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A New Extension of Optimal Auxiliary Function Method to Fractional Non-Linear Coupled ITO System and Time Fractional Non-Linear KDV System

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Abstract: In this article, we investigate the utilization of Riemann–Liouville's fractional integral and the Caputo derivative in the application of the Optimal Auxiliary Function Method (OAFM). The extended OAFM is employed to analyze fractional non-linear coupled ITO systems and non-linear KDV systems, which feature equations of a fractional order in time. We compare the results obtained for the ITO system with those derived from the Homotopy Perturbation Method (HPM) and the New Iterative Method (NIM), and for the KDV system with the Laplace Adomian Decomposition Method (LADM). OAFM demonstrates remarkable convergence with a single iteration, rendering it highly effective. In contrast to other existing analytical approaches, OAFM emerges as a dependable and efficient methodology, delivering high-precision solutions for intricate problems while saving both computational resources and time. Our results indicate superior accuracy with OAFM in comparison to HPM, NIM, and LADM. Additionally, we enhance the accuracy of OAFM through the introduction of supplementary auxiliary functions.

Keywords: optimal auxiliary function method (OAFM); riemann–liouville integral; modified riemann–liouville derivatives; time fractional coupled ITO system; non-linear KDV system of time fractional order; Caputo fractional derivative

MSC: 35R11; 35G50

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1. Introduction

Fractional calculus has applications in several areas, including dynamical system manipulation, electrical community optics and signal processing, which may be successfully described using linear or non-linear fractional differential equations (FDEs). The concept of fractional derivatives and integrals was first introduced by Riemann and Liouville [1]. In the past 200 years, it has gained significant scientific attention globally in theory and applications with research on integrals and derivatives of fractional orders. In particular, some of the areas include remarkable applications in a variety of technical and medicinal fields.

The discipline of fractional calculus has seen a large number of innovative and intriguing models develop throughout time [2–4]. In order to demonstrate various scientific phenomena in the fields of fluid mechanics, plasma physics, solid-state physics, population dynamics, chemical kinetics, nonlinear optics, soliton theory, protein chemistry, etc, many of the above-mentioned applied science fields [5–11] involve a variety of processes for which nonlinear models are essential. The most elected type of equations which have been used are partial differential equations (PDEs), however, it makes practical sense to model a variety of complicated processes in applied sciences by using fractional partial differential equations (FPDEs) due to the fractional nature and allowable generalisation. The El Nino-Southern oscillation mannequin and groundwater oat are two significant models that are represented by FPDEs. These extended FPDE mathematical models underpin why it is crucial to use them for understanding natural processes. Researchers have attempted to numerically and analytically solve these models in order to analyse the precise dynamics of the stated processes [12,13]. There are a few numerical and analytical techniques which have been recommended for such problems, e.g., the control volume scheme, Laplace transformation method, finite element method (FVM), a domain decomposition method (ADM) [5], variation iteration method (VIM), homotopy analysis method (HAM) [14], and many more. These methods have many advantages but are still not applicable for this class of problem. The suggested method presented by Vasile Marinca is a powerful technique called the optimal auxilury function method (OAFM) [6]. This approach was developed to use minimal processing and obtain an accurate result after just one repetition. Fractional order issues are also solved using numerical techniques. S. Momani et al. compared linear and differential fractional order issues numerically. The numerical techniques for resolving partial differential equations of a fractional order were expanded by Obedit et al. The Korteweg-de Vries (KDV) equation has been used in a wide range of physical processes as a model for the development and interaction of nonlinear waves. It was initially developed as an evolution equation for a shallow water channel where it is one-dimensional, with a small-amplitude and long surface gravity waves [15,16]. More recently, the KDV equation may be found in a wide range of additional physical contexts, such as collision-less hydro-magnetic waves, stratified internal waves, ion-acoustic waves, plasma physics, lattice dynamics, and more. Some quantum mechanical theoretical physics events can be explained using the KDV model. It serves as a model for mass transport in fluid dynamics, aerodynamics, and continuum mechanics as well as the formation of shock waves, solitons, turbulence, boundary layer behaviour, and many more phenomena. Since all of the physical events are non-conservative, fractional differential equations may be used to explain them. However, the formulation of fractional differential equations (FDEs) as well as their solutions have grown to be important mathematical-physical issues [17,18]. The FDEs are the fractional generalisation of ordinary and partial differential equations. It is a very efficient and effective tool, which is not affected by any small or large parameter changes as with the perturbation method; it also controls error management. Furthermore, it deals with linear and non-linear problems without any limitation and loss of generality. This method was applied by many researchers for solutions of different types of differential equations [19–22].

In the recent past, the modified form of an optimal auxiliary function to the PDEs has been utilised for the solution of Lax's seventh order KDV and the Sawada-Kotera equation which was introduced by Laiq Zada et al. [23]. The same problem has been solved by Rashid Nawaz by converting it to a fractional order and it was utilised for higher dimensions [24]. Several recent methods of fractional differential equations and their applications were studied and investigated by many researchers, see [25–29].

The objective of this article is to extend OAFM to the solution of fractional non-linear coupled ITO system and time fractional non-linear KDV system. The ITO system is quite sophisticated and was introduced by Masaaki Ito. In the present work, OAFM has been extended to the solution of fractional non-linear coupled ITO systems and non-linear KDV systems of time fractional order equations. The following suggested ITO system will be very helpful to understand the continuous quantum measurement and estimation. Consider the following FPDE system,

$$\begin{aligned}
&\frac{\partial^{\gamma}\sigma(\varpi,\psi)}{\partial\psi^{\gamma}}-\frac{\partial\nu(\varpi,\psi)}{\partial\varpi}=0,\ 0<\gamma\leq 1\,,\\
&\frac{\partial^{\beta}\nu(\varpi,\psi)}{\partial\psi^{\beta}}+2\bigg(\frac{\partial^{3}\nu(\varpi,\psi)}{\partial\varpi^{3}}+3\sigma(\varpi,\psi)\frac{\partial\nu(\varpi,\psi)}{\partial\varpi}+3\ (\varpi,\psi)\frac{\partial\sigma(\varpi,\psi)}{\partial\varpi}\bigg)+12\vartheta(\varpi,\psi)\frac{\partial\vartheta(\varpi,\psi)}{\partial\varpi}=0,\\
&\frac{\partial^{\eta}\vartheta(\varpi,\psi)}{\partial\psi^{\eta}}-\frac{\partial^{3}\vartheta(\varpi,\psi)}{\partial\varpi^{3}}-3\sigma(\varpi,\psi)\frac{\partial\vartheta(\varpi,\psi)}{\partial\varpi}=0.
\end{aligned}\tag{1}$$

Coupled with the initial conditions

$$\sigma_0(\varpi,0)=\frac{d\varpi}{3z},\ \ \nu_0(\varpi,0)=-\frac{d^2\varpi^2}{2z^2},\ \ \vartheta_0(\varpi,0)=0.\tag{2}$$

The non-linear KDV system of time fractional order equation is expressed as

$$\begin{aligned}
&\frac{\partial^{\gamma}\sigma(\varpi,\psi)}{\partial\psi^{\gamma}}+d\frac{\partial^{3}\sigma(\varpi,\psi)}{\partial\varpi^{3}}+6d\sigma(\varpi,\psi)\frac{\partial\sigma(\varpi,\psi)}{\partial\varpi}-6\nu(\varpi,\psi)\frac{\partial\nu(\varpi,\psi)}{\partial\varpi}=0,\\
&\frac{\partial^{\beta}\nu(\varpi,\psi)}{\partial\psi^{\beta}}+d\frac{\partial^{3}\nu(\varpi,\psi)}{\partial\varpi^{3}}+3d\sigma(\varpi,\psi)\frac{\partial\nu(\varpi,\psi)}{\partial\varpi}=0.
\end{aligned}\tag{3}$$

Additionally, coupled with the following initial conditions

$$\begin{aligned}
\sigma(\varpi,0)&=\chi^2 sech^2\Big(\frac{z}{2}+\frac{\chi\varpi}{2}\Big)\\
\nu(\varpi,0)&=\sqrt{\frac{d}{2}}\chi^2 sech^2\Big(\frac{z}{2}+\frac{\chi\varpi}{2}\Big)
\end{aligned}\tag{4}$$

The contents of this article are organised as follows. In Section 2, the basic terminologies are discussed. The section thereafter, Section 3, investigates the suggested scheme for obtaining the solutions of the current model. In Section 4, different problems are tested while the results are discussed and the conclusions are given in Section 5.

2. Preliminaries

To understand the concept of OAFM, the following definitions are the basic terminologies required [7,24,30].

Definition 1. *Let $f(\varpi)$ be a piece-wise continous $(0,\infty)$ and integrable on any subinterval, then the Riemann–Liouville (R-L) fractional integral of order γ is defined as*

$$\mathrm{I}^{\gamma}_{\varpi}f(\varpi)=\begin{cases}\frac{1}{\Gamma(\gamma)}\int\limits_0^{\varpi}(\varpi-r)^{\gamma-1}f(r)dr & if\ \gamma>0,\ \varpi>0,\\ f(r) & if\ \gamma=0,\end{cases}\tag{5}$$

where gamma function is the special function denoted by Γ.

Definition 2. *The fractional derivative of a function f of order γ in the Riemann–Liouville sense is defined as*

$$D^{\gamma}_{\varpi}f(\varpi)=\frac{1}{\Gamma(q-\gamma)}\frac{d^q}{d\varpi^q}\int\limits_0^{\varpi}(\varpi-r)^{q-\gamma-1}\ f(r)\,dr\ \ if\ \gamma>0,\ \varpi>0.\tag{6}$$

Here, q *is a non negative integer which satisfies*

$$q - 1 < \gamma \leq q.$$

Definition 3. *The fractional derivative of order* γ *in the Caputo sense, is defined as*

$$D_r^{\gamma} f(\varpi) = \begin{cases} I^{q-\gamma}\left[\frac{\partial^q}{\partial r^q} f(r)\right] & if\; q-1 < \gamma \leq q,\;\; q \in \mathbb{N} \\ \frac{d^{\gamma}}{dr^{\gamma}} f(r) & if\;\; \gamma \in \mathbb{N}. \end{cases} \tag{7}$$

where

$$q \in \mathbb{N}, \varpi > 0, \quad r \geq -1 \;\; and \;\; \varphi \in T_r.$$

3. Mathematical Model Formulation

The general form of the fractional order partial differential equation is given as

$$\frac{\partial^{\gamma}\sigma(\varpi,\psi)}{\partial\psi^{\gamma}} = \wp(\varpi,\psi) + M(\sigma(\varpi,\psi)) \tag{8}$$

coupled with the boundary conditions

$$\begin{array}{ll} D_0^{\gamma-k}\sigma(\varpi,0) = h_k(\varpi). \;\; (k = 0,1,\ldots\ldots\ldots, j-1), & D_0^{\gamma-n}\sigma(\varpi,0) = 0, \;\; j = [\gamma]. \\ D_0^{k}\sigma(\varpi,0) = g_k(\varpi). \;\;\; (k = 0,1,\ldots\ldots\ldots, j-1) & D_0^{n}\sigma(\varpi,0) = 0, \;\; j = [\gamma]. \end{array} \tag{9}$$

The partial derivative $\frac{\partial^{\gamma}}{\partial\psi^{\gamma}}$ indicates the Caputo or R-L operator, where $\sigma(\varpi,\psi)$ stands for unknown function and $\wp(\varpi,\psi)$ stands for known analytic function. The following comprise the construction of the proposed method listed in seven steps:

Step 1: To find the estimated result of Equation (8), two component forms will be taken under consideration, which can be given as

$$\tilde{\sigma}(\varpi,\psi) = \sigma_0(\varpi,\psi) + \sigma_1(\varpi,\psi,T_i), \quad i = 1,2,3,\ldots,\ell. \tag{10}$$

Step 2: By putting Equation (10) in Equation (8), we obtain the solution of the zero- and first-order, which is given as

$$\frac{\partial^{\gamma}\sigma_0(\varpi,\psi)}{\partial\psi^{\gamma}} + \frac{\partial^{\gamma}\sigma_1(\varpi,\psi)}{\partial\psi^{\gamma}} + \wp(\varpi,\psi) + M\left[\frac{\partial^{\gamma}\sigma_0(\varpi,\psi)}{\partial\psi^{\gamma}} + \frac{\partial^{\gamma}\sigma_1(\varpi,\psi),T_i}{\partial\psi^{\gamma}}\right] = 0. \tag{11}$$

Step 3: With the help of linear equations, we obtain an initial approximation of the form given below because non-linear equations are complicated and obtaining their solution is not possible, so we take the linear part and find its solution, then use that solution as our initial estimation.

$$\frac{\partial^{\gamma}\sigma_0(\varpi,\psi)}{\partial\psi^{\gamma}} + \wp(\varpi,\psi) = 0. \tag{12}$$

Using the inverse operator, we obtain $\sigma_0(\varpi,\psi)$ as the following,

$$\sigma_0(\varpi,\psi) = \wp(\varpi,\psi). \tag{13}$$

Step 4: The expanded form of the non-linear term presented in Equation (11) is

$$M\left[\frac{\partial^{\gamma}\sigma_0(\varpi,\psi)}{\partial\psi^{\gamma}} + \frac{\partial^{\gamma}\sigma_1(\varpi,\psi,T_i)}{\partial\psi^{\gamma}}\right] = M[\sigma_0(\varpi,\psi)] + \sum_{k=1}^{\infty}\frac{\sigma_1^k}{k!}\mathrm{N}^{(k)}[\sigma_0(\varpi,\psi)]. \tag{14}$$

Step 5: To interpret Equation (14) smoothly and increase the convergence of the first-order approximation, we introduce an expression which is given below

$$\frac{\partial^{\gamma}\sigma_1(\varpi,\psi,T_i)}{\partial\psi^{\gamma}} = -B_1[\sigma_0(\varpi,\psi)]M[\sigma_0(\varpi,\psi)] - B_2\big[\sigma_0(\varpi,\psi),T_j\big], \tag{15}$$

where B_1 and B_2 are the auxiliary functions depending on $\sigma_0(\varpi,\psi)$ and the convergence control parameter T_i and T_j, where $i = 1,2,3,4,\ldots,\ell$ and $j = \ell+1,\ell+2,\ldots,k$.

Remark 1. *B_1 and B_2 are of the form $\sigma_0(\varpi,\psi)$ or $M[\sigma_0(\varpi,\psi)]$ or the combination of both $\sigma_0(\varpi,\psi)$ and $M[\sigma_0(\varpi,\psi)]$ but they are not unique.*

For example, if $\sigma_0(\varpi,\psi)$ or $M[\sigma_0(\varpi,\psi)]$ is a polynomial function, then B_1 and B_2 are sums of polynomial functions; if $\sigma_0(\varpi,\psi)$ or $M[\sigma_0(\varpi,\psi)]$ is an exponential function, then B_1 and B_2 are sums of exponential functions; if $\sigma_0(\varpi,\psi)$ or $M[\sigma_0(\varpi,\psi)]$ is a trigonometric function, then B_1 and B_2 are sums of trigonometric function and so on. In a special case, $M[\sigma_0(\varpi,\psi)=0]$, then it is clear that $\sigma_0(\varpi,\psi)$ is an exact solution.

Step 6: Using the inverse operator (Definition 1), after the substitution of the auxiliary function to Equation (15), we achieve the first-order solution $\sigma_1(\varpi,\psi)$ by OAFM.

Step 7: For finding the square of the residual error to obtain the values of T_i and T_j, we use either the Collocation method, Galerkin's method, Ritz method, or the Least Square method.

$$B\big(T_i,T_j\big) = \int_0^{\psi}\int_{\Omega} R^2\big(\varpi,\psi;T_i,T_j\big)d\varpi d\psi. \tag{16}$$

Here, R is the residual and it is defined as follows:

$$\begin{aligned} R\big(\varpi,\psi,T_i,T_j\big) &= \frac{\partial^{\gamma}\widetilde{\sigma}(\varpi,\psi),T_i,T_j}{\partial\psi} + \wp(\varpi,\psi) + M\big[\sigma\big(\varpi,\psi,T_i,T_j\big)\big] \\ & \quad i = 1,2,3,\ldots,\ell, \quad j = \ell+1,\ell+2,\ell+3,\ldots,k. \end{aligned} \tag{17}$$

For the convergence control parameter, the following system will occur

$$\frac{\partial J}{\partial T_1} = \frac{\partial J}{\partial T_2} = \frac{\partial J}{\partial T_3} = \frac{\partial J}{\partial T_i} = 0,\ i = 1,2,\ldots \tag{18}$$

4. Applications

The motivation of this part is to take some examples to show the effectiveness and accuracy of the suggested method in the previous section.

Problem 1:

Take the time-fractional ITO system [31] for $0<\gamma\le 1$, $0<\beta\le 1$ and $0<\eta\le 1$ to be

$$\begin{aligned} &\tfrac{\partial^{\gamma}\sigma(\varpi,\psi)}{\partial\psi^{\gamma}} - \tfrac{\partial\nu(\varpi,\psi)}{\partial\varpi} = 0, \\ &\tfrac{\partial^{\beta}\nu(\varpi,\psi)}{\partial\psi^{\beta}} + 2\Big(\tfrac{\partial^{3}\nu(\varpi,\psi)}{\partial\varpi^{3}} + 3\sigma(\varpi,\psi)\tfrac{\partial\nu(\varpi,\psi)}{\partial\varpi} + 3\nu(\varpi,\psi)\tfrac{\partial\sigma(\varpi,\psi)}{\partial\varpi}\Big) + 12\vartheta(\varpi,\psi)\tfrac{\partial\vartheta(\varpi,\psi)}{\partial\varpi} = 0, \\ &\tfrac{\partial^{\eta}\vartheta(\varpi,\psi)}{\partial\psi^{\eta}} - \tfrac{\partial^{3}\vartheta(\varpi,\psi)}{\partial\varpi^{3}} - 3\sigma(\varpi,\psi)\tfrac{\partial\vartheta(\varpi,\psi)}{\partial\varpi} = 0. \end{aligned} \tag{19}$$

coupled with the subsidiary boundary conditions

$$\sigma_0(\varpi, 0) = \frac{d\varpi}{3z}, \ \nu_0(\varpi, 0) = -\frac{d^2\varpi^2}{2z^2}, \ \vartheta_0(\varpi, 0) = 0. \tag{20}$$

An accurate solution of Equation (19) when $\gamma = \beta = \eta = 1$ is

$$\sigma(\varpi, \psi) = \frac{d\varpi}{3(3b\psi + z)}, \ \nu(\varpi, \psi) = \frac{d^2\varpi^2}{2(3b\psi + z)^2}, \ \vartheta(\varpi, \psi) = 0. \tag{21}$$

In Equation (19), we consider linear and non-linear terms as

$$\begin{cases} L(\sigma(\varpi, \psi)) = \frac{\partial^\gamma \sigma(\varpi,\psi)}{\partial \psi^\gamma} \\ M(\sigma(\varpi, \psi)) = -\frac{\partial \nu(\varpi,\psi)}{\partial \varpi} \end{cases}$$
$$\begin{cases} L(\nu(\varpi, \psi)) = \frac{\partial^\beta \nu(\varpi,\psi)}{\partial \psi^\beta} \\ M(\nu(\varpi, \psi)) = 2\Big(\frac{\partial^3 \nu(\varpi,\psi)}{\partial \varpi^3} + 3\sigma(\varpi, \psi)\frac{\partial \nu(\varpi,\psi)}{\partial \varpi} + 3\nu(\varpi, \psi)\frac{\partial \sigma(\varpi,\psi)}{\partial \varpi}\Big) + \\ 12\vartheta(\varpi, \psi)\frac{\partial \vartheta(\varpi,\psi)}{\partial \varpi}, \\ L(\vartheta(\varpi, \psi)) = \frac{\partial^\eta \sigma((\varpi,\psi)}{\partial \psi^\eta}, \\ M(\vartheta(\varpi, \psi)) = -\frac{\partial^3 \vartheta(\varpi,\psi)}{\partial \varpi^3} - 3\sigma(\varpi, \psi)\frac{\partial \vartheta(\varpi,\psi)}{\partial \varpi}. \end{cases} \tag{22}$$

As stated by the OAFM, the initial condition will help to obtain the zero-order problem. The zero-order system can be stated as follows

$$\begin{aligned} \frac{\partial^\gamma \sigma(\varpi,\psi)}{\partial \psi^\gamma} &= 0, \quad \sigma_0(\varpi, 0) = \frac{d\varpi}{3(3b\psi+z)}; \\ \frac{\partial^\beta \nu(\varpi,\psi)}{\partial \psi^\beta} &= 0, \quad \nu_0(\varpi, 0) = \frac{d^2\varpi^2}{2(3b\psi+z)^2}, \\ \frac{\partial^\eta \vartheta(\varpi,\psi)}{\partial \psi^\eta} &= 0, \ \vartheta_0(\varpi, 0) = 0. \end{aligned} \tag{23}$$

By using Definition 1, we obtain the zero-order solution for the system (23) as,

$$\sigma_0(\varpi, \psi) = \frac{d\varpi}{3(3b\psi + z)}. \ \nu_0(\varpi, \psi) = \frac{d^2\varpi^2}{2(3b\psi + z)^2}. \ \vartheta_0(\varpi, \psi) = 0. \tag{24}$$

Then, by using Equation (24) and substituting it into Equation (22), the non-linear terms become

$$\begin{aligned} M(\sigma_0(\varpi, \psi)) &= -\frac{\partial v_0(\varpi,\psi)}{\partial \varpi}, \\ M(\nu_0(\varpi, \psi)) &= 2\Big(\frac{\partial^3 \nu_0(\varpi,\psi)}{\partial \varpi^3} + 3\sigma_0(\varpi, \psi)\frac{\partial \nu_0(\varpi,\psi)}{\partial \varpi} + 3\nu_0(\varpi, \psi)\frac{\partial \sigma_0(\varpi,\psi)}{\partial \varpi}\Big) + 12\vartheta(\varpi, \psi)\frac{\partial \vartheta_0(\varpi,\psi)}{\partial \varpi}, \\ M(\vartheta_0(\varpi, \psi)) &= -\frac{\partial^3 \vartheta_0(\varpi,\psi)}{\partial \varpi^3} - 3\sigma_0(\varpi, \psi)\frac{\partial \vartheta_0(\varpi,\psi)}{\partial \varpi}. \end{aligned} \tag{25}$$

Applying OAFM, the first order approximation can be obtained as

$$\begin{aligned} \frac{\partial^\gamma \sigma_1(\varpi,\psi)}{\partial \vartheta^\gamma} &= -B_1[\sigma_0(\varpi, \psi), T_\iota]M[\sigma_0(\varpi, \psi)] - B_2\big[\sigma_0(\varpi, \psi), T_j\big]. \\ \frac{\partial^\beta \nu_1(\varpi,\psi)}{\partial \vartheta^\beta} &= -B_3[\nu_0(\varpi, \psi), T_\iota]M[\nu_0(\varpi, \psi)] - B_4\big[\nu_0(\varpi, \psi), T_j\big]. \\ \frac{\partial^\eta \vartheta_1(\varpi,\psi)}{\partial \vartheta^\eta} &= -B_5[\vartheta_0(\varpi, \psi), T_\iota]M[\vartheta_0(\varpi, \psi)] - B_6\big[\vartheta_0(\varpi, \psi), T_j\big]. \end{aligned} \tag{26}$$

Thus, we obtain B_1, B_2, B_3, B_4, B_5, B_6, using the initial approximation,

$$\begin{aligned}
B_1 &= -\left(T_1\left(1-\frac{3b\psi^1}{z}+\frac{9\left(d^2\right)\psi^2}{z^2}-\frac{27\left(d^3\right)\psi^3}{z^3}+\frac{81\left(d^4\right)\psi^4}{z^4}-\frac{243\left(d^5\right)\psi^5}{z^5}\right)\right),\\
B_2 &= 0,\\
B_3 &= -\left(T_2\left(1-\frac{9d\psi}{2z}+\frac{18d^2\psi^2}{z^2}-\frac{135d^3\psi^3}{2z^3}+\frac{243d^4\psi^4}{z^4}-\frac{1701d^5\psi^5}{2z^5}\right)\right),\\
B_4 &= 0,\\
B_5 &= -T_3,\\
B_6 &= 0.
\end{aligned} \tag{27}$$

From here, if we use Equations (25) and (27) and then substituting them in Equation (26), we obtain the first approximation as

$$\begin{aligned}
\sigma_1(\varpi,\psi) &= \frac{b^2T_1\psi^\gamma(-z+3d\psi)\left(z^2-3dz\psi+9d^2\psi^2\right)\left(z^2+3bz\psi+9d^2\psi^2\right)\varpi}{z^7\Gamma(1+\gamma)}.\\
\nu_1(\varpi,\psi) &= \frac{3bdT_3\psi^\gamma\left(-2z^5+9dz^4\psi-36d^2z^3\psi^2+135d^3z^2\psi^3-486b^4dz\psi^4+1701d^5\psi^5\right)\varpi^2}{2z^8\Gamma(1+\gamma)}.\\
\vartheta_1(\varpi,\psi) &= 0.
\end{aligned} \tag{28}$$

By adding Equations (27) and (28), we obtain the first-order approximate solution as

$$\begin{aligned}
\widetilde{\sigma}(\varpi,\psi) &= \sigma_0(\varpi,\psi)+\sigma_1(\varpi,\psi,T_1).\\
\widetilde{\nu}(\varpi,\psi) &= \nu_0(\varpi,\psi)+\nu_1(\varpi,\psi,T_2).\\
\widetilde{\vartheta}(\varpi,\psi) &= \vartheta_0(\varpi,\psi)+\vartheta_1(\varpi,\psi,T_3).
\end{aligned} \tag{29}$$

$$\begin{aligned}
\widetilde{\sigma}(\varpi,\psi) &= \frac{b\varpi}{3(3b\psi+z)}+\frac{d^2T_1\psi^\gamma(-z+3d\psi)\left(z^2-3dz\psi+9b^2\psi^2\right)\left(z^2+3dz\psi+9bd\psi^2\right)\varpi}{z^7\Gamma(1+\gamma)}.\\
\widetilde{\nu}(\varpi,\psi) &= \frac{d^2\varpi^2}{2(3b\psi+z)^2}+\frac{3d^3T_2{}^\gamma\left(-2z^5+9dz^4\psi-36d^2z^3\psi^2+135d^3z^2\psi^3-486d^4z\psi^4+1701d^5\psi^5\right)\varpi^2}{2z^8\Gamma(1+\gamma)}.\\
\widetilde{\vartheta}(\varpi,\psi) &= 0.
\end{aligned} \tag{30}$$

The results of Problem 1 are presented in Tables 1–3, and visualized in Figures 1–8.

Table 1. Convergence control parameters for different values of γ for Problem 1.

	$\gamma = 1$	$\gamma = 0.75$	$\gamma = 0.5$
T_1	1.00000000000000005	1.1106291281240243	1.57032523295656778
T_2	−0.999999993169958	−1.0987316755026958	−1.5570896342028568

Table 2. Comparison of absolute error obtained by the OAFM solution with the HPM and NIM solution $\sigma(\varpi,\psi)$ for Problem 1, when $\gamma = \beta = 1$ and z = 9.

(ϖ,ψ)	OAFM Solution at $\gamma = 0.5$	OAFM Solution $\gamma = 0.75$	$\sigma(\varpi,\psi)$ $\gamma = 1$	Exact Solution	Absolute Error HPM [31]	Absolute Error NIM [31]	Absolute Error OAFM
(0.1,0.1)	0.0030652	0.00344689	0.0035843	0.0035842	4.42497×10^{-9}	2.90082×10^{-9}	1.63888×10^{-13}
(0.3,0.1)	0.0091677	0.01034448	0.0107526	0.0107529	1.32731×10^{-8}	8.70278×10^{-9}	4.91643×10^{-13}
(0.5,0.1)	0.0151789	0.01723468	0.0179217	0.017927	2.21288×10^{-8}	1.45031×10^{-8}	8.19489×10^{-13}
(0.1,0.2)	0.0027889	0.00328556	0.0034724	0.0034725	6.85856×10^{-8}	4.42077×10^{-8}	2.03231×10^{-11}
(0.3,0.2)	0.0083496	0.00985699	0.0104162	0.0104164	2.05787×10^{-7}	1.32697×10^{-7}	6.09690×10^{-11}
(0.5,0.2)	0.0139345	0.01642709	0.0173665	0.0173617	3.42976×10^{-7}	2.21056×10^{-7}	1.01633×10^{-11}
(0.1,0.3)	0.0026764	0.00315397	0.003356	0.0033679	3.3677×10^{-7}	2.13289×10^{-7}	3.36787×10^{-10}
(0.3,0.3)	0.0078438	0.00946156	0.0101037	0.0101012	1.0121×10^{-6}	6.39743×10^{-7}	1.01109×10^{-10}
(0.5,0.3)	0.0130722	0.01576956	0.016889	0.0168351	1.6865×10^{-6}	1.06633×10^{-6}	1.68333×10^{-10}

Table 3. Comparison of absolute error obtained by the OAFM solution with the HPM and NIM solution $\nu(\varpi, \psi)$ for Problem 1, when $\gamma = \beta = 1$ and z = 9.

(ϖ, ψ)	OAFM β= 0.5	OAFM β= 0.75	$\nu(\varpi, \psi)$ β= 1	Exact Solution	Absolute error HPM [31]	Absolute Error NIM [31]	Absolute Error OAFM
(0.1,0.1)	−0.000789	−0.000983	−0.000578	−0.000458	3.6876×10^{-11}	1.2940×10^{-11}	4.8516×10^{-15}
(0.3,0.1)	−0.003596	−0.008090	−0.005209	−0.005207	3.4567×10^{-10}	1.1746×10^{-10}	4.3678×10^{-14}
(0.5,0.1)	−0.009789	−0.013999	−0.015852	−0.014488	9.1597×10^{-10}	3.2142×10^{-10}	1.2123×10^{-13}
(0.1,0.2)	−0.000783	−0.000223	−0.000545	−0.000556	5.6445×10^{-10}	1.9564×10^{-10}	2.7675×10^{-13}
(0.3,0.2)	−0.002956	−0.045352	−0.004834	−0.007882	5.0767×10^{-9}	1.7232×10^{-9}	2.4667×10^{-12}
(0.5,0.2)	−0.008090	−0.009909	−0.001763	−0.001565	1.4450×10^{-8}	4.7936×10^{-9}	6.8567×10^{-12}
(0.1,0.3)	−0.000566	−0.000784	−0.000589	−0.000340	2.7556×10^{-9}	9.0244×10^{-10}	4.4566×10^{-12}
(0.3,0.3)	−0.009962	−0.006798	−0.007861	−0.004891	2.4233×10^{-8}	8.1490×10^{-9}	4.0420×10^{-11}
(0.5,0.3)	−0.07640	−0.098705	−0.013753	−0.012753	6.6770×10^{-7}	2.2609×10^{-7}	1.1144×10^{-9}

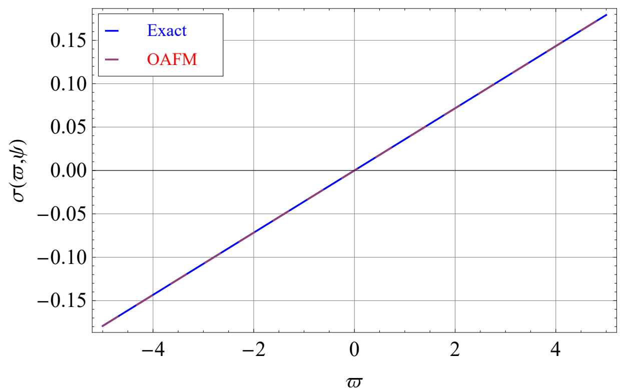

Figure 1. Two-dimensional plots of exact and OAFM solution $\sigma(\varpi, \psi)$ of Problem 1.

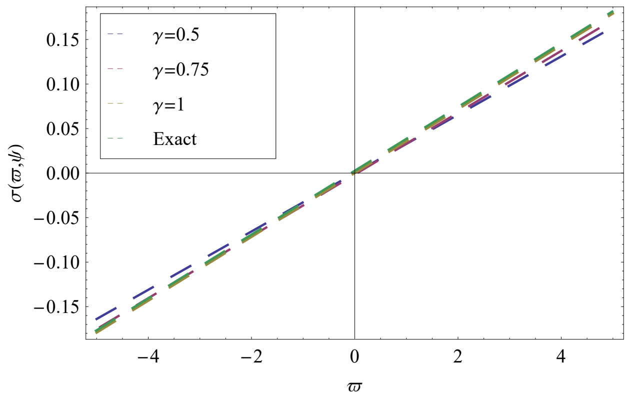

Figure 2. Impact of γ on OAFM solution of Problem 1.

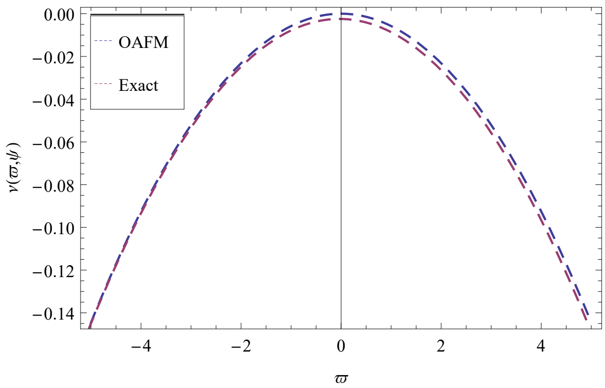

Figure 3. Two-dimensional plots of exact and OAFM solution $\nu(\varpi, \psi)$ of Problem 1.

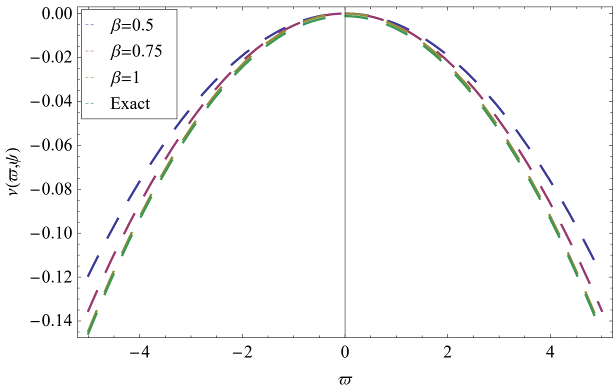

Figure 4. Impact of β on OAFM solution $\nu(\varpi, \psi)$ for Problem 1.

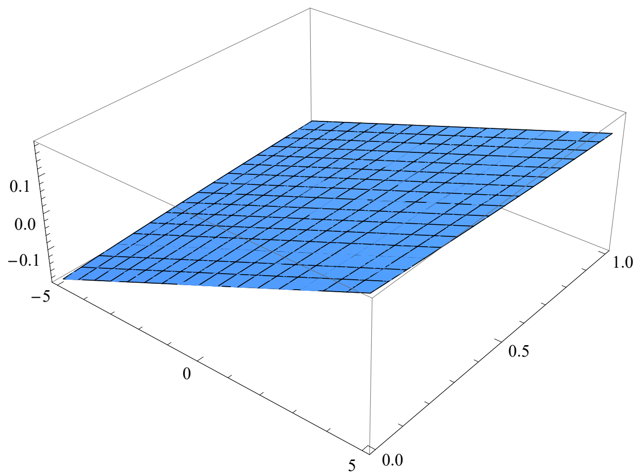

Figure 5. The OAFM solution of $\sigma(\varpi, \psi)$ for Problem 1.

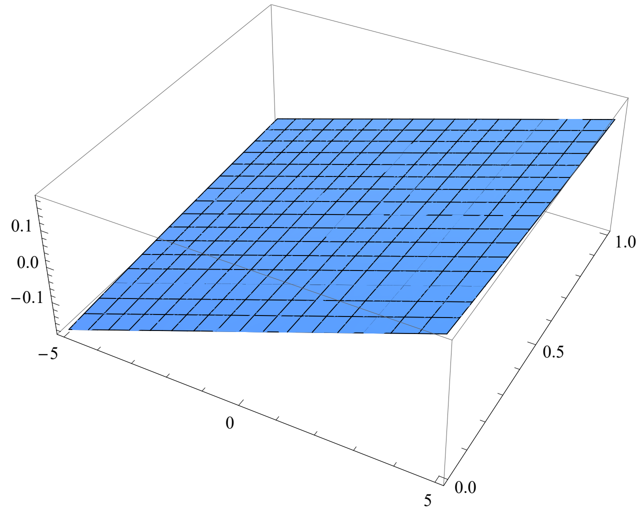

Figure 6. The exact solution of $\sigma(\varpi, \psi)$ for Problem 1.

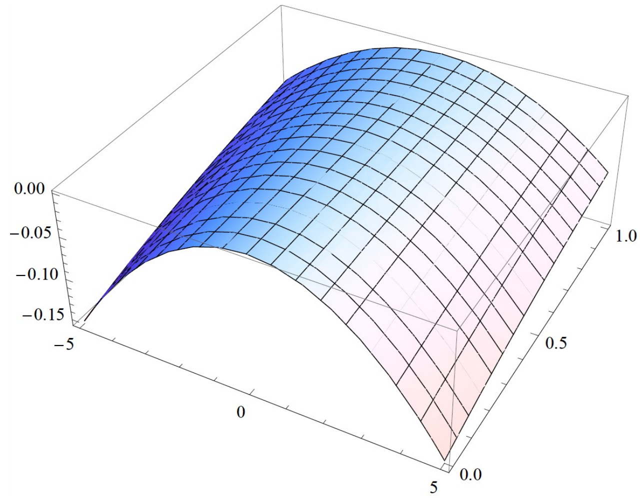

Figure 7. The OAFM solution of $\nu(\varpi, \psi)$ for Problem 1.

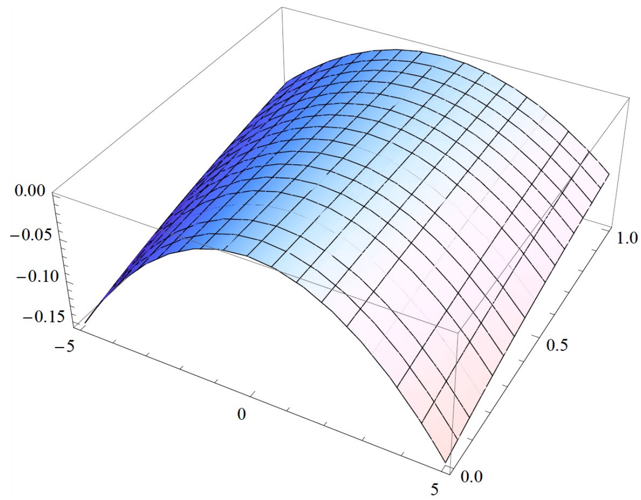

Figure 8. The exact solution of $\nu(\varpi, \psi)$ for Problem 1.

Problem 2:

Consider the non-linear KDV system of time-fractional order equation [32]

$$\begin{aligned} &\frac{\partial^{\gamma}\sigma(\varpi,\psi)}{\partial\psi^{\gamma}} + d\frac{\partial^{3}\sigma(\varpi,\psi)}{\partial\varpi^{3}} + 6d\sigma(\varpi,\psi)\frac{\partial\sigma(\varpi,\psi)}{\partial\varpi} - 6\nu(\varpi,\psi)\frac{\partial\nu(\varpi,\psi)}{\partial\varpi} = 0, \\ &\frac{\partial^{\beta}\nu(\varpi,\psi)}{\partial\psi^{\beta}} + d\frac{\partial^{3}\nu(\varpi,\psi)}{\partial\varpi^{3}} + 3d\sigma(\varpi,\psi)\frac{\partial\nu(\varpi,\psi)}{\partial\varpi} = 0. \end{aligned} \tag{31}$$

coupled with the initial conditions

$$\begin{aligned} \sigma(\omega,0) &= \chi^{2}sech^{2}\left(\tfrac{z}{2} + \tfrac{\chi\omega}{2}\right) \\ \nu(\omega,0) &= \sqrt{\tfrac{d}{2}}\chi^{2}sech^{2}\left(\tfrac{z}{2} + \tfrac{\chi\omega}{2}\right) \end{aligned} \tag{32}$$

An accurate result of Equation (31) when $\gamma = \beta = 1$ is

$$\begin{aligned} \sigma(\varpi,\psi) &= \chi^{2}sech^{2}\left(\tfrac{z}{2} + \tfrac{\chi\varpi}{2} - \tfrac{d\chi^{3}\psi}{2}\right). \\ \nu(\varpi,\psi) &= \sqrt{\tfrac{d}{2}}\chi^{2}sech^{2}\left(\tfrac{z}{2} + \tfrac{\chi\varpi}{2} - \tfrac{d\chi^{3}\psi}{2}\right) \end{aligned} \tag{33}$$

In Equation (32), we take the linear and non-linear terms as

$$\begin{cases} L(\nu(\omega,\psi)) = \frac{\partial^{\gamma}\sigma(\varpi,\psi)}{\partial\psi^{\gamma}} \\ M(\sigma(\varpi,\psi)) = d\frac{\partial^{3}\sigma(\varpi,\psi)}{\partial\omega^{3}} + 6d\sigma(\varpi,\psi)\frac{\partial\sigma(\varpi,\psi)}{\partial\omega} - 6\nu(\varpi,\psi)\frac{\partial\nu(\varpi,\psi)}{\partial\omega} \end{cases}$$
$$\begin{cases} L(\nu(\omega,\psi)) = \frac{\partial^{\beta}\nu(\varpi,\psi)}{\partial\psi^{\beta}} \\ M(\nu(\omega,\psi)) = d\frac{\partial^{3}\nu(\omega,\psi)}{\partial\omega^{3}} + 3d\sigma(\omega,\psi)\frac{\partial\nu(\omega,\psi)}{\partial\omega}. \end{cases} \tag{34}$$

According to the OAFM, the zero-order problem can be obtained as follows

$$\begin{aligned} &\frac{\partial^{\gamma}\sigma_{0}(\varpi,\psi)}{\partial\psi^{\gamma}} = 0, \quad \sigma_{0}(\varpi,0) = \chi^{2}sech^{2}\left(\tfrac{z}{2} + \tfrac{\chi\varpi}{2}\right). \\ &\frac{\partial^{\beta}\nu_{0}(\varpi,\psi)}{\partial\psi^{\beta}} = 0, \quad \nu_{0}(\varpi,0) = \sqrt{\tfrac{d}{2}}\chi^{2}sech^{2}\left(\tfrac{z}{2} + \tfrac{\chi\varpi}{2}\right) \end{aligned} \tag{35}$$

Using the R-L operator on Equation (35), its solution becomes

$$\begin{aligned} \sigma_{0}(\varpi,0) &= \chi^{2}sech^{2}\left(\tfrac{z}{2} + \tfrac{\chi\varpi}{2}\right). \\ \nu_{0}(\varpi,0) &= \sqrt{\tfrac{d}{2}}\chi^{2}sech^{2}\left(\tfrac{z}{2} + \tfrac{\chi\varpi}{2}\right) \end{aligned} \tag{36}$$

By using Equation (36) and substituting into Equation (34), then the non-linear operator becomes

$$\begin{aligned} M(\sigma_{0}(\varpi,\psi)) &= d\frac{\partial^{3}\sigma_{0}(\varpi,\psi)}{\partial\varpi^{3}} + 6d\sigma_{0}(\varpi,\psi)\frac{\partial\sigma_{0}(\varpi,\psi)}{\partial\varpi} - 6\nu_{0}(\varpi,\psi)\frac{\partial\nu_{0}(\varpi,\psi)}{\partial\varpi}, \\ M(\nu_{0}(\varpi,\psi)) &= d\frac{\partial^{3}\nu_{0}(\varpi,\psi)}{\partial\varpi^{3}} + 3d\sigma_{0}(\varpi,\psi)\frac{\partial\nu_{0}(\varpi,\psi)}{\partial\varpi}. \end{aligned} \tag{37}$$

According to the OAFM, the first-order problem can be obtained as

$$\begin{aligned} \frac{\partial^{\gamma}\sigma_{1}(\varpi,\psi)}{\partial\psi^{\gamma}} &= -B_{1}[\sigma_{0}(\varpi,\psi),T_{\iota}]M[\sigma_{0}(\varpi,\psi)] - B_{2}\left[\sigma_{0}(\varpi,\psi),T_{j}\right], \\ \frac{\partial^{\beta}\nu_{1}(\varpi,\psi)}{\partial\psi^{\beta}} &= -B_{3}[\nu_{0}(\varpi,\psi),T_{\iota}]M[\nu_{0}(\varpi,\psi)] - B_{4}\left[\nu_{0}(\varpi,\psi),T_{j}\right]. \end{aligned} \tag{38}$$

Now, we select B_1 and B_2 as according to our initial approximation,

$$B_{1} = -(T_{1}), B_{2} = 0; B_{3} = -(T_{2}); B_{4} = 0 \tag{39}$$

By using Equation (36), Equation (39) and substituting into Equation (38), then we obtain the first approximation as

$$\begin{aligned}\sigma(\varpi,\psi) &= -\frac{8bT_1\psi^{\gamma}\chi^5 cosech^3(z+\varpi\chi)sinh^4\left(\frac{1}{2}(z+\varpi\chi)\right)}{\Gamma(1+\gamma)}.\\ \nu(\varpi,\psi) &= -\frac{4\sqrt{2}d^{\frac{3}{2}}T_2\psi^{\beta}\chi^5 cosech^3(z+\varpi\chi)sinh^4\left(\frac{1}{2}(z+\varpi\chi)\right)}{\Gamma(1+\gamma)}\end{aligned} \tag{40}$$

By combining Equations (36) and (40), we obtain the first-order approximate solution as

$$\begin{aligned}\widetilde{\sigma}(\varpi,\psi) &= \sigma_0(\varpi,\psi)+\sigma_1(\varpi,\psi,T_1).\\ \widetilde{\nu}(\varpi,\psi) &= \nu_0(\varpi,\psi)+\nu_1(\varpi,\psi,T_2).\end{aligned} \tag{41}$$

$$\begin{aligned}\widetilde{\sigma}(\varpi,\psi) &= \chi^2 sech^2\left(\frac{z}{2}+\frac{\chi\varpi}{2}\right)-\frac{8bT_1\psi^{\gamma}\chi^5 cosech^3(z+\varpi\chi)sinh^4\left(\frac{1}{2}(z+\varpi\chi)\right)}{\Gamma(1+\gamma)}\\ \widetilde{\nu}(\varpi,\psi) &= \sqrt{\frac{d}{2}}\chi^2 sech^2\left(\frac{z}{2}+\frac{\chi\varpi}{2}\right)-\frac{4\sqrt{2}d^{\frac{3}{2}}T_2\psi^{\beta}\chi^5 cosech^3(z+\varpi\chi)sinh^4\left(\frac{1}{2}(z+\varpi\chi)\right)}{\Gamma(1+\gamma)}.\end{aligned} \tag{42}$$

The results of Problem 2 are presented in Tables 4–6, and visualized in Figures 9–16.

Table 4. Convergence control parameters for different values of γ for Problem 2.

	$\gamma = 1$	$\gamma = 0.75$	$\gamma = 0.5$
T_1	−0.999999999999999	−1.1107207345376926	−1.5707944267948912
T_2	−1	−1.1107207343495915	−1.5707966767948968

Table 5. Comparison of absolute error obtained by the OAFM solution $\sigma(\varpi,\psi)$ with the LADM solution for Problem 2, when γ = 1.

$\sigma(\varpi,\psi)$	$\sigma(\varpi,\psi)$ $\gamma = 0.5$	$\sigma(\varpi,\psi)$ $\gamma = 0.75$	$\sigma(\varpi,\psi)$ $\gamma = 1$	**Exact Solution**	**Absolute Error Of LADM σ [32]**	**Absolute Error OAFM σ**
(−10,0.1)	0.0156829	0.0179767	0.0175892	0.0179965	7.449×10^{-6}	3.0897×10^{-7}
(−10,0.3)	0.0170899	0.0171973	0.0177834	0.0173434	2.238×10^{-5}	2.76564×10^{-6}
(−10,0.5)	0.0166835	0.0145741	0.0171706	0.0171892	3.727×10^{-5}	7.6875×10^{-6}
(0,0.1)	0.1983382	0.1979706	0.1971987	0.1971780	3.975×10^{-5}	6.94561×10^{-7}
(0,0.3)	0.2001215	0.1999765	0.1988955	0.1984591	1.192×10^{-4}	6.34342×10^{-6}
(0,0.5)	0.2099428	0.2008858	0.1994672	0.1967333	1.987×10^{-4}	1.7789×10^{-5}
(10,0.1)	0.0067212	0.0025761	0.0024988	0.0024898	1.073×10^{-6}	4.75781×10^{-8}
(10,0.3)	0.0088613	0.0025981	0.0025785	0.0025789	3.221×10^{-6}	4.29878×10^{-7}
(10,0.5)	0.0034889	0.0089657	0.0025782	0.0025884	5.368×10^{-6}	1.19453×10^{-6}

Table 6. Comparison of absolute error obtained by the OAFM solution $\nu(\varpi,\psi)$ with the LADM solution for Problem 2, when β = 1.

(ϖ,ψ)	$\nu(\varpi,\psi)$ $\beta = 0.5$	$\nu(\varpi,\psi)$ $\beta = 0.75$	$\nu(\varpi,\psi)$ $\beta = 1$	**Exact Solution**	**Absolute Error LADM [32]**	**Absolute Error of OAFM ν**
(−10,0.1)	0.0087814	0.0087883	0.0086781	0.0087792	4.928×10^{-7}	1.53935×10^{-8}
(−10,0.3)	0.0083424	0.0084566	0.0035717	0.0086731	1.478×10^{-6}	1.38052×10^{-6}
(−10,0.5)	0.0023067	0.0084450	0.0078653	0.0085689	2.464×10^{-6}	3.82125×10^{-6}
(0,0.1)	0.0913191	0.0988753	0.0984598	0.0985867	2.629×10^{-6}	3.47491×10^{-7}
(0,0.3)	0.1023607	0.0996782	0.0976577	0.0991675	7.889×10^{-6}	3.17276×10^{-6}
(0,0.5)	0.1005674	0.1008929	0.0989256	0.0997176	1.314×10^{-5}	8.93948×10^{-6}
(10,0.1)	0.0018906	0.0013480	0.0012409	0.0012429	7.100×10^{-8}	2.37791×10^{-8}
(10,0.3)	0.0017806	0.0012670	0.0016762	0.0012764	2.131×10^{-7}	2.14889×10^{-7}
(10,0.5)	0.0012964	0.0012848	0.0012796	0.0012782	3.552×10^{-7}	5.99366×10^{-7}

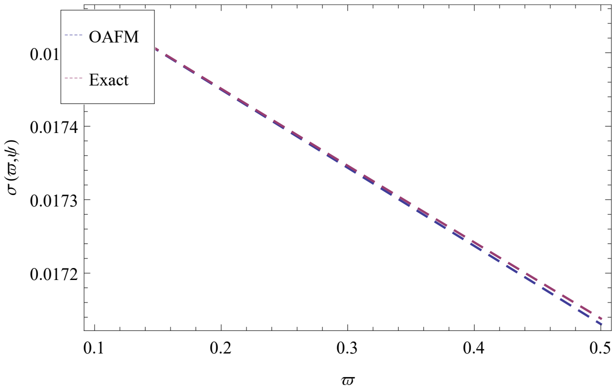

Figure 9. Two-dimensional plots of exact and OAFM solution $\sigma(\varpi, \psi)$of Problem 2.

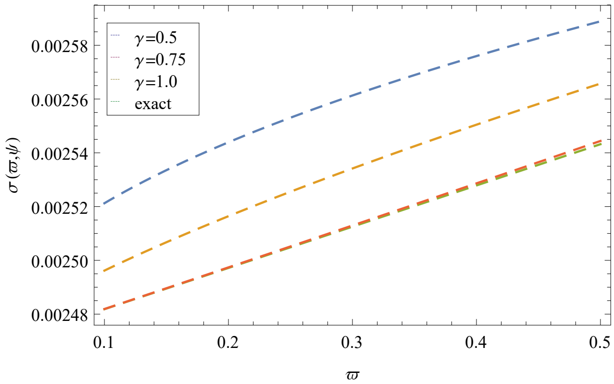

Figure 10. Impact of γ on OAFM solution For Problem 2.

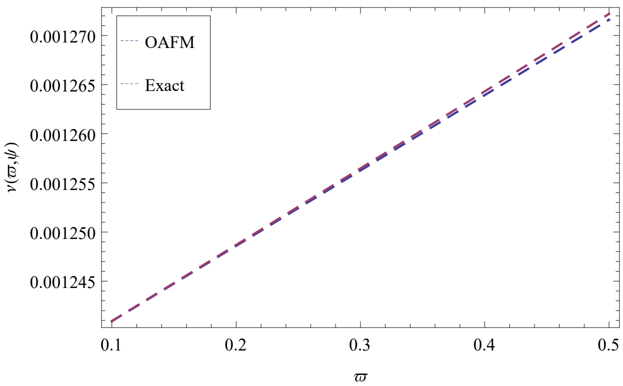

Figure 11. Two-dimensional plots of exact and OAFM solution $\nu(\varpi, \psi)$ of Problem 2.

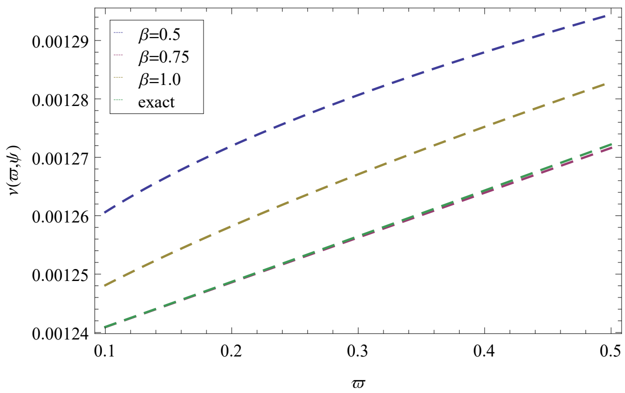

Figure 12. Impact of β on OAFM solution for Problem 2.

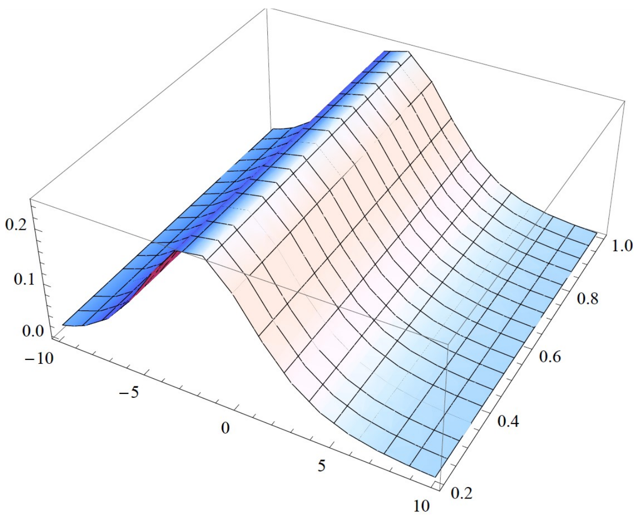

Figure 13. The OAFM solution of $\sigma(\varpi, \psi)$ for Problem 2.

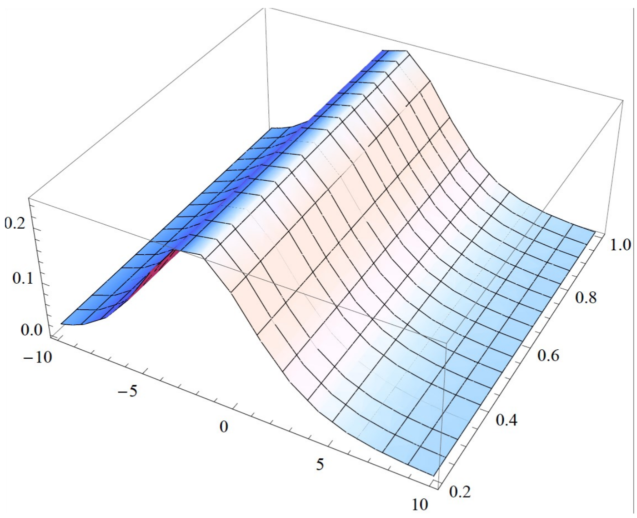

Figure 14. The exact solution of $\sigma(\varpi, \psi)$ for Problem 2.

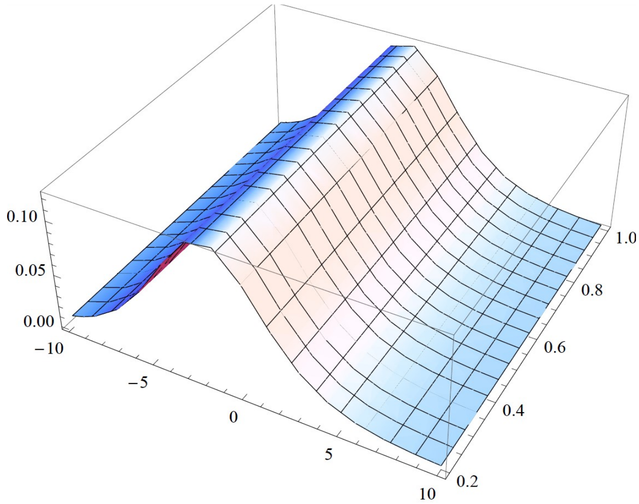

Figure 15. The OAFM solution of $\nu(\varpi, \psi)$ for Problem 2.

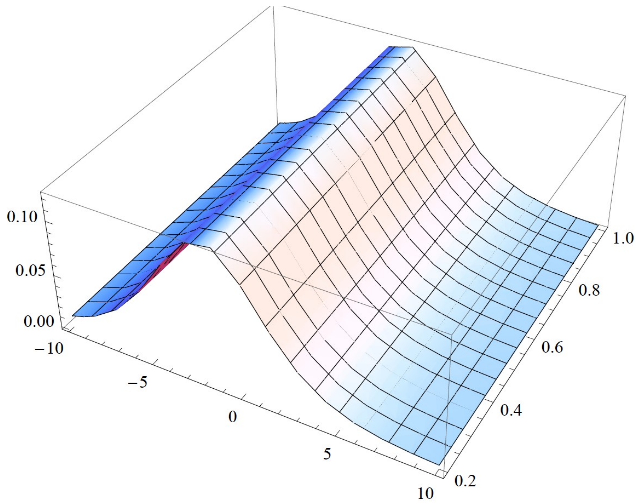

Figure 16. The exact solution of $\nu(\varpi, \psi)$ for Problem 2.

5. Discussion and Conclusions

The extended OAFM was used to solve the time-fractional order equations of the non-linear KDV system and the time-fractional ITO system. The findings produced by OAFM for the time-fractional order equation with HPM, NIM, and LADM are presented in Section 4, with the associated tables and figures for the ITO system and non-linear KDV system.

In Problem 1, Table 1 shows the different values of T_1 and T_2, Tables 2 and 3 display the comparison of the absolute errors of the first-order OAFM solution for $\sigma(\varpi, \psi)$ and $\nu(\varpi, \psi)$ of time fractional coupled ITO equation with a third-order HPM solution and second-order NIM solution using $\gamma = \beta = 1$.

Figures 1 and 3 depict 2D plots as a comparison of the exact and OAFM solutions of $\sigma(\varpi, \psi)$ and $\nu(\varpi, \psi)$ for Problem 1 at $\gamma = 1$. Figures 2 and 4 show the impact of γ and β on OAFM solution, respectively. Figures 5–8 display the 3D plot of the OAFM and the exact solutions of $\sigma(\varpi, \psi)$ and $\nu(\varpi, \psi)$ for Problem 1 at $\gamma = 1$.

In Problem 2, Table 4 shows the different values of T_1 and T_2 for convergence control parameters. Tables 5 and 6 show the comparisons of the absolute errors of the first-order OAFM solution with LADM for $\sigma(\varpi, \psi)$ and $\nu(\varpi, \psi)$ of a coupled non-linear KDV system of time fractional equations. Additionally, Figures 9 and 11 depict the 2D plots of the OAFM and the exact solutions of $\sigma(\varpi, \psi)$ and $\nu(\varpi, \psi)$ while Figures 10 and 12 display the 2D plots obtained by the OAFM solution of $\sigma(\varpi, \psi)$ and $\nu(\varpi, \psi)$ for different values of γ

and β. Figures 13–16 display the 3D plots of the OAFM and the exact solutions of $\sigma(\varpi, \psi)$ and $\nu(\varpi, \psi)$ for Problem 2 at $\gamma = 1$.

In this work, we have carefully proposed a consistent modification to the Optimal Auxiliary Function Method (OAFM) for addressing fractional differential equations. The objective of this paper is to apply the OAFM using the Riemann-–Liouville fractional Integral (R-L) and the Caputo derivative. It is observed that the proposed modification in OAFM has rendered it more effective, efficient, and influential than before in finding analytical as well as numerical solutions for a wide range of linear and non-linear fractional differential equations. The key advantage compared to other approaches is that OAFM requires fewer computational resources, making it accessible even on machines with modest specifications. There are currently no limitations on the applicability of this approach, making it suitable for addressing more complex models arising from real-world challenges in the future. In summary, it is a quick and efficient method. Based on the tables and figures presented, we can conclude that the suggested technique is advantageous and straightforward for solving fractional order partial differential equations.

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