^p} + \|\phi\|^q) + \|g\|_{\mathbb{L}^p}\right),
\end{aligned} \tag{15}$$

since

$$\begin{aligned}
\sup_{0\leq t\leq T}\mathbb{E}\|x(t-\tau)\|^q &\leq \max\left\{\sup_{-\tau\leq t\leq 0}\mathbb{E}\|\phi(t)\|^q,\ \sup_{0\leq t\leq T}\mathbb{E}\|x(t)\|^q\right\} \\
&= \max\left\{\|\phi\|^q,\ \|x\|^q_{\mathbb{H}^q}\right\} \leq \|\phi\|^q + \|x\|^q_{\mathbb{H}^q}.
\end{aligned}$$

For I_4, by using (H1), (H2), Cauchy–Schwarz inequality, Ito's isometry, Lemma 1, and Jensen inequality, we have

$$\begin{aligned}
I_4 &= 5^{p-1}\mathbb{E}\left(\left(\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\kappa(s,x(s),x(s-\sigma))dW(s)\right\|^2\right)^{\frac{p}{2}}\right) \\
&\leq 5^{p-1}\mathbb{E}\left(\int_0^t \|X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\|^2\|\kappa(s,x(s),x(s-\sigma))\|^2 ds\right)^{\frac{p}{2}} \\
&\leq 5^{p-1}\mathbb{E}\left(\left(\int_0^t 1^{\frac{p}{p-2}}ds\right)^{\frac{p-2}{p}} \cdot \left(\int_0^t \|X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\|^p\|\kappa(s,x(s),x(s-\sigma))\|^p ds\right)^{\frac{2}{p}}\right)^{\frac{p}{2}} \\
&\leq 5^{p-1}T^{\frac{p}{2}-1}\mathbb{E}\left(\int_0^t \|X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\|^p\|\kappa(s,x(s),x(s-\sigma))\|^p ds\right) \\
&\leq 5^{p-1}T^{\frac{p}{2}-1}2^{p-1}E_{\alpha,\alpha}(\gamma T^{\alpha})^p\mathbb{E}\left(\int_0^t (t-s)^{p\alpha-p}[C_1^p(\|x(s)\|^p + \|x(s-\sigma)\|^p) + \|\kappa(s,0,0)\|^p]ds\right) \\
&\leq \frac{10^{p-1}T^{p\alpha-\frac{p}{2}}}{p\alpha-p+1}E_{\alpha,\alpha}(\gamma T^{\alpha})^p(C_1^p(2\|x\|^p_{\mathbb{H}^p} + \|\phi\|^p) + \|\kappa(\cdot,0,0)\|^p_{\infty}).
\end{aligned} \tag{16}$$

For I_5, using (H1), (H2), Lemmas 1, 3 and Jensen inequality, we obtain

$$
\begin{aligned}
I_5 &= 5^{p-1}\mathbb{E}\left(\left\|\int_0^t\int_V X^{A,B}_{\sigma,\alpha,\alpha}(t-s)f(s,x(s),x(s-\sigma),v)\bar{N}(ds,dv)\right\|^p\right)\\
&\le 5^{p-1}D_p\mathbb{E}\left(\int_0^t\int_V |X^{A,B}_{\sigma,\alpha,\alpha}(t-s)|^2 f^2(s,x(s),x(s-\sigma),v)\lambda(dv)ds\right)^{\frac{p}{2}}\\
&\quad + 5^{p-1}D_p\mathbb{E}\left(\int_0^t\int_V |X^{A,B}_{\sigma,\alpha,\alpha}(t-s)|^p f^p(s,x(s),x(s-\sigma),v)\lambda(dv)ds\right)\\
&\le 5^{p-1}D_p(T^{\frac{p}{2}-1}+1)\mathbb{E}\left(\int_0^t |X^{A,B}_{\sigma,\alpha,\alpha}(t-s)|^p\int_V f^p(s,x(s),x(s-\sigma),v)\lambda(dv)ds\right)\\
&\le 5^{p-1}D_p(T^{\frac{p}{2}-1}+1)2^{p-1}E_{\alpha,\alpha}(\gamma T^\alpha)^p\\
&\qquad\cdot\mathbb{E}\left(\int_0^t (t-s)^{p\alpha-p}[C_1^p(\|x(s)\|^p+\|x(s-\sigma)\|^p)+\|f(s,0,0,0)\|^p]ds\right)\\
&\le \frac{10^{p-1}D_p(T^{\frac{p}{2}-1}+1)T^{p\alpha-p+1}}{p\alpha-p+1}E_{\alpha,\alpha}(\gamma T^\alpha)^p(C_1^p(2\|x\|^p_{\mathbb{H}^p}+\|\phi\|^p)+\|f(\cdot,0,0,0)\|^p_\infty).
\end{aligned}
\tag{17}
$$

Submitting (13)–(17) into (12) implies that $\|\mathcal{T}x\|_{\mathbb{H}^p}<\infty$. Thus, the operator $\mathcal{T}$ is well-defined. □

Theorem 1. *Let $1-\frac{1}{p}<\alpha<1$. Assume that $(H1)$ and $(H2)$ hold, then (1) has a unique solution $x\in\mathbb{H}^p([0,T])$.*

Proof. For $T>0$, we choose and fix a constant $\mu>0$ such that

$$
\mu > 2\cdot 3^{p-1}C_1^pE_{\alpha,\alpha}(\gamma T^\alpha)^p(T^{\frac{p}{q}}+(D_p+1)T^{\frac{p}{2}-1}+1)\Gamma(p\alpha-p+1). \tag{18}
$$

On the space $\mathbb{H}^p([0,T])$, we define a weighted norm $\|\cdot\|_\mu$ as below

$$
\|x\|_\mu := \sup_{t\in[0,T]}\left(\frac{\mathbb{E}(\|x(t)\|^p)}{E_{p\alpha-p+1,1}(\mu t^{p\alpha-p+1})}\right)^{\frac{1}{p}},\quad \forall x\in\mathbb{H}^p([0,T]).
$$

Similarly to Theorem 1 in [18], it is easy to know that the norms $\|\cdot\|_{\mathbb{H}^p}$ and $\|\cdot\|_\mu$ are equivalent. Hence, $(\mathbb{H}^p([0,T]),\|\cdot\|_\mu)$ is a Banach space. We can easily prove that $\mathcal{T}:\mathbb{H}^p([0,T])\to\mathbb{H}^p([0,T])$ defined in (10) is uniformly bounded operator by Lemma 5. Next, we only check that $\mathcal{T}$ is a contraction operator.

Firstly, by using Hölder inequality (H1) and Lemma 1, we obtain

$$
\begin{aligned}
&\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)(g(s,x(s),x(s-\sigma))-g(s,y(s),y(s-\sigma)))ds\right\|^p\\
&\le \left(\int_0^t 1^q ds\right)^{\frac{p}{q}}\cdot\int_0^t \|X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\|^p\|g(s,x(s),x(s-\sigma))-g(s,y(s),y(s-\sigma))\|^p ds\\
&\le t^{\frac{p}{q}}\int_0^t (t-s)^{p(\alpha-1)}E_{\alpha,\alpha}(\gamma(t-s)^\alpha)^p\|g(s,x(s),x(s-\sigma))-g(s,y(s),y(s-\sigma))\|^p ds\\
&\le T^{\frac{p}{q}}E_{\alpha,\alpha}(\gamma T^\alpha)^pC_1^p\int_0^t (t-s)^{p(\alpha-1)}(\|x(s)-y(s)\|^p+\|x(s-\sigma)-y(s-\sigma)\|^p)ds.
\end{aligned}
\tag{19}
$$

Secondly, similarly to the Proof of (16), one has

$$\begin{aligned}
&\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)(\kappa(s,x(s),x(s-\sigma))-\kappa(s,y(s),y(s-\sigma)))dW(s)\right\|^p \\
&\leq T^{\frac{p}{2}-1}\int_0^t \|X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\|^p\|\kappa(s,x(s),x(s-\sigma))-\kappa(s,y(s),y(s-\sigma))\|^p ds \\
&\leq T^{\frac{p}{2}-1}E_{\alpha,\alpha}(\gamma T^\alpha)^p\int_0^t (t-s)^{p\alpha-p}[C_1^p(\|x(s)-y(s)\|^p+\|x(s-\sigma)-y(s-\sigma)\|^p)ds \\
&\leq T^{\frac{p}{2}-1}C_1^pE_{\alpha,\alpha}(\gamma T^\alpha)^p\int_0^t (t-s)^{p\alpha-p}(\|x(s)-y(s)\|^p+\|x(s-\sigma)-y(s-\sigma)\|^p)ds.
\end{aligned} \tag{20}$$

Thirdly, similarly to the Proof of (17), we obtain

$$\begin{aligned}
&\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\int_V(f(s,x(s),x(s-\sigma),v)-f(s,y(s),y(s-\sigma),v))\bar{N}(ds,dv)\right\|^p \\
&\leq D_p\left(\int_0^t \|X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\|^2\int_V\|f(s,x(s),x(s-\sigma),v)-f(s,y(s),y(s-\sigma),v)\|^2\lambda(dv)ds\right)^{\frac{p}{2}} \\
&+D_p\int_0^t |X^{A,B}_{\sigma,\alpha,\alpha}(t-s)|^p\int_V|f(s,x(s),x(s-\sigma),v)-f(s,y(s),y(s-\sigma),v)|^p\lambda(dv)ds \\
&\leq D_p(T^{\frac{p}{2}-1}+1)\int_0^t \|X^{A,B}_{\sigma,\alpha,\alpha}(T-s)\|^p\int_V\|f(s,x(s),x(s-\sigma),v)-f(s,y(s),y(s-\sigma),v)\|^p\lambda(dv)ds \\
&\leq D_pC_1^p(T^{\frac{p}{2}-1}+1)E_{\alpha,\alpha}(\gamma T^\alpha)^p\int_0^t (t-s)^{p\alpha-p}(\|x(s)-y(s)\|^p+\|x(s-\sigma)-y(s-\sigma\|^p)ds.
\end{aligned} \tag{21}$$

For each $x,y\in\mathbb{H}^p([0,T])$, from (10), (11), and (19)-(21), we have

$$\begin{aligned}
&\mathbb{E}(\|\mathcal{T}x(t)-\mathcal{T}y(t)\|^p) \\
&\leq 3^{p-1}\mathbb{E}\left(\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)(g(s,x(s),x(s-\sigma))-g(s,y(s),y(s-\sigma)))ds\right\|^p\right) \\
&+3^{p-1}\mathbb{E}\left(\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)(\kappa(s,x(s),x(s-\sigma))-\kappa(s,y(s),y(s-\sigma)))dW(s)\right\|^p\right) \\
&+3^{p-1}\mathbb{E}\left(\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\int_V(f(s,x(s),x(s-\sigma),v)-f(s,y(s),y(s-\sigma),v))\bar{N}(ds,dv)\right\|^p\right) \\
&\leq 3^{p-1}T^{\frac{p}{q}}C_1^pE_{\alpha,\alpha}(\gamma T^\alpha)^p\int_0^t (t-s)^{p(\alpha-1)}\mathbb{E}(\|x(s)-y(s)\|^p+\|x(s-\sigma)-y(s-\sigma\|^p)ds \\
&+3^{p-1}T^{\frac{p}{2}-1}C_1^pE_{\alpha,\alpha}(\gamma T^\alpha)^p\int_0^t (t-s)^{p\alpha-p}\mathbb{E}(\|x(s)-y(s)\|^p+\|x(s-\sigma)-y(s-\sigma\|^p)ds \\
&+3^{p-1}C_1^pD_p(T^{\frac{p}{2}-1}+1)E_{\alpha,\alpha}(\gamma T^\alpha)^p\int_0^t (t-s)^{p\alpha-p}\mathbb{E}(\|x(s)-y(s)\|^p+\|x(s-\sigma)-y(s-\sigma\|^p)ds \\
&=\omega\int_0^t (t-s)^{p\alpha-p}\mathbb{E}(\|x(s)-y(s)\|^p+\|x(s-\sigma)-y(s-\sigma\|^p)ds,
\end{aligned} \tag{22}$$

where

$$\omega:=3^{p-1}C_1^pE_{\alpha,\alpha}(\gamma T^\alpha)^p(T^{\frac{p}{q}}+(D_p+1)T^{\frac{p}{2}-1}+D_p).$$

For $t>\sigma$, one has

$$\begin{aligned}
\int_0^t (t-s)^{p(\alpha-1)}\|x(s-\sigma)-y(s-\sigma)\|^p ds &= \int_0^\sigma+\int_\sigma^t (t-s)^{p(\alpha-1)}\|x(s-\sigma)-y(s-\sigma)\|^p ds \\
&=\int_\sigma^t (t-s)^{p(\alpha-1)}\|x(s-\sigma)-y(s-\sigma)\|^p ds \\
&=\int_0^{t-\sigma}(t-\sigma-u)^{p(\alpha-1)}\|x(u)-y(u)\|^p du.
\end{aligned} \tag{23}$$

From Lemma 2, combining (22) and (23) for each $t \in [0, T]$, we obtain

$$\begin{aligned}
&\frac{\mathbb{E}(\|\mathcal{T}x(t) - \mathcal{T}y(t)\|^p)}{E_{p\alpha-p+1,1}(\mu t^{p\alpha-p+1})} \\
&\quad \le \frac{\omega}{E_{p\alpha-p+1,1}(\mu t^{p\alpha-p+1})} \int_0^t (t-s)^{p\alpha-p} E_{p\alpha-p+1,1}(\mu s^{p\alpha-p+1}) ds \|x-y\|_\mu^p \\
&\qquad + \frac{\omega}{E_{p\alpha-p+1,1}(\mu (t-\sigma)^{p\alpha-p+1})} \int_0^{t-\sigma} (t-\sigma-u)^{p\alpha-p} E_{p\alpha-p+1,1}(\mu u^{p\alpha-p+1}) du \|x-y\|_\mu^p \\
&\quad \le \frac{2\omega\Gamma(p\alpha-p+1)}{\mu} \|x-y\|_\mu^p,
\end{aligned}$$

which implies that

$$\|\mathcal{T}x - \mathcal{T}y\|_\mu \le \rho \|x-y\|_\mu,$$

where $\rho = \left(\frac{2\omega\Gamma(p\alpha-p+1)}{\mu}\right)^{\frac{1}{p}}$.

Based on (18), one can obtain $\rho < 1$ and the operator $\mathcal{T}$ is a contractive. Thus, (1) has a unique solution using the Banach fixed-point theorem. This completes the proof of Theorem 1. □

4. An Averaging Principle

To show the averaging principle for FSDDEs (1), let us consider the following standard form of (1)

$$\begin{aligned}
x_\epsilon(t) = {} & X_{\sigma,\alpha,1}^{A,B}(t+\sigma)\phi(-\sigma) + \int_{-\sigma}^0 X_{\sigma,\alpha,\alpha}^{A,B}(t-s)[({}^C D_{-\sigma^+}^\alpha \phi)(s) - A\phi(s)] ds \\
& + \epsilon \int_0^t X_{\sigma,\alpha,\alpha}^{A,B}(t-s) g(s, x_\epsilon(s), x_\epsilon(s-\sigma)) ds \\
& + \sqrt{\epsilon} \int_0^t X_{\sigma,\alpha,\alpha}^{A,B}(t-s) \kappa(s, x_\epsilon(s), x_\epsilon(s-\sigma)) dW(s) \\
& + \sqrt{\epsilon} \int_0^t X_{\sigma,\alpha,\alpha}^{A,B}(t-s) \int_V f(s, x_\epsilon(s), x_\epsilon(s-\sigma), v) \bar{N}(ds, dv),
\end{aligned} \tag{24}$$

where $\epsilon \in (0, \epsilon_0]$ is a positive small parameter with ϵ_0 being a fixed number.

Consider the averaged form which corresponds to the standard form (24) as follows :

$$\begin{aligned}
y_\epsilon(t) = {} & X_{\sigma,\alpha,1}^{A,B}(t+\sigma)\phi(-\sigma) + \int_{-\sigma}^0 X_{\sigma,\alpha,\alpha}^{A,B}(t-s)[({}^C D_{-\sigma^+}^\alpha \phi)(s) - A\phi(s)] ds \\
& + \epsilon \int_0^t X_{\sigma,\alpha,\alpha}^{A,B}(t-s) \hat{g}(s, y_\epsilon(s), y_\epsilon(s-\sigma)) ds \\
& + \sqrt{\epsilon} \int_0^t X_{\sigma,\alpha,\alpha}^{A,B}(t-s) \hat{\kappa}(s, y_\epsilon(s), y_\epsilon(s-\sigma)) dW(s) \\
& + \sqrt{\epsilon} \int_0^t X_{\sigma,\alpha,\alpha}^{A,B}(t-s) \int_V \hat{f}(s, y_\epsilon(s), y_\epsilon(s-\sigma), v) \bar{N}(ds, dv),
\end{aligned} \tag{25}$$

where $\hat{g} : R^n \times R^n \to R^n$, $\hat{\kappa} : R^n \times R^n \to R^{n\times m}$, and $\hat{f} : R^n \times R^n \times V \to R^n$ satisfying the following averaging condition:

(H3) For each $t \in J$, $x, y \in \mathbb{R}^n$, and $p \ge 2$, there exists a positive bounded function $\varphi_i(\cdot)$, $i = 1, 2, 3$ such that

$$\frac{1}{t} \int_0^t \|g(s,x,y) - \hat{g}(x,y)\|^p ds \le \varphi_1(t)(1 + \|x\|^p + \|y\|^p),$$

$$\frac{1}{t}\int_0^t \|(t-s)^{\alpha-1}(\kappa(s,x,y)-\hat{\kappa}(x,y))\|^p ds \le \varphi_2(t)(1+\|x\|^p+\|y\|^p),$$

$$\frac{1}{t}\int_0^t \left(\int_V \|(t-s)^{\alpha-1}(f(s,x,y,v)-\hat{f}(x,y,v))\|^p \lambda(dv)\right) ds \le \varphi_3(t)(1+\|x\|^p+\|y\|^p),$$

where $\lim_{t\to\infty}\varphi_i(t)=0$, $i=1,2,3$.

Theorem 2. *Assume that (H1)–(H3) are satisfied. Then, for a given arbitrary small number $\delta>0$, $p\ge 2$ with $1-\frac{1}{p}<\alpha<1$, there exist $L>0$, $\epsilon_1\in(0,\epsilon_0]$ and $\beta\in(0,1)$ such that*

$$\mathbb{E}\left(\sup_{t\in[-\tau,L\epsilon^{-\beta}]}|x_\epsilon(t)-y_\epsilon(t)|^p\right)\le\delta, \tag{26}$$

for all $\epsilon\in(0,\epsilon_1]$.

Proof. If $p=2$, it is easy to prove that (26) holds using method similarly to that in [20]. In the following, we will only consider the case of $p>2$. From Equations (25), (26), and inequality (11), we obtain

$$\begin{aligned}\|x_\epsilon(t)-y_\epsilon(t)\|^p \le{}& 3^{p-1}\epsilon^p\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)[g(s,x_\epsilon(s),x_\epsilon(s-\sigma))-\hat{g}(y_\epsilon(s),y_\epsilon(s-\sigma))]ds\right\|^p\\ &+3^{p-1}\epsilon^{\frac{p}{2}}\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)[\kappa(s,x_\epsilon(s),x_\epsilon(s-\sigma))-\hat{\kappa}(y_\epsilon(s),y_\epsilon(s-\sigma))]dW(s)\right\|^p\\ &+3^{p-1}\epsilon^{\frac{p}{2}}\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\int_V[f(s,x_\epsilon(s),x_\epsilon(s-\sigma),v))-\hat{f}(x_\epsilon(s),x_\epsilon(s-\sigma),v))]\bar{N}(ds,dv)\right\|^p.\end{aligned}\tag{27}$$

For any $t\in[0,u]\subset[0,T]$, taking the expectation on both sides of Equation (27), we have

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le t\le u}\|x_\epsilon(t)-y_\epsilon(t)\|^p\right)\\ &\le 3^{p-1}\epsilon^p\mathbb{E}\left(\sup_{0\le t\le u}\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)[g(s,x_\epsilon(s),x_\epsilon(s-\sigma))-\hat{g}(y_\epsilon(s),y_\epsilon(s-\sigma))]ds\right\|^p\right)\\ &\quad+3^{p-1}\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sup_{0\le t\le u}\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)[\kappa(s,x_\epsilon(s),x_\epsilon(s-\sigma))-\hat{\kappa}(y_\epsilon(s),y_\epsilon(s-\sigma))]dW(s)\right\|^p\right)\\ &\quad+3^{p-1}\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sup_{0\le t\le u}\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\int_V[f(s,x_\epsilon(s),x_\epsilon(s-\sigma),v))-\hat{f}(x_\epsilon(s),x_\epsilon(s-\sigma),v))]\bar{N}(ds,dv)\right\|^p\right)\\ &=I_1+I_2+I_3.\end{aligned}\tag{28}$$

Applying Jensen's inequality, we obtain

$$\begin{aligned}I_1\le{}& 6^{p-1}\epsilon^p\cdot\mathbb{E}\left(\sup_{0\le t\le u}\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)[g(s,x_\epsilon(s),x_\epsilon(s-\sigma))-g(s,y_\epsilon(s),y_\epsilon(s-\sigma))]ds\right\|^p\right)\\ &+6^{p-1}\epsilon^p\cdot\mathbb{E}\left(\sup_{0\le t\le u}\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)[g(s,y_\epsilon(s),y_\epsilon(s-\sigma))-\hat{g}(y_\epsilon(s),y_\epsilon(s-\sigma))]ds\right\|^p\right)\\ ={}&I_{11}+I_{12}.\end{aligned}\tag{29}$$

Thanks to Hölder inequality and (H2), we obtain

$$
\begin{aligned}
I_{11} &\leq 6^{p-1}\epsilon^p \left(\int_0^u 1^q ds\right)^{\frac{p}{q}} \\
&\quad \cdot \mathbb{E}\left(\sup_{0\leq t\leq u}\int_0^t \|X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\|^p \|g(s,x_\epsilon(s),x_\epsilon(s-\sigma))-g(s,y_\epsilon(s),y_\epsilon(s-\sigma))\|^p ds\right) \\
&\leq 6^{p-1}\epsilon^p u^{p-1} C_1^p E_{\alpha,\alpha}(\gamma u^\alpha)^p \\
&\quad \cdot \mathbb{E}\left(\sup_{0\leq t\leq u}\int_0^t (t-s)^{p(\alpha-1)}[\|x_\epsilon(s)-y_\epsilon(s)\|^p + \|x_\epsilon(s-\sigma))-y_\epsilon(s-\sigma))\|^p]ds\right) \\
&\leq 6^{p-1}\epsilon^p u^{p-1} C_1^p E_{\alpha,\alpha}(\gamma u^\alpha)^p \\
&\quad \cdot \int_0^u (u-s)^{p(\alpha-1)}\left[\mathbb{E}\left(\sup_{0\leq\theta\leq s}\|x_\epsilon(\theta)-y_\epsilon(\theta)\|^p\right)+\mathbb{E}\left(\sup_{0\leq\theta\leq s}\|x_\epsilon(\theta-\sigma)-y_\epsilon(\theta-\sigma)\|^p\right)\right]ds \\
&\leq 2\cdot 6^{p-1}\epsilon^p u^{p-1} C_1^p E_{\alpha,\alpha}(\gamma u^\alpha)^p \int_0^u (t-s)^{p(\alpha-1)}\mathbb{E}\left(\sup_{0\leq\theta\leq s}\|x_\epsilon(\theta)-y_\epsilon(\theta)\|^p\right)ds,
\end{aligned} \tag{30}
$$

since

$$
\sup_{0\leq\theta\leq s}\|x_\epsilon(\theta-\tau)-y_\epsilon(\theta-\tau)\|^p \leq \sup_{0\leq\theta\leq s}\|x_\epsilon(\theta)-y_\epsilon(\theta)\|^p.
$$

Applying Hölder inequality, we obtain

$$
\begin{aligned}
I_{12} &\leq 6^{p-1}\epsilon^p \cdot \mathbb{E}\left(\sup_{0\leq t\leq u}\left(\left(\int_0^t \|X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\|^q ds\right)^{\frac{p}{q}}\right.\right. \\
&\qquad \left.\left.\cdot\int_0^t \|g(s,y_\epsilon(s),y_\epsilon(s-\sigma))-\hat{g}(y_\epsilon(s),y_\epsilon(s-\sigma))\|^p ds\right)\right) \\
&\leq 6^{p-1}\epsilon^p E_{\alpha,\alpha}(\gamma u^\alpha)^p \left(\frac{u^{q\alpha-q+1}}{q\alpha-q+1}\right)^{\frac{p}{q}} \\
&\qquad \cdot u\|\varphi_1\|_\infty\left[1+\mathbb{E}\left(\sup_{0\leq t\leq u}\|y_\epsilon(t)\|^p\right)+\mathbb{E}\left(\sup_{0\leq t\leq u}\|y_\epsilon(t-\sigma)\|^p\right)\right] \\
&= 6^{p-1}\|\varphi_1\|_\infty M_1 (q\alpha-q+1)^{-(p-1)}\epsilon^p E_{\alpha,\alpha}(\gamma u^\alpha)^p u^{p\alpha},
\end{aligned} \tag{31}
$$

here $\|\varphi_1\|_\infty = \sup_{t\in[0,u]}|\varphi_1(t)|$, $M_1 = 1+\mathbb{E}\left(\sup_{0\leq t\leq u}\|y_\epsilon(t)\|^p\right)+\mathbb{E}\left(\sup_{0\leq t\leq u}\|y_\epsilon(t-\tau)\|^p\right)$.

For the second term I_2, we have

$$
\begin{aligned}
I_2 &\leq 6^{p-1}\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sup_{0\leq t\leq u}\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)[\kappa(s,x_\epsilon(s),x_\epsilon(s-\sigma))-\kappa(s,y_\epsilon(s),y_\epsilon(s-\sigma))]dW(s)\right\|^p\right) \\
&\quad + 6^{p-1}\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sup_{0\leq t\leq u}\left\|\int_0^t X^{A,B}_{\sigma,\alpha,\alpha}(t-s)[\kappa(s,y_\epsilon(s),y_\epsilon(s-\sigma))-\hat{\kappa}(y_\epsilon(s),y_\epsilon(s-\sigma))]dW(s)\right\|^p\right) \\
&= I_{21}+I_{22}.
\end{aligned} \tag{32}
$$

In view of the Burkholder–Davis–Gundy's inequality, Hölder's inequality and Doob's martingale inequality, and (H1), one has

$$
\begin{aligned}
I_{21} &\leq 6^{p-1}\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sup_{0\leq t\leq u}\int_0^t \|X_{\sigma,\alpha,\alpha}^{A,B}(t-s)\|^2\|\kappa(s,x_\epsilon(s),x_\epsilon(s-\sigma))-\kappa(s,y_\epsilon(s),y_\epsilon(s-\sigma))\|^2 ds\right)^{\frac{p}{2}} \\
&\leq 6^{p-1}\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sup_{0\leq t\leq u}\left(\left(\int_0^t 1^{\frac{p}{p-2}}ds\right)^{\frac{p-2}{p}\cdot\frac{p}{2}}\right.\right. \\
&\quad \left.\left.\cdot\int_0^t \|X_{\sigma,\alpha,\alpha}^{A,B}(t-s)\|^p\|\kappa(s,x_\epsilon(s),x_\epsilon(s-\sigma))-\kappa(s,y_\epsilon(s),y_\epsilon(s-\sigma))\|^p ds\right)\right) \\
&\leq 6^{p-1}\epsilon^{\frac{p}{2}}C_1^p u^{\frac{p}{2}-1}E_{\alpha,\alpha}(\gamma u^\alpha)^p \\
&\quad \cdot\int_0^u (u-s)^{p\alpha-p}\left[\mathbb{E}\left(\sup_{0\leq\theta\leq s}\|x_\epsilon(\theta)-y_\epsilon(\theta)\|^p\right)+\mathbb{E}\left(\sup_{0\leq\theta\leq s}\|x_\epsilon(\theta-\sigma)-y_\epsilon(\theta-\sigma)\|^p\right)\right]ds \\
&\leq 2\cdot 6^{p-1}\epsilon^{\frac{p}{2}}C_1^p u^{\frac{p}{2}-1}E_{\alpha,\alpha}(\gamma u^\alpha)^p\int_0^u (u-s)^{p\alpha-p}\mathbb{E}\left(\sup_{0\leq\theta\leq s}\|x_\epsilon(\theta)-y_\epsilon(\theta)\|^p\right)ds.
\end{aligned} \tag{33}
$$

Applying (H3) and an estimation method similar to Equation (33), we obtain

$$
\begin{aligned}
I_{22} &\leq 6^{p-1}\epsilon^{\frac{p}{2}}u^{\frac{p}{2}-1}\mathbb{E}\left(\sup_{0\leq\theta\leq u}\int_0^t \|X_{\sigma,\alpha,\alpha}^{A,B}(t-s)\|^p\|\kappa(s,y_\epsilon(s),y_\epsilon(s-\sigma))-\hat{\kappa}(y_\epsilon(s),y_\epsilon(s-\sigma))\|^p ds\right) \\
&\leq 6^{p-1}\epsilon^{\frac{p}{2}}u^{\frac{p}{2}-1}E_{\alpha,\alpha}(\gamma u^\alpha)^p u\|\varphi_2\|_\infty\left[1+\mathbb{E}\left(\sup_{0\leq t\leq u}|y_\epsilon(t)|^p\right)+\mathbb{E}\left(\sup_{0\leq t\leq u}|y_\epsilon(t-\sigma)|^p\right)\right] \\
&= 6^{p-1}M_1\|\varphi_2\|_\infty\epsilon^{\frac{p}{2}}E_{\alpha,\alpha}(\gamma u^\alpha)^p u^{\frac{p}{2}}.
\end{aligned} \tag{34}
$$

For the third term I_3, we have

$$
\begin{aligned}
I_3 &\leq 3^{p-1}\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sup_{0\leq t\leq u}\int_0^t\left\|\int_V X_{\sigma,\alpha,\alpha}^{A,B}(t-s)[f(s,x_\epsilon(s),x_\epsilon(s-\sigma),v)-f(s,y_\epsilon(s),y_\epsilon(s-\sigma),v)]\bar{N}(ds,dv)\right\|^p\right) \\
&\quad +3^{p-1}\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sup_{0\leq t\leq u}\int_0^t\left\|\int_V X_{\sigma,\alpha,\alpha}^{A,B}(t-s)[f(s,y_\epsilon(s),y_\epsilon(s-\sigma),v)-\hat{f}(y_\epsilon(s),y_\epsilon(s-\sigma),v)]\bar{N}(ds,dv)\right\|^p\right) \\
&= I_{31}+I_{32}.
\end{aligned} \tag{35}
$$

From Lemma 3, similarly to the Proof of (17), one has

$$
\begin{aligned}
I_{31} &\leq 3^{p-1}\epsilon^{\frac{p}{2}}D_p\mathbb{E}\left(\sup_{0\leq t\leq u}\int_0^t\int_V \|X_{\sigma,\alpha,\alpha}^{A,B}(t-s)\|^2\|f(s,x_\epsilon(s),x_\epsilon(s-\sigma),v)-f(s,y_\epsilon(s),y_\epsilon(s-\sigma),v)\|^2\lambda(dv)ds\right)^{\frac{p}{2}} \\
&\quad +3^{p-1}\epsilon^{\frac{p}{2}}\mathbb{E}\left(\sup_{0\leq t\leq u}\int_0^t\int_V \|X_{\sigma,\alpha,\alpha}^{A,B}(t-s)\|^p\|f(s,x_\epsilon(s),x_\epsilon(s-\sigma),v)-f(s,y_\epsilon(s),y_\epsilon(s-\sigma),v)\|^p\lambda(dv)ds\right) \\
&\leq 3^{p-1}\epsilon^{\frac{p}{2}}(D_p u^{\frac{p}{2}-1}+1) \\
&\quad \cdot\mathbb{E}\left(\sup_{0\leq t\leq u}\int_0^t\int_V \|X_{\sigma,\alpha,\alpha}^{A,B}(t-s)\|^p\|f(s,x_\epsilon(s),x_\epsilon(s-\sigma),v)-f(s,y_\epsilon(s),y_\epsilon(s-\sigma),v)\|^p\lambda(dv)ds\right) \\
&\leq 3^{p-1}\epsilon^{\frac{p}{2}}(D_p u^{\frac{p}{2}-1}+1)E_{\alpha,\alpha}(\gamma u^\alpha)^p C_1^p \\
&\quad \cdot\int_0^u (u-s)^{p\alpha-p}\left[\mathbb{E}\left(\sup_{0\leq\theta\leq s}\|x_\epsilon(\theta)-y_\epsilon(\theta)\|^p\right)+\mathbb{E}\left(\sup_{0\leq\theta\leq s}\|x_\epsilon(\theta-\sigma)-y_\epsilon(\theta-\sigma)\|^p\right)\right]ds \\
&\leq 2\cdot 3^{p-1}C_1^p\epsilon^{\frac{p}{2}}(D_p u^{\frac{p}{2}-1}+1)E_{\alpha,\alpha}(\gamma u^\alpha)^p\int_0^u (t-s)^{p\alpha-p}\mathbb{E}\left(\sup_{0\leq\theta\leq s}\|x_\epsilon(\theta)-y_\epsilon(\theta)\|^p\right)ds.
\end{aligned} \tag{36}
$$

Moreover, by (H3), we also have

$$
\begin{aligned}
I_{32} &\leq 3^{p-1}\epsilon^{\frac{p}{2}} D_p \mathbb{E}\left(\sup_{0\leq t\leq u} \int_0^t \int_V \|X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\|^2 \|f(s,y_\epsilon(s),y_\epsilon(s-\sigma),v) - \hat{f}(y_\epsilon(s),y_\epsilon(s-\sigma),v)\|^2 \lambda(dv)ds \right)^{\frac{p}{2}} \\
&+ 3^{p-1}\epsilon^{\frac{p}{2}} \mathbb{E}\left(\sup_{0\leq t\leq u} \int_0^t \int_V \|X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\|^p \|f(s,y_\epsilon(s),y_\epsilon(s-\sigma),v) - \hat{f}(y_\epsilon(s),y_\epsilon(s-\sigma),v)\|^p \lambda(dv)ds \right) \\
&\leq 3^{p-1}\epsilon^{\frac{p}{2}} (D_p u^{\frac{p}{2}-1} + 1) \\
&\cdot \mathbb{E}\left(\sup_{0\leq t\leq u} \int_0^t \int_V \|X^{A,B}_{\sigma,\alpha,\alpha}(t-s)\|^p \|f(s,y_\epsilon(s),y_\epsilon(s-\sigma),v) - \hat{f}(y_\epsilon(s),y_\epsilon(s-\sigma),v)\|^p \lambda(dv)ds \right) \\
&\leq 3^{p-1}\epsilon^{\frac{p}{2}} (D_p u^{\frac{p}{2}-1} + 1) E_{\alpha,\alpha}(\gamma u^\alpha)^p u \|\varphi_3\|_\infty \left[1 + \mathbb{E}\left(\sup_{0\leq t\leq u} \|y_\epsilon(t)\|^p \right) + \mathbb{E}\left(\sup_{0\leq t\leq u} \|y_\epsilon(t-\sigma)\|^p \right) \right] \\
&\leq 3^{p-1} M_1 \|\varphi_3\|_\infty \epsilon^{\frac{p}{2}} E_{\alpha,\alpha}(\gamma u^\alpha)^p (D_p u^{\frac{p}{2}} + u).
\end{aligned} \tag{37}
$$

From (28)–(37), for $u \in (0,T]$, we obtain

$$
\mathbb{E}\left(\sup_{0\leq t\leq u} \|x_\epsilon(t) - y_\epsilon(t)\|^p \right) \leq A(u) + B(u) \int_0^u (u-s)^{p\alpha - p} \mathbb{E}\left(\sup_{0\leq \theta\leq s} \|x_\epsilon(\theta) - y_\epsilon(\theta)\|^p \right) ds, \tag{38}
$$

where

$$
\begin{aligned}
A(u) &= 6^{p-1}\|\varphi_1\|_\infty M_1 (q\alpha - q + 1)^{-(p-1)} \epsilon^p E_{\alpha,\alpha}(\gamma u^\alpha)^p u^{p\alpha} \\
&+ 6^{p-1} M_1 \|\varphi_2\|_\infty \epsilon^{\frac{p}{2}} E_{\alpha,\alpha}(\gamma u^\alpha)^p u^{\frac{p}{2}} \\
&+ 3^{p-1} M_1 \|\varphi_3\|_\infty \epsilon^{\frac{p}{2}} E_{\alpha,\alpha}(\gamma u^\alpha)^p (D_p u^{\frac{p}{2}} + u),
\end{aligned}
$$

and

$$
\begin{aligned}
B(u) &= 2\cdot 6^{p-1} C_1^p \epsilon^p E_{\alpha,\alpha}(\gamma u^\alpha)^p u^{p-1} + 2\cdot 6^{p-1} C_1^p \epsilon^{\frac{p}{2}} E_{\alpha,\alpha}(\gamma u^\alpha)^p u^{\frac{p}{2}-1} \\
&+ 2\cdot 3^{p-1} C_1^p \epsilon^{\frac{p}{2}} E_{\alpha,\alpha}(\gamma u^\alpha)^p (D_p u^{\frac{p}{2}-1} + 1).
\end{aligned}
$$

By using of Lemma 4, we obtain

$$
\mathbb{E}\left(\sup_{0\leq t\leq u} \|x_\epsilon(t) - y_\epsilon(t)\|^p \right) \leq A(u) E_{p(\alpha-1)+1}\left(B(u)\Gamma(p(\alpha-1)+1) u^{p(\alpha-1)+1} \right).
$$

Choose $L > 0$ and $\beta \in (0,1)$ such that, for all $t \in (0, L\epsilon^{-\beta}] \subset (0,T]$ satisfies the following

$$
\mathbb{E}\left(\sup_{0< t\leq L\epsilon^{-\beta}} \|x_\epsilon(t) - y_\epsilon(t)\|^p \right) \leq \bar{A}(\epsilon) E_{p(\alpha-1)+1}(\bar{B}(\epsilon)\Gamma(p(\alpha-1)+1))\epsilon^{1-\beta},
$$

where

$$
\begin{aligned}
\bar{A}(\epsilon) &= 6^{p-1} E_{\alpha,\alpha}(\gamma T^\alpha)^p \|\varphi_1\|_\infty M_1 (q\alpha - q + 1)^{-(p-1)} L^{p\alpha} \epsilon^{p(1-\alpha\beta)} \\
&+ 6^{p-1} M_1 \|\varphi_2\|_\infty E_{\alpha,\alpha}(\gamma T^\alpha)^p L^{\frac{p}{2}} \epsilon^{\frac{p}{2}(1-\beta)} \\
&+ 3^{p-1} M_1 \|\varphi_3\|_\infty E_{\alpha,\alpha}(\gamma T^\alpha)^p (D_p L^{\frac{p}{2}} \epsilon^{\frac{p}{2}(1-\beta)} + L\epsilon^{\frac{p}{2}-\beta}),
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{B}(\epsilon) &= 2\cdot 6^{p-1} C_1^p E_{\alpha,\alpha}(\gamma T^\alpha)^p L^{p-1} \epsilon^{p-(p-1)\beta} \\
&+ 2\cdot 6^{p-1} C_1^p E_{\alpha,\alpha}(\gamma T^\alpha)^p L^{\frac{p}{2}-1} \epsilon^{\frac{p}{2}-(\frac{p}{2}-1)\beta} \\
&+ 2\cdot 3^{p-1} C_1^p E_{\alpha,\alpha}(\gamma T^\alpha)^p (D_p L^{\frac{p}{2}-1} \epsilon^{\frac{p}{2}-(\frac{p}{2}-1)\beta} + \epsilon^{\frac{p}{2}}),
\end{aligned}
$$

are two constants. Thus, for any given number $\delta > 0$, there exists $\epsilon_1 \in (0, \epsilon_0]$ such that, for each $\epsilon \in (0, \epsilon_1]$ and $t \in [0, L\epsilon^{-\beta}] \subset J$,

$$\mathbb{E}\left(\sup_{t\in[0,L\epsilon^{-\beta}]} \|x_\epsilon(t) - y_\epsilon(t)\|^p\right) \leq \delta.$$

□

Remark 1. *If $p = 2$ and $f \equiv 0$, then FSDDEs (1) reduces to FSDDSs (1.1) in [16]. Therefore, Theorems 1 and 2 generalize the main results of [16].*

By using Theorem 2 and Chebyshev–Markov inequality, we can obtain the following corollary.

Corollary 1. *Assume that (H1)–(H3) are satisfied. Then, for a given arbitrary small number $\delta > 0$, $p \geq 2$ with $1 - \frac{1}{p} < \alpha < 1$, then for arbitrarily number $\bar{\delta} > 0$ such that for $L > 0$, $\epsilon_1 \in (0, \epsilon_0]$ and $\beta \in (0, 1)$ satisfying for all $\epsilon \in (0, \epsilon_1]$*

$$\lim_{\epsilon\to 0} \mathbb{P}\left(\sup_{t\in[0,L\epsilon^{-\beta}]} \|x_\epsilon(t) - y_\epsilon\|^p > \bar{\delta}\right) = 0.$$

5. Applications

In this section, we will provide two examples to illustrate the application of our main results.

Example 1. *Consider the following Caputo-type FSDDSs with Poisson jumps:*

$$\begin{cases} ({}^{C}D_0^{0.9}x)(t) = Ax(t) + Bx(t-0.4) + g(t, x(t), x(t-0.4)) + \kappa(t, x(t), x(t-0.4))\frac{dW(t)}{dt} \\ \qquad\qquad + \int_V f(t, x(t), x(t-0.4), v)\tilde{N}(dt, dv), \quad t \in J, \\ x(t) = \phi(t), \quad -0.4 \leq t \leq 0, \end{cases} \tag{39}$$

where $\alpha = 0.9$, $\sigma = 0.4$, $J = [0, 4]$, $x(t) = (x_1(t), x_2(t))^T$, and

$$A = \begin{pmatrix} 0.3 & 0.1 \\ 0.15 & 0.2 \end{pmatrix}, \quad B = \begin{pmatrix} 0.2 & 0.1 \\ 0.15 & 0.25 \end{pmatrix}, \quad \phi(t) = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix},$$

and

$$g(t, x(t), x(t-0.4)) = \begin{pmatrix} \frac{1}{3}e^{-2t}\sin(x_1(t)) + \frac{1}{4}e^{-t}\sin^3 t \arctan(x_1(t-0.4)) + \frac{1}{7} \\ \frac{1}{3}e^{-2t}\cos(x_2(t)) + \frac{1}{4}e^{-t}\cos^3 t \arctan(x_2(t-0.4)) + \frac{1}{6} \end{pmatrix},$$

and

$$\kappa(t, x(t), x(t-0.4)) = \begin{pmatrix} \frac{1}{4}e^{-t}\arctan(x_1(t)) + \frac{1}{3}e^{-2t}\cos^2 t \sin(x_1(t-0.4)) + \frac{1}{3} \\ \frac{1}{4}e^{-t}\sin(x_2(t)) + \frac{1}{3}e^{-t}\sin^2 t \arctan(x_2(t-0.4)) + \frac{1}{6} \end{pmatrix},$$

and

$$f(t, x(t), x(t-0.4), v) = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}.$$

For each $x(t), y(t) \in \Upsilon$ and $t \in [0, T]$, we have

$$\begin{aligned}
&\|g(t,x(t),x(t-0.4))-g(t,y(t),y(t-0.4))\| \\
&\leq \frac{1}{3}|x_1(t)-y_1(t)|+\frac{1}{4}|x_1(t-0.4)-y_1(t-0.4)|+\frac{1}{3}|x_2(t)-y_2(t)|+\frac{1}{4}|x_2(t-0.4)-y_2(t-0.4)| \\
&\leq \frac{1}{3}(\|x(t)-y(t)\|+\|x(t-0.4)-y(t-0.4)\|).
\end{aligned}$$

Thus

$$\|g(t,x(t),x(t-0.4))-g(t,y(t),y(t-0.4))\|^3 \leq \frac{2^2}{3^3}(\|x(t)-y(t)\|^3+\|x(t-0.4)-y(t-0.4)\|^3),$$

which implies that the function g satisfies the assumption (H1). Similarly, we can obtain that the functions κ and f satisfy the assumptions (H1) and (H2).

Let $p=3$. By calculation, we have $\gamma=\|A\|+\|B\|=0.8$, $\|g\|_{\mathbb{L}^p}=\int_0^4\|g(t,0,0)\|^3dt=0.0651$, $\|\kappa(\cdot,0,0)\|_\infty=\frac{1}{2}$, $\|f(\cdot,0,0,0)\|_\infty=\frac{5}{6}$, $C_1=\frac{4}{27}$ and

$$\begin{aligned}
\Xi &= \left(\int_{-0.4}^{0}\|({}^CD^{0.9}_{-0.4^+}\phi)(s)-A\phi(s)\|^{\frac{3}{2}}ds\right)^2 \\
&\leq \left(\sqrt{2}\int_{-0.4}^{0}(\|({}^CD^{0.9}_{-0.4^+}\phi)(s)\|^{\frac{3}{2}}+\|A\phi(s)\|^{\frac{3}{2}})ds\right)^2 \\
&\leq \left(\sqrt{2}\int_{-0.4}^{0}\left(\left\|\begin{pmatrix}\frac{1}{\Gamma(0.1)}\int_{-0.4}^{s}(s-t)^{-0.9}dt \\ \frac{1}{2\Gamma(0.1)}\int_{-0.4}^{s}(s-t)^{-0.9}dt\end{pmatrix}\right\|^{\frac{3}{2}}+\left\|\begin{pmatrix}0.35\\0.25\end{pmatrix}\right\|^{\frac{3}{2}}\right)\right)^2 \\
&\leq 2\left(\int_{-0.4}^{0}\left(\left(\frac{3}{2\Gamma(1.1)}\right)^{\frac{3}{2}}(s+0.4)^{0.15}+0.6^{1.5}\right)ds\right)^2=1.5722.
\end{aligned}$$

Hence, we may choose a suitable value $\mu>0$ such that

$$2\cdot 3^2C_1^3E_{0.9,0.9}(0.8\cdot 4^{0.9})^3(4^2+(D_3+1)2+1)\Gamma(0.7)<\mu.$$

By Theorem 1, FSDDEs (39) have a unique solution $x\in\mathbb{H}^3([0,4])$.

Example 2. *In the following, we consider the standard form of (39) as follows*

$$\begin{cases}({}^CD_0^{0.9}x_\epsilon)t)=Ax_\epsilon(t)+Bx_\epsilon(t-0.4)+\epsilon g(t,x_\epsilon(t),x_\epsilon(t-0.4))+\sqrt{\epsilon}\kappa(t,x_\epsilon(t),x_\epsilon(t-0.4))\frac{dW(t)}{dt} \\ \qquad +\sqrt{\epsilon}\int_V f(t,x_\epsilon(t),x_\epsilon(t-0.4),v)\bar{N}(dt,dv), \quad t\in J, \\ x_\epsilon(t)=\phi(t), \quad -0.4\leq t\leq 0,\end{cases} \tag{40}$$

where $x_\epsilon(t)=(x_{1,\epsilon}(t),x_{2,\epsilon}(t))^T$, and

$$g(t,x_\epsilon(t),x_\epsilon(t-0.4))=\begin{pmatrix}\frac{1}{3}e^{-2t}\sin(x_{1,\epsilon}(t))+\frac{1}{4}e^{-t}\sin^3 t\arctan(x_{1,\epsilon}(t-0.4))+\frac{1}{7} \\ \frac{1}{3}e^{-2t}\cos(x_{2,\epsilon}(t))+\frac{1}{4}e^{-t}\cos^3 t\arctan(x_{2,\epsilon}(t-0.4))+\frac{1}{6}\end{pmatrix},$$

and

$$\kappa(t,x_\epsilon(t),x_\epsilon(t-0.4))=\begin{pmatrix}\frac{1}{4}e^{-t}\arctan(x_{1,\epsilon}(t))+\frac{1}{3}e^{-2t}\cos^2 t\sin(x_{1,\epsilon}(t-0.4))+\frac{1}{3} \\ \frac{1}{4}e^{-t}\sin(x_{2,\epsilon}(t))+\frac{1}{3}e^{-2t}\sin^2 t\arctan(x_{2,\epsilon}(t-0.4))+\frac{1}{6}\end{pmatrix},$$

and

$$f(t,x_\epsilon(t),x_\epsilon(t-0.4),v)=\begin{pmatrix}\frac{1}{2}\\ \frac{1}{3}\end{pmatrix}.$$

We can easily check that the conditions (H1) and (H2) hold, and according to Theorem 1, FSDDSs (40) have a unique solution x_ϵ given by

$$
\begin{aligned}
x_\epsilon(t) = {} & X^{A,B}_{0.4,0.9,1}(t+0.4)\phi(-0.4) + \int_{-0.4}^{0} X^{A,B}_{0.4,0.9,0.9}(t-s)[{}^{C}D^{0.9}_{-0.4^+}\phi)(s) - A\phi(s)]ds \\
& +\epsilon \int_0^t X^{A,B}_{0.4,0.9,0.9}(t-s)g(s, x_\epsilon(s), x_\epsilon(s-0.4))ds \\
& +\sqrt{\epsilon}\int_0^t X^{A,B}_{0.4,0.9,0.9}(t-s)\kappa(s, x_\epsilon(s), x_\epsilon(s-0.4))dW(s) \\
& +\sqrt{\epsilon}\int_0^t X^{A,B}_{0.4,0.9,0.9}(t-s)\int_V f(s, x_\epsilon(s), x_\epsilon(s-0.4), v)\tilde{N}(ds, dv).
\end{aligned} \tag{41}
$$

By calculation, one has

$$\hat{g}(x_\varepsilon(t), x_\varepsilon(t-\sigma)) = \lim_{t\to\infty}\frac{1}{t}\int_0^t f(s, x_\epsilon(s), x_\varepsilon(s-\sigma))ds = \begin{pmatrix} \frac{1}{7} \\ \frac{1}{6} \end{pmatrix},$$

$$\hat{\kappa}(x_\varepsilon(t), x_\varepsilon(t-\sigma)) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma(s, x_\varepsilon(s), x_\varepsilon(s-\sigma))ds = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{6} \end{pmatrix},$$

$$\hat{f}(x_\varepsilon(t), x_\varepsilon(t-\sigma), v) = \lim_{t\to\infty}\frac{1}{t}\int_0^t g(s, x_\varepsilon(s), x_\varepsilon(s-\sigma), v)ds = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}.$$

We are now checking that condition (H3) holds. In fact, one has

$$
\begin{aligned}
& \frac{1}{t}\int_0^t \|g(s, x_\varepsilon(s), x_\varepsilon(s-\sigma)) - \hat{g}(x_\varepsilon(s), x_\varepsilon(s-\sigma))\|^p\, ds \\
& = \frac{1}{t}\int_0^t \left\| \begin{pmatrix} \frac{1}{3}e^{-2s}\sin(x_{1,\epsilon}(s)) + \frac{1}{4}e^{-s}\sin^3 s\arctan(x_{1,\epsilon}(s-0.4)) \\ \frac{1}{3}e^{-2s}\cos(x_{2,\epsilon}(s)) + \frac{1}{4}e^{-s}\cos^3 s\arctan(x_{2,\epsilon}(s-0.4)) \end{pmatrix} \right\|^p ds \\
& = \frac{1}{3^p t}\int_0^t (e^{-s}(|x_{1,\epsilon}(s)| + |x_{2,\epsilon}(s)|) + e^{-s}(|x_{1,\epsilon}(s-0.4)| + |x_{2,\epsilon}(s-0.4)|))^p\, ds \\
& = \frac{1}{3^p t}\int_0^t (e^{-ps}(\|x_\epsilon(s)\| + \|x_\epsilon(s-0.4)\|)^p\, ds \\
& \le \frac{2^{p-1}}{3^p t}(\|x_\epsilon(s)\|^p + \|x_\epsilon(s-0.4)\|^p)\int_0^t e^{-ps}ds \\
& = \frac{2^{p-1}(1-e^{-pt})}{3^p pt}(1 + \|x_\epsilon(s)\|^p + \|x_\epsilon(s-0.4)\|^p),
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{t}\int_0^t \|(t-s)^{\alpha-1}(\kappa(s, x_\epsilon(s), x_\epsilon(s-\sigma)) - \hat{\kappa}(x_\epsilon(s), x_\epsilon(s-\sigma)))\|^p\, ds \\
& = \frac{1}{t}\int_0^t \left\| \begin{pmatrix} \frac{1}{4}(t-s)^{\alpha-1}e^{-s}\arctan(x_{1,\epsilon}(s)) + \frac{1}{3}(t-s)^{\alpha-1}e^{-2s}\cos^2 s\sin(x_{1,\epsilon}(s-0.4)) \\ \frac{1}{4}(t-s)^{\alpha-1}e^{-s}\sin(x_{2,\epsilon}(s)) + \frac{1}{3}(t-s)^{\alpha-1}e^{-s}\sin^2 s\arctan(x_{2,\epsilon}(s-0.4)) \end{pmatrix} \right\|^p ds \\
& \le \frac{2^{p-1}}{3^p t}(\|x_\epsilon(s)\|^p + \|x_\epsilon(s-0.4)\|^p)\int_0^t (t-s)^{p\alpha-p}ds \\
& = \frac{2^{p-1}}{(p\alpha-p+1)3^p}t^{p\alpha-p}(1 + \|x_\epsilon(s)\|^p + \|x_\epsilon(s-0.4)\|^p),
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{1}{t}\int_0^t \left(\int_V \|(t-s)^{\alpha-1}(f(s, x_\epsilon(s), x_\epsilon(s-0.4), v) - \hat{f}(x_\epsilon(s), x_\epsilon(s-0.4), v)\|^p \lambda(dv)\right)ds \\
& = \frac{1}{t}\int_0^t\int_V \left\| \begin{pmatrix} \frac{1}{2}(t-s)^{\alpha-1} \\ \frac{1}{3}(t-s)^{\alpha-1} \end{pmatrix} \right\|^p \lambda(dv)ds \\
& = \frac{1}{t}\left(\frac{5}{6}\right)^p \lambda(V)\int_0^t (t-s)^{p(\alpha-1)}ds \\
& \le \frac{5^p\lambda(V)}{(p\alpha-p+1)6^p}t^{p\alpha-p}(1 + \|x_\epsilon(s)\|^p + \|x_\epsilon(s-0.4)\|^p).
\end{aligned}
$$

Thus, (H3) is satisfied with

$$\varphi_1(t) = \frac{2^{p-1}(1-e^{-pt})}{3^p pt}, \quad \varphi_2(t) = \frac{2^{p-1}}{(p\alpha - p + 1)3^p} t^{p\alpha - p}, \quad \text{and } \varphi_3(t) = \frac{5^p \lambda(V)}{(p\alpha - p + 1)6^p} t^{p\alpha - p}.$$

Therefore, the conditions of Theorem 2 and Corollary 1 are satisfied. So, as $\epsilon \to 0$, the original solution $x_\epsilon(\cdot) \to y_\epsilon(\cdot)$ in the sense of p square ($p = 3$) and in the probability, where

$$\begin{aligned} y_\epsilon(t) \ &= X^{A,B}_{0.4,0.9,1}(t+0.4)\phi(-0.4) + \int_{-0.4}^{0} X^{A,B}_{0.4,0.9,0.9}(t-s)[{}^C D^{0.9}_{-0.4^+}\phi)(s) - A\phi(s)]ds \\ &+\epsilon \int_0^t X^{A,B}_{0.4,0.9,0.9}(t-s)\hat{g}(y_\epsilon(s), y_\epsilon(s-0.4))ds \\ &+\sqrt{\epsilon} \int_0^t X^{A,B}_{0.4,0.9,0.9}(t-s)\hat{\kappa}(y_\epsilon(s), y_\epsilon(s-0.4))dW(s) \\ &+\sqrt{\epsilon} \int_0^t X^{A,B}_{0.4,0.9,0.9}(t-s) \int_V \hat{f}(y_\epsilon(s), y_\epsilon(s-0.4), v)\tilde{N}(ds, dv). \end{aligned} \tag{42}$$

6. Conclusions

In this article, we established and proved the existence and uniqueness theorem for solutions of Caputo-type fractional stochastic delay differential systems (FSDDSs) with Poisson jumps. By utilizing Burkholder–Davis–Gundy's inequality, Doob's martingale inequality, fractional Gronwall's inequality, Hölder's inequality, and Jensen's inequality, we proved the averaging principle for FSDDSs in the sense of L^p. This provides an effective stochastic approximation of the solutions of FSDDSs. Our method for fractional averaging will be beneficial for the study of the dynamics behavior of FSDDSs. Our results enrich the research field of fractional-order stochastic delay differential equations. Finally, we provided two examples to show the usefulness of our results.

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