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Minimal and Primitive Terracini Loci of a Four-Dimensional Projective Space

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Abstract: We study two quite different types of Terracini loci for the order d-Veronese embedding of an n-dimensional projective space: the minimal one and the primitive one (defined in this paper). The main result is that if $n = 4$, $d \geq 19$ and $x \leq 2d$, no subset with x points is a minimal Terracini set. We give examples that show that the result is sharp. We raise several open questions.

Keywords: Terracini locus; double points; Veronese embedding; four-dimensional projective space; primitive Terracini locus; minimal Terracini locus

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1. Introduction

Let $X \subset \mathbb{P}^r$ be an integral and nondegenerate variety defined over an algebraically closed field $\mathbb{K}$ with characteristic zero. For any positive integer x, let $S(X_{\mathrm{reg}}, x)$ denote the set of all $S \subset X_{\mathrm{reg}}$ such that $\#S = x$. For any $p \in X_{\mathrm{reg}}$, let $2p$ denote the closed subscheme of X with $(\mathcal{I}_p)^2$ as its ideal sheaf. For each $S \in S(X_{\mathrm{reg}}, x)$, set $2S := \cup_{p \in S} 2p$. We have $\deg(2S) = x(\dim X + 1)$. By the Terracini lemma ([1] Cor. 1.10), the zero-dimensional scheme $2S$ is a key player for the study of the secant varieties of X.

Let $\mathcal{T}(X, x)$ denote the set of all $S \in S(X_{\mathrm{reg}}, x)$ such that $\langle 2S \rangle \neq \mathbb{P}^r$ and $\dim\langle 2S \rangle \leq \deg(2S) - 2 = x(\dim X + 1) - 2$, where $\langle \ \rangle$ denotes the linear span. The set $\mathcal{T}(X, x)$ is called a *Terracini set* of the embedding $X \subset \mathbb{P}^r$ or the x-Terracini locus of X. These Terracini loci are the sets at which a certain morphism connected to the x-secant variety ramifies (see Section 6 for motivations). The integer $x(\dim X + 1) - \dim\langle 2S \rangle + 1$ is the dimension of the kernel of the differential of this map at a point associated with S ([1] Cor. 1.10). Knowing that $S \notin \mathcal{T}(X, x)$ guarantees that S is an isolated solution for the X-rank problem of a sufficiently general $p \in \langle S \rangle$ (Section 6). In previous works, it was clear that these sets $\mathcal{T}(X, x)$ (which are algebraic subsets for the Zariski topology) have a rich geometry ([2]). Sometimes $\mathcal{T}(X, x) = \emptyset$ for all $x > 0$, e.g., if X is a rational normal curve ([2] Th. 1.1(i)). Assuming $\mathcal{T}(X, x) \neq \emptyset$ for some x, the minimal integer x, $t(X, \min)$, such that $\mathcal{T}(X, x) \neq \emptyset$ is certainly important (see Remark 1 for an easy lower bound on it).

But not all Terracini sets are "equally interesting". Take $S \in \mathcal{T}(X, x)$. Quite often, there are integers $y > x$ and $S_1 \in S(X_{\mathrm{reg}}, y - x)$ such that $S \cap S_1 = \emptyset$ and $S \cup S_1 \in \mathcal{T}(X, y)$. Indeed, the condition "$\dim\langle 2(S \cup S_1)\rangle \leq \deg(2(S \cup S_1)) - 2 = y(\dim X + 1) - 2$" is true by the corresponding inequality for $2S$ and we only need $\langle 2(S \cup S_1)\rangle \neq \mathbb{P}^r$. This is true for all $p \in X_{\mathrm{reg}} \setminus S$ if $\dim\langle 2S \rangle \leq r - \dim X - 2$, and this inequality is very often satisfied in interesting ranges of integers x. In many cases, from a single $S \in \mathcal{T}(X, x)$, we obtain larger sets that give $\mathcal{T}(X, y) \neq \emptyset$ for all $y > x$ ([2] Th. 1.1(iii)). Take $S \in \mathcal{T}(X, x)$. We say that S is *minimal* if $\dim\langle 2S' \rangle = \deg(2S') - 1$ for all $S' \subsetneq S$, i.e., if no proper subset of S is Terracini. Let $\mathcal{T}(X, x)'$ denote the set of all minimal $S \in \mathcal{T}(X, x)$. It is easy to check that $\mathcal{T}(X, x)' = \emptyset$ for all $x \gg 0$ (Remark 4). Thus, minimal Terracini sets are more important. Of course, elements of $\mathcal{T}(X, t(X, \min))$ are minimal.

Now assume that X is the image $v_d(\mathbb{P}^n)$, $n > 1$, of the d-Veronese embedding $v_d : \mathbb{P}^n \to \mathbb{P}^r$, $r = \binom{n+d}{n} - 1$ of $\mathbb{P}^n$. We take finite sets in $\mathbb{P}^n$ instead of $v_d(\mathbb{P}^n)$. For any positive integer x, let $\mathcal{T}_1(n, d; x)$ denote the set of all $S \in S(\mathbb{P}^n, x)$ such that $v_d(S) \in \mathcal{T}(X, x)$. Fix a line $L \subset \mathbb{P}^n$ and take $S \subset L$ such that $\#S = 1 + \lceil d/2 \rceil$. Since $h^0(\mathcal{O}_{\mathbb{P}^1}(d)) = d + 1$, $\dim\langle 2v_d(S)\rangle \le \deg(2S) - 2$. Since $n > 1$ and $d \ge 2$, $(1 + \lceil d/2 \rceil)(n + 1) \le \binom{n+d}{n}$. Thus, $S \in \mathcal{T}_1(n, d; 1 + \lceil d/2 \rceil)$. This set S is not very interesting; it lies in a tiny part of $\mathbb{P}^n$, and this (many points in a tiny part) is the only reason to be an element of $\mathcal{T}_1(n, d; x)$. Let $\mathcal{T}(n, d; x)$ denote the set of $S \in \mathcal{T}_1(n, d; x)$ such that $\langle S \rangle = \mathbb{P}^n$, i.e., $\mathbb{P}^n$ is the minimal projective space containing S. A similar notation and assumption is available for Segre embeddings of multiprojective spaces. Take $S \in \mathcal{T}(n, d; x)$. We say that S is *minimally Terracini* and write $S \in \mathcal{T}(n, d; x)'$ if $\dim\langle v_d(S')\rangle = (n + 1)\#S - 1$ for all $S' \subsetneq S$ ([2]) . Note that $v_d(\mathcal{T}(n, d; x)') = \mathcal{T}(X, x)' \cap v_d(\mathcal{T}(n, d; x))$.

In Section 5, we consider the following set: $\tilde{\mathcal{T}}(n, d; x)$. Let $\tilde{\mathcal{T}}(n, d; x)$ be the set of all $S \in \mathcal{T}(n, d; x)$ such that $h^1(\mathcal{I}_{2A}(d)) = 0$ for all $A \subsetneq S$ such that $\langle A \rangle = \mathbb{P}^n$. We say that $\tilde{\mathcal{T}}(n, d; x)$ is the *primitive Terracini loci* of the Veronese variety $v_d(\mathbb{P}^n)$. Obviously,

$$\mathcal{T}(n, d; x)' \subseteq \tilde{\mathcal{T}}(n, d; x) \subseteq \mathcal{T}(n, d; x).$$

In our opinion, the minimal Terracini locus is the most important one, and in [2] it was shown how different $\mathcal{T}(n, d; x)'$ and $\mathcal{T}(n, d; x)$ are. In the second part of this paper (Section 5), we show how different the primitive Terracini locus is with respect to the other ones. By the semicontinuity theorem for cohomology, the sets $\mathcal{T}(n, d; x)'$, $\tilde{\mathcal{T}}(n, d; x)$ and $\mathcal{T}(n, d; x)$ are locally closed subsets of the set $S(\mathbb{P}^n, x)$ of all subsets of $\mathbb{P}^n$ of cardinality x. In particular, it makes sense to speak about their dimension.

We prove the following result:

Theorem 1. *Fix positive integers d, x such that $x \le 2d$. If $x \le 2d - 1$, assume $d \ge 17$. If $x = 2d$, assume $d \ge 19$. Then, $\mathcal{T}(4, d; x)' = \emptyset$.*

We have $\mathcal{T}(4, d; 2d + 1)' \neq \emptyset$ and we classify all $S \in \mathcal{T}(4, d; 2d + 1)'$ contained in a (reducible) rational normal curve (Remark 8 and Proposition 2).

Remark 1. *Take an integral and nondegenerate variety $X \subset \mathbb{P}^r$. Recall the positive integer $t(X, \min)$, which is a key invariant of the embedding $X \subset \mathbb{P}^r$, and that every $S \in \mathcal{T}(X, \min)$ is minimal. Now assume $X = v_d(\mathbb{P}^n)$, $n \ge 2$, $d \ge 4$. With this general definition (as discussed in [2]), we would obtain $t(X, \min) = \lceil (d + 2)/2 \rceil$ for all $n > 1$. Obviously, this is not interesting, since in the applications, to see the difficulty of a problem, it is better to consider only multivariate polynomials which are "concise", i.e., such that there is no linear change in coordinates making the polynomials depending on a smaller number of variables. With this restriction, Theorem 1, Remark 8 and Proposition 2 solve the minimal x-problem for $n = 4$. For $n = 2, 3$, the problem was solved in [2]. The case $n = 2$ was easy, while the proof of the case $n = 3$ was longer. In our opinion, the proof of Theorem 1 is as short as possible with our technology. We stress that these tools are useful for other problems (see the discussion of Question 5 at the end of Section 2).*

A similar restriction should be added in the study of other important (for the applications) embedded varieties, e.g., the multiprojective spaces (which are associated with tensors and partially symmetric tensors) and Grassmannians (which are associated with antisymmetric tensors). For tensors and partially symmetric tensors, it is equivalent to study exactly the concise tensors.

We recall the following conjecture ([2] Conjecture 1.2):

Conjecture 1: we have $\mathcal{T}(n, d; x)' = \emptyset$ for all $x \le \lfloor (nd + 1)/2 \rfloor$.

Question 1: Assume d large. Are all $S \in \mathcal{T}(4, d; 2d + 1)'$ contained in a (reducible) rational normal curve?

We know that $\mathcal{T}(4, d; \lceil 5d/2 \rceil)' \neq \emptyset$ (Remark 9).

Question 2: Assume d large. Is $\mathcal{T}(4, d; x)' = \emptyset$ for all $2d + 1 < x < \lceil 5d/2 \rceil$?

Question 3: Assume d large. Is $\mathcal{T}(4, d; 1 + \lceil 5d/2 \rceil)' = \emptyset$?

Question 4: Fix a positive integer e. Is there an integer $d(e)$ such that $\mathcal{T}(4,d;x)' = \emptyset$ for all $d \geq d(e)$ and all $\lceil 5d/2 \rceil < x \leq e + \lceil 5d/2 \rceil$?

Conjecture 2: For all n, there is an integer $d_0(n)$ that has $\mathcal{T}(n,d;2+\lfloor (nd+1)/2 \rfloor)' = \emptyset$ for all $d \geq d_0(n)$.

In Section 2, we give the key definition (*critical scheme*) used in [2] and give several results (and a question) on the Hilbert function of a zero-dimensional scheme $Z \subset \mathbb{P}^n$.

Section 3 is devoted to the proof of Theorem 1. An outline of the proof is presented at the beginning of the section.

In Section 4, we consider the range $x > 2d$: the range of the last four questions.

In Section 5, we consider $\tilde{\mathcal{T}}(n,d;x)$. We give conditions on n, d and x in order to have $\tilde{\mathcal{T}}(n,d;x) \neq \emptyset$ and other conditions implying $\tilde{\mathcal{T}}(n,d;x) = \emptyset$. We classify the sets $\tilde{\mathcal{T}}(n,d;x)$ if $x < 3d/2$ (Theorem 3). The main difference between $\mathcal{T}(n,d;x)$ and $\tilde{\mathcal{T}}(n,d;x)$ is that $\tilde{\mathcal{T}}(n,d;x) = \emptyset$ for $x \gg 0$ (Theorem 2).

In Section 6, we give the main motivations for the study of the Terracini loci. Several tools used here and in [2] (in particular zero-dimensional schemes, not just finite sets) are useful for other topics, e.g., the description of evaluation codes and the computation of their minimum distance and higher Hamming weights ([3,4]).

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2. Preliminary Results

For any sequence $\{w_i\}$, $i \geq 1$ of non-negative integers, we say that $\{w_i\}$ is *weakly decreasing* if $w_i \geq w_{i+1}$ for all $i \geq 1$.

A rational normal curve $C \subset \mathbb{P}^r$ is an integral and nondegenerate curve of degree $\deg(C) = r$. All rational normal curves of $\mathbb{P}^r$ are smooth and rational and they are the nondegenerate curves of $\mathbb{P}^r$ of minimal degree.

For any projective scheme M, any effective Cartier divisor D of M and any zero-dimensional scheme $Z \subset M$ of the residual scheme $\mathrm{Res}_D(Z)$ of Z with respect to D is the zero-dimensional subscheme of M with $\mathcal{I}_{Z,M} : \mathcal{I}_{D,M}$ as its ideal sheaf. We have $\mathrm{Res}_D(Z) \subseteq Z$ and $\deg(Z) = \deg(D \cap Z) + \deg(\mathrm{Res}_D(Z))$. For any line bundle $\mathcal{L}$ on M, we have an exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes \mathcal{L}(-D) \to \mathcal{I}_Z \otimes \mathcal{L} \to \mathcal{I}_{Z\cap D,D} \otimes \mathcal{L}_{|D} \to 0 \tag{1}$$

Often, we will say that (1) is the residual exact sequence of D without mentioning Z and $\mathcal{L}$. Let Z_{red} denote the reduction in Z, i.e., the set of all $p \in M$ such that $\{p\} \subseteq Z$. The set Z_{red} is finite, and $\#S \leq \deg(Z)$ and Z_{red} and Z have the same number of connected components.

A key definition used in [2] is the notion of a *critical scheme*.

Definition 1. *Take a finite set $S \subset \mathbb{P}^n$ such that $h^1(\mathcal{I}_{2S}(d)) > 0$. A zero-dimensional scheme $Z \subset \mathbb{P}^n$ such that $Z_{\mathrm{red}} \subseteq S$, each connected component of Z has degree ≤ 2, $h^1(\mathcal{I}_Z(d)) > 0$ and $h^1(\mathcal{I}_{Z'}(d)) = 0$ for all $Z' \subsetneq Z$ is called a critical scheme of S.*

Remark 2. *Take a finite set $S \subset \mathbb{P}^n$ such that $h^1(\mathcal{I}_{2S}(d)) > 0$. There is a critical scheme of S ([5,6] and [2] Lemma 2.8 and Definition 2.9). If $S \in \mathcal{T}(n,d;\#S)'$, then $Z_{\mathrm{red}} = S$ ([2] Lemma 2.11). Let $D \subset \mathbb{P}^n$ be a hypersurface. Set $t := \deg(D)$. Assume $Z \not\subseteq D$, i.e., assume $Z \cap D \neq Z$. Since Z is critical, $h^1(\mathcal{I}_{D\cap Z}(d)) = 0$. Hence, $h^1(D, \mathcal{I}_{Z\cap D,D}(d)) = 0$. The residual exact sequence of D gives $h^1(\mathcal{I}_{\mathrm{Res}_D(Z)}(d-t)) > 0$.*

Using Remark 2, we obtain the following lower bound for the integer $t(X, \min)$:

Lemma 1. *Let $X \subset \mathbb{P}^r$ be an integral and nondegenerate variety. Let $\gamma(X)$ be the minimal degree of a zero-dimensional scheme $Z \subset X_{\mathrm{reg}}$ such that $\dim\langle Z \rangle \leq \deg(Z) - 2$ and $\langle Z \rangle \neq \mathbb{P}^r$, with the convention $\gamma(X) = \infty$ if there is no such Z (e.g., for a rational normal curve). Then, $t(X, \min) \geq \lceil \gamma(X)/2 \rceil$ if $\gamma(X)$ is finite while $\mathcal{T}(X,x) = \emptyset$ for all x if $\gamma(X) = \infty$.*

Remark 3. *Often, it is easy to compute $\gamma(X)$. For instance, if $X \subset \mathbb{P}^r$ is the image of the d-Veronese embedding of $\mathbb{P}^n$, $d \geq 3$, then $\gamma(X) = d+2$.*

Remark 4. *Let $X \subset \mathbb{P}^r$ be an integral and nondegenerate variety. Set $n := \dim X$. By the Terracini lemma ([1] Cor. 1.10), we have $\mathcal{T}(X,x)' = \emptyset$ for all $x > \lceil (r+1)/(n+1) \rceil$ (see [2] Prop. 3.5) for the case of the Veronese varieties. Now assume that X is secant-defective and let k be the first integer such that the k-secant variety has dimensions of at most $k(n+1)-2$. In this case, $\mathcal{T}(X,k)'$ contains a general $S \in S(X,k)$ by the Terracini lemma ([1] Cor. 1.10 (b)). Thus, in this case, k is the maximal integer y such that $\mathcal{T}(X,y)' \neq \emptyset$. By the semicontinuity theorem for cohomology, we have $\mathcal{T}(X,k) = S(X_{\mathrm{reg}},k)$.*

Remark 5. *Fix positive integers d, z and a zero-dimensional scheme $Z \subset \mathbb{P}^2$ such that $\deg(Z) = z$ and $h^1(\mathcal{I}_Z(d)) > 0$.*

(a) Assume $z \leq 3d$. Then, Z is in one of the following cases ([7] Rem. (i) at. p. 116):

(i) There is a line L such that $\deg(L \cap Z) \geq d+2$;
(ii) There is a conic D such that $\deg(D \cap Z) \geq 2d+2$;
(iii) $z = 3d$ and Z is the complete intersection of a plane curve of degree three and a plane curve of degree d.

(b) Assume $z \leq 4d-4$ and $z \geq 16$. Then, either Z is as in one of the cases (i), (ii) or (iii) of part (a) or there is a plane cubic C such that $\deg(Z \cap C) \geq 3d+1$ or $z = 4d-4$ and Z is the complete intersection of a plane curve of degree four and a plane curve of degree $d-1$ (case $s = 4$ of [7] Cor. 2).

(c) Assume $z \geq 25$ and $z \leq 5d-11$. Then, either Z is as in case (b) or there is $W \subseteq Z$ such that $4d-4 \leq \deg(W) \leq 4d+2$, $h^1(\mathcal{I}_W(d)) > 0$ and W is contained in a plane curve of degree four (case $s = 5$ of [7] Cor. 2).

(d) Assume $z \geq 36$ and $z \leq 6d-19$. Then, either Z is as in case (c) or there is $W \subseteq Z$ such that $4d-4 \leq \deg(W) \leq 4d+2$, $h^1(\mathcal{I}_W(d)) > 0$ and W is contained in a plane curve of degree four or there is $E \subseteq Z$ such that $5d-10 \leq \deg(E) \leq 5d$, $h^1(\mathcal{I}_E(d)) > 0$ and E is contained in a plane curve of degree five (case $s = 6$ of [7] Cor. 2).

The following result is well-known for finite sets, but we need it for certain very mild zero-dimensional schemes:

Proposition 1. *Assume $d \geq 12$. Let $Z \subset \mathbb{P}^3$ be a zero-dimensional scheme such that $\langle Z \rangle = \mathbb{P}^3$, its connected components have degree ≤ 2, $z := \deg(Z) \leq 3d+3$ and no line contains at least $\lceil (d+2)/2 \rceil + 1$ points of Z_{red}. Assume $h^1(\mathcal{I}_Z(d)) > 0$. Then, one of the following cases occurs:*

(i) There is a line $L \subset \mathbb{P}^3$ such that $\deg(L \cap Z) \geq d+2$;
(ii) There is a conic D such that $\deg(D \cap Z) \geq 2d+2$;
(iii) There is a plane cubic T such that $\deg(T \cap Z) = 3d$ and $T \cap Z$ is the complete intersection of T and a degree d plane curve;
(iv) There is a plane cubic T' such that $\deg(T' \cap Z) \geq 3d+1$;
(v) There is a (reducible) rational normal curve $C \subset \mathbb{P}^3$ such that $\deg(C \cap Z) \geq 3d+2$.

Proof. If $z \leq 3d+1$, then Proposition 1 is just [2] Proposition 6.1. Assume $3d+2 \leq z \leq 3d+3$. Since $\dim |\mathcal{O}_{\mathbb{P}^3}(2)| = 10$, any zero-dimensional scheme of degree ≤ 9 is contained in a quadric.

Take a plane H_1 such that $z_1 := \deg(Z \cap H_1)$ is maximal. If $h^1(\mathcal{I}_{H_1 \cap Z}(d)) > 0$, then we use Remark 5. Thus, we may assume $h^1(\mathcal{I}_{H_1 \cap Z}(d)) = 0$. The residual exact sequence of H_1 gives $h^1(\mathcal{I}_{\mathrm{Res}_{H_1}(Z)}(d-1)) > 0$.

(a) Assume $z_1 \geq 5$. Thus, $\deg(\mathrm{Res}_{H_1}(Z)) = z - z_1 \leq 3(d-1)+1$. If $\langle \mathrm{Res}_{H_1}(Z) \rangle = \mathbb{P}^3$, we use [2] Proposition 6.1 for the integer $d-1$. If $\dim \langle \mathrm{Res}_{H_1}(Z) \rangle \leq 2$, we use Remark 5 for the integer $d-1$. In both cases, we are in one of the cases (i), (ii) or (iii) of Proposition 1 for the integer $d-1$, and all cases are contained in a plane. Hence, in this case, we have $\deg(\mathrm{Res}_{H_1}(Z)) \leq z - z_1 \leq z/2$. Thus, there is a line L such that $\deg(\mathrm{Res}_{H_1}(Z) \cap L) \geq d+1$.

If $\deg(L \cap Z) \geq d+2$, then we are in case (i) for the integer d. Thus, we may assume $\deg(Z \cap L) = d+1$. Take a plane $M \supset L$ such that $z' := \deg(M \cap Z)$ is maximal. Note that $z' \geq d+2$ and hence $z - z' \leq 2d+1 \leq 3(d-1)+1$. If $h^1(\mathcal{I}_{M\cap Z}(d)) > 0$, then we conclude using Remark 5 (more precisely, d is odd and the conic is singular). Thus, we may assume $h^1(\mathcal{I}_{M\cap Z}(d)) = 0$. The residual exact sequence of M gives $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(d-1)) > 0$. As above, we obtain the existence of a line R such that $\deg(Z \cap R) = d+1$ and $Z \cap R = \mathrm{Res}_M(Z) \cap R$. If $R \cap L \neq \emptyset$ (we allow the case $R = L$, but it does not occur by the assumption on Z_{red}), then we are in case (ii) with a singular conic because $L \subset M$ and $\deg(R \cap \mathrm{Res}_M(Z)) = d+1$. Thus, we may assume $R \cap L = \emptyset$. Take a general $Q \in |\mathcal{I}_{R\cup L}(2)|$. Since $\mathcal{I}_{R\cup L}(2)$ is globally generated, $Z \cap Q = Z \cap (R \cup L)$ and hence $h^1(\mathcal{I}_{Q\cap Z}(d)) = 0$. The residual exact sequence of Q gives $h^1(\mathcal{I}_{\mathrm{Res}_Q(Z)}(d-2)) > 0$. Since $\deg(\mathrm{Res}_Q(Z)) \leq d+1$, there is a line J such that $\deg(J \cap \mathrm{Res}_Q(Z)) \geq d$. Assume for the moment that J, R and L are pairwise disjoint. Let Q' be the only quadric containing $J \cup L \cup R$. The quadric Q is smooth and J, L and R are in the same ruling of Q', say the ruling $|\mathcal{O}_{Q'}(1,0)|$. Since $\deg(\mathrm{Res}_{Q'}(Z)) \leq 2$, the residual exact of Q' gives $h^1(Q, \mathcal{I}_{Z\cap Q',Q'}(d)) > 0$. Since $H^1(\mathcal{O}_{Q'}(d-3,d)) = 0$ by the Künneth formula, the restriction map $H^0(\mathcal{O}_{Q'}(d)) \to H^0(\mathcal{O}_{L\cup J\cup R}(d))$ is surjective. Hence, $h^1(\mathcal{I}_Z(d)) = 0$, unless one of the lines L, J or R is as in case (i). Now assume $J \cap (L \cup R) \neq \emptyset$, say $J \cap L \neq \emptyset$. Taking the plane $\langle L \cup J \rangle$, we obtain a contradiction, unless $J \cup R \cup L$ is a reducible rational normal curve.

(b) By step (a), we are allowed to assume $z_1 \leq 4$. Hence, Z is contained in no reducible quadric.

Take any quadric Q such that $\deg(Z \cap Q) \geq 8$.

(b1) Assume $h^1(\mathcal{I}_{\mathrm{Res}_Q(Z)}(d-2)) > 0$. Since $\deg(\mathrm{Res}_Q(Z)) \leq 3d+3-8 = 3(d-2)+1$, we may apply either Remark 5 or ([2] Proposition 6.1). We obtain $z_1 \geq d$, contradicting one of our assumptions.

(b2) Assume $h^1(\mathcal{I}_{Z\cap Q}(d)) > 0$. If $\deg(Z \cap Q) \leq 3d+1$, then we may apply either Remark 5 or [2] Proposition 6.1 to the scheme $Z \cap Q$. Thus, we may assume $\deg(Z \cap Q) \in \{3d+2, 3d+3\}$. Taking $Z \cap Q$ instead of Z, we may assume $Z \subset Q$. We may also assume that Z is contained in every quadric containing at least a degree eight subscheme of Z. Thus, Z is contained in ∞^1 quadrics. Since $z_1 \leq 4$, Z is not contained in a reducible quadric. Thus, Z is contained in the complete intersection T of two quadric surfaces. Note that the restriction map $H^0(\mathcal{O}_{\mathbb{P}^3}(d)) \to H^0(\mathcal{O}_T(d))$ is surjective. Thus, $h^1(T, \mathcal{I}_{Z,T}(d)) > 0$. Looking at the connected components of T, we reduce to the case T connected. If T is irreducible, it is sufficient to use that $\deg(Z) - \#Z_{\mathrm{red}} = 1$ and $\deg(Z) < 4d$ ([2] Remark 3.1). If T is reducible, it is sufficient to perform all possible decompositions of T and use the assumption on Z_{red} (this is performed in [2] Lemma 6.4 for all $z < 4d$, with minor additional assumptions). □

Remark 6. *We take the assumptions on d and z as in Proposition 1, except that now we assume $Z \subset \mathbb{P}^4$ and $\langle Z \rangle = \mathbb{P}^4$. The thesis of Proposition 1 is true and it is easy to prove it by using Proposition 1 and starting with a hyperplane H_1 such that $\deg(Z \cap H_1)$ is maximal.*

Remark 7. *Fix integers n, d and z such that $n \geq 2$, $d \geq 4$ and $z < 3d$. Let $Z \subset \mathbb{P}^n$ be a zero-dimensional scheme such that $\deg(Z) = z$, each connected component of Z has degree ≤ 2 and $h^1(\mathcal{I}_Z(d)) > 0$. By Remark 5, we may assume $n > 2$ and use induction on n. As in the argument of Remark 6, by using hyperplanes instead of planes, we obtain that either there is a line $L \subset \mathbb{P}^3$ such that $\deg(Z \cap L) \geq d+2$ or there is a conic D such that $\deg(D \cap Z) \geq 2d+2$.*

Question 5: Take a large integer d. Are Proposition 1 or Remarks 6 and 7 still true for any degree z zero-dimensional scheme? Are they true at least for curvilinear zero-dimensional schemes, i.e., for the zero-dimensional schemes whose connected components may be embedded in a smooth curve?

Solving the last question would be a key step for the study of the cactus rank of homogeneous polynomials ([8–10]) and would also have nice consequences for computing the higher Hamming weights of certain evaluation codes ([3,4] and several papers quoting them). When the evaluation code comes from a smooth curve, it would be sufficient to study Question 5 for curvilinear schemes because all zero-dimensional subschemes of a smooth curve are curvilinear.

3. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. We outline its proof. Assume, by contradiction, that $\mathcal{T}(4,d;x)' \neq \emptyset$ and fix $S \in \mathcal{T}(4,d;x)'$. Let Z be a critical scheme of S (Definition 1). The proof is divided into two parts.

In step (a), we consider the case $x \le 2d-1$. We prove this case by taking a general linear projection Z from a general point of $\mathbb{P}^4$ into $\mathbb{P}^3$ and applying several results of [2] to the image of Z. A key ingredient of step (a) is the definition of the set $\mathcal{V}$ of allowable points of projection.

In step (b), we consider the case $x = 2d$. We take $Q \in |\mathcal{I}_Z(2)|$ containing at least five connected components of Z. If $Z \nsubseteq Q$, we reduce to step (a) for the integer $d-2$. If $Z \subset Q$, we use several residual exact sequences with respect to hyperplanes of $\mathbb{P}^4$ and use Proposition 1 for the intersection of Z with these hyperplanes.

Proof of Theorem 1: Set $z := \deg(Z)$. Since each connected component of Z has a degree at most of two, $z \le 2x$.

(a) Assume $x \le 2d-1$. For any $o \in \mathbb{P}^4$, let $\ell_o : \mathbb{P}^4 \setminus \{o\} \to \mathbb{P}^3$ denote the linear projection from o. Since each connected component of Z has degree ≤ 2, there is a nonempty open subset $\mathcal{U}$ of $\mathbb{P}^4 \setminus S$ such that for all $o \in \mathcal{U}$, the morphism $\ell_{o|Z}$ is an embedding. By the semicontinuity theorem for cohomology restricting if necessary $\mathcal{U}$, we may assume that all degree z schemes $\ell_o(Z)$, $o \in \mathcal{U}$, have the same Hilbert function. Fix $o \in \mathcal{U}$ and set $Z' := \ell_o(Z)$ and $S' := \ell_o(S)$. Since $\langle S \rangle = \mathbb{P}^4$, we have $\langle S' \rangle = \mathbb{P}^3$. Take homogeneous coordinates x_0, x_1, x_2, x_3, x_4 of $\mathbb{P}^4$ such that $o = [0:0:0:0:1]$. For any $p = [p_0 : p_1 : p_2 : p_3 : p_4] \neq o$, we have $\ell_o(p) = [p_0 : p_1 : p_2 : p_3]$. For any constant $t \neq 0$, define $h_t \in \mathrm{Aut}(\mathbb{P}^4)$ by the formula $h_t([p_0 : p_1 : p_2 : p_3 : p_4]) = [p_0 : p_1 : p_2 : p_3 : tp_4]$. We see that $\ell_o(Z)$ is a flat limit of the family $h_t(Z)$ of projectively equivalent schemes. The semicontinuity theorem for cohomology gives $h^0(\mathbb{P}^3, \mathcal{I}_{\ell_o(Z)}(d)) > 0$. We have $h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)) = \binom{d+3}{3}$. Since $4x < \binom{d+3}{3}$, $h^0(\mathbb{P}^3, \mathcal{I}_{\ell_o(Z)}(d)) > 0$. Thus, $S' \in \mathcal{T}(3,d;x)$. Our assumptions on d and x are the ones made in [2] Theorem 1.4. Thus, $S' \notin \mathcal{T}(3,d,x)'$. Since the critical scheme Z of S has only finitely many subschemes, there is a nonempty open subset $\mathcal{V}$ of $\mathcal{U}$ such that $\dim\langle \ell_o(Z'') \rangle = \min\{3, \dim\langle Z'' \rangle\}$ for all $o \in \mathcal{V}$. From now on, we assume $o \in \mathcal{V}$. Since $S \in \mathcal{T}(4,d;3)'$, we obtain that there is scheme $A \subset Z$ such that $\dim\langle \ell_o(A) \rangle \le 2$ and $h^1(\mathcal{I}_{\ell_o(A)}(d)) > 0$. Hence, a minimal $Z'' \subset Z'$ with $h^1(\mathbb{P}^3, \mathcal{I}_{Z''}(d)) > 0$ has $\langle Z'' \rangle = \mathbb{P}^3$. Set $S'' := Z''_{\mathrm{red}}$ and $G := S \cap \ell_o^{-1}(S'')$. The minimality of Z'' gives $h^1(\langle Z'' \rangle, \mathcal{I}_{Z'',\langle Z'' \rangle}(d)) = 1$ and $h^1(\langle Z'' \rangle, \mathcal{I}_{Z'_1,\langle Z'' \rangle}(d)) = 0$ for every $Z'_1 \subsetneq Z''$.

(a1) Assume that $M' := \langle S'' \rangle$ has dimension ≤ 2. We exclude the case $\dim M' = 1$ because the minimality of S implies that S does not contain $\lceil d/2 \rceil + 1$ collinear points. Thus, M' is a plane. The definition of $\mathcal{V}$ gives that $\langle G \rangle$ is a plane and $\ell_{o|G} : G \to S''$ is a linear isomorphism of plane subsets. Consider the residual exact sequence of M' in $\mathbb{P}^3$:

$$0 \to \mathcal{I}_{S'',\mathbb{P}^3}(d-1) \to \mathcal{I}_{(2S'',\mathbb{P}^3),\mathbb{P}^3}(d) \to \mathcal{I}_{(2S'',M'),M'}(d) \to 0 \tag{2}$$

By assumption, $h^1(\mathcal{I}_{(2S'',\mathbb{P}^3)}(d)) > 0$. If $h^1(M', \mathcal{I}_{(2S'',M'),M'}(d)) > 0$, we obtain the inequality $h^1(\mathcal{I}_{2G}(d)) > 0$, and hence S is not minimal. By (2), we obtain $h^1(\mathcal{I}_{S''}(d-1)) > 0$, and hence $h^1(\mathcal{I}_G(d-1)) > 0$. Since $\#G \le 2(d-1)+1$, there is a line containing at least $d+1$ points of G ([9] Lemma 34). Thus, S is not minimal.

(a2) Assume $\langle S'' \rangle = \mathbb{P}^3$. Since $d \ge 17$, $4x < \binom{d+3}{3}$. Hence, $h^0(\mathbb{P}^3, \mathcal{I}_{2S'',\mathbb{P}^3}(d)) > 0$. Recall that Z'' is minimal and $\mathcal{T}(3,d;y)' = \emptyset$ for all $y \le 2d-1$ such that $y \neq \lceil 3d/2 \rceil + 1$. By [2] Theorem 1.3, there is a rational normal curve C containing Z'' and $\deg(Z'') \in \{3d+2, 3d+3\}$. A rational normal curve $C \subset \mathbb{P}^3$ such that $\#(\ell_o(Z) \cap C) \ge 3d+2$ must be the linear projection of a rational normal curve of a hyperplane of $\mathbb{P}^4$ because Z has finitely many subschemes and we may take o outside the finitely many rational normal curves of $\mathbb{P}^4$ containing a subschemes of Z of degree at least $3d+2$.

First assume $Z' \neq Z' \cap C$. Since $\mathcal{I}_{C,\mathbb{P}^3}(2)$ is globally generated and each connected component of Z' has degree ≤ 2, $Q \cap Z' = C \cap Z'$ for a general $Q \in |\mathcal{I}_{C,\mathbb{P}^3}(2)|$. Let $Q_1 \subset \mathbb{P}^4$ denote the quadric cone with a vertex containing o such that $\ell_o(Q_1 \setminus \{o\}) = Q$. Since $Z' \nsubseteq Q$, $Z \nsubseteq Q_1$. Since S is minimal, $h^1(\mathcal{I}_{Z\cap Q_1}(d)) = 0$. The residual exact sequence of Q_1 gives $h^1(\mathcal{I}_{\mathrm{Res}_{Q_1}(Z)}(d-2)) > 0$. Since $\deg(\mathrm{Res}_{Q_1}(Z)) \le 4d-2-3d-2 \le d-1$, we obtain a contradiction. Now assume $Z' = Z' \cap C$. Since C may depend on o, we call it $C(o)$. The rational normal curve $C(o)$ is unique (for a fixed $o \in \mathcal{V}$) because $\deg(Z') > 6$. Call C_o the cone with vertex o and $C(o)$ as its base. Since this is the only remaining case to consider for $x < 2d$, this would occur for all $o \in \mathcal{V}$. We obtain that Z is contained in all two-dimensional cubic cones C_o, $o \in \mathcal{V}$. Fix $o, a \in \mathcal{V}$ such that $a \neq o$. Since C_a is cut out by quadrics, $C_a \cap C_o$ is strictly contained in the complete intersection of C_o and a quadric, which is a degree six scheme, counting the multiplicities of its connected components. Since $\deg(Z) > 16$, the set $\Delta := \cap_{u\in\mathcal{V}} C_u$ contains a curve, T_1, (maybe with multiple components) with $\deg(T_1) < 6$. Taking $u \in \mathcal{V} \setminus C_o \cap \mathcal{V}$, we obtain that $C_u \cap C_o$ contains no line. Since $\ell_o(C_o \setminus \{o\}) = C(o)$, we obtain that T_1 has no component of degree 2 and that either it is linearly isomorphic to $C(o)$ (and hence it must be a linear section of C_o not containing o) or it is a rational normal curve of $\mathbb{P}^4$ containing o. Taking a general $u \in \mathcal{V}$ instead of o, we exclude the latter case.

Take a general $u \in \mathcal{V}$ and a general $Q \in |\mathcal{I}_{C_u}(2)|$. We obtain $C_o \cap Q = T_1 \cup T_2$ with T_2, another hyperplane section of C_o. Taking a different general $Q' \in |\mathcal{I}_{C_u}(2)|$, we obtain that $Z \setminus Z \cap T_1$ has a degree ≤ 6. Since $\langle S \rangle = \mathbb{P}^4$, the residual exact sequence of $\langle T_1 \rangle$ gives that S is not minimal.

(b) Assume $x = 2d$ and $d \ge 19$. Fix $A \subset S$ such that $\#A = 5$ and call E the union of the connected components of Z with a point of A as their reduction. We have $\dim |\mathcal{I}_E(2)| \ge 4$. Take a general $U \in |\mathcal{I}_E(2)|$.

(b1) Assume $Z \nsubseteq U$. Since S is minimal and $Z \nsubseteq U$, the residual exact sequence of U gives $h^1(\mathcal{I}_{\mathrm{Res}_U(Z)}(d-2)) > 0$. Set $F := \mathrm{Res}_U(Z)_{\mathrm{red}}$. Since $F \subseteq S \setminus A$, hence, $\#F \le 2(d-2)-1$. Thus, we may apply part (a) and [2] Th. 1.3, 1.4, 1.5 to F. As in step (a), we take a general $o \in \mathbb{P}^4$ and set $Z' := \ell_o(Z)$ with $\deg(Z') = z$. Since $h^1(\mathcal{I}_{\mathrm{Res}_U(Z)}(d-2)) > 0$, we may apply step (a) for the integer $d-2$. Thus, we obtain that one of the following possibilities occurs:

1. There is $F_1 \subseteq F$ such that $\#F_1 = \lceil d/2 \rceil$, $L_1 := \langle F_1 \rangle$ is a line and $\deg(W_1 \cap L_1) \ge d$, where W_1 is the union of the connected components of Z with a point of F_1 as their reduction;
2. There is a conic C_2 containing $W_2 \subset \ell_o(\mathrm{Res}_U(Z))$ with $\deg(W_2) \ge 2d-2$;
3. There is a plane cubic C_3 such that $W_3 := C_3 \cap \ell_o(\mathrm{Res}_U(Z))$ is the complete intersection of C_3 and a degree $d-2$ curve of $\langle C_3 \rangle$;
4. There is a plane cubic containing $W_4 \subset \ell_o(\mathrm{Res}_U(Z))$ with $\deg(W_4) \ge 3(d-2)+1$;
5. There is a rational normal curve C_5 of a hyperplane of $\mathbb{P}^4$ containing $W_5 \subset \ell_o(\mathrm{Res}_U(Z))$ with $\deg(W_5) \ge 3d-4$.

We recall that (as in step (a)) for any $A \subseteq Z$, we have $\dim\langle \ell_o(Z) \rangle = \min\{3, \dim\langle \ell_o(Z) \rangle\}$, and hence each subscheme of Z' contained in a plane (resp. a line) comes from a subscheme of Z contained in a plane (resp. a line).

(b1.1) Assume the existence of W_5. Since a rational normal curve C_5 of a hyperplane of $\mathbb{P}^4$ is scheme-theoretically cut out by quadric hypersurfaces and each connected component of Z has degree ≤ 2, $Q \cap Z = C_5 \cap Z$ for a general $Q \in |\mathcal{I}_{C_5}(2)|$. If $\deg(Z \cap C_5) \ge 3d+4$, then S is not minimal. If $Z \nsubseteq C_5$, then $\deg(\mathrm{Res}_Q(Z)) \le d-2$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_Q(Z)}(d-2)) = 0$, contradicting the minimality of S.

(b1.2) Assume the existence of W_i with $i \in \{3,4\}$ and call C_3 the plane cubic containing W_i. Take a general hyperplane H' containing $\langle C_3 \rangle$. Since $\langle S \rangle = \mathbb{P}^4$ and S is minimal, $h^1(\mathcal{I}_{\mathrm{Res}_{H'}(Z)}(d-1)) > 0$. Since $\deg(\mathrm{Res}_{H'}(Z)) \le 2(d-1)+1$, there is a line L' such that

$\deg(L' \cap \mathrm{Res}_{H'}(Z)) \ge d+1$. Since S is minimal, $\deg(L' \cap Z) = d+1$. For a general H' containing $\langle C_3 \rangle$, we have $Z \cap H' = Z \cap \langle C_3 \rangle$. For a general hyperplane $H'' \supset L'$ we have $H'' \cap Z = L \cap Z$. Thus, $Z \subset L' \cup \langle C_3 \rangle$ and $Z \cap L' \cap \langle C_3 \rangle = \emptyset$. We also obtain that d is odd, $\#(S \cap L') = (d+1)/2$ and $\#(S \cap \langle C_3 \rangle) = (3d-1)/2$. Since $\langle S \rangle = \mathbb{P}^4$, we obtain $L' \cap \langle C_3 \rangle = \emptyset$. Thus, there is a hyperplane $H_1 \supset \langle C_3 \rangle$ containing exactly one point of $S \cap L_1$. Since $\deg(\mathrm{Res}_{H_1}(Z)) \le d$, the residual exact sequence of H_1 contradicts the minimality of S.

(b1.3) Assume the existence of W_2. Take a general hyperplane H_2 containing $\langle W_2 \rangle$. Since $\deg(\mathrm{Res}_{H_2}(Z)) \le 2d+2$ and $h^1(\mathcal{I}_{\mathrm{Res}_{H_2}(Z)}(d-1)) > 0$, either there is a conic D'' such that $\deg(D'' \cap \mathrm{Res}_{H_2}(Z)) \ge 2d$ or there is a line L'' such that $\deg(L'' \cap \mathrm{Res}_{H_2}(Z)) \ge d+1$.

(b1.3.1) Assume the existence of D''. Taking a general hyperplane H_3 containing D'', we obtain that either $\deg(W_2) \ge 2d$ or there is a line R such that $\deg(\mathrm{Res}_{H_3}(Z) \cap R) \ge d+1$. First assume $\deg(W_2) \ge 2d$. Since $\langle S \rangle = \mathbb{P}^4$, $h^1(\mathcal{I}_W(d-1)) = 0$ for every scheme of degree $\le d$ and S is minimal, we first obtain $S \cap \langle W_2 \rangle \cap \langle D' \rangle = \emptyset$ and then $Z \subset D'' \cup C_2$ with $\deg(Z \cap D'') = \deg(Z \cap C_2) = 2d$. Take a hyperplane M containing $\langle D'' \rangle$ and a point of F_2. Since $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(d-1)) > 0$, we obtain that C_2 is reducible with one of its components, R', such that $\deg(Z \cap R') = d+1$. However, taking as M a hyperplane containing a point of $S \cap R'$, we obtain a contradiction.

Now assume the existence of R. Since $z \le 4d$, R must be a component of C_2. Since S is minimal, $R \cap C'' = \emptyset$. Taking a general quadric Q containing $R \cup D''$, we obtain a contradiction because $W_2 \nsubseteq Q$ and $\deg(\mathrm{Res}_Q(Z)) \le d-1$.

(b1.3.2) Assume the existence of L''. Since S is minimal, $\deg(L'' \cap Z) = d+1$. Take a general hyperplane U_2 containing L''. Since $Z \cap (H_2 \cup U_2) = Z \cap (\langle W_2 \rangle \cup L'')$, $Z \nsubseteq H_2 \cup U_2$. Since $\deg(\mathrm{Res}_{H_2 \cup U_2}(Z)) \le d+3$, there is a line J such that $\deg(J \cap \mathrm{Res}_{H_2 \cup U_2}(Z)) \ge d$. Take a hyperplane U_3 containing $L'' \cup J$. Since $\deg(\mathrm{Res}_{U_3}(Z)) \le 2d-1$, either $\deg(W_2 \setminus W_2 \cap U_3) \ge 2d$ (and we excluded this case in step (b1.3.1)) or there is a line J_1 such that $\deg(\mathrm{Res}_{U_3}(Z) \cap J_1) \ge d+1$ (it may be an irreducible component of C_2). Taking a general quadric hypersurface containing $L'' \cup J \cup L_1$, we obtain a contradiction.

(b1.4) Assume the existence of F_1. Set $L_1 := \langle F_1 \rangle$. Since S is minimal, $d \le \deg(Z \cap L_1) \le d+1$. Take a hyperplane H containing the line L_1 and spanned by $Z \cap H$. Note that $\deg(H \cap Z) \ge d+2$. Since S is minimal, $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)) > 0$. Assume for the moment $\dim\langle \mathrm{Res}_H(Z) \rangle \ge 3$. Since $\deg(\mathrm{Res}_H(Z)) \le 3d-3 = 3(d-1)$ and $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)) > 0$, either there is a line L_2 such that $\deg(L_2 \cap \mathrm{Res}_H(Z)) \ge d+1$ or there is a conic D such that $\deg(D \cap \mathrm{Res}_H(Z)) \ge 2d$ or there is a plane cubic D' with $\deg(D' \cap \mathrm{Res}_H(Z)) = 3d-3$. We excluded the existence of D in step (b1.3) and the existence of D' in step (b1.2). Thus, L_2 exists. Since S is minimal, $\deg(L_2 \cap Z) = d+1$ and hence $S \cap L_1 \cap L_2 = \emptyset$. Take a general hyperplane H_1 containing $L_1 \cup L_2$. Since $\langle S \rangle = \mathbb{P}^4$, $\mathrm{Res}_{H_1}(Z) \ne \emptyset$. Since S is minimal, $h^1(\mathcal{I}_{\mathrm{Res}_{H_1}(Z)}(d-1)) > 0$. Since $\deg(\mathrm{Res}_{H_1}(Z)) \le 4d-2d-1 = 2(d-1)+1$, there is a line L_2 such that $\deg(\mathrm{Res}_{H_2}(Z)) = \emptyset$. Since S is minimal, we first obtain $Z \cap L_2 \cap L_3 = \emptyset$ and then $L_2 \cap L_3 = \emptyset$. Call H_3 the hyperplane $\langle L_2 \cup L_3 \rangle$. Since S is minimal, $L_1 \nsubseteq H_3$ and hence $\deg(\mathrm{Res}_{H_3}(Z)) \ge d-1$. Since $\deg(\mathrm{Res}_{H_3}(Z)) \le 2d-2$ and $h^1(\mathcal{I}_{\mathrm{Res}_{H_3}(Z)}(d-1)) > 0$, we obtain $\deg(Z \cap L_1) = d+1$ and $Z \cap L_1 \cap H_3 = \emptyset$. Since S is minimal, we also obtain $L_1 \cap L_2 = L_1 \cap L_3 = \emptyset$. Thus, $\mathcal{I}_{L_1 \cup L_2 \cup L_3}(2)$ is globally generated and hence $Z \cap Q = Z \cap (L_1 \cup L_2 \cup L_3)$ has degree $3d+3$. Thus, $Z \nsubseteq Q$ and $h^1(\mathcal{I}_{\mathrm{Res}_Q(Z)}(d-2)) = 0$, a contradiction.

Now assume $\dim\langle \mathrm{Res}_H(Z) \rangle \le 2$. Since $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)) > 0$, either there is a line R such that $\deg(R \cap \mathrm{Res}_H(Z)) \ge d+1$ or there is a conic R_2 such that $\deg(R_2 \cap \mathrm{Res}_H(Z)) \ge 2d$ or there is a plane cubic R_3 such that $\deg(R_3 \cap \mathrm{Res}_H(Z)) \ge 3d-3$ (Remark 5). We excluded the existence of R_2 and R_3 in steps (b1.2) and (b1.3). Thus, there is R. We use L_1 and R as we used L_1 and L_2 in the first part of step (b1.4).

(b2) Assume $Z \subset U$. Since U is general in $|\mathcal{I}_E(2)|$, we obtain $|\mathcal{I}_Z(2)| = |\mathcal{I}_E(2)|$. By part (b1), we have $|\mathcal{I}_Z(2)| = |\mathcal{I}_E(2)|$ for all $A \subset S$ such that $\#A = 5$. Assume for the moment that S is not in a linear general position, i.e., there is a hyperplane H containing at least five points of S. Take $A \subseteq S \cap H$ such that $\#A = 5$ and take the quadric $2H$. Since $Z \subset 2H$ by the assumption of step (b2), we obtain $S \subset H$, a contradiction. Thus, S is in the linear general position and hence every hyperplane contains at most a degree eight subscheme of Z. Set $Z_0 := Z$. Let H_1 be a hyperplane such that $z_1 := \deg(Z \cap H_1)$ is maximal. Set $Z_1 := \mathrm{Res}_{H_1}(Z)$. Fix an integer $i \geq 2$ and assume the hyperplane H_j, the integer z_j and the scheme Z_j for all $j = 1, \dots, i-1$. Take a hyperplane H_i such that $z_i := \deg(H_i \cap Z_{i-1})$ is maximal and set $Z_i := \mathrm{Res}_{H_i}(Z_{i-1})$. The sequence $\{z_i\}$ is weakly decreasing and $4d = 2x \geq z = \sum_{i \geq 1} z_i$. Since $\dim |\mathcal{O}_{\mathbb{P}^4}(1)| = 4$, $Z_i = \emptyset$ if $z_i \leq 4$. Thus, there is a minimal integer $e \leq d$ such that $\deg(Z_e) \leq 1$. Since $h^1(\mathcal{I}_{Z_e}(d-e)) = 0$ and S is minimal, $Z \subset H_1 \cup \cdots \cup H_e$. Set $T := H_1 \cup \cdots \cup H_{e-1}$. We have $Z_{e-1} = \mathrm{Res}_T(Z)$ and $z_e = \deg(Z_{e-1}) \geq 2$. Note that $z_{e-1} \geq z_e$ and that $z_{e-1} \geq z_e + 3 - m$ if $\dim\langle Z_{e-1}\rangle = m$. Since S is minimal, $h^1(\mathcal{I}_{Z_{e-1}}(d-e+1)) > 0$ and hence $z_e \geq d-e+3$. Moreover, either $z_e \geq 3(d-e+1)+1$ or $d-e+4 \leq z_e \leq 3(d-e+1)$ and Z_{e-1} is contained in a plane or $z_e = d-e+3$ and Z_{e-1} spans a line. Thus, in all cases $z_{e-1} \geq d-e+5$. Recall that $z \geq (e-1)z_{e-1} + z_e$ and $z_{e-1} \leq z_1 \leq 8$. Since $e \leq d$ and $d \geq 19$, we first obtain $e \in \{d-2, d-1, d\}$ and then $z \geq 5d$, a contradiction. □

4. Beyond Theorem 1

We recall the definition of a reducible rational normal curve ([2] §4.1). Let $T \subset \mathbb{P}^n$ be a reduced, connected and degree n curve spanning $\mathbb{P}^n$. If T is irreducible, then it is a rational normal curve. Now assume that T has $s \geq 2$ irreducible components. Since T is connected, there is an ordering $T_1, \dots, T_s$ of the irreducible components of T (called a *good ordering*) such that each $T_1 \cup \cdots \cup T_i$, $1 \leq i \leq s$, is connected. Set $d_i := \deg(T_i)$. For any reduced and connected curve M, let $p_a(M)$ denote its arithmetic genus, i.e., set $p_a(M) := h^1(\mathcal{O}_M)$. We have $p_a(T_1 \cup \cdots \cup T_i) = 0$, the linear span of each $T_1 \cup \cdots T_i$ has dimension $d_1 + \cdots + d_i$ and $T_1 \cup \cdots \cup T_i$ is a (reducible) rational normal curve in its linear span. Note that $\mathrm{Sing}(T)$ is the set of all points of T contained in at least two irreducible components of T. An irreducible component T_i of T is said to be a *final* component if $\#(\mathrm{Sing}(T) \cap T_i) = 1$. In any good ordering $T_1, \dots, T_s$ of T, T_1 and T_s are final components, but there may be other final components (e.g., take at T the union of n general lines though some point of $\mathbb{P}^n$). By [2] Proposition 4.7, there are very strong restrictions for the existence of $S \in \mathcal{T}(n,d;\lceil (nd+2)/2 \rceil)'$ contained in a reducible normal rational curve T: n is even, d is odd, $S \subset T_{\mathrm{reg}}$, all final components have an odd degree and $\#(S \cap T_i) = (d_i d+1)/2$ for all i.

Now assume $n = 4$. We have $d_1 + \cdots + d_s = 4$ and hence $2 \leq s \leq 4$. Take a reducible rational normal curve $T \subset \mathbb{P}^4$. To have some $S \in \mathcal{T}(4,d;2d+1)'$ with $S \subset T$, we also need d the be odd, $S \subset T_{\mathrm{reg}}$ and $\#(S \cap T_i) = d_i(d+1)/2$ for all final components T_i ([2] Proposition 6.1). Thus, either $s = 2$ and $\{d_1, d_2\} = \{1, 3\}$ or $3 \leq s \leq 4$ and all final components are lines. All $s \in \{2,3,4\}$ and $d_1, \dots, d_s$ with $d_1 + \cdots + d_s = 4$ occur for some reducible normal curve of $\mathbb{P}^4$ (Remark 8 and Proposition 2). If $d_i = d_j = 1$ and $\#(S \cap T_i) = \#(S \cap T_j) = (d+1)/2$ for some $i \neq j$, then $T_i \cap T_j = \emptyset$, because no reducible conic contained in T contains $d+1$ points of S. If we also prescribe that all final components of T have an odd integer as $\#(S \cap T_i)$ ([2] Proposition 6.1), then we obtain the following list:

Remark 8. *Assume $d_1 \geq d_2$ if $s = 2$. With this restriction, we only have the following cases:*

1. $s = 2$, $d_1 = 3$, $d_2 = 1$;
2. $s = 3$, $d_1 = d_3 = 1$, $d_2 = 2$;
3. $s = 4$, $d_1 = d_2 = d_3 = d_4 = 1$, *$T_1$ and T_4 are final components, T_2 and T_3 are not final components;*
4. $s = 4$, $d_1 = d_2 = d_3 = d_4 = 1$; *$T$ is nodal; and T_1, T_2, T_4 are final components, while* $\#(T_3 \cap T_i) = 1$ *for* $i = 1,2,4$.

Take one of the four cases just listed. Take $S \in \mathcal{T}(4,d;2d+1)'$ such that $S \subset T_{\mathrm{reg}}$ and set $x_i := \#(S \cap T_i)$. Since $S \subset T_{\mathrm{reg}}$, we have $x_1 + \cdots + x_4 = 2d+1$. In case (1), we have $d_1 = (3d+1)/2$ and $d_2 = (d+1)/2$. In case (2), we have $x_1 = x_3 = (d+1)/2$ and $x_2 = d$.

In case (4), we have $x_1 = x_2 = x_4 = (d+1)/2$ and $x_3 = (d-1)/2$. Now we exclude case (3). We must have $x_1 = x_4 = (d+1)/2$ and hence $x_2 + x_3 = d$. With no loss of generality, we may assume $x_2 \geq x_3$. Since d is odd, we have $x_2 \geq (d+1)/2$ and hence the reducible conic $T_1 \cup T_2$ contains at least $d+1$ points of S. Thus, S is not minimal.

Proposition 2. *Take an odd d, $d \geq 5$ and a reducible rational normal curve $T = T_1 \cup \cdots \cup T_s \subset \mathbb{P}^4$ as in case (1) or (2) or (4) of Remark 8 and take $S \subset T_{\mathrm{reg}}$. Set $x_i := \#(S \cap T_i)$. In case (1), assume $x_1 = (3d+1)/2$ and $x_2 = (d+1)/2$. In case (2), assume $x_1 = x_2 = (d+1)/2$ and $x_3 = d$. In case (4), assume $x_1 = x_2 = x_4 = (d+1)/2$ and $x_3 = (d-1)/2$. Then, $S \in \mathcal{T}(4, d; d+1)'$.*

Proof. Obviously, $\langle S \rangle = \mathbb{P}^4$. Since $h^0(\mathcal{O}_T(d)) = 4d+1$ and $\deg(2S \cap T) = 4d+2$, $h^1(\mathcal{I}_{2S}(d)) > 0$. Thus, $S \in \mathcal{T}(4, d; 2d+1)$. Fix $S' \subset S$ such that $\#S = 2d$ and set $A_i := S' \cap T_i$ and $y_i := \#(A_i)$. Note that $y_i = x_i$ for $s-1$ irreducible components of T and $y_i = x_i - 1$ for the other component of T. To prove that S is minimal, it is sufficient to prove that $h^1(\mathcal{I}_{2S'}(d)) = 0$. Since the restriction map $H^0(\mathcal{O}_{\mathbb{P}^4}(d)) \to H^0(\mathcal{O}_T(d))$ is surjective, it is sufficient to prove that $h^1(T, \mathcal{I}_{A,T}(d)) = 0$. This is performed by using $s-1$ Mayer–Vietoris exact sequences. $\square$

Remark 9. *The lowest integer $x \geq 2d+2$ that we know such that $\mathcal{T}(4, d; x)' \neq \emptyset$ is the integer $\lceil 5d/2 \rceil$. We construct an element of $\mathcal{T}(4, d; \lceil 5d/2 \rceil)'$ in the following way. Let $C \subset \mathbb{P}^4$ be an integral and linearly normal degree five curve such that $p_a(C) = 1$. If d is odd, any $S \subset Y_{\mathrm{reg}}$ such that $\#S = (5d+1)/2$ is an element of $\mathcal{T}(4, d; (5d+1)/2)'$. If d is even, take $S \in |\mathcal{L}|$, where $\mathcal{L}$ is a degree $5d/2$ line bundle on Y such that $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_Y(5)$.*

5. Primitive Terracini Loci

Remark 10. *Since $\mathcal{T}_1(1, d; x) = \emptyset$ for all d and x ([2] Lemma 3.4), $\tilde{\mathcal{T}}(1, d; x) = \emptyset$ for all d and x.*

Remark 11. *Since $\mathcal{T}(n, 2; x) = \emptyset$ for all x ([2] Lemma 3.6), $\tilde{\mathcal{T}}(n, 2; x) = \emptyset$ for all n and x.*

By Remarks 10 and 11, we may assume $n \geq 2$ and $d \geq 3$.

The following result is the main difference between $\tilde{\mathcal{T}}(n, d; x)$ and $\mathcal{T}(n, d; x)$. The latter is nonempty for all $x \gg 0$ if $(n, d) \neq (2, 3)$ by [2] Th. 1.1(iii).

Theorem 2. *Fix integers $n \geq 2$, $d \geq 3$ and $x \geq 1 + \lceil (\binom{n+d}{n} + 1)/(n+1) \rceil$. Then, $\tilde{\mathcal{T}}(n, d; x) = \emptyset$*

Proof. Assume $\tilde{\mathcal{T}}(n, d; x) \neq \emptyset$ and take $S \in \tilde{\mathcal{T}}(n, d; x)$. Take $E \subset S$ such that $\langle E \rangle = \mathbb{P}^n$ and $\#E = n+1$. Take $B \subset S \setminus E$ such that $\#B = x - n - 2$ and set $A := E \cup B$. We have $\#A = x - 1$ and hence $\deg(2A) > \binom{n+d}{n}$. Thus, $h^1(\mathcal{I}_{2A}(d)) > 0$. Since $\langle A \rangle = \mathbb{P}^n$, S is not primitive, a contradiction. $\square$

The following result shows that, often, $\mathcal{T}(n, d; x)'$ and $\tilde{\mathcal{T}}(n, d; x)$ are quite different.

Theorem 3. *Fix integers $n \geq 2$ and $d \geq 3$.*

(a) If $x \leq \lceil d/2 \rceil + n - 1$, then $\tilde{\mathcal{T}}(n, d; x) = \emptyset$.

(b) We have $\tilde{\mathcal{T}}(n, d; \lceil d/2 \rceil + n) \neq \emptyset$. For any $S \in \tilde{\mathcal{T}}(n, d; \lceil d/2 \rceil + n)$, there is a line L such that $\#(S \cap L) = \lceil d/2 \rceil + 1$.

(c) Assume $d \geq 5$ and $\lceil d/2 \rceil + n < x \leq d + n - 2$. We have $\tilde{\mathcal{T}}(n, d; x) = \emptyset$.

(d) Assume $d \geq 5$. We have $\tilde{\mathcal{T}}(n, d; d+n-1) \neq \emptyset$. For any $S \in \tilde{\mathcal{T}}(n, d; d+n-1)$, there is a reduced conic D such that $\#(S \cap D) = d+1$; if d is even, then D is smooth; if d is odd, then D may be reducible with each component containing $(d+1)/2$ points of S.

Proof. Take $S \in \tilde{\mathcal{T}}(n, d; x)$ and let Z be a critical scheme of S. Set $z := \deg(Z)$. We have $z \leq 2x$ and $h^1(\mathcal{I}_Z(d)) = 1$. Since $2x \leq 2d+1$, there is a line $L \subset \mathbb{P}^n$ such that $\deg(Z \cap L) \geq d+2$ ([9] Lemma 34). Hence, $\#(S \cap L) \geq \lceil d/2 \rceil + 1$. Since $\langle S \rangle = \mathbb{P}^n$, there is

$E \subsetneq S \setminus S \cap L$ such that $\#E = n-1$ and $\langle E \cup L\rangle = \mathbb{P}^n$. We obtain part (a). Taking any $F \subset L$ such that $\#F = \lceil d/2\rceil + 1$ and taking $S := E \cup F$, we also obtain part (b).

Using Remark 7 instead of [9] Lemma 34, we obtain parts (c) and (d). □

The following observation gives another difference between $\tilde{\mathcal{T}}(n,d;x)$ and $\mathcal{T}(n,d;x)'$ ([2] Lemma 2.11).

Remark 12. *Fix $S \in \tilde{\mathcal{T}}(n,d;x)$ and let Z be a critical scheme of S. Part (b) of Theorem 3 shows that sometimes $Z_{\mathrm{red}} \neq S$.*

Lemma 2. *Fix positive integers n and d. Take a finite set $S \subset \mathbb{P}^n$. If $h^1(\mathcal{I}_{2S}(d)) = 0$, then $h^1(\mathcal{I}_S(d-1)) = 0$.*

Proof. Assume that the lemma fails for n and d and take $S \subset \mathbb{P}^n$ with minimal cardinality for which it fails. Note that $S \neq \emptyset$. Fix $p \in S$ and set $A := S \setminus \{p\}$. Since $A \subset S$, $h^1(\mathcal{I}_{2A}(d)) \le h^1(\mathcal{I}_{2S}(d)) = 0$. Hence, $h^0(\mathcal{I}_{2S}(d)) = h^0(\mathcal{I}_{2A}(d)) - n - 1$. The minimality assumption for S implies $h^1(\mathcal{I}_A(d)) = 0$. Thus, $h^1(\mathcal{I}_S(d-1)) = 1$ and $|\mathcal{I}_A(d-1)| = |\mathcal{I}_S(d-1)|$. Take a system of homogeneous coordinates $x_0, \dots, x_n$ of $\mathbb{P}^n$. Fix $f \in H^0(\mathcal{I}_{2A}(d))$. Take $o \in A$ and $i \in \{0,\dots,n\}$. Since f vanishes on $2o$, $f_i := \partial f/\partial_{x_i}$ vanishes at o. Since $|\mathcal{I}_A(d-1)| = |\mathcal{I}_S(d-1)|$, $f_i(p) = 0$. Since $\sum_{i=0}^n x_i f_i = df$, $f(p) = 0$. Thus, p is in the base locus of $|\mathcal{I}_{2A}(d)|$. Hence, $h^0(\mathcal{I}_{2S}(d)) \ge h^0(\mathcal{I}_{2A}(d)) - n$, a contradiction. □

Proposition 3. *If $\tilde{\mathcal{T}}(n-1,d;x-1) \neq \emptyset$, then $\tilde{\mathcal{T}}(n,d;x) \neq \emptyset$. Moreover, $\dim \tilde{\mathcal{T}}(n,d;x) \ge 2n + \dim \tilde{\mathcal{T}}(n,d;x-1)$.*

Proof. Fix a hyperplane $H \subset \mathbb{P}^n$ and $p \in \mathbb{P}^n \setminus H$. Take $A \in \tilde{\mathcal{T}}(n-1,d;x-1)$. Identify $\mathbb{P}^{n-1}$ with the hyperplane H and hence see A as a subset of H. Set $S := A \cup \{p\}$. All cones with a vertex at p and singular at all points of A are singular at all points of S. Thus, $h^0(\mathcal{I}_{2S}(d)) \ge h^0(H, \mathcal{I}_{2A,H}(d)) > 0$. Since $h^1(H, \mathcal{I}_{2A,H}(d)) > 0$ and $(2A, H) \subset 2S$, $h^1(\mathcal{I}_{2S}(d)) > 0$. Take $E \subsetneq S$ such that $\langle E\rangle$. Obviously, $p \in E$. Set $B := S \setminus \{p\}$. We have $\langle B\rangle = H$. To prove that $S \in \tilde{\mathcal{T}}(n,d;x)$ (and hence to prove both assertions of Proposition 3), it is sufficient to prove that $h^1(\mathcal{I}_{2E}(d)) = 0$. Consider the residual exact sequence of H:

$$0 \to \mathcal{I}_{B\cup 2p}(d-1) \to \mathcal{I}_{2E}(d) \to \mathcal{I}_{(2B,H),H}(d) \to 0.$$

Since $A \in \tilde{\mathcal{T}}(n-1,d;x-1)$ and $\langle B\rangle = H$, $h^1(H, \mathcal{I}_{(2H,H),H}(d)) = 0$. Hence, it is sufficient to prove that $h^1(\mathcal{I}_{B\cup 2p}(d-1)) = 0$. Since $d \ge 2$, the residual exact sequence of H gives $h^1(\mathcal{I}_{B\cup 2p}(d-1)) \le h^1(H, \mathcal{I}_B(d-1))$. Lemma 2 applied to H and B gives $h^1(H, \mathcal{I}_B(d-1)) = 0$. □

Proposition 4. *Fix integers $n \ge 2$ and $d \ge 4$. If $d = 4$, assume $n \ge 6$. Set $x := \lceil (\binom{n+d-1}{n-1} + 1)/n\rceil + 1$. Then, $\tilde{\mathcal{T}}(n,d;x) \neq \emptyset$ and $\dim \tilde{\mathcal{T}}(n,d;x) \ge 2n + (n-1)(x-1)$.*

Proof. Fix a hyperplane $H \subset \mathbb{P}^n$, $p \in \mathbb{P}^n \setminus H$ and a general $A \in S(H, x-1)$. Set $S := A \cup \{p\}$. Since $x - 1 \ge n$ and A is general, $\langle S\rangle = \mathbb{P}^n$. Since $d \ge 4$, the scheme $2S$ is contained in the singular locus of a degree d hypersurface, the union of $2H$ and a degree $d-2$ hypersurface singular at p. Thus, $h^0(\mathcal{I}_{2S}(d)) > 0$. Since $n(x-1) > h^0(\mathcal{O}_H(d))$, $h^1(H, \mathcal{I}_{(2A,H),H}(d)) > 0$. Since $(2A, H) \subset 2S$, $h^1(\mathcal{I}_{2S}(d)) > 0$. Take $B \subsetneq S$ such that $\langle B\rangle = \mathbb{P}^n$. Obviously, $p \in B$. Set $E := B \setminus \{p\}$. Since $n(x-2) \le h^0(\mathcal{O}_H(d))$ and $\mathcal{O}_H(d)$ is not secant defective ([11–13]), $h^1(H, \mathcal{I}_{(2E,H}(d)) = 0$. Applying Lemma 2 as in the proof of Proposition 3, we obtain $h^1(\mathcal{I}_{2B}(d)) = 0$. Thus, S is primitive. □

Proposition 5. *Fix integers $d \ge 3$, $n \ge 2$ and $x > 0$ such that $\tilde{\mathcal{T}}(n-1,d;x-1) = 0$, $n(x-1) \le \binom{n+d-1}{n-1}$ and $\mathcal{T}(n,d;x)' = \emptyset$. Then, $\tilde{\mathcal{T}}(n,d;x) = \emptyset$.*

Proof. Assume the existence of $S \in \tilde{\mathcal{T}}(n,d;x)$. Since $(n+1)(x-1) < \binom{n+d}{n}$, $h^0(\mathcal{I}_{2E}(d)) > 0$ for all $E \subsetneq S$. Let $\mathcal{A}$ denote the set of all $A \subsetneq S$ such that $h^1(\mathcal{I}_{2A}(d)) > 0$. Since $\mathcal{T}(n,d;x)' = \emptyset$, $\mathcal{A} \neq \emptyset$. Since $S \in \tilde{\mathcal{T}}(n,d;x)$, $\langle A\rangle \neq \mathbb{P}^n$ for all $A \in \mathcal{A}$. The set $\mathcal{A}$ is partially ordered by inclusion. If $A \subseteq B \subsetneq S$ and $A \in \mathcal{A}$, then $B \in \mathcal{A}$. Thus, there is $B \in \mathcal{A}$ such that $\#B = x-1$. Since $\langle S\rangle = \mathbb{P}^n$ and $S \in \tilde{\mathcal{T}}(n,d;x)$, $H := \langle B\rangle$ has dimension $n-1$. Set $\{p\} := S \setminus H$. Lemma 2 and the residual exact sequence of H imply $h^1(H, \mathcal{I}_{(2B,H),H}(d)) > 0$. Since $\tilde{\mathcal{T}}(n-1,d;x-1)) = 0$, there is $E \subsetneq B$ such that $\langle E\rangle = H$ and $h^1(H, \mathcal{I}_{(2E,H),H}(d)) > 0$. Since $\langle \{p\} \cup H\rangle = \mathbb{P}^n$, the set $E \cup \{p\}$ gives $S \notin \tilde{\mathcal{T}}(n,d;x)$, a contradiction. $\square$

6. Motivations

In this section, we give the original motivation for the study of Terracini loci. Just to fix the notation, we conduct it in the set-up of the Veronese embeddings of a projective space.

Fix the positive integers n, d and x and let $\nu_d : \mathbb{P}^n \to \mathbb{P}^r$, $r = -1 + \binom{n+d}{n}$ denote the Veronese embedding. Let $x_0, \dots, x_n$ be homogeneous coordinates. Let $\mathbb{K}[x_0, \dots, x_n]_d$ denote the $\binom{n+d}{n}$-vector space of all degree d forms in $n+1$ variables, i.e., $\mathbb{K}[x_0, \dots, x_n]_d = H^0(\mathcal{O}_{\mathbb{P}^n}(d))$. Thus, elements of $\mathbb{P}^r$ correspond to equivalent classes of nonzero forms. Fix $f \in \mathbb{K}[x_0, \dots, x_n]_d$, $f \neq 0$ and let $[f] \in \mathbb{P}^r$ denote its equivalence class. An additive decomposition of f with exactly x addenda is an equality $f = \sum_{i=1}^{x} \ell_i^d$ with each ℓ_i a linear form. This decomposition is equivalent to the existence of $S \in S(\mathbb{P}^n, x)$ such that $[f] \in \langle S\rangle$ and $[f] \notin \langle S'\rangle$ for any $S' \subsetneq S$. The set $\mathcal{S}([f])$ of the additive decompositions of $[f]$ is often called the *solution set* of $[f]$. The set $\mathcal{S}([f]) \subset S(\mathbb{P}^n, x)$ has a topology, the restriction to it of the Zariski topology of $S(\mathbb{P}^n, x)$. If $h^1(\mathcal{I}_{2S}(d)) = 0$, then the Terracini lemma says that S is the unique additive decomposition of $[f]$ which is "near" to S, i.e., S is an isolated point of $\mathcal{S}([f])$ for the Zariski topology. Moreover, if $h^1(\mathcal{I}_{2S}(d)) = 0$, then $h^1(\mathcal{I}_{2A}(d)) = 0$ for all $A \in S(\mathbb{P}^n, x)$ that are near S in the Zariski topology, and we may recover in this way (varying A) all points of $\mathbb{P}^r$ in a Zariski neighborhood of $[f]$. Given S, it is possible to quickly check the value of $h^1(\mathcal{I}_{2S}(d))$ by using software (it is a linear algebra problem). In all cases with $(n+1)x < \binom{n+d}{n}$, we have $h^0(\mathcal{I}_{2E}(d)) > 0$ for all $E \in S(\mathbb{P}^n, x)$. Hence, only the value of $h^1(\mathcal{I}_{2S}(d))$ matters, and usually $\langle S\rangle = \mathbb{P}^n$. If $h^1(\mathcal{I}_{2S}(d)) > 0$, it is easy to check all $S' \subsetneq S$ to see if S is minimal or primitive.

7. Conclusions

Our main results are negative (certain Terracini loci are empty), but we discuss in Section 6 how the emptiness results are used. Among the existing results, we stress the ones with the lowest possible number of points for multivariate forms of fixed degrees in a given number of variables.

We raised several open questions in the introduction and listed another one (on the Hilbert function of zero-dimensional schemes) at the end of Section 2 with a discussion of its possible applications, for instance, to evaluation codes ([3,4]).

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