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On the Modified Numerical Methods for Partial Differential Equations Involving Fractional Derivatives

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Abstract: This paper provides both analytical and numerical solutions of (PDEs) involving time-fractional derivatives. We implemented three powerful techniques, including the modified variational iteration technique, the modified Adomian decomposition technique, and the modified homotopy analysis technique, to obtain an approximate solution for the bounded space variable ν. The Laplace transformation is used in the time-fractional derivative operator to enhance the proposed numerical methods' performance and accuracy and find an approximate solution to time-fractional Fornberg–Whitham equations. To confirm the accuracy of the proposed methods, we evaluate homogeneous time-fractional Fornberg–Whitham equations in terms of non-integer order and variable coefficients. The obtained results of the modified methods are shown through tables and graphs.

Keywords: fractional derivatives; partial differential equation; numerical methods; nonlinear equation; integral transforms; Laplace transformation

MSC: 65N99

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1. Introduction

Fractional calculus is a branch of mathematics that stands out in modeling problems involving non-locality and memory effect concepts that are not well explained by classical calculus. The historical development of fractional calculus dates back several centuries and involves contributions from multiple mathematicians. The concept of fractional derivatives (FD) has more than 300 years of history, yet it is still an open research topic, and many interested researchers are actively working on this topic. It can be traced back to a letter from ĹHôpital to Leibnitz in 1695, on the meaning of the derivative of a function of order 1/2. Later investigations and further developments were made by other mathematicians, such as Euler in 1730, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Abel in 1823–1826, Liouville in 1832–1873, Riemann in 1847, Holmgren in 1865–1867, and Gruñwald in 1867–1872. These are early mathematicians who explored the possibility of extending differentiation and integration to fractional orders [1–4].

Consider the fractional-order nonlinear Fornberg–Whitham Equation (FWE)

$$\psi_{\varsigma}^{\vartheta}(\nu,\varsigma) - \psi_{\nu\nu\varsigma}(\nu,\varsigma) + \psi_{\nu}(\nu,\varsigma) + \psi(\nu,\varsigma)\psi_{\nu}(\nu,\varsigma) - \psi(\nu,\varsigma)\psi_{\nu\nu\nu}(\nu,\varsigma) - 3\psi_{\nu}\psi_{\nu\nu}(\nu,\varsigma) = 0, \quad (1)$$

where $\psi(\nu,\varsigma)$ represents the fluid velocity, $0 < \vartheta \leq 1$ is the fractional order, $\varsigma > 0$ stands for time, and ν represents the spatial coordinate. In particular, for $\vartheta = 1$, Equation (1) transforms into the classical (FWE), which Whitham initially introduced in 1967 to investigate wave breaking phenomena [5]. In 1978, Fornberg and Whitham [6] discovered a peaked solution of the form $\psi(\nu,\varsigma) = \mathcal{C}\, exp(-|\nu/2 - 2\varsigma/3|)$, with $\mathcal{C}$ representing an arbitrary constant.

In the literature, various numerical techniques exist for solving fractional differential equations (FDEs). Among these are the variational iteration technique (VIM) [7–10], which is a powerful mathematical tool for solving both linear and nonlinear equations, and the Adomian decomposition technique (ADM) [11,12], to possess a wide class of nonlinear problems. Another powerful and efficient method for nonlinear problems is the homotopy analysis technique (HAM), which was proposed by Liao [13–16]. However, the cost related to resolving extensive nonlinear systems, as well as the subsequent huge linear systems following linearization, may differ based on several factors, in addition to the system's complexity and the technique utilized for its resolution. To solve the extensive linear systems that arise from fractional partial differential equations (FPDEs), several discretization methods have been proposed, including spectral analysis [17], the finite volume method [18], and the finite element multigrid method [19]. These approaches effectively solve linear equations' systems and show promising results in terms of stability and convergence.

Over the last few years, fractional numerical methods have been developed and improved to obtain an accurate solution to time-fractional partial differential equations. The modified methods are based on a combination of a numerical method with an appropriate transform operator, including Laplace transform (LT) [20], Shehu transform (ShT) [21], Sumudu transform (SuT) [22], and Elzaki transform (ET) [23], in order to achieve an analytical or numerical solution to fractional partial differential equations (FPDEs) in the Atangana–Baleanu (AB) sense. The use of (FPDEs) presents a more accurate description of diffusion processes in diverse scientific domains, and is increasingly crucial in the representation of real-world situations. For example, the nonlinear (FWE) Equation (1) carries substantial physical meaning, as it functions as a mathematical framework for describing the behavior of nonlinear dispersive waves in the field of fluid dynamics and wave propagation.

Moreover, the stability, existences, and uniqueness are important to show for any nonlinear partial differential equations. In [24], Gao et al. have studied the stability of solutions for the (FWE) in $L^1(\mathbb{R})$ space. The existence and uniqueness of the solution for the fractional-order nonlinear (FWE) were demonstrated [25,26]. In [25], Kumar et al. have employed the Laplace decomposition method (LDM) to find an approximate solution for the (FWE), which includes the (AB) fractional derivative with a fractional order acting on the function $\psi(\nu, \varsigma)$. In [26], Sartanpara et al. have utilized the (q-HAMShT) technique to derive an approximate analytical solution for the (FWE) involving a fractional-order derivative in the Caputo sense. In [27], Shah et al. have utilized adapted approaches, specifically the (ADMShT) approach and the (VIMShT) approach, in order to attain an approximate analytical solution for the (FWE), with consideration given to the Caputo sense of non-integer order derivatives. In [28], Iqbal et al. have effectively utilized two adapted techniques to explore an approximate solution for the (FWE), which incorporates fractional-order derivatives with a Mittag–Leffler kernel. In [29], Haroon et al. have applied both the Adomian decomposition transform (ADT) and variational iteration transform (VIT) techniques, which incorporate the (AB)–Caputo fractional-order derivatives. The (ET) is used in the (AB) derivative to find an approximate solution for the (FWE). In [30], Alderremy et al. have employed the natural transform decomposition (NTD) approach to derive an approximate numerical solution for the (FWE) involving fractional-order derivatives with the Caputo derivative. In [31], Shah et al. have introduced an analytical approach for solving the Benney equation using the (HAM) transform approach, with consideration given to the fractional-order (AB) derivative in the Riemann–Liouville sense. In [32], Nonlaopon et al. have implemented the Laplace homotopy perturbation transform method (LHPTM) for the results of fractional-order Whitham–Broer–Kaup equations. In [33], Mofarreh et al. have combined the idea of the Adomian decomposition method (ADM) with Laplace transform (LT) for solving fractional-order heat equations with the help of the Caputo–Fabrizio operator. In [34], Sunitha et al. have used the q-homotopy analysis method (q-HAM) combined with the Elzaki transform (ET) to investigate the two-dimensional advection–dispersion (AD) problem. These equations are mainly used to describe the fate of pollutants in aquifers. In [35], Alsidrani et al. have successfully utilized three powerful techniques, including the (VIM) technique, the (ADM) technique, and the (HAM) technique, to approximate a solution for the (FWE) with variable coefficients, in view of the Caputo operator. In [36], Alshammari et al. have used two approaches, (LADM) and (VITM), to solve one-dimensional and three-dimensional diffusion equations with a fractional-order derivative.

In this paper, we consider the (FPDE) with variable coefficients involving the Atangana–Baleanu (AB) derivatives

$$ {}^{ABC}\mathcal{D}_{\varsigma}^{\vartheta}\psi - \alpha(\nu)\frac{\partial^3\psi}{\partial\nu^2\partial\varsigma} + 2\kappa(\nu)\frac{\partial\psi}{\partial\nu} + \beta(\nu)\psi\frac{\partial\psi}{\partial\nu} - \gamma(\nu)\psi\frac{\partial^3\psi}{\partial\nu^3} - 3\omega(\nu)\frac{\partial\psi}{\partial\nu}\frac{\partial^2\psi}{\partial\nu^2} = 0, \tag{2} $$

with the initial condition

$$ \psi(\nu,0) = \phi(\nu),\ \ \psi(l,\varsigma) = \psi(L,\varsigma) = 0,\ \ \varsigma > 0,\ \ l \le \nu \le L, \text{and } l, L \in \mathbb{R}, \tag{3} $$

where ${}^{ABC}\mathcal{D}_{\varsigma}^{\vartheta}$ represents the Atangana–Baleanu–Caputo derivative; $\alpha(\nu), \kappa(\nu), \beta(\nu), \gamma(\nu)$, and $\omega(\nu)$ are continuous functions; and $s - 1 < \vartheta \le s$ for $(s \in \mathbb{N})$.

Including variable coefficients is intended to improve the model's precision when representing the propagation of waves. In addition to showing the accuracy of the proposed methods in finding the approximate solutions of (TFPDEs), we consider the fractional-order operator with a Mittag–Leffler kernel. In particular, this holds significant importance and is a valuable tool when examining dispersive wave phenomena. The first approach (LVIM) combines the Laplace transform and the variational iteration technique, which is an analytical technique which requires one to determine a Lagrange multiplier for solving differential equations without discretization or linearization. The second approach (LADM) combines the Laplace transform and the Adomian decomposition technique. This technique is a straightforward and effective approach to obtain an analytical approximation for a wide class of both linear and nonlinear equations without linearization or discretization techniques, which typically lead to extensive numerical computation. The third approach (LHAM) combines the Laplace transform and the homotopy analysis technique. The advantage of this method is that it contains the auxiliary parameter $\hbar$, which provides a simple way to adjust and control the convergence region and rate of solution series. To enhance these methods, the Laplace transformation operator is used. The Atangana–Baleanu (AB) fractional derivative is a nonlocal derivative more suitable for modeling anomalous diffusion phenomena than classical fractional derivatives.

We structure this paper as follows. In Section 2, we provide preliminary definitions and discuss certain properties. Section 3 focuses on establishing an analysis of (LVIM) for the (TFPDE). Section 4 is dedicated to establishing an analysis of (LADM) for the (TFPDE). Section 5 is dedicated to establishing an analysis of (LHAM) for the (TFPDE). Section 7 demonstrates the techniques for solving fractional-order nonlinear (PDEs) with an appropriate initial condition.

2. Preliminary Concepts

In this section, we present the definitions of partial fractional derivative operators and their properties, which will be used later.

Definition 1 ([37])**.** *The Riemann–Liouville fractional integral of order ϑ with respect to ς is defined by*

$$ {}_{a}\mathcal{I}_{\varsigma}^{\vartheta}(\psi(\nu,\varsigma)) = \frac{1}{\Gamma(\vartheta)}\int_{a}^{\varsigma}\psi(\nu,\xi)(\varsigma-\xi)^{\vartheta-1}d\xi, \tag{4} $$

where $\Gamma(\vartheta)$ is a Gamma function.

Definition 2 ([37]). *Let $s-1<\vartheta\leq s, s\in\mathbb{N}$, $\psi\in\mathbb{L}^1(a,b)$, and $a<b$. The partial Riemann–Liouville fractional derivative of order ϑ of $\psi(\nu,\varsigma)$ with respect to ς is defined by*

$$ {}_a\mathcal{D}_\varsigma^\vartheta(\psi(\nu,\varsigma))=\frac{\partial^s}{\partial\varsigma^s}\frac{1}{\Gamma(s-\vartheta)}\int_a^\varsigma\psi(\nu,\xi)(\varsigma-\xi)^{s-\vartheta-1}\mathrm{d}\xi. \tag{5}$$

Definition 3 ([38,39]). *For s to be the smallest integer that exceeds ϑ, the Caputo fractional derivative operator of order $\vartheta>0$ is defined by*

$$ {}^C\mathcal{D}_\varsigma^\vartheta(\psi(\nu,\varsigma))=\begin{cases}\frac{1}{\Gamma(s-\vartheta)}\int_0^\varsigma\frac{\partial^s\psi(\nu,\xi)}{\partial\xi^s}(\varsigma-\xi)^{s-\vartheta-1}\mathrm{d}\xi, & s-1<\vartheta<s,\\ \frac{\partial^s}{\partial\zeta^s}\psi(\nu,\varsigma) & \vartheta=s\in\mathbb{N}.\end{cases} \tag{6}$$

Definition 4 ([29,40]). *The Atangana–Baleanu fractional derivative operator in the Caputo sense (ABC) and the Atangana–Baleanu fractional derivative operator in the Riemann–Liouville sense (ABR) of order $\vartheta, 0<\vartheta\leq 1$, and $\psi\in\mathbb{L}^1(a,b), a<b$, respectively, are defined by*

$$ {}_a^{ABC}\mathcal{D}_\varsigma^\vartheta(\psi(\nu,\varsigma))=\frac{\mathcal{P}(\vartheta)}{1-\vartheta}\int_a^\varsigma\frac{\partial\psi(\nu,\xi)}{\partial\xi}\mathcal{E}_\vartheta\big[-\frac{\vartheta(\varsigma-\xi)^\vartheta}{1-\vartheta}\big]\mathrm{d}\xi, \tag{7}$$

and

$$ {}_a^{ABR}\mathcal{D}_\varsigma^\vartheta(\psi(\nu,\varsigma))=\frac{\mathcal{P}(\vartheta)}{1-\vartheta}\frac{d}{d\varsigma}\int_a^\varsigma\psi(\nu,\xi)\mathcal{E}_\vartheta\big[-\frac{\vartheta(\varsigma-\xi)^\vartheta}{1-\vartheta}\big]\mathrm{d}\xi, \tag{8}$$

where $\mathcal{P}(\vartheta)$ denotes a normalization function such that $\mathcal{P}(0)=\mathcal{P}(1)=1$, and $\mathcal{E}_\vartheta(\xi)=\sum_{k=0}^\infty\frac{\xi^k}{\Gamma(k\vartheta+1)}$ represents the Mittag–Leffler function as given in [41].

Definition 5 ([29,40]). *The Atangana–Baleanu fractional integral operator of order $\vartheta\in(0,1)$ and a function $\psi\in\mathbb{L}^1(a,b), a<b$, is defined by*

$$ {}_a^{AB}\mathcal{I}_\varsigma^\vartheta(\psi(\nu,\varsigma))=\frac{1-\vartheta}{\mathcal{P}(\vartheta)}(\psi(\nu,\varsigma))+\frac{\vartheta}{\Gamma(\vartheta)\mathcal{P}(\vartheta)}\int_a^\varsigma\psi(\nu,\xi)(\varsigma-\xi)^{\vartheta-1}\mathrm{d}\xi. \tag{9}$$

Definition 6 ([40]). *The Laplace transformation operator connected with the (ABC) and (ABR) operators with respect to ς, are, respectively, defined by*

$$ \mathrm{L}_\varsigma[{}^{ABC}\mathcal{D}_\varsigma^\vartheta\psi(\nu,\varsigma)](\mathfrak{y})=\frac{\mathfrak{y}^\vartheta\mathcal{P}(\vartheta)\{\mathrm{L}_\varsigma[\psi(\nu,\varsigma)](\mathfrak{y})-\frac{1}{\mathfrak{y}}\psi(\nu,0)\}}{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}, \tag{10}$$

and

$$ \mathrm{L}_\varsigma[{}^{ABR}\mathcal{D}_\varsigma^\vartheta\psi(\nu,\varsigma)](\mathfrak{y})=\frac{\mathfrak{y}^\vartheta\mathcal{P}(\vartheta)\mathrm{L}_\varsigma[\psi(\nu,\varsigma)](\mathfrak{y})}{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}. \tag{11}$$

Proposition 1. *For $\psi(\nu,\varsigma)$ defined in $[a,b]$, $\varsigma>0$, $s-1<\vartheta\leq s$, and $s\in\mathbb{N}$. The operator ${}_a^{AB}\mathcal{I}_\varsigma^\vartheta$ satisfies the following properties, which were verified in [42].*

1. ${}_a^{ABC}\mathcal{D}_\varsigma^\vartheta(\psi(\nu,\varsigma))=0$, *if $\psi(\nu,\varsigma)$ is a constant function.*
2. ${}_a^{ABR}\mathcal{D}_\varsigma^\vartheta\ {}_a^{AB}\mathcal{I}_\varsigma^\vartheta(\psi(\nu,\varsigma))=\psi(\nu,\varsigma)$.
3. ${}_a^{AB}\mathcal{I}_\varsigma^\vartheta\ {}_a^{ABC}\mathcal{D}_\varsigma^\vartheta(\psi(\nu,\varsigma))=\psi(\nu,\varsigma)-\sum_{k=0}^{s}\frac{(\varsigma-a)^k}{k!}\frac{\partial^k\psi(\nu,a)}{\partial\varsigma^k}$.
4. ${}_a^{AB}\mathcal{I}_\varsigma^\vartheta\ {}_a^{ABR}\mathcal{D}_\varsigma^\vartheta(\psi(\nu,\varsigma))=\psi(\nu,\varsigma)-\sum_{k=0}^{s-1}\frac{(\varsigma-a)^k}{k!}\frac{\partial^k\psi(\nu,a)}{\partial\varsigma^k}$.

Remark 1. *From Definition 6, we have the relation between the AB–Caputo and AB–Riemann–Liouville operators, which was verified in [40]*

$$ {}_{a}^{ABC}\mathcal{D}_{\varsigma}^{\vartheta}(\psi(\nu,\varsigma)) = {}_{a}^{ABR}\mathcal{D}_{\varsigma}^{\vartheta}(\psi(\nu,\varsigma)) - \frac{\mathcal{P}(\vartheta)}{1-\vartheta}\,\psi(\nu,a)\,\mathcal{E}_{\vartheta}\Big(-\frac{\vartheta(\varsigma-a)^{\vartheta}}{1-\vartheta}\Big). \tag{12}$$

Remark 2. *The following are some significant advantages of the fractional derivatives in Definitions 2–4.*

1. *Both Caputo and Riemann–Liouville fractional derivatives have singular kernels.*
2. *Both AB–Caputo and AB–Riemann–Liouville fractional derivatives have non-singular and non-local kernels.*
3. *Both Caputo and AB–Caputo fractional derivatives of a constant function are zero.*
4. *Both Riemann–Liouville and AB–Riemann–Liouville fractional derivatives of a constant function do not equal zero.*

3. Conceptualization of (Lvim)

Here, we demonstrate the (LVIM) solution for the (FNPDEs) with variable coefficients.

Step 1: Consider the following fractional-order nonlinear (PDE)

$$ {}^{ABC}\mathcal{D}_{\varsigma}^{\vartheta}\psi(\nu,\varsigma) + \mathcal{L}\psi(\nu,\varsigma) + \mathcal{N}\psi(\nu,\varsigma) = \mathcal{G}(\nu,\varsigma), \tag{13}$$

with the initial condition

$$ \psi(\nu,0) = \phi(\nu),\ \ \psi(l,\varsigma) = \psi(L,\varsigma) = 0,\ \ \varsigma > 0,\ \ l < \nu < L,\ \ l, L \in \mathbb{R}, \tag{14}$$

where ${}^{ABC}\mathcal{D}_{\varsigma}^{\vartheta} = \frac{\partial^{\vartheta}}{\partial\varsigma^{\vartheta}}$ is the Atangana–Baleanu–Caputo derivative of order $s-1 < \vartheta \le s$, $(s \in \mathbb{N})$, $\mathcal{L}, \mathcal{N}$ are linear and nonlinear operators, respectively, and $\mathcal{G}(\nu,\varsigma)$ is the source term.

Step 2: Taking the Laplace transformation operator L_{ς} on both sides of Equation (13), we obtain

$$ \mathrm{L}_{\varsigma}[{}^{ABC}\mathcal{D}_{\varsigma}^{\vartheta}\psi(\nu,\varsigma)] + \mathrm{L}_{\varsigma}[\mathcal{L}\psi(\nu,\varsigma) + \mathcal{N}\psi(\nu,\varsigma)] = \mathrm{L}_{\varsigma}[\mathcal{G}(\nu,\varsigma)]. \tag{15}$$

By Definition (6) and the Laplace differentiation property, we obtain

$$ \frac{\mathfrak{y}^{\vartheta}}{\mathfrak{y}^{\vartheta}(1-\vartheta)+\vartheta}\Big[\mathrm{L}_{\varsigma}[\psi(\nu,\varsigma)](\mathfrak{y}) - \frac{1}{\mathfrak{y}}\psi(\nu,0)\Big] + \mathrm{L}_{\varsigma}[\mathcal{L}\psi(\nu,\varsigma) + \mathcal{N}\psi(\nu,\varsigma)] = \mathrm{L}_{\varsigma}[\mathcal{G}(\nu,\varsigma)]. \tag{16}$$

This method requires one to determine a Lagrange multiplier $\lambda(\mathfrak{y})$, which can be defined as

$$ \lambda(\mathfrak{y}) = -\frac{\mathfrak{y}^{\vartheta}(1-\vartheta)+\vartheta}{\mathfrak{y}^{\vartheta}}. \tag{17}$$

Step 3: Applying the inverse Laplace transform operator $\mathrm{L}_{\varsigma}^{-1}$ in Equation (16), we obtain

$$ \psi(\nu,\varsigma) = \phi(\nu) - \mathrm{L}_{\varsigma}^{-1}\left[\frac{\mathfrak{y}^{\vartheta}(1-\vartheta)+\vartheta}{\mathfrak{y}^{\vartheta}}\cdot \mathrm{L}_{\varsigma}\big[\mathcal{L}\psi(\nu,\varsigma) + \mathcal{N}\psi(\nu,\varsigma) - \mathcal{G}(\nu,\varsigma)\big]\right]. \tag{18}$$

Step 4: According to the variational iteration method [9,10], the correction functional of Equation (13) can be constructed as follows

$$ \psi_{m+1}(\nu,\varsigma) = \psi_m(\nu,0) - \mathrm{L}_{\varsigma}^{-1}\left[\frac{\mathfrak{y}^{\vartheta}(1-\vartheta)+\vartheta}{\mathfrak{y}^{\vartheta}}\cdot \mathrm{L}_{\varsigma}\big[\mathcal{L}\psi_m(\nu,\varsigma) + \mathcal{N}\psi_m(\nu,\varsigma) - \mathcal{G}(\nu,\varsigma)\big]\right]. \tag{19}$$

Step 5: The series form solution for Equation (19) can be obtained as follows

$$\begin{aligned}
\psi_0(\nu,\varsigma) &= \phi(\nu) + \mathrm{L}_\varsigma^{-1}\left[\frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma[\mathcal{G}(\nu,\varsigma)]\right],\\
\psi_1(\nu,\varsigma) &= \psi_0(\nu,\varsigma) - \mathrm{L}_\varsigma^{-1}\left[\frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma[\mathcal{L}\psi_0(\nu,\varsigma)+\mathcal{N}\psi_0(\nu,\varsigma)]\right],\\
&\vdots\\
\psi_{m+1}(\nu,\varsigma) &= \psi_m(\nu,\varsigma) - \mathrm{L}_\varsigma^{-1}\left[\frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma[\mathcal{L}\psi_m(\nu,\varsigma)+\mathcal{N}\psi_m(\nu,\varsigma)]\right],\ m\geq 1.
\end{aligned} \tag{20}$$

Therefore, Equation (19) will yield a series of approximations, $\psi_k(\nu,\varsigma)=\sum_{m=0}^{k-1}\psi_m(\nu,\varsigma)$, and the solution of Equation (13) is given by

$$\psi(\nu,\varsigma) = \lim_{k\to\infty}\psi_k(\nu,\varsigma). \tag{21}$$

4. Conceptualization of (Ladm)

Here, we demonstrate the (LADM) solution for the (FNPDEs) with variable coefficients. Step 1: Consider the nonlinear (PDE), Equation (13), with the initial condition Equation (14).

Step 2: Taking the Laplace transformation operator L_ς on both sides of Equation (13), we obtain

$$\mathrm{L}_\varsigma[{}^{ABC}\mathcal{D}_\varsigma^\vartheta\psi(\nu,\varsigma)] + \mathrm{L}_\varsigma[\mathcal{L}\psi(\nu,\varsigma)+\mathcal{N}\psi(\nu,\varsigma)] = \mathrm{L}_\varsigma[\mathcal{G}(\nu,\varsigma)]. \tag{22}$$

By Definition (6) and the Laplace differentiation property, we obtain

$$\frac{\mathfrak{y}^\vartheta}{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}\left[\mathrm{L}_\varsigma[\psi(\nu,\varsigma)](\mathfrak{y}) - \frac{1}{\mathfrak{y}}\psi(\nu,0)\right] + \mathrm{L}_\varsigma[\mathcal{L}\psi(\nu,\varsigma)+\mathcal{N}\psi(\nu,\varsigma)] = \mathrm{L}_\varsigma[\mathcal{G}(\nu,\varsigma)]. \tag{23}$$

Step 3: Applying the inverse Laplace transform $\mathrm{L}_\varsigma^{-1}$ on Equation (23), we obtain

$$\begin{aligned}
\psi(\nu,\varsigma) = \phi(\nu) - \mathrm{L}_\varsigma^{-1}\Bigg[&\frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma\left[\mathcal{L}\psi(\nu,\varsigma)+\mathcal{N}\psi(\nu,\varsigma)\right]\\
&+\frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma[\mathcal{G}(\nu,\varsigma)]\Bigg].
\end{aligned} \tag{24}$$

Since the (ADM) procedure is in the form

$$\psi(\nu,\varsigma) = \sum_{m=0}^{\infty}\psi_m(\nu,\varsigma), \tag{25}$$

and the nonlinear term $\mathcal{N}\psi(\nu,\varsigma)$ can be decomposed into an infinite series of polynomials, given by

$$\mathcal{N}\psi(\nu,\varsigma) = \sum_{m=0}^{\infty}\mathcal{B}_m(\nu,\varsigma), \tag{26}$$

where $\mathcal{B}_m(\nu,\varsigma)$ are the Adomian polynomials of $\psi_0,\psi_1,\cdots,\psi_m$ defined by

$$\mathcal{B}_m(\psi_0,\psi_1,\cdots,\psi_m) = \frac{1}{m!}\frac{d^m}{dq^m}\left[\mathcal{N}\left(\sum_{k=0}^{\infty}\psi_k q^k\right)\right]_{q=0},\ m=0,1,\ldots \tag{27}$$

Therefore, the first Adomian polynomials for $\mathcal{N}\psi(\nu,\varsigma)$ are defined as follows

$$
\begin{aligned}
\mathcal{B}_0 &= \mathcal{N}(\psi_0),\\
\mathcal{B}_1 &= \psi_1\left(\frac{d\mathcal{N}(\psi_0)}{d\psi_0}\right),\\
\mathcal{B}_2 &= \psi_2\left(\frac{d\mathcal{N}(\psi_0)}{d\psi_0}\right) + \frac{\psi_1^2}{2!}\left(\frac{d^2\mathcal{N}(\psi_0)}{d\psi_0^2}\right),\\
\mathcal{B}_3 &= \psi_3\left(\frac{d\mathcal{N}(\psi_0)}{d\psi_0}\right) + \psi_1\psi_2\left(\frac{d^2\mathcal{N}(\psi_0)}{d\psi_0^2}\right) + \frac{\psi_1^3}{3!}\left(\frac{d^3\mathcal{N}(\psi_0)}{d\psi_0^3}\right),
\end{aligned}
\tag{28}
$$

Step 4: Substituting Equations (25) and (26) into Equation (24), we obtain

$$
\begin{aligned}
\sum_{m=0}^{\infty}\psi_m(\nu,\varsigma) = \phi(\nu) - \mathrm{L}_\varsigma^{-1}\Bigg[\frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma\big[\mathcal{L}\big(\sum_{m=0}^{\infty}\psi_m(\nu,\varsigma)\big)\\
+\sum_{m=0}^{\infty}\mathcal{B}_m(\nu,\varsigma)\big] + \frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma\big[\mathcal{G}(\nu,\varsigma)\big]\Bigg].
\end{aligned}
\tag{29}
$$

Step 5: The (ADM) transforms Equation (29) into a set of recursive relations, given by

$$
\begin{aligned}
\psi_0(\nu,\varsigma) &= \phi(\nu) + \mathrm{L}_\varsigma^{-1}\left[\frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma\big[\mathcal{G}(\nu,\varsigma)\big]\right],\\
\psi_1(\nu,\varsigma) &= -\mathrm{L}_\varsigma^{-1}\left[\frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma\big[\mathcal{L}(\psi_0(\nu,\varsigma)) + \mathcal{B}_0(\nu,\varsigma)\big]\right],\\
&\ \ \vdots\\
\psi_{m+1}(\nu,\varsigma) &= -\mathrm{L}_\varsigma^{-1}\left[\frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma\big[\mathcal{L}(\psi_m(\nu,\varsigma)) + \mathcal{B}_m(\nu,\varsigma)\big]\right],\ m\geq 1.
\end{aligned}
\tag{30}
$$

Let the expression $\psi_k(\nu,\varsigma) = \sum_{m=0}^{k-1}\psi_m(\nu,\varsigma)$ be the k-term approximation of $\psi(\nu,\varsigma)$, and using the above Equation (30) yields the approximate solution of Equation (13)

$$
\psi(\nu,\varsigma) = \lim_{m\to\infty}\psi_k(\nu,\varsigma). \tag{31}
$$

5. Conceptualization of (Lham)

Here, we demonstrate the (LHAM) solution for the (FNPDEs) with variable coefficients.

Step 1: Consider the nonlinear (PDE) Equation (13) with the initial condition Equation (14).

Step 2: Taking the Laplace transformation operator L_ς on both sides of Equation (13), we obtain

$$
\mathrm{L}_\varsigma[{}^{ABC}\mathcal{D}_\varsigma^\vartheta\psi(\nu,\varsigma)] + \mathrm{L}_\varsigma[\mathcal{L}\psi(\nu,\varsigma) + \mathcal{N}\psi(\nu,\varsigma)] = \mathrm{L}_\varsigma[\mathcal{G}(\nu,\varsigma)]. \tag{32}
$$

By Definition (6) and the Laplace differentiation property, we obtain

$$
\frac{\mathfrak{y}^\vartheta}{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}\left[\mathrm{L}_\varsigma[\psi(\nu,\varsigma)](\mathfrak{y}) - \frac{1}{\mathfrak{y}}\psi(\nu,0)\right] + \mathrm{L}_\varsigma[\mathcal{L}\psi(\nu,\varsigma) + \mathcal{N}\psi(\nu,\varsigma)] = \mathrm{L}_\varsigma[\mathcal{G}(\nu,\varsigma)]. \tag{33}
$$

On simplifying Equation (33), we obtain

$$
\begin{aligned}
\mathrm{L}_\varsigma\big[\psi(\nu,\varsigma)\big] = \frac{1}{\mathfrak{y}}\psi(\nu,0) - \frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma\big[\mathcal{L}\psi(\nu,\varsigma) + \mathcal{N}\psi(\nu,\varsigma)\big]\\
+\frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma[\mathcal{G}(\nu,\varsigma)].
\end{aligned}
\tag{34}
$$

Step 3: The nonlinear operator $\mathcal{N}(\nu,\varsigma)$ can be determined by

$$\begin{aligned}\mathcal{N}\big[\varphi(\nu,\varsigma;\varrho)\big] = \mathrm{L}_\varsigma\big[\varphi(\nu,\varsigma;\varrho)\big] - \frac{1}{\mathfrak{y}}\varphi_0(\nu;\varrho) + \frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma\big[\mathcal{L}\varphi(\nu,\varsigma;\varrho) \\ + \mathcal{N}\varphi(\nu,\varsigma;\varrho)\big] - \frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma[\mathcal{G}(\nu,\varsigma)],\end{aligned} \quad (35)$$

with ${}^{ABC}\mathcal{D}_\varsigma^\vartheta(\mathcal{C}) = 0$.

Step 4: According to Liao [14], we can construct the homotopy for Equation (34) as follows

$$(1-n\,\varrho)\mathrm{L}_\varsigma\big[\varphi(\nu,\varsigma;\varrho) - \psi_0(\nu,\varsigma)\big] = \varrho\hbar\mathcal{H}(\nu,\varsigma)\mathcal{N}\big[\varphi(\nu,\varsigma;\varrho)\big], \quad (36)$$

where $\varrho \in [0, \frac{1}{n}], n \geq 1$ is an embedding parameter, L_ς is the Laplace transformation operator, $\varphi(\nu,\varsigma;\varrho)$ is a mapping function for $\psi(\nu,\varsigma)$, $\psi_0(\nu,\varsigma)$ is an initial guess of $\psi(\nu,\varsigma)$, $\hbar \neq 0$, and $\mathcal{H}(\nu,\varsigma) \neq 0$. It is clear that, for $\varrho = 0$ and $\varrho = \frac{1}{n}$, we obtain

$$\varphi(\nu,\varsigma;0) = \psi_0(\nu,\varsigma), \text{and } \varphi\big(\nu,\varsigma;\frac{1}{n}\big) = \psi(\nu,\varsigma). \quad (37)$$

Step 5: As ϱ moves from 0 to $\frac{1}{n}$, the solution $\varphi(\nu,\varsigma;\varrho)$ varies from the initial guess $\psi_0(\nu,\varsigma)$ to the solution $\psi(\nu,\varsigma)$. Expanding $\varphi(\nu,\varsigma;\varrho)$ into the Taylor series with respect to the embedding parameter ϱ, we obtain

$$\varphi(\nu,\varsigma;\varrho) = \psi_0(\nu,\varsigma) + \sum_{m=1}^{\infty}\psi_m(\nu,\varsigma)\,\varrho^m, \quad (38)$$

where

$$\psi_m(\nu,\varsigma) = \frac{1}{m!}\frac{\partial^m\varphi(\nu,\varsigma;\varrho)}{\partial\varrho^m}\bigg|_{\varrho=0}. \quad (39)$$

If $\psi_0(\nu,\varsigma)$, n, $\hbar$, and $\mathcal{H}(\nu,\varsigma)$ are so properly chosen, the series $\varphi(\nu,\varsigma;\varrho)$ in Equation (38) will converge at $\varrho = \frac{1}{n}$

$$\varphi\big(\nu,\varsigma;\frac{1}{n}\big) = \psi(\nu,\varsigma) = \psi_0(\nu,\varsigma) + \sum_{m=1}^{\infty}\psi_m(\nu,\varsigma)\big(\frac{1}{n}\big)^m, \quad (40)$$

which is one of the solutions of the (PDE), demonstrated by Liao [16].

For $\hbar = -1$ and $\mathcal{H}(\nu,\varsigma) = 1$, Equation (36) becomes

$$(1-n\,\varrho)\mathrm{L}_\varsigma\big[\varphi(\nu,\varsigma;\varrho) - \psi_0(\nu,\varsigma)\big] + \varrho\mathcal{N}\big[\varphi(\nu,\varsigma;\varrho)\big] = 0. \quad (41)$$

Define the vector

$$\vec{\psi}_m(\nu,\varsigma) = \{\psi_0(\nu,\varsigma), \psi_1(\nu,\varsigma), \ldots, \psi_m(\nu,\varsigma)\}.$$

Step 6: By differentiating Equation (36) m times with respect to ϱ, then letting $\varrho = 0$ and dividing by $m!$ with the assumption $\mathcal{H}(\nu,\varsigma) = 1$, we obtain the mth-order deformation equation

$$\mathrm{L}_\varsigma[\psi_m(\nu,\varsigma) - \mathcal{Q}_m\,\psi_{m-1}(\nu,\varsigma)] = \hbar\mathcal{R}_m[\vec{\psi}_{m-1}(\nu,\varsigma)]. \quad (42)$$

Step 7: Applying the inverse Laplace transform $\mathrm{L}_\varsigma^{-1}$ on Equation (42), we obtain the solution of the above mth-order deformation equation

$$\psi_m(\nu,\varsigma) = \mathcal{Q}_m\,\psi_{m-1}(\nu,\varsigma) + \hbar\mathrm{L}_\varsigma^{-1}\big[\mathcal{R}_m[\vec{\psi}_{m-1}(\nu,\varsigma)]\big], \quad (43)$$

where

$$\mathcal{R}_m[\vec{\psi}_{m-1}(\nu,\varsigma)] = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\varphi(\nu,\varsigma;\varrho)]}{\partial\varrho^{m-1}}\bigg|_{\varrho=0}, \quad (44)$$

and

$$\mathcal{Q}_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases} \tag{45}$$

Step 8: By using Equation (43) with the initial condition Equation (14), we obtain the first terms of the (LHAM) approximate series solutions

$$\begin{aligned} \psi_0(\nu,\varsigma) &= \phi(\nu), \\ \psi_1(\nu,\varsigma) &= \hbar \mathrm{L}_{\varsigma}^{-1}\left[\mathcal{R}_1[\vec{\psi}_0(\nu,\varsigma)]\right], \\ \psi_2(\nu,\varsigma) &= \psi_1(\nu,\varsigma) + \hbar \mathrm{L}_{\varsigma}^{-1}\left[\mathcal{R}_2[\vec{\psi}_1(\nu,\varsigma)]\right], \\ &\vdots \\ \psi_m(\nu,\varsigma) &= \mathcal{Q}_m\, \psi_{m-1}(\nu,\varsigma) + \hbar \mathrm{L}_{\varsigma}^{-1}\left[\mathcal{R}_m[\vec{\psi}_{m-1}(\nu,\varsigma)]\right]. \end{aligned} \tag{46}$$

Therefore, we obtain an accurate approximation of the series solution of Equation (13) as follows

$$\psi(\nu,\varsigma) = \sum_{k=0}^{m} \psi_k(\nu,\varsigma). \tag{47}$$

6. Convergence Analysis

Theorem 1 ([43])**.** *Let $\psi(\nu,\varsigma)$ be the approximate series solution that was found for the finite series $\sum_{k=0}^{m} \psi_k(\nu,\varsigma)$. Assuming $R \in (0,1)$ such that $\|\psi_{k+1}(\nu,\varsigma)\| \leq R\|\psi_k(\nu,\varsigma)\|$, the maximum absolute error is estimated by*

$$\|\psi(\nu,\varsigma) - \sum_{k=0}^{m} \psi_k(\nu,\varsigma)\| \leq \frac{R^{m+1}}{1-R}\|\psi_0(\nu,\varsigma)\|. \tag{48}$$

Proof. Let the series $\sum_{k=0}^{m} \psi_k(\nu,\varsigma)$ be finite, which implies that $\sum_{k=0}^{m} \psi_k(\nu,\varsigma) < \infty$.

$$\begin{aligned} \|\psi(\nu,\varsigma) - \sum_{k=0}^{m} \psi_k(\nu,\varsigma)\| = \|\sum_{k=m+1}^{\infty} \psi_k(\nu,\varsigma)\| &\leq \sum_{k=m+1}^{\infty} \|\psi_k(\nu,\varsigma)\| \\ &\leq \sum_{k=m+1}^{\infty} R^k \|\psi_0(\nu,\varsigma)\| \\ &\leq R^{m+1}(1 + R + R^2 + R^3 + \cdots)\|\psi_0(\nu,\varsigma)\| \\ &\leq \frac{R^{m+1}}{1-R}\|\psi_0(\nu,\varsigma)\|, \end{aligned} \tag{49}$$

which completes the proof of the theorem. □

Theorem 2 ([44])**.** *If the series solution $\psi(\nu,\varsigma) = \sum_{k=0}^{\infty} \psi_k(\nu,\varsigma)$ converges, then it is an exact solution of the nonlinear problem* (13)*.*

7. Implementation of Techniques

In this section, we employ the (LVIM), (LADM), and (LHAM) techniques to derive approximate solutions for the fractional-order nonlinear partial differential Equation (FNPDE), with consideration given to the fractional order in the Atangana–Baleanu (AB) derivative and variable coefficients.

Example 1. *Consider the following fractional-order nonlinear (PDE) with variable coefficients*

$$
\begin{aligned}
&{}^{ABC}\mathcal{D}_{\varsigma}^{\vartheta}\psi(\nu,\varsigma) - \nu\,\frac{\partial^3\psi(\nu,\varsigma)}{\partial\nu^2\partial\varsigma} + 2\nu\,\frac{\partial\psi(\nu,\varsigma)}{\partial\nu} + \nu\,\psi(\nu,\varsigma)\frac{\partial\psi(\nu,\varsigma)}{\partial\nu} \\
&- \nu\,\psi(\nu,\varsigma)\frac{\partial^3\psi(\nu,\varsigma)}{\partial\nu^3} - 3\nu\,\frac{\partial\psi(\nu,\varsigma)}{\partial\nu}\frac{\partial^2\psi(\nu,\varsigma)}{\partial\nu^2} = 0,
\end{aligned}
\tag{50}
$$

and the initial condition

$$
\psi(\nu,0) = \mathrm{e}^{(\nu/2)},\ \psi(-1,\varsigma) = \psi(1,\varsigma) = 0,\ \varsigma > 0,\ -1 \le \nu \le 1. \tag{51}
$$

Taking the Laplace transform of Equation (50),

$$
\begin{aligned}
&\frac{\mathfrak{y}^{\vartheta}}{\mathfrak{y}^{\vartheta}(1-\vartheta)+\vartheta}\left[\mathrm{L}_{\varsigma}[\psi(\nu,\varsigma)](\mathfrak{y}) - \frac{1}{\mathfrak{y}}\psi(\nu,0)\right] + \mathrm{L}_{\varsigma}[-\nu\,\frac{\partial^3\psi(\nu,\varsigma)}{\partial\nu^2\partial\varsigma} + 2\nu\,\frac{\partial\psi(\nu,\varsigma)}{\partial\nu} \\
&+ \nu\,\psi(\nu,\varsigma)\frac{\partial\psi(\nu,\varsigma)}{\partial\nu} - \nu\,\psi(\nu,\varsigma)\frac{\partial^3\psi(\nu,\varsigma)}{\partial\nu^3} - 3\nu\,\frac{\partial\psi(\nu,\varsigma)}{\partial\nu}\frac{\partial^2\psi(\nu,\varsigma)}{\partial\nu^2}] = 0.
\end{aligned}
\tag{52}
$$

Applying the inverse Laplace transform in Equation (52),

$$
\begin{aligned}
\psi(\nu,\varsigma) = \mathrm{e}^{(\nu/2)} + \mathrm{L}_{\varsigma}^{-1}\Bigg[&\frac{\mathfrak{y}^{\vartheta}(1-\vartheta)+\vartheta}{\mathfrak{y}^{\vartheta}}\cdot\mathrm{L}_{\varsigma}\big[\nu\,\frac{\partial^3\psi(\nu,\varsigma)}{\partial\nu^2\partial\varsigma} - 2\nu\,\frac{\partial\psi(\nu,\varsigma)}{\partial\nu} \\
&- \nu\,\psi(\nu,\varsigma)\frac{\partial\psi(\nu,\varsigma)}{\partial\nu} + \nu\,\psi(\nu,\varsigma)\frac{\partial^3\psi(\nu,\varsigma)}{\partial\nu^3} + 3\nu\,\frac{\partial\psi(\nu,\varsigma)}{\partial\nu}\frac{\partial^2\psi(\nu,\varsigma)}{\partial\nu^2}\big]\Bigg].
\end{aligned}
\tag{53}
$$

Apply the variational iteration method to obtain the series form solution. The iteration formula for Equation (50) can be constructed as

$$
\begin{aligned}
\psi_{m+1}(\nu,\varsigma) = \psi_m(\nu,0) + \mathrm{L}_{\varsigma}^{-1}\Bigg[&\frac{\mathfrak{y}^{\vartheta}(1-\vartheta)+\vartheta}{\mathfrak{y}^{\vartheta}}\cdot\mathrm{L}_{\varsigma}\big[\nu\,\frac{\partial^3\psi_m(\nu,\varsigma)}{\partial\nu^2\partial\varsigma} - 2\nu\,\frac{\partial\psi_m(\nu,\varsigma)}{\partial\nu} \\
&- \nu\,\psi_m(\nu,\varsigma)\frac{\partial\psi_m(\nu,\varsigma)}{\partial\nu} + \nu\,\psi_m(\nu,\varsigma)\frac{\partial^3\psi_m(\nu,\varsigma)}{\partial\nu^3} + 3\nu\,\frac{\partial\psi_m(\nu,\varsigma)}{\partial\nu}\frac{\partial^2\psi_m(\nu,\varsigma)}{\partial\nu^2}\big]\Bigg].
\end{aligned}
\tag{54}
$$

By using Equation (54), we obtain the (LVIM) approximate series solution, where

$$
\psi_0(\nu,\varsigma) = \psi(\nu,0) = \mathrm{e}^{(\nu/2)}. \tag{55}
$$

For $m = 0$, we obtain

$$
\begin{aligned}
\psi_1(\nu,\varsigma) &= \psi_0(\nu,\varsigma) + \mathrm{L}_{\varsigma}^{-1}\Bigg[\frac{\mathfrak{y}^{\vartheta}(1-\vartheta)+\vartheta}{\mathfrak{y}^{\vartheta}}\cdot\mathrm{L}_{\varsigma}\big[\nu\,\frac{\partial^3\psi_0(\nu,\varsigma)}{\partial\nu^2\partial\varsigma} - 2\nu\,\frac{\partial\psi_0(\nu,\varsigma)}{\partial\nu} \\
&\quad - \nu\,\psi_0(\nu,\varsigma)\frac{\partial\psi_0(\nu,\varsigma)}{\partial\nu} + \nu\,\psi_0(\nu,\varsigma)\frac{\partial^3\psi_0(\nu,\varsigma)}{\partial\nu^3} + 3\nu\,\frac{\partial\psi_0(\nu,\varsigma)}{\partial\nu}\frac{\partial^2\psi_0(\nu,\varsigma)}{\partial\nu^2}\big]\Bigg] \\
&= \mathrm{e}^{(\nu/2)} + \mathrm{L}_{\varsigma}^{-1}\left[\frac{\mathfrak{y}^{\vartheta}(1-\vartheta)+\vartheta}{\mathfrak{y}^{\vartheta}}\cdot\mathrm{L}_{\varsigma}\big[0 - \nu\mathrm{e}^{(\nu/2)} - \frac{\nu\mathrm{e}^{\nu}}{2} + \frac{\nu\mathrm{e}^{\nu}}{8} + \frac{3\nu\mathrm{e}^{\nu}}{8}\big]\right] \\
&= \mathrm{e}^{(\nu/2)}\left[1 - \nu + \nu\,\vartheta - \frac{\nu\,\vartheta\,\varsigma^{\vartheta}}{\Gamma(\vartheta+1)}\right].
\end{aligned}
\tag{56}
$$

For $m = 1$, we obtain

$$\begin{aligned}
\psi_2(\nu,\varsigma) &= \psi_1(\nu,\varsigma) + \mathrm{L}_\varsigma^{-1}\Bigg[\frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma\big[\nu\,\frac{\partial^3\psi_1(\nu,\varsigma)}{\partial\nu^2\partial\varsigma} - 2\nu\,\frac{\partial\psi_1(\nu,\varsigma)}{\partial\nu} \\
&\quad - \nu\,\psi_1(\nu,\varsigma)\frac{\partial\psi_1(\nu,\varsigma)}{\partial\nu} + \nu\,\psi_1(\nu,\varsigma)\frac{\partial^3\psi_1(\nu,\varsigma)}{\partial\nu^3} + 3\nu\,\frac{\partial\psi_1(\nu,\varsigma)}{\partial\nu}\frac{\partial^2\psi_1(\nu,\varsigma)}{\partial\nu^2}\big]\Bigg] \\
&= \mathrm{e}^{(\nu/2)}\left(1+\nu^2\,(\vartheta-1)^2 + 2\,\nu\,\vartheta^2 - 2\,\nu\,\vartheta\right) - \nu\,\mathrm{e}^{\nu}\left(2\,\nu(\vartheta-1)^3 + 3\,\vartheta^3 - 7\,\vartheta^2 - 5\,\vartheta - 1\right) \\
&\quad + \frac{\nu\,\mathrm{e}^{\nu}\,\vartheta^3\,\Gamma(2\vartheta+1)\,(2\nu+3)\,\varsigma^{3\vartheta}}{\Gamma(\vartheta+1)^2\,\Gamma(3\vartheta+1)} + \frac{\nu\,\mathrm{e}^{(\nu/2)}\,\vartheta^2\,(\vartheta-1)\,(\nu+4)\,\varsigma^{(\vartheta-1)}}{4\Gamma(\vartheta+1)} \\
&\quad - \frac{4\,\nu\,\mathrm{e}^{(\nu/2)}\,\vartheta^2\left(\frac{(\nu+\frac{3}{2})\,(\vartheta-1)\,\mathrm{e}^{(\nu/2)}\,\Gamma(2\vartheta+1)}{2} + \Gamma(\vartheta+1)^2(\mathrm{e}^{(\nu/2)}\,(\nu\,\vartheta-\nu+\frac{3\vartheta}{2}-1) - \frac{\nu}{4} - \frac{1}{2})\right)\varsigma^{2\vartheta}}{\Gamma(2\vartheta+1)\,\Gamma(\vartheta+1)^2} \\
&\quad + \frac{6\,\nu\,\mathrm{e}^{(\nu/2)}\vartheta\left((\vartheta-1)\,\mathrm{e}^{(\nu/2)}\,(\nu\,\vartheta+\frac{3\,\vartheta}{2}-\nu-\frac{5}{6}) - \frac{\nu\,\vartheta}{3} - \frac{2\,\vartheta}{3} + \frac{\nu}{3} + \frac{1}{3}\right)\varsigma^{\vartheta}}{\Gamma(\vartheta+1)} \\
&\quad - \frac{\nu\,\mathrm{e}^{(\nu/2)}\,\Gamma(\vartheta)\,\vartheta^3\,(\nu+4)\,\varsigma^{(2\vartheta-1)}}{4\Gamma(\vartheta+1)\,\Gamma(2\vartheta)}.
\end{aligned} \tag{57}$$

Therefore, the approximate (LVIM) series solution of Equation (50) is

$$\begin{aligned}
\psi(\nu,\varsigma) &= 3\,\mathrm{e}^{(\nu/2)} + \nu\,\mathrm{e}^{(\nu/2)}\,\vartheta\,(4+\nu)\left(\frac{-\vartheta\,\varsigma^{(2\vartheta-1)}}{4\Gamma(2\vartheta)} + \frac{(\vartheta-1)\,\varsigma^{(\vartheta-1)}}{4\Gamma(\vartheta)}\right) \\
&\quad + \nu\,\mathrm{e}^{(\nu/2)}\left(\mathrm{e}^{(\nu/2)}\,(\vartheta-1)\,(2\,\nu\,\vartheta-2\,\nu+3\,\vartheta-1) - \nu\,\vartheta+\nu-2\,\vartheta-1\right)\left(1-\vartheta+\frac{\vartheta\,\varsigma^{\vartheta}}{\Gamma(\vartheta+1)}\right) \\
&\quad + \nu\,\mathrm{e}^{(\nu/2)}\,\vartheta\left(2\,\mathrm{e}^{(\nu/2)}\,(-2\nu\,\vartheta+2\nu-3\vartheta+2) + (\nu+2)\right)\left(\frac{\vartheta\,\varsigma^{2\vartheta}}{\Gamma(2\vartheta+1)} - \frac{(\vartheta-1)\,\varsigma^{\vartheta}}{\Gamma(\vartheta+1)}\right) \\
&\quad + \frac{\nu\,\mathrm{e}^{\nu}\,\vartheta^2\,\Gamma(2\vartheta+1)\,(2\nu+3)\left(\frac{\vartheta\,\varsigma^{3\vartheta}}{\Gamma(3\vartheta+1)} - \frac{\varsigma^{2\vartheta}\,(\vartheta-1)}{\Gamma(2\vartheta+1)}\right)}{\Gamma(1+\vartheta)^2}.
\end{aligned} \tag{58}$$

Taking the Laplace transform of Equation (50),

$$\begin{aligned}
&\frac{\mathfrak{y}^\vartheta}{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}\left[\mathrm{L}_\varsigma[\psi(\nu,\varsigma)](\mathfrak{y}) - \frac{1}{\mathfrak{y}}\,\mathrm{e}^{(\nu/2)}\right] + \mathrm{L}_\varsigma\Big[-\nu\,\frac{\partial^3\psi(\nu,\varsigma)}{\partial\nu^2\partial\varsigma} + 2\,\nu\,\frac{\partial\,\psi(\nu,\varsigma)}{\partial\nu} \\
&+\nu\,\psi(\nu,\varsigma)\frac{\partial\,\psi(\nu,\varsigma)}{\partial\nu} - \nu\,\psi(\nu,\varsigma)\frac{\partial^3\,\psi(\nu,\varsigma)}{\partial\nu^3} - 3\,\nu\,\frac{\partial\,\psi(\nu,\varsigma)}{\partial\nu}\frac{\partial^2\,\psi(\nu,\varsigma)}{\partial\nu^2}\Big] = 0.
\end{aligned} \tag{59}$$

Applying the inverse Laplace transform in Equation (59),

$$\begin{aligned}
\psi(\nu,\varsigma) &= \mathrm{e}^{(\nu/2)} - \mathrm{L}_\varsigma^{-1}\Bigg[\frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma\Big[-\nu\,\frac{\partial^3\,\psi(\nu,\varsigma)}{\partial\nu^2\partial\varsigma} + 2\,\nu\,\frac{\partial\,\psi(\nu,\varsigma)}{\partial\nu} \\
&\quad +\nu\,\psi(\nu,\varsigma)\frac{\partial\,\psi(\nu,\varsigma)}{\partial\nu} - \nu\,\psi(\nu,\varsigma)\frac{\partial^3\,\psi(\nu,\varsigma)}{\partial\nu^3} - 3\,\nu\,\frac{\partial\,\psi(\nu,\varsigma)}{\partial\nu}\frac{\partial^2\,\psi(\nu,\varsigma)}{\partial\nu^2}\Big]\Bigg].
\end{aligned} \tag{60}$$

Let $\psi(\nu,\varsigma) = \sum_{m=0}^{\infty}\psi_m(\nu,\varsigma)$ and define $\mathcal{N}\psi(\nu,\varsigma) = \sum_{m=0}^{\infty}\mathcal{B}_m(\nu,\varsigma)$, where $\mathcal{B}_m(\nu,\varsigma)$ are the Adomian polynomials of $\psi_0,\psi_1,\cdots,\psi_m$

$$\mathcal{N}\psi(\nu,\varsigma) = \nu\,\psi(\nu,\varsigma)\frac{\partial\psi(\nu,\varsigma)}{\partial\nu} - \nu\,\psi(\nu,\varsigma)\frac{\partial^3\psi(\nu,\varsigma)}{\partial\nu^3} - 3\nu\,\frac{\partial\psi(\nu,\varsigma)}{\partial\nu}\frac{\partial^2\psi(\nu,\varsigma)}{\partial\nu^2},$$

we have

$$\begin{aligned}
\sum_{m=0}^{\infty}\psi_m(\nu,\varsigma) &= \mathrm{e}^{(\nu/2)} - \mathrm{L}_\varsigma^{-1}\Bigg[\frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma\Big[-\nu\,\mathcal{L}_{\nu\nu\varsigma}\sum_{m=0}^{\infty}\psi_m(\nu,\varsigma) \\
&\quad + 2\,\nu\,\mathcal{L}_\nu\sum_{m=0}^{\infty}\psi_m(\nu,\varsigma) + \sum_{m=0}^{\infty}\mathcal{B}_m(\nu,\varsigma)\Big]\Bigg].
\end{aligned} \tag{61}$$

The first Adomian polynomials for $\mathcal{N}\psi(\nu,\varsigma)$ are defined as

$$\mathcal{B}_0 = \mathcal{N}(\psi_0) = 0,\;\; \mathcal{B}_1 = \psi_1\big(\mathcal{N}(\psi_0)\big)_\nu = 0,\;\; \mathcal{B}_2 = \psi_2\big(\mathcal{N}(\psi_0)\big)_\nu + \frac{\psi_1^2}{2!}\big(\mathcal{N}(\psi_0)\big)_{\nu\nu} = 0. \tag{62}$$

Therefore, Equation (61) becomes

$$\begin{aligned}\psi_0(\nu,\varsigma) &= \mathrm{e}^{(\nu/2)},\\ \psi_{m+1}(\nu,\varsigma) &= -\,\mathrm{L}_\varsigma^{-1}\bigg[\frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma\big[-\nu\,\mathcal{L}_{\nu\nu\varsigma}\psi_m(\nu,\varsigma)+2\,\nu\,\mathcal{L}_\nu\psi_m(\nu,\varsigma)+\mathcal{B}_m(\nu,\varsigma)\big]\bigg],\;\; m\geq 0.\end{aligned} \tag{63}$$

By using Equation (63), we obtain the (LADM) approximate series solution
For $m=0$, we obtain

$$\begin{aligned}\psi_1(\nu,\varsigma) &= -\,\mathrm{L}_\varsigma^{-1}\bigg[\frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma\big[-\nu\,\mathcal{L}_{\nu\nu\varsigma}\psi_0(\nu,\varsigma)+2\,\nu\,\mathcal{L}_\nu\psi_0(\nu,\varsigma)+\mathcal{B}_0(\nu,\varsigma)\big]\bigg]\\ &= \mathrm{e}^{(\nu/2)}\bigg[-\nu+\nu\,\vartheta-\frac{\nu\,\vartheta\,\varsigma^\vartheta}{\Gamma(\vartheta+1)}\bigg].\end{aligned} \tag{64}$$

For $m=1$, we obtain

$$\begin{aligned}\psi_2(\nu,\varsigma) &= -\mathrm{L}_\varsigma^{-1}\bigg[\frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot \mathrm{L}_\varsigma\big[-\nu\mathcal{L}_{\nu\nu\varsigma}\psi_1(\nu,\varsigma)+2\nu\mathcal{L}_\nu\psi_1(\nu,\varsigma)+\mathcal{B}_1(\nu,\varsigma)\big]\bigg]\\ &= \nu\mathrm{e}^{(\nu/2)}(\vartheta-1)^2(2+\nu)-\frac{\nu\mathrm{e}^{(\nu/2)}\vartheta^3\Gamma(\vartheta)(4+\nu)\varsigma^{(2\vartheta-1)}}{4\Gamma(1+\vartheta)\Gamma(2\vartheta)}+\frac{\nu\mathrm{e}^{(\nu/2)}\vartheta^2(2+\nu)\,\varsigma^{2\vartheta}}{\Gamma(2\vartheta+1)}\\ &+\frac{\nu\mathrm{e}^{(\nu/2)}\vartheta^2(\vartheta-1)(4+\nu)\varsigma^{(\vartheta-1)}}{4\Gamma(1+\vartheta)}-\frac{2\nu\mathrm{e}^{(\nu/2)}\vartheta(\vartheta-1)(2+\nu)\varsigma^\vartheta}{\Gamma(\vartheta+1)}.\end{aligned} \tag{65}$$

Therefore, the approximate (LADM) series solution of Equation (50) is

$$\begin{aligned}\psi(\nu,\varsigma) &= \mathrm{e}^{(\nu/2)}(1+\nu^2\,(\vartheta-1)^2+2\,\nu\,\vartheta^2-3\,\nu\,\vartheta+\nu)-\frac{2\,\nu\,\mathrm{e}^{(\nu/2)}\,\vartheta\,(2\,\vartheta+\nu\,\vartheta-\nu-\frac{3}{2})\,\varsigma^\vartheta}{\Gamma(\vartheta+1)}\\ &-\frac{\nu\,\mathrm{e}^{(\nu/2)}\,\vartheta^3\,\Gamma(\vartheta)\,(4+\nu)\varsigma^{(2\vartheta-1)}}{4\Gamma(\vartheta+1)\,\Gamma(2\vartheta)}+\frac{\nu\,\mathrm{e}^{(\nu/2)}\,\vartheta^2\,(\vartheta-1)\,(4+\nu)\,\varsigma^{(\vartheta-1)}}{4\Gamma(\vartheta+1)}\\ &+\frac{\nu\,\mathrm{e}^{(\nu/2)}\,\vartheta^2\,(2+\nu)\,\varsigma^{2\vartheta}}{\Gamma(2\vartheta+1)}.\end{aligned} \tag{66}$$

Taking the Laplace transform of Equation (50),

$$\begin{aligned}&\frac{\mathfrak{y}^\vartheta}{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}\big[\mathrm{L}_\varsigma[\psi(\nu,\varsigma)](\mathfrak{y})-\frac{1}{\mathfrak{y}}\,\mathrm{e}^{(\nu/2)}\big]+\mathrm{L}_\varsigma\big[-\nu\,\frac{\partial^3\psi(\nu,\varsigma)}{\partial\nu^2\partial\varsigma}+2\,\nu\,\frac{\partial\psi(\nu,\varsigma)}{\partial\nu}\\ &+\nu\,\psi(\nu,\varsigma)\frac{\partial\psi(\nu,\varsigma)}{\partial\nu}-\nu\,\psi(\nu,\varsigma)\frac{\partial^3\psi(\nu,\varsigma)}{\partial\nu^3}-3\,\nu\,\frac{\partial\psi(\nu,\varsigma)}{\partial\nu}\frac{\partial^2\psi(\nu,\varsigma)}{\partial\nu^2}\big]=0.\end{aligned} \tag{67}$$

On simplifying Equation (67), we obtain

$$\begin{aligned}\mathrm{L}_\varsigma[\psi(\nu,\varsigma)](\mathfrak{y}) &= \frac{1}{\mathfrak{y}}\,\mathrm{e}^{(\nu/2)}-\frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot\mathrm{L}_\varsigma\big[-\nu\,\frac{\partial^3\psi(\nu,\varsigma)}{\partial\nu^2\partial\varsigma}+2\,\nu\,\frac{\partial\psi(\nu,\varsigma)}{\partial\nu}\\ &+\nu\,\psi(\nu,\varsigma)\frac{\partial\psi(\nu,\varsigma)}{\partial\nu}-\nu\,\psi(\nu,\varsigma)\frac{\partial^3\psi(\nu,\varsigma)}{\partial\nu^3}-3\,\nu\,\frac{\partial\psi(\nu,\varsigma)}{\partial\nu}\frac{\partial^2\psi(\nu,\varsigma)}{\partial\nu^2}\big],\end{aligned} \tag{68}$$

where the nonlinear operator can be written as

$$\begin{aligned}\mathcal{N}\big[\varphi(\nu,\varsigma;\varrho)\big] &= \mathrm{L}_\varsigma\big[\varphi(\nu,\varsigma;\varrho)\big]-\frac{1}{\mathfrak{y}}\,\mathrm{e}^{(\nu/2)}+\frac{\mathfrak{y}^\vartheta(1-\vartheta)+\vartheta}{\mathfrak{y}^\vartheta}\cdot\mathrm{L}_\varsigma\big[-\nu\,\frac{\partial^3\varphi(\nu,\varsigma;\varrho)}{\partial\nu^2\partial\varsigma}+2\,\nu\,\frac{\partial\varphi(\nu,\varsigma;\varrho)}{\partial\nu}\\ &+\nu\,\varphi(\nu,\varsigma;\varrho)\frac{\partial\varphi(\nu,\varsigma;\varrho)}{\partial\nu}-\nu\,\varphi(\nu,\varsigma;\varrho)\frac{\partial^3\varphi(\nu,\varsigma;\varrho)}{\partial\nu^3}-3\nu\,\frac{\partial\varphi(\nu,\varsigma;\varrho)}{\partial\nu}\frac{\partial^2\varphi(\nu,\varsigma;\varrho)}{\partial\nu^2}\big].\end{aligned} \tag{69}$$

According to Liao [14], the mth-order deformation equation is defined by

$$\mathrm{L}_{\varsigma}[\psi_m(\nu,\varsigma) - \mathcal{Q}_m\,\psi_{m-1}(\nu,\varsigma)] = \hbar\,\mathcal{H}(\nu,\varsigma)\,\mathcal{R}_m[\vec{\psi}_{m-1}(\nu,\varsigma)]. \tag{70}$$

Applying the inverse Laplace transform in Equation (70) with $\mathcal{H}(\nu,\varsigma) = 1$,

$$\psi_m(\nu,\varsigma) = \mathcal{Q}_m\,\psi_{m-1}(\nu,\varsigma) + \hbar\,\mathrm{L}_{\varsigma}^{-1}\big[\mathcal{R}_m[\vec{\psi}_{m-1}(\nu,\varsigma)]\big], \tag{71}$$

where

$$\begin{aligned}\mathcal{R}_m[\vec{\psi}_{m-1}(\nu,\varsigma)] &= \mathrm{L}_{\varsigma}[\psi_{m-1}(\nu,\varsigma)] - (1-\frac{\mathcal{Q}_m}{n})\frac{1}{\mathfrak{y}}\mathrm{e}^{(\nu/2)} + \frac{\mathfrak{y}^{\vartheta}(1-\vartheta)+\vartheta}{\mathfrak{y}^{\vartheta}}\cdot\mathrm{L}_{\varsigma}\Big[-\nu\frac{\partial^3\psi_{m-1}(\nu,\varsigma)}{\partial\nu^2\partial\varsigma} \\ &+ 2\nu\frac{\partial\psi_{m-1}(\nu,\varsigma)}{\partial\nu} + \sum_{k=0}^{m-1}\Big(\nu\psi_k(\nu,\varsigma)\frac{\partial\psi_{m-1-k}(\nu,\varsigma)}{\partial\nu} - \nu\psi_k(\nu,\varsigma)\frac{\partial^3\psi_{m-1-k}(\nu,\varsigma)}{\partial\nu^3} \\ &- 3\nu\frac{\partial\,\psi_k(\nu,\varsigma)}{\partial\nu}\frac{\partial^2\,\psi_{m-1-k}(\nu,\varsigma)}{\partial\nu^2}\Big)\Big].\end{aligned} \tag{72}$$

Therefore, Equation (71) can be expressed as

$$\begin{aligned}\psi_m(\nu,\varsigma) = \mathcal{Q}_m\psi_{m-1}(\nu,\varsigma) &+ \hbar\mathrm{L}_{\varsigma}^{-1}\Big\{\mathrm{L}_{\varsigma}[\psi_{m-1}(\nu,\varsigma)] - (1-\frac{\mathcal{Q}_m}{n})\frac{1}{\mathfrak{y}}\mathrm{e}^{(\nu/2)} \\ &+ \frac{\mathfrak{y}^{\vartheta}(1-\vartheta)+\vartheta}{\mathfrak{y}^{\vartheta}}\cdot\mathrm{L}_{\varsigma}\Big[-\nu\frac{\partial^3\psi_{m-1}(\nu,\varsigma)}{\partial\nu^2\partial\varsigma} + 2\nu\frac{\partial\psi_{m-1}(\nu,\varsigma)}{\partial\nu} \\ &+ \sum_{k=0}^{m-1}\Big(\nu\psi_k(\nu,\varsigma)\frac{\partial\psi_{m-1-k}(\nu,\varsigma)}{\partial\nu} - \nu\psi_k(\nu,\varsigma)\frac{\partial^3\psi_{m-1-k}(\nu,\varsigma)}{\partial\nu^3} \\ &- 3\nu\frac{\partial\psi_k(\nu,\varsigma)}{\partial\nu}\frac{\partial^2\psi_{m-1-k}(\nu,\varsigma)}{\partial\nu^2}\Big)\Big]\Big\}.\end{aligned} \tag{73}$$

By using Equation (73), we obtain the (LHAM) approximate series solution, where

$$\psi_0(\nu,\varsigma) = \mathrm{e}^{(\nu/2)}. \tag{74}$$

For $m = 1$, we obtain

$$\begin{aligned}\psi_1(\nu,\varsigma) &= \hbar\mathrm{L}_{\varsigma}^{-1}\Big[\frac{\mathfrak{y}^{\vartheta}(1-\vartheta)+\vartheta}{\mathfrak{y}^{\vartheta}}\cdot\mathrm{L}_{\varsigma}\big[0+\nu\mathrm{e}^{(\nu/2)} + \frac{\nu\mathrm{e}^{\nu}}{2} - \frac{\nu\mathrm{e}^{\nu}}{8} - \frac{3\nu\mathrm{e}^{\nu}}{8}\big]\Big] \\ &= \hbar\nu\mathrm{e}^{(\nu/2)}\Big[1-\vartheta+\frac{\vartheta\varsigma^{\vartheta}}{\Gamma(\vartheta+1)}\Big].\end{aligned} \tag{75}$$

For $m = 2$, we obtain

$$\begin{aligned}\psi_2(\nu,\varsigma) &= \mathrm{e}^{(\nu/2)}\hbar\big(-2\nu\mathrm{e}^{(\nu/2)}\hbar(\vartheta-1)^2 + \hbar\nu^2(\vartheta-1)^2 + \nu\hbar(2\vartheta^2-5\vartheta+3) - \nu\vartheta + \nu - 1 + n^{-1}\big) \\ &- \frac{2\nu\mathrm{e}^{(\nu/2)}\vartheta\hbar\big(-2\mathrm{e}^{(\nu/2)}\hbar(\vartheta-1) - \frac{1}{2} + ((2+\nu)\vartheta - \nu - \frac{5}{2})\hbar\big)\varsigma^{\vartheta}}{\Gamma(\vartheta+1)} \\ &+ \frac{\nu\mathrm{e}^{(\nu/2)}\vartheta^2\hbar^2(\vartheta-1)(4+\nu)\varsigma^{(\vartheta-1)}}{4\Gamma(\vartheta+1)} - \frac{\nu\mathrm{e}^{(\nu/2)}\vartheta^3\hbar^2\Gamma(\vartheta)(-4+\nu)\varsigma^{(2\vartheta-1)}}{4\Gamma(\vartheta+1)\Gamma(2\vartheta)} \\ &+ \frac{\nu\mathrm{e}^{(\nu/2)}\vartheta^2\hbar^2(-2\mathrm{e}^{(\nu/2)}+\nu+2)\varsigma^{2\vartheta}}{\Gamma(2\vartheta+1)}.\end{aligned} \tag{76}$$

Therefore, the approximate (LHAM) series solution of Equation (50) is

$$\psi(\nu,\varsigma)=e^{(\nu/2)}\Bigg[\bigg(-2\nu e^{(\nu/2)}\hbar^2(\vartheta-1)^2+\big(1+\nu\hbar^2(\nu\vartheta-\nu+2\vartheta-3)(\vartheta-1)+\hbar(2\nu-2\nu\vartheta-1)\big)+n^{-1}\hbar\bigg)+\frac{\nu\vartheta^2\hbar^2(\vartheta-1)(4+\nu)\varsigma^{(\vartheta-1)}}{4\Gamma(\vartheta+1)}-\frac{\nu\vartheta^3\hbar^2\Gamma(\vartheta)(-4+\nu)\varsigma^{(2\vartheta-1)}}{4\Gamma(\vartheta+1)\Gamma(2\vartheta)}+\frac{\nu\vartheta^2\hbar^2(-2e^{(\nu/2)}+\nu+2)\varsigma^{2\vartheta}}{\Gamma(2\vartheta+1)}+\frac{\nu\vartheta\big(4e^{(\nu/2)}\hbar^2(\vartheta-1)-(2\nu\vartheta+4\vartheta-2\nu-5)\hbar^2+2\hbar\big)\varsigma^{\vartheta}}{\Gamma(\vartheta+1)}\Bigg]. \quad (77)$$

The present research work aims to find analytical and numerical solutions for (FWEs) and implements efficient analytical techniques. In Tables 1–3, the $|Exact-\psi(\nu,\varsigma)|$ of the LVIM, LADM, and LHAM techniques at various fractional-order derivatives are shown. There are agreements between the numerical results with the consideration of the assumptions $n=1$ and $\hbar=-1$. In Figures 1–3, the 3D graphs for $|Exact-\psi(\nu,\varsigma)|$ and the 2D plots for $|Exact-\psi(\nu,\varsigma)|$ of LVIM, LADM and LHAM, respectively, show the behavior of the approximate solutions with respect to different fractional-order values ϑ and $\nu\in[-1,1]$. It is confirmed that the LVIM, LADM, and LHAM plots are in strong agreement with each other.

Table 1. Numerical values for $|Exact-\psi(\nu,\varsigma)|$ for various ϑ and ν with LVIM at $\varsigma=1$ for Problem 1.

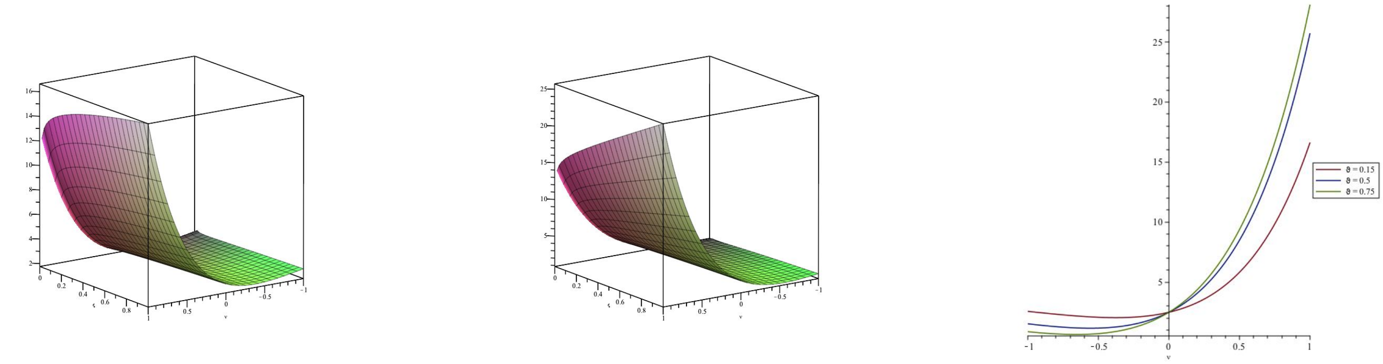
ν	$\vartheta=0.15$	$\vartheta=0.5$	$\vartheta=0.75$	$\vartheta=0.9$	$\vartheta=1$
−1	2.5608997	1.5243783	0.86575929	0.40880488	0.04587202
−0.9	2.4391575	1.3960871	0.76228210	0.33540222	0.00334838
−0.6	2.1324828	1.1627305	0.64564224	0.33060677	0.08109736
−0.3	2.0521899	1.3781686	1.0617518	0.89122639	0.75621919
0	2.4865829	2.4865829	2.4865829	2.4865829	2.4865829
0.3	3.9237917	5.2158466	5.6856178	5.8602898	6.0000555
0.6	7.1665349	10.732416	11.866131	12.169850	12.418299
0.9	13.506698	20.874006	22.903632	23.192297	23.447494
1	16.633409	25.737892	28.122363	28.346017	28.560704

Table 2. Numerical values for $|Exact-\psi(\nu,\varsigma)|$ for various ϑ and ν with LADM at $\varsigma=1$ for Problem 1.

ν	$\vartheta=0.15$	$\vartheta=0.5$	$\vartheta=0.75$	$\vartheta=0.9$	$\vartheta=1$
−1	0.30174365	0.47301413	0.75801599	0.94465579	1.0532908
−0.9	0.25833658	0.42310542	0.70823490	0.89945100	1.0132440
−0.6	0.18505643	0.31900487	0.57581633	0.75791444	0.87163404
−0.3	0.23836754	0.31985520	0.48993265	0.61546219	0.69638419
0	0.48658288	0.48658288	0.48658288	0.48658288	0.48658288
0.3	1.0209246	0.90100711	0.61502581	0.39227107	0.24291962
0.6	1.9618160	1.6716411	0.94158718	0.36204438	0.03160977
0.9	3.4667163	2.9409261	1.5544244	0.43648878	0.33078392
1	4.1274189	3.5054335	1.8414253	0.49356178	0.43430132

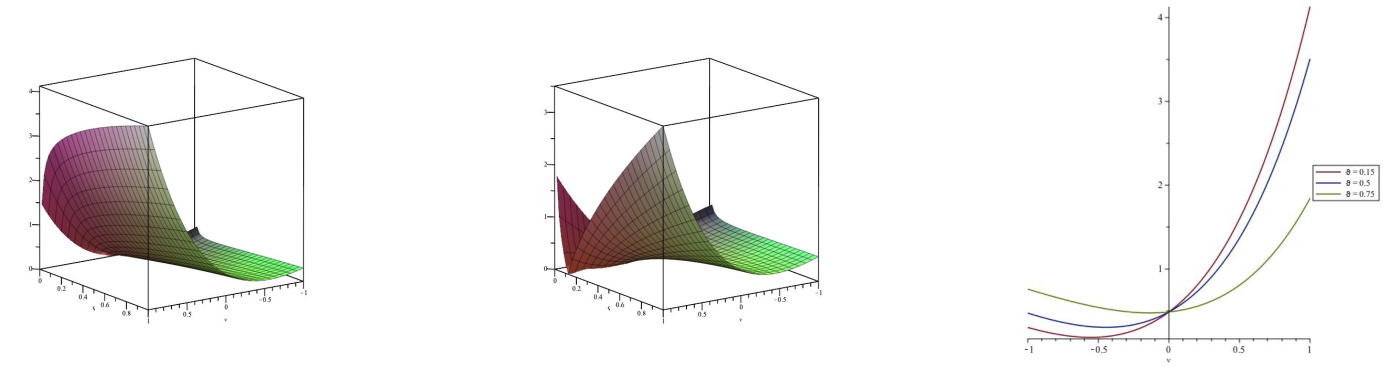
Table 3. Numerical values for $|Exact-\psi(\nu,\varsigma)|$ for various ϑ and ν with LHAM at $\varsigma=1$ for Problem 1.

ν	$\vartheta=0.15$	$\vartheta=0.5$	$\vartheta=0.75$	$\vartheta=0.9$	$\vartheta=1$
−1	0.29525006	0.29525654	0.29525677	0.29525308	0.29524874
−0.9	0.31037496	0.31038109	0.31038130	0.31037782	0.31037371
−0.6	0.36055931	0.36056406	0.36056422	0.36056153	0.36055834
−0.3	0.41885797	0.41886072	0.41886082	0.41885925	0.41885741
0	0.48658288	0.48658288	0.48658288	0.48658288	0.48658288
0.3	0.56525811	0.56525441	0.56525431	0.56525641	0.56525891
0.6	0.65665448	0.65664578	0.65664548	0.65665038	0.65665618
0.9	0.76282858	0.76281348	0.76281288	0.76282148	0.76283158
1	0.80190628	0.80188868	0.80188808	0.80189798	0.80190988



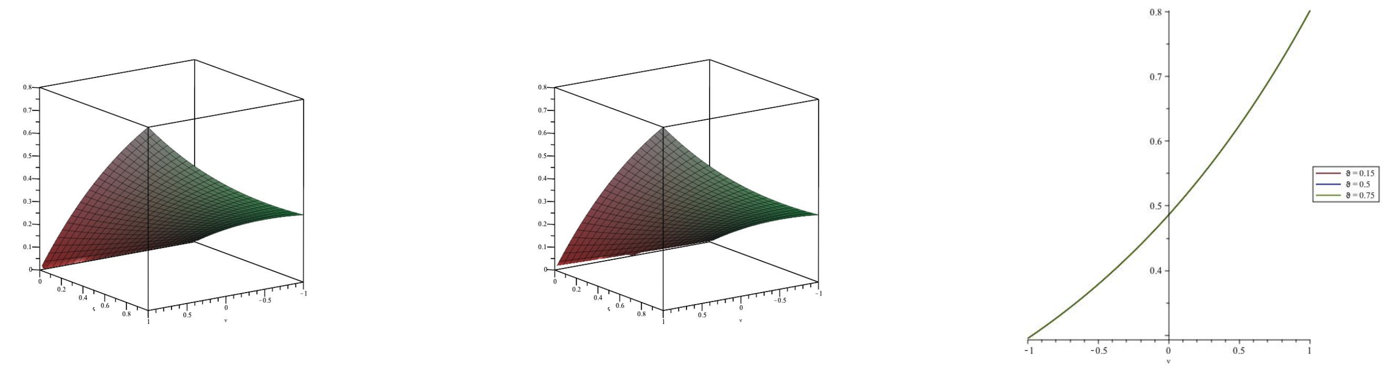
(**a**) (**b**) (**c**)

Figure 1. (**a**) Three-dimensional surface for $|Exact - \psi(\nu, \varsigma)|$ of LVIM Equation (58) at $\vartheta = 0.15$; (**b**) surface for $|Exact - \psi(\nu, \varsigma)|$ of LVIM Equation (58) at $\vartheta = 0.5$; (**c**) two-dimensional plots for $|Exact - \psi(\nu, \varsigma)|$ of LVIM Equation (58) with respect to $\varsigma = 1$ at different values of ϑ in Example 1.



(**a**) (**b**) (**c**)

Figure 2. (**a**) Three-dimensional surface for $|Exact - \psi(\nu, \varsigma)|$ of LADM Equation (66) at $\vartheta = 0.15$; (**b**) surface for $|Exact - \psi(\nu, \varsigma)|$ of LADM Equation (66) at $\vartheta = 0.5$; (**c**) two-dimensional plots for $|Exact - \psi(\nu, \varsigma)|$ of LADM Equation (66) with respect to $\varsigma = 1$ at different values of ϑ in Example 1.



(**a**) (**b**) (**c**)

Figure 3. (**a**) Three-dimensional surface for $|Exact - \psi(\nu, \varsigma)|$ of LHAM Equation (77) at $\vartheta = 0.15$, $\hbar = -0.0001$, and $n = 1$; (**b**) surface for $|Exact - \psi(\nu, \varsigma)|$ of LHAM Equation (77) at $\vartheta = 0.5$, $\hbar = -0.0001$, and $n = 1$; (**c**) two-dimensional plots for $|Exact - \psi(\nu, \varsigma)|$ of LHAM Equation (77) with respect to $\varsigma = 1$, $\hbar = -0.0001$, and $n = 1$ at different values of ϑ in Example 1.

8. Conclusions

In this paper, we have presented the numerical results for a fractional-order nonlinear partial differential equations with variable coefficients. To achieve an accurate approximate solution for the fractional-order nonlinear (FWE), three modified techniques, including the LVIM technique, the LADM technique, and the LHAM technique, were implemented successfully. The fractional derivatives were considered in the form of (AB) derivatives of order $\vartheta = (0, 1]$. As ϑ moves from 0 to 1, the fractional order derivative has a significant impact on the approximate solutions, which has been demonstrated in Tables 1–3. These modified methods are powerful tools for solving complex nonlinear partial differential equations with fractional orders.

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