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On a Backward Problem for the Rayleigh–Stokes Equation with a Fractional Derivative

Songshu Liu [1,*], Tao Liu [1] and Qiang Ma [2]

[1] School of Mathematics and Statistics, Northeastern University at Qinhuangdao, Qinhuangdao 066004, China; liutao@neuq.edu.cn
[2] Department of Mathematics, Harbin Institute of Technology at Weihai, Weihai 264209, China; hitmaqiang@hit.edu.cn
* Correspondence: liusongshu@neuq.edu.cn

Abstract: The Rayleigh–Stokes equation with a fractional derivative is widely used in many fields. In this paper, we consider the inverse initial value problem of the Rayleigh–Stokes equation. Since the problem is ill-posed, we adopt the Tikhonov regularization method to solve this problem. In addition, this paper not only analyzes the ill-posedness of the problem but also gives the conditional stability estimate. Finally, the convergence estimates are proved under two regularization parameter selection rules.

Keywords: Rayleigh–Stokes equation with a fractional derivative; backward problem; Tikhonov regularization method; convergence estimate

MSC: 35R25; 35R30

1. Introduction

Fractional derivatives and integrals provide a good tool to describe phenomena with non-locality and memory characteristics. Fractional derivatives and fractional equations are also widely used in many scientific fields such as engineering, physics, finance, and hydrology [1–4]. So far, fractional integrals and derivatives have taken many forms, such as the Riemann–Liouville, Grünwald–Letnikov, Riesz, Caputo, Hadamard, and Caputo–Fabrizio. As a generalized form of integral calculus, fractional calculus has been paid more attention to by scholars because it is more in line with the actual phenomenon and has unique advantages compared with integral calculus. Fractional differential equations have important applications in the fields of fluid mechanics, economics, and control theory. Although fractional differential equation can describe the actual phenomenon more accurately [2,5,6], it is difficult to obtain the analytical solution of a fractional differential equation because of the non-local property of the fractional derivative. Therefore, it is necessary to find an effective numerical method to solve fractional differential equations.

In recent years, the Rayleigh–Stokes equation for a heated generalized second-grade fluid has played an important role in describing the practical problems of non-Newtonian fluid mechanics, which have attracted much attention from many researchers. Many achievements have been made in the study of the direct problems of Rayleigh–Stokes equation. In [7], Fourier coefficients transform and the fractional Laplace transform are used to solve the exact solution of the Rayleigh–Stokes problem. In [8], the exact solution of some oscillatory motions of the generalized Rayleigh–Stokes problem is discussed, and the velocity field and corresponding analytical expressions of infinite plate oscillating flow are given. The vibration caused by the oscillatory pressure gradient is determined by the Fourier sine transform and the Laplace transform. In [9], the authors use the fractional derivative method to solve the Rayleigh–Stokes problem on the boundary. In addition, some scholars have used numerical methods to study the Rayleigh–Stokes problem. In [10],

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the authors used implicit and explicit difference numerical methods to obtain numerical solutions of second-order generalized thermal fluid Rayleigh–Stokes problems with fractional derivatives. In [11], an approximate numerical method is proposed for the Rayleigh–Stokes problem of generalized second-order fluids in a bounded domain. In [12], the numerical methods with fourth-order spatial accuracy for Rayleigh–Stokes' first problem is studied. In [13,14], the authors study the numerical solutions of Rayleigh–Stokes problems for generalized second-order thermal fluids with fractional derivatives. The other numerical methods for solving Rayleigh–Stokes problems can be seen in the cited works [15–17].

However, in practical problems, the parameters used in most model equations, such as physical parameters, source terms, initial conditions, and boundary conditions are unknown, and these unknown parameters need to be identified through measurement data. Thus, it leads to the inverse problems of the Rayleigh–Stokes equation for second-grade fluids. According to the current research status, the research on the inverse problem of the Rayleigh–Stokes equation is still limited. In [18], an inverse problem to estimate an unknown order of a Riemann–Liouville fractional derivative for a fractional Stokes' first problem is considered. In [19], the authors use the filter regularization method to analyze the Rayleigh–Stokes inverse problem with Gaussian random noise. In [20], the authors use the filter regularization method to identify the unknown source term of the Rayleigh–Stokes problem with Gaussian random noise and prove the error estimation between the regularized solution and the exact solution. But the regularization parameter is an a priori choice rule, which depends on an unknown priori bound. In [21], the authors provide the existence and regularity of the inverse problem for the nonlinear fractional Rayleigh–Stokes equations. In [22,23], the authors give a Tikhonov regularization method and filter regularization method to identify the source term for the Rayleigh–Stokes problem. In [24], the authors use the trigonometric method in nonparametric regression associated to regularize the instable solution of the initial inverse problem for the nonlinear fractional Rayleigh–Stokes equation with random discrete data. In [25], the authors consider the regularity of the solution for a final value problem for the Rayleigh–Stokes equation.

In the following, we consider the backward problem for the Rayleigh–Stokes equation in a general bounded domain. Let $T > 0$ be a given positive number, and Ω be a bounded domain in $\mathbb{R}^d$. The mathematical problem is given by

$$\begin{cases} \partial_t u(x,t) - (1+\gamma\partial_t^\alpha)\Delta u(x,t) = 0, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T], \\ u(x,T) = g(x), & x \in \Omega, \end{cases} \tag{1}$$

where $\gamma > 0$ is a constant, u is the velocity distribution. $\partial_t = \partial/\partial t$, and ∂_t^α is the Riemann–Liouville fractional derivative of order $\alpha \in (0,1)$ defined by [1]

$$\partial_t^\alpha u(x,t) = \frac{d}{dt}\int_0^t \omega_{1-\alpha}(t-s)u(x,s)ds, \quad \omega_\alpha = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 < \alpha < 1. \tag{2}$$

The backward problem is to find the initial data $u(x,0) = f(x)$ from the given measured data at the final condition $u(x,T) = g(x)$. In practice, the exact data g are approximated by the noisy observation data g^δ, which are assumed to satisfy

$$\|g^\delta - g\| \le \delta, \tag{3}$$

where $\|\cdot\|$ denotes the $L^2(\Omega)$-norm, and the constant $\delta > 0$ is a noise level.

In this paper, the Tikhonov regularization method is used to study the backward problem of the Rayleigh–Stokes equation with a fractional derivative. This method has dealt with a number of inverse problems, such as the backward problem [26,27] and the inverse unknown source problem [28–30]. We prove the error estimate between the regularized solution and the exact solution under a priori and a posteriori regularization parameter selection rules. The posteriori regularization parameter selection rules only depend on the measured data and do not depend on the priori bound of the exact solution.

The structure of this paper is as follows. Section 2 introduces some preliminary results. Section 3 gives the ill-posedness of problem (1) and the conditional stability of problem (1). In Section 4, the Tikhonov regularization method is used to deal with the backward problem, and the error estimates between the exact solution and the regularized solution are obtained under a priori and a posteriori regularization parameter choice rules.

2. Preliminary Results

Throughout this article, we use the following definitions.

Definition 1. *Let $\{\lambda_n, \phi_n\}$ be the Dirichlet eigenvalues and corresponding eigenvectors of the Laplacian operator $-\Delta$ in Ω. The family of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ satisfies $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$, where $\lambda_n \to \infty$ as $n \to \infty$:*

$$\begin{cases} \Delta\phi_n(x) = -\lambda_n\phi_n(x), & x \in \Omega, \\ \phi_n(x) = 0, & x \in \partial\Omega. \end{cases} \tag{4}$$

Definition 2. *For $k > 0$, we define*

$$H^k(\Omega) := \left\{ f \in L^2(\Omega) \middle| \sum_{n=1}^{\infty} \lambda_n^{2k} |(f, \phi_n)|^2 < +\infty \right\}, \tag{5}$$

equipped with the norm

$$\|f\|_{H^k(\Omega)} = \left(\sum_{n=1}^{\infty} \lambda_n^{2k} |(f, \phi_n)|^2 \right)^{\frac{1}{2}}, \quad k > 0. \tag{6}$$

In the following, we present the solution of the direct problem of the Rayleigh–Stokes equation

$$\begin{cases} \partial_t u(x,t) - (1 + \gamma\partial_t^{\alpha})\Delta u(x,t) = 0, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T], \\ u(x,0) = f(x), & x \in \Omega. \end{cases} \tag{7}$$

Indeed, suppose that the direct problem (7) has a solution $u(x,t) \in C([0,T]; L^2(\Omega)) \cap C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$, and using the Equation (2.21) in [31], we obtain

$$u(x,t) = \sum_{n=1}^{\infty} f_n u_n(t) \phi_n(x). \tag{8}$$

Here, $f_n = (f(x), \phi_n(x))$ is the Fourier coefficient, and the function $u_n(t)$ satisfies

$$u_n(t) = \int_0^{\infty} e^{-st} B_n(s) ds, \tag{9}$$

where

$$B_n(s) = \frac{\gamma}{\pi} \frac{\lambda_n s^{\alpha} \sin \alpha\pi}{(-s + \lambda_n \gamma s^{\alpha} \cos \alpha\pi + \lambda_n)^2 + (\lambda_n \gamma s^{\alpha} \sin \alpha\pi)^2}.$$

According to the condition $u(x,T) = g(x)$, and using (9), we obtain

$$g(x) = \sum_{n=1}^{\infty} f_n u_n(T) \phi_n(x) := Kf(x), \tag{10}$$

or equivalently,

$$g_n = f_n u_n(T), \tag{11}$$

where $g_n = (g(x), \phi_n(x))$ is the Fourier coefficient. Here, the linear operator $K : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$Kf(x) = \sum_{n=1}^{\infty} \left[\int_0^{\infty} e^{-sT} B_n(s) ds \right] (f(x), \phi_n(x)) \phi_n(x) = \int_{\Omega} k(x,\omega) f(\omega) d\omega, \tag{12}$$

where
$$k(x,\omega)=\sum_{n=1}^{\infty}\left[\int_0^{\infty}e^{-sT}B_n(s)ds\right]\phi_n(x)\phi_n(\omega).$$
Then, we can obtain the solution of the backward problem (1) as follows
$$f(x)=\sum_{n=1}^{\infty}\frac{g_n}{u_n(T)}\phi_n(x). \tag{13}$$

3. Ill-Posedness and Conditional Stability Estimate

To analyze the ill-posedness and give the conditional stability estimate of the backward problem, we need to provide the following lemmas.

Lemma 1 ([31])**.** *The functions $u_n(t), n=1,2,\cdots$ have the following properties:*

(a) $u_n(0)=1, \quad 0<u_n(t)\le 1, \quad t\ge 0;$
(b) *$u_n(t)$ are completely monotone for $t\ge 0$;*
(c) $|\lambda_n u_n(t)|\le c\min\{t^{-1},t^{\alpha-1}\}, \quad t>0;$
(d) $\int_0^T |u_n(t)|dt<\frac{1}{\lambda_n}, \quad T>0,$

where the constant c does not depend on n and t.

Lemma 2 ([19])**.** *Let us assume that $\alpha\in(0,1)$. The following estimate holds for all $t\in[0,T]$*
$$u_n(t)\ge\frac{C(\gamma,\alpha,\lambda_1)}{\lambda_n}, \tag{14}$$
where
$$C(\gamma,\alpha,\lambda_1)=\gamma\sin\alpha\pi\int_0^{+\infty}\frac{e^{-sT}s^{\alpha}ds}{\gamma^2s^{2\alpha}+\frac{s^2}{\lambda_1^2}+1}. \tag{15}$$

Now, we will prove that the backward problem is ill-posed. By using the result in Lemma 1, for $t>0$, we have
$$\frac{1}{u_n(T)}\ge\frac{\lambda_n}{c\min\{T^{-1},T^{\alpha-1}\}}. \tag{16}$$
Hence, we know that $\frac{1}{u_n(T)}$ is a completely monotonic increasing function with respect to λ_n. Then, the small error in the high-frequency components for $g^{\delta}(x)$ will be amplified by the factor $\frac{1}{u_n(T)}$. So, the initial data $u(x,0)=f(x)$ from the given measured data $g^{\delta}(x)$ are ill-posed.

In the following, we introduce a conditional stability estimate of the backward problem for the fractional Rayleigh–Stokes Equation (1).

Theorem 1. *Let $f\in H^k(\Omega)$ be such that*
$$\|f\|_{H^k(\Omega)}\le E, \tag{17}$$
for some $E>0$. Then, we have the following estimate
$$\|f\|_{L^2(\Omega)}\le C_1E^{\frac{1}{k+1}}\|g\|^{\frac{k}{k+1}}, \tag{18}$$
where $C_1=C^{-\frac{k}{k+1}}(\gamma,\alpha,\lambda_1)$.

Proof. From Formula (13), and applying the Hölder inequality, we know

$$\begin{aligned}
\|f\|^2_{L^2(\Omega)} &= \sum_{n=1}^{\infty} \left| \frac{(g(x),\phi_n(x))}{u_n(T)} \right|^2 \\
&= \sum_{n=1}^{\infty} \frac{|(g(x),\phi_n(x))|^{\frac{2}{k+1}} |(g(x),\phi_n(x))|^{\frac{2k}{k+1}}}{|u_n(T)|^2} \\
&\le \left[\sum_{n=1}^{\infty} \frac{|(g(x),\phi_n(x))|^2}{|u_n(T)|^{2k+2}} \right]^{\frac{1}{k+1}} \left[\sum_{n=1}^{\infty} |(g(x),\phi_n(x))|^2 \right]^{\frac{k}{k+1}} \\
&\le \left[\sum_{n=1}^{\infty} \frac{|(f(x),\phi_n(x))|^2}{|u_n(T)|^{2k}} \right]^{\frac{1}{k+1}} \|g\|^{\frac{2k}{k+1}}_{L^2(\Omega)}. \qquad (19)
\end{aligned}$$

By using Lemma 2, we obtain

$$\sum_{n=1}^{\infty} \frac{|(f(x),\phi_n(x))|^2}{|u_n(T)|^{2k}} \le \sum_{n=1}^{\infty} \frac{\lambda_n^{2k}|(f(x),\phi_n(x))|^2}{C^{2k}(\gamma,\alpha,\lambda_1)} = \frac{\|f\|^2_{H^k(\Omega)}}{C^{2k}(\gamma,\alpha,\lambda_1)}. \qquad (20)$$

Combining Formulas (19) and (20), we obtain

$$\|f\|^2_{L^2(\Omega)} \le \frac{\|f\|^{\frac{2}{k+1}}_{H^k(\Omega)}}{C^{\frac{2k}{k+1}}(\gamma,\alpha,\lambda_1)} \|g\|^{\frac{2k}{k+1}}_{L^2(\Omega)}.$$

Hence, we have

$$\|f\|_{L^2(\Omega)} \le C_1 E^{\frac{1}{k+1}} \|g\|^{\frac{k}{k+1}},$$

where $C_1 = C^{-\frac{k}{k+1}}(\gamma,\alpha,\lambda_1)$. $\square$

Remark 1. Essentially, Theorem 1 provides the following conditional stability estimate

$$\|f_1 - f_2\|_{L^2(\Omega)} \le C_1 \|f_1 - f_2\|^{\frac{1}{k+1}}_{H^k(\Omega)} \|Kf_1 - Kf_2\|^{\frac{k}{k+1}}.$$

4. Tikhonov Regularization Method and Convergence Estimates

In this section, we solve the backward problem (1) by using the Tikhonov regularization method, which minimizes the function

$$\|Kf - g\|^2 + \beta^2 \|f\|^2; \qquad (21)$$

here, β is a regularization parameter. By Theorem 2.12 in [32], we know that its minimizer $f_\beta(x)$ satisfies

$$K^* K f_\beta(x) + \beta^2 f_\beta(x) = K^* g(x). \qquad (22)$$

Due to the singular value decomposition for a compact self-adjoint operator, we have

$$f_\beta(x) = \sum_{n=1}^{\infty} \frac{u_n(T)}{\beta^2 + u_n^2(T)} (g,\phi_n)\phi_n. \qquad (23)$$

If the observed data $g^\delta(x)$ are noise-contaminated, we have

$$f_\beta^\delta(x) = \sum_{n=1}^{\infty} \frac{u_n(T)}{\beta^2 + u_n^2(T)} (g^\delta,\phi_n)\phi_n. \qquad (24)$$

4.1. A Priori Choice Rule

We first give two lemmas.

Lemma 3. *Assume condition (3) holds, and we have the following estimate*

$$\|f_\beta^\delta(x) - f_\beta(x)\| \le \frac{\delta}{2\beta}. \qquad (25)$$

Proof. According to the Formulas (3), (23), and (24), we have

$$\begin{aligned}
\|f_\beta^\delta(x) - f_\beta(x)\|^2 &= \left\| \sum_{n=1}^{\infty} \frac{u_n(T)}{\beta^2 + u_n^2(T)} (g^\delta, \phi_n)\phi_n - \sum_{n=1}^{\infty} \frac{u_n(T)}{\beta^2 + u_n^2(T)} (g, \phi_n)\phi_n \right\|^2 \\
&= \left\| \sum_{n=1}^{\infty} \frac{u_n(T)}{\beta^2 + u_n^2(T)} (g^\delta - g, \phi_n)\phi_n \right\|^2 \\
&= \sum_{n=1}^{\infty} \left(\frac{u_n(T)}{\beta^2 + u_n^2(T)} \right)^2 (g_n^\delta - g_n)^2 \\
&\leq \delta^2 (\sup_{n\geq 1} A(n))^2, \qquad (26)
\end{aligned}$$

where,

$$A(n) = \frac{|u_n(T)|}{\beta^2 + u_n^2(T)} \leq \frac{1}{2\beta}.$$

Thus, we obtain

$$\|f_\beta^\delta(x) - f_\beta(x)\| \leq \frac{\delta}{2\beta}. \qquad (27)$$

The proof of Lemma 3 is complete. □

Lemma 4. *Assume that the condition (17) holds; then, we have*

$$\|f(x) - f_\beta(x)\| = \begin{cases} \beta E \frac{\lambda_1^{1-k}}{2C(\gamma,\alpha,\lambda_1)}, & 0 < k < 1, \\ \beta^k E \sqrt{(\frac{1}{2C(\gamma,\alpha,\lambda_1)})^2 + 1}, & k \geq 1. \end{cases} \qquad (28)$$

Proof. From Formulas (13) and (23), we know

$$\begin{aligned}
\|f(x) - f_\beta(x)\|^2 &= \sum_{n=1}^{\infty} \left(\frac{1}{u_n(T)} - \frac{u_n(T)}{\beta^2 + u_n^2(T)} \right)^2 g_n^2 \\
&= \sum_{n=1}^{\infty} \left(\frac{\beta^2}{(\beta^2 + u_n^2(T))u_n(T)} \right)^2 g_n^2 \\
&= \sum_{n=1}^{\infty} \left(\frac{\beta^2 \lambda_n^k \lambda_n^{-k}}{(\beta^2 + u_n^2(T))u_n(T)} \right)^2 g_n^2 \\
&\leq (\sup_{n\geq 1} B(n))^2 \sum_{n=1}^{\infty} \frac{\lambda_n^{2k} g_n^2}{u_n^2(T)} \\
&= (\sup_{n\geq 1} B(n))^2 \|f\|_{H^k(\Omega)}^2. \qquad (29)
\end{aligned}$$

Here,

$$B(n) = \frac{\beta^2 \lambda_n^{-k}}{\beta^2 + u_n^2(T)}. \qquad (30)$$

Now, by using Lemma 2, we estimate $B(n)$,

$$B(n) \leq \frac{\beta^2 \lambda_n^{-k}}{2\beta u_n(T)} = \frac{\beta \lambda_n^{-k}}{2u_n(T)} \leq \frac{\beta \lambda_n^{1-k}}{2C(\gamma, \alpha, \lambda_1)}. \qquad (31)$$

We divide this into the two following cases:

Case 1: If $k \geq 1$, we know

$$\lambda_n^{1-k} = \frac{1}{\lambda_n^{k-1}} \leq \frac{1}{\lambda_1^{k-1}} = \lambda_1^{1-k}. \qquad (32)$$

Combining (29), (31), and (32), we obtain

$$\|f(x)-f_\beta(x)\| \le \frac{\beta\lambda_1^{1-k}}{2C(\gamma,\alpha,\lambda_1)}\|f\|_{H^k(\Omega)} \le \beta E\frac{\lambda_1^{1-k}}{2C(\gamma,\alpha,\lambda_1)}. \tag{33}$$

Case 2: If $0<k<1$, we choose any $\eta\in(0,1)$ and rewrite $\mathbb{N}=\mathcal{A}_1\cup\mathcal{A}_2$, where

$$\mathcal{A}_1=\{n\in\mathbb{N},\lambda_n^{1-k}\le\beta^{-\eta}\},\quad \mathcal{A}_2=\{n\in\mathbb{N},\lambda_n^{1-k}>\beta^{-\eta}\}. \tag{34}$$

From Formula (31), we have

$$\begin{aligned}\|f(x)-f_\beta(x)\|^2 &\le \sup_{n\in\mathcal{A}_1}\left(\frac{\beta\lambda_n^{1-k}}{2C(\gamma,\alpha,\lambda_1)}\right)^2\sum_{n\in\mathcal{A}_1}\lambda_n^{2k}(f(x),\phi_n(x))^2\\ &\quad+\sup_{n\in\mathcal{A}_2}\left(\frac{\beta^2\lambda_n^{-k}}{\beta^2+u_n^2(T)}\right)^2\sum_{n\in\mathcal{A}_2}\lambda_n^{2k}(f(x),\phi_n(x))^2\\ &\le \left(\frac{1}{2C(\gamma,\alpha,\lambda_1)}\right)^2\beta^{2-2\eta}\|f\|^2_{H^k(\Omega)}+\sup_{n\in\mathcal{A}_2}\lambda_n^{-2k}\|f\|^2_{H^k(\Omega)}\\ &\le \left(\frac{1}{2C(\gamma,\alpha,\lambda_1)}\right)^2\beta^{2-2\eta}\|f\|^2_{H^k(\Omega)}+\beta^{\frac{2\eta k}{1-k}}\|f\|^2_{H^k(\Omega)}.\end{aligned} \tag{35}$$

Choosing $\eta=1-k$, and the Formula (17), we obtain

$$\begin{aligned}\|f(x)-f_\beta(x)\|^2 &\le \left(\frac{1}{2C(\gamma,\alpha,\lambda_1)}\right)^2\beta^{2-2\eta}\|f\|^2_{H^k(\Omega)}+\beta^{\frac{2\eta k}{1-k}}\|f\|^2_{H^k(\Omega)}\\ &=\beta^{2k}E^2\left(\left(\frac{1}{2C(\gamma,\alpha,\lambda_1)}\right)^2+1\right).\end{aligned} \tag{36}$$

This means

$$\|f(x)-f_\beta(x)\|\le\beta^kE\sqrt{\left(\frac{1}{2C(\gamma,\alpha,\lambda_1)}\right)^2+1}. \tag{37}$$

The proof of Lemma 4 is complete. □

Theorem 2. *Suppose that a priori condition (17) and the noise assumption (3) hold; then,*

(1) If $k\ge1$, and we choose $\beta=(\frac{\delta}{E})^{\frac12}$, we have the convergence estimate

$$\|f_\beta^\delta(x)-f(x)\|\le\frac12\delta^{\frac12}E^{\frac12}\left(1+\frac{\lambda_1^{1-k}}{C(\gamma,\alpha,\lambda_1)}\right). \tag{38}$$

(2) If $0<k<1$, and we choose $\beta=(\frac{\delta}{E})^{\frac{1}{k+1}}$, we obtain the convergence estimate

$$\|f_\beta^\delta(x)-f(x)\|\le\delta^{\frac{k}{k+1}}E^{\frac{1}{k+1}}\left(\frac12+\sqrt{\left(\frac{1}{2C(\gamma,\alpha,\lambda_1)}\right)^2+1}\right). \tag{39}$$

Proof. According to the triangle inequality and Lemmas 3 and 4, we know

$$\|f_\beta^\delta(x)-f(x)\|\le\|f_\beta^\delta(x)-f_\beta(x)\|+\|f_\beta(x)-f(x)\|.$$

Hence, we can easily obtain the conclusion to Theorem 2. □

4.2. A Posteriori Choice Rule

In this subsection, we derive the convergence estimate by using a posteriori regularization choice rule (namely Morozov's discrepancy principle).

According to Morozov's discrepancy principle [32], we choose the regularization parameter β as the solution of the following equation

$$\|Kf_\beta^\delta(x)-g^\delta(x)\|=\tau\delta, \tag{40}$$

where $\tau > 1$ is a constant.

Lemma 5. *Set $\rho(\beta) = \|Kf_\beta^\delta(x) - g^\delta(x)\|$. Then, the following results hold*

(a) $\rho(\beta)$ *is a continuous function;*
(b) $\lim_{\beta\to 0}\rho(\beta) = 0$;
(c) $\lim_{\beta\to +\infty}\rho(\beta) = \|g^\delta(x)\|$;
(d) $\rho(\beta)$ *is a strictly increasing function over* $(0, +\infty)$.

Proof. The proof follows from the straightforward results using the expressions of

$$Kf_\beta^\delta(x) = \sum_{n=1}^{\infty} \frac{u_n^2(T)}{\beta^2 + u_n^2(T)}(g^\delta(x), \phi_n(x))\phi_n(x), \tag{41}$$

and

$$\rho(\beta) = \|Kf_\beta^\delta(x) - g^\delta(x)\| = \left(\sum_{n=1}^{\infty}\left(\frac{\beta^2}{\beta^2 + u_n^2(T)}\right)^2 (g^\delta(x), \phi_n(x))^2\right)^{\frac{1}{2}}. \tag{42}$$

□

Remark 2. *According to Lemma 5, we know there exists a unique solution for Equation (40) if* $\|g^\delta\| > \tau\delta > 0$.

Lemma 6. *If β is the solution of Equation (40), we can obtain the following inequality*

$$\frac{1}{\beta} \le \begin{cases} \left(\frac{C_2}{\tau-1}\right)^{\frac{1}{k+1}}\left(\frac{E}{\delta}\right)^{\frac{1}{k+1}}, & 0 < k < 1, \\ \left(\frac{C_3}{\tau-1}\right)^{\frac{1}{2}}\left(\frac{E}{\delta}\right)^{\frac{1}{2}}, & k \ge 1, \end{cases} \tag{43}$$

where $C_2 = \frac{1}{2}M(k+1)^{\frac{k+1}{2}}(1-k)^{\frac{1-k}{2}}C^{-k-1}(\gamma, \alpha, \lambda_1)$ *and* $C_3 = \frac{M\lambda_1^{1-k}}{C^2(\gamma,\alpha,\lambda_1)}$ *are independent of s.*

Proof. From Equation (40), we have

$$\begin{aligned} \tau\delta &= \left\|\sum_{n=1}^{\infty}\frac{\beta^2}{\beta^2 + u_n^2(T)}(g^\delta(x), \phi_n(x))\phi_n(x)\right\| \\ &\le \left\|\sum_{n=1}^{\infty}\frac{\beta^2}{\beta^2 + u_n^2(T)}(g^\delta(x) - g(x), \phi_n(x))\phi_n(x)\right\| \\ &\quad + \left\|\sum_{n=1}^{\infty}\frac{\beta^2}{\beta^2 + u_n^2(T)}(g(x), \phi_n(x))\phi_n(x)\right\| \\ &\le \delta + \left\|\sum_{n=1}^{\infty}\frac{\beta^2}{\beta^2 + u_n^2(T)}(g(x), \phi_n(x))\phi_n(x)\right\|. \end{aligned} \tag{44}$$

Then, we obtain

$$(\tau - 1)\delta \le \left\|\sum_{n=1}^{\infty}\frac{\beta^2}{\beta^2 + u_n^2(T)}(g(x), \phi_n(x))\phi_n(x)\right\|. \tag{45}$$

Using the a priori bound condition of $f(x)$, we obtain

$$\begin{aligned}
&\left\| \sum_{n=1}^{\infty} \frac{\beta^2}{\beta^2 + u_n^2(T)}(g(x), \phi_n(x))\phi_n(x) \right\| \\
\leq\ & \left\| \sum_{n=1}^{\infty} \frac{\beta^2 u_n(T)\lambda_n^{-k}}{\beta^2 + u_n^2(T)} \frac{\lambda_n^k (g(x), \phi_n(x))\phi_n(x)}{u_n(T)} \right\| \\
\leq\ & \sup_{n\geq 1} \frac{\beta^2 u_n(T)\lambda_n^{-k}}{\beta^2 + u_n^2(T)} \left[\sum_{n=1}^{\infty} \frac{\lambda_n^{2k} g_n^2(x)}{u_n^2(T)} \right]^{\frac{1}{2}} \\
=\ & \sup_{n\geq 1} \frac{\beta^2 u_n(T)\lambda_n^{-k}}{\beta^2 + u_n^2(T)} \|f\|_{H^k(\Omega)},
\end{aligned} \tag{46}$$

where

$$H(n) = \frac{\beta^2 u_n(T)\lambda_n^{-k}}{\beta^2 + u_n^2(T)}. \tag{47}$$

Due to Lemma 2 and Formula (16), we obtain

$$H(n) = \frac{\beta^2 u_n(T)\lambda_n^{-k}}{\beta^2 + u_n^2(T)} \leq \frac{\beta^2 \frac{c\min\{T^{-1}, T^{\alpha-1}\}}{\lambda_n}\lambda_n^{-k}}{\beta^2 + (\frac{C(\gamma,\alpha,\lambda_1)}{\lambda_n})^2} = \frac{\beta^2 c \min\{T^{-1}, T^{\alpha-1}\}\lambda_n^{1-k}}{\beta^2\lambda_n^2 + C^2(\gamma,\alpha,\lambda_1)}. \tag{48}$$

Let $s = \lambda_n, M = c\min\{T^{-1}, T^{\alpha-1}\}$; then, we set

$$G(s) = \frac{\beta^2 M s^{1-k}}{\beta^2 s^2 + C^2(\gamma,\alpha,\lambda_1)}. \tag{49}$$

We divide this into the two following cases:
Case 1: If $0 < k < 1$, then we have $\lim_{s\to 0} G(s) = \lim_{s\to\infty} G(s) = 0$; thus, we know

$$G(s) \leq \sup_{s\in(0,+\infty)} G(s) \leq G(s_0),$$

where $s_0 \in (0, +\infty)$ such that $G'(s_0) = 0$. It is easy to prove that $s_0 = \sqrt{\frac{1-k}{k+1}}\frac{C(\gamma,\alpha,\lambda_1)}{\beta} > 0$; thus, we have

$$G(s) \leq G(s_0) = \frac{1}{2}M(k+1)^{\frac{k+1}{2}}(1-k)^{\frac{1-k}{2}}C^{-k-1}(\gamma,\alpha,\lambda_1)\beta^{k+1} := C_2\beta^{k+1}. \tag{50}$$

Case 2: If $k \geq 1$, then we have

$$G(s) \leq \frac{\beta^2 M s^{1-k}}{C^2(\gamma,\alpha,\lambda_1)} \leq \frac{\beta^2 M \lambda_1^{1-k}}{C^2(\gamma,\alpha,\lambda_1)} := C_3\beta^2. \tag{51}$$

Combining Formulas (45) and (50) with (51), we obtain

$$(\tau - 1)\delta \leq \begin{cases} C_2\beta^{k+1}E, & 0 < k < 1, \\ C_3\beta^2 E, & k \geq 1. \end{cases} \tag{52}$$

This yields

$$\frac{1}{\beta} \leq \begin{cases} (\frac{C_2}{\tau-1})^{\frac{1}{k+1}}(\frac{E}{\delta})^{\frac{1}{k+1}}, & 0 < k < 1, \\ (\frac{C_3}{\tau-1})^{\frac{1}{2}}(\frac{E}{\delta})^{\frac{1}{2}}, & k \geq 1. \end{cases}$$

Thus, the proof of Lemma 6 is complete. □

Theorem 3. *Suppose a priori condition (17) and the noise assumption (3) hold, and we take the solution of Equation (40) as the regularization parameter; then,*
(1) If $k \geq 1$, we obtain the following convergence estimate

$$\|f_\beta^\delta(x) - f(x)\| \le C_1(\tau+1)^{\frac{k}{k+1}}\delta^{\frac{k}{k+1}}E^{\frac{1}{k+1}} + \frac{1}{2}\left(\frac{C_3}{\tau-1}\right)^{\frac{1}{2}}\delta^{\frac{1}{2}}E^{\frac{1}{2}}. \tag{53}$$

(2) If $0 < k < 1$, we obtain the following convergence estimate

$$\|f_\beta^\delta(x) - f(x)\| \le \left[C_1(\tau+1)^{\frac{k}{k+1}} + \frac{1}{2}\left(\frac{C_2}{\tau-1}\right)^{\frac{1}{k+1}}\right]\delta^{\frac{k}{k+1}}E^{\frac{1}{k+1}}, \tag{54}$$

where $C_1 = C^{-\frac{k}{k+1}}(\gamma,\alpha,\lambda_1)$, $C_2 = \frac{1}{2}M(k+1)^{\frac{k+1}{2}}(1-k)^{\frac{1-k}{2}}C^{-k-1}(\gamma,\alpha,\lambda_1)$ and $C_3 = \frac{M\lambda_1^{1-k}}{C^2(\gamma,\alpha,\lambda_1)}$ are independent of s.

Proof. Due to the triangle inequality, we have

$$\|f_\beta^\delta(x) - f(x)\| \le \|f_\beta^\delta(x) - f_\beta(x)\| + \|f_\beta(x) - f(x)\|. \tag{55}$$

Firstly, we give an estimate for the second term on the right side of Formula (55),

$$\begin{aligned} Kf_\beta(x) - Kf(x) &= \sum_{n=1}^{\infty}\frac{-\beta^2}{\beta^2+u_n^2(T)}(g(x),\phi_n(x))\phi_n(x) \\ &= \sum_{n=1}^{\infty}\frac{-\beta^2}{\beta^2+u_n^2(T)}(g(x)-g^\delta(x),\phi_n(x))\phi_n(x) \\ &+ \sum_{n=1}^{\infty}\frac{-\beta^2}{\beta^2+u_n^2(T)}(g^\delta(x),\phi_n(x))\phi_n(x). \end{aligned} \tag{56}$$

Combining Formulas (3) and (40), we obtain

$$\|Kf_\beta(x) - Kf(x)\| \le \delta + \tau\delta = (\tau+1)\delta. \tag{57}$$

In addition, by applying a priori bound condition of $f(x)$, we obtain

$$\begin{aligned} \|f_\beta(x) - f(x)\|^2_{H^k(\Omega)} &= \sum_{n=1}^{\infty}\left(\frac{\beta^2}{\beta^2+u_n^2(T)}\right)^2\frac{\lambda_n^{2k}|(g(x),\phi_n(x))|^2}{u_n^2(T)} \\ &\le \sum_{n=1}^{\infty}\frac{\lambda_n^{2k}|(g(x),\phi_n(x))|^2}{u_n^2(T)} = \|f\|^2_{H^k(\Omega)} \le E^2. \end{aligned} \tag{58}$$

By Theorem 1 and Formula (57), we have

$$\|f_\beta(x) - f(x)\| \le C_1(\tau+1)^{\frac{k}{k+1}}\delta^{\frac{k}{k+1}}E^{\frac{1}{k+1}}. \tag{59}$$

Now, we give an estimate for the first term on the right side of Formula (55); similar to Formula (25), we have

$$\|f_\beta^\delta(x) - f_\beta(x)\| \le \frac{\delta}{2\beta}. \tag{60}$$

Substituting Formula (43) into Formula (60), we obtain

$$\|f_\beta^\delta(x) - f_\beta(x)\| \le \begin{cases} \frac{1}{2}(\frac{C_2}{\tau-1})^{\frac{1}{k+1}}\delta^{\frac{k}{k+1}}E^{\frac{1}{k+1}}, & 0<k<1, \\ \frac{1}{2}(\frac{C_3}{\tau-1})^{\frac{1}{2}}\delta^{\frac{1}{2}}E^{\frac{1}{2}}, & k \ge 1. \end{cases} \tag{61}$$

Combining Formula (59) with Formula (61), we conclude

$$\|f_\beta^\delta(x) - f(x)\| \le \begin{cases} [C_1(\tau+1)^{\frac{k}{k+1}} + \frac{1}{2}(\frac{C_2}{\tau-1})^{\frac{1}{k+1}}]\delta^{\frac{k}{k+1}}E^{\frac{1}{k+1}}, & 0<k<1, \\ C_1(\tau+1)^{\frac{k}{k+1}}\delta^{\frac{k}{k+1}}E^{\frac{1}{k+1}} + \frac{1}{2}(\frac{C_3}{\tau-1})^{\frac{1}{2}}\delta^{\frac{1}{2}}E^{\frac{1}{2}}, & k \ge 1. \end{cases} \tag{62}$$

The proof of Theorem 3 is complete. □

5. Conclusions

This paper studies the inverse problem of the Rayleigh–Stokes equation and adopts the Tikhonov regularization method to solve this inverse problem. Based on the conditional stability results, the corresponding convergence estimates are obtained under a priori and a posteriori regularization parameter choice rules, respectively. However, this paper provides a theoretical proof. In future, the validity and stability of the proposed method will be verified numerically. Moreover, we are currently considering the one parameter inversion problem, and next we will consider multi-parameter inversion problems.

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