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Investigation of the Oscillatory Properties of Solutions of Differential Equations Using Kneser-Type Criteria

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Abstract: This study investigates the oscillatory properties of a fourth-order delay functional differential equation. This study's methodology is built around two key tenets. First, we propose optimized relationships between the solution and its derivatives by making use of some improved monotonic features. By using a comparison technique to connect the oscillation of the studied equation with some second-order equations, the second aspect takes advantage of the significant progress made in the study of the oscillation of second-order equations. Numerous applications of functional differential equations of the neutral type served as the inspiration for the study of a subclass of these equations.

Keywords: delay differential equations; oscillatory behavior; Kneser-type criteria; comparison theorems

MSC: 34C10; 34K11



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1. Introduction

In this study, we consider the functional differential equation with p -Laplacian-like operators

$$\frac{d}{dt} \left[a_0(t) \phi \left(\frac{d^3}{dt^3} x(t) \right) \right] + a_1(t) \phi \left(\frac{d^3}{dt^3} x(t) \right) + a_2(t) \phi(x(g(t))) = 0, \quad (1)$$

where $t \in \mathbb{I} := [t_0, \infty)$, $\phi(u) = |u|^{p-2}u$, and the following assumptions are satisfied:

- (A1) $p > 1$ is a constant;
- (A2) $a_0 \in C^1(\mathbb{I}, \mathbb{R}^+)$, $a_i \in C(\mathbb{I}, [0, \infty))$ for $i = 1, 2$, $a_0'(t) \geq 0$, and $a_2(t) > 0$;
- (A3) $g \in C(\mathbb{I}, \mathbb{R})$, $g(t) \leq t$, $g'(t) \geq 0$, and $\lim_{t \rightarrow \infty} g(t) = \infty$;
- (A4) $\lim_{t \rightarrow \infty} \mathcal{A}_0(t) = \infty$, where

$$\mathcal{A}_0(t) := \int_{t_1}^t \left(\frac{\widehat{a}(\zeta)}{a_0(\zeta)} \right)^{\frac{1}{p-1}} d\zeta,$$

and

$$\widehat{a}(t) := \exp \left[- \int_{t_1}^t \frac{a_1(\zeta)}{a_0(\zeta)} d\zeta \right].$$

Functional differential equations (FDEs) are used in the natural sciences, engineering technology, and automatic control, as stated by Hale [1–5]. According to [6], the p -Laplace FDE has a wide variety of applications in continuum mechanics.

The great development witnessed by various sciences has been accompanied by many nonlinear mathematical models. However, it is difficult to find solutions to these models using traditional methods. Therefore, researchers resort to obtaining approximate solutions through numerical methods, or studying the properties of the solutions of these equations.

Many biological, chemical, and physical phenomena have mathematical models that use differential equations of the fourth-order delay. Examples of these applications include soil settlement and elastic issues. The oscillatory traction of a muscle, which takes place when the muscle is subjected to an inertial force, is one model that can be modeled using a fourth-order oscillatory equation with delay.

The qualitative study of differential equations contributes significantly to understanding and analyzing phenomena and problems without obtaining solutions. Qualitative studies have been developed in many theoretical and numerical ways. The qualitative studies include the study of stability, control, oscillation, bifurcation, periodicity, boundedness, and others.

One type of differential equation in which oscillatory behavior is frequent is the class of FDEs. It is known that deviating arguments that express the phenomenon's prior and present times are present in equations of this type when they deal with the aftereffects of life phenomena, which increase the likelihood that oscillatory solutions will exist (see [7]). One of the fundamental subclasses of FDEs is the delayed functional differential equation, also known as the delay differential equation. This type is based on the past and present values of the temporal derivatives, which results in forecasts for the future that are more precise and successful.

Oscillation theory, as one of the branches of qualitative theory, is interested in investigating the asymptotic and oscillatory properties of the solutions of FDEs. Studies in oscillation theory began by relating the oscillatory behavior of the linear differential equation to complex solutions of the characteristic equation, see [8,9]. Then, many methods and techniques have been developed that investigate the oscillatory behavior of different FDEs, which include delay, advanced, neutral, and mixed, as well as in canonical and noncanonical cases, see [10,11].

Here, we mention the basic definitions and some elementary previous results that we use to prove our results.

Definition 1. A function $x \in C^{(n-1)}([t_*, \infty), \mathbb{R})$, $t_* \in \mathbb{I}$, is said to be a solution of (1) if $a_0 \cdot \phi(x^{(n-1)}) \in C^1([t_*, \infty), \mathbb{R})$, x satisfies (1), and $\sup\{|x(t)| : t \geq t_1\} > 0$ for $t_1 \geq t_*$.

Definition 2. Such a solution x is called nonoscillatory if x is positive or negative, eventually; otherwise, x is called oscillatory.

Definition 3. FDE (1) is called oscillatory if every solution to it is oscillatory.

Next, we review some of the previous results that contributed to the development of the oscillation theory for equations of the middle term and for equations of the fourth order.

In 1979, Onose [12] studied the oscillation of the FDEs

$$\frac{d^2}{dt^2} \left[a_0(t) \frac{d^2}{dt^2} x(t) \right] + w(t, x(g(t))) = 0$$

and

$$\frac{d^2}{dt^2} \left[a_0(t) \frac{d^2}{dt^2} x(t) \right] + w(t, x(g(t))) = r(t),$$

under the condition

$$\int_{t_0}^{\infty} a_0^{-1}(z) dz = \infty.$$

In [13], Grace et al. presented some oscillation conditions for the FDE

$$\frac{d^3}{dt^3} \left[a_0(t) \frac{d}{dt} x(t) \right] + a_2(t) w(x(g(t))) = 0.$$

Wu [14] and Kamo and Usami [15] addressed the oscillatory properties of the equation

$$\frac{d^2}{dt^2} \left[a_0(t) \left| \frac{d^2}{dt^2} x(t) \right|^{\alpha-1} \frac{d^2}{dt^2} x(t) \right] + a_2(t) |x(t)|^{\beta-1} x(t) = 0,$$

where $\alpha, \beta \in \mathbb{R}^+$.

For even-order equations, Zhang et al. [16,17] and Baculikova et al. [18] studied the FDE

$$\frac{d}{dt} \left[a_0(t) \left(\frac{d^{n-1}}{dt^{n-1}} x(t) \right)^\alpha \right] + q(t) f(x(g(t))) = 0, \quad (2)$$

where $\alpha > 0$ is a quotient of odd integers. In [16,17], under the condition

$$\int_{t_0}^{\infty} a_0^{-1/\alpha}(z) dz < \infty, \quad (3)$$

Zhang et al. used the Riccati approach, and provided some oscillation criteria for Equation (2) when $f(x) = x^\beta$, $\beta \leq \alpha$, whereas Baculikova et al. [18] used the comparison technique to test the oscillation of FDE (1), and considered the two cases (3) and

$$\int_{t_0}^{\infty} a_0^{-1/\alpha}(z) dz = \infty.$$

For equations with a middle term, Grace [19] inspected the oscillatory behavior of the FDE

$$\frac{d}{dt} \left[a_0(t) \frac{d}{dt} x(t) \right] + a_1(t) x(h(t)) + a_2(t) w(x(g(t))) = 0. \quad (4)$$

In [20], Saker et al. obtained Kamenev-type criteria for FDE (4), and improved results in [19]. Tunc and Kaymaz [21] studied the neutral FDE

$$\frac{d^2}{dt^2} z(t) + a_1(t) \frac{d}{dt} z(t) + a_2(t) x(g(t)) = 0,$$

under the condition

$$\int_{t_0}^{\infty} \exp \left(- \int_{t_0}^t a_1(z) dz \right) dt = \infty,$$

where $z(t) = x(t) + a_3(t)x(h(t))$, and $h(t) \leq t$. Graef et al. [22] studied the oscillation of the mixed neutral FDE

$$\frac{d}{dt} \left[a_0(t) \frac{d}{dt} z(t) \right] + a_1(t) \frac{d}{dt} z(t) + a_2(t) x(g(t)) = 0, \quad (5)$$

under the condition

$$\int_{t_0}^{\infty} a_0^{-1}(z) \exp \left(- \int_{t_0}^z \frac{a_1(s)}{a_0(t)} ds \right) dz = \infty, \quad (6)$$

where

$$z(t) = x(t) + c_0(t)x(h_0(t)) + c_1(t)x(h_1(t)), \quad h_0(t) < t, \text{ and } h_1(t) > t.$$

Jadlovská and Džurina [23] derived Kneser-type criteria to test the oscillation of the FDE

$$\frac{d}{dt} \left[a_0(t) \phi \left(\frac{d}{dt} x(t) \right) \right] + a_2(t) \phi(x(g(t))) = 0. \quad (7)$$

Theorem 1 ([23], Theorem 2). Assume that $p \geq 2$ and

$$\alpha := \liminf_{t \rightarrow \infty} \frac{\eta(t)}{\eta(g(t))} < \infty.$$

FDE (7) is oscillatory if

$$\liminf_{t \rightarrow \infty} \left[a_0^{1/(p-1)}(t) \eta^{p-1}(g(t)) \eta(t) a_2(t) \right] > \delta,$$

where

$$\delta = (p-1) \max \left\{ \frac{\ell(1-\ell)^{p-1}}{\alpha^{(p-1)\ell}} : \ell \in (0,1) \right\}$$

and

$$\eta(t) := \int_{t_0}^t a_0^{-1/(p-1)}(s) ds.$$

Using the comparison method with second-order equations, Elabbasy et al. [24] studied the oscillation of FDE (1) when $\phi(u) = u$.

Theorem 2 ([24], Theorem 2). *If the differential equations*

$$\frac{d}{dt} \left(a_0(t) \frac{d}{dt} w(t) \right) + \frac{\kappa}{2} a_2(t) g^2(t) w(t) = 0$$

and

$$\frac{d^2}{dt^2} w(t) + w(t) \int_t^\infty \left[\frac{1}{a_0(s)} \int_s^\infty a_2(u) \frac{g^2(u)}{u^2} du \right] ds = 0$$

are oscillatory, where $\kappa \in (0,1)$, then FDE (1) is oscillatory.

2. Main Results

Assume first that x is an eventually positive solution of FDE (1), i.e., $x(t) > 0$ for $t \geq t_1 \in \mathbb{I}$. According to Lemma 4 in [25], we have, eventually,

$$x'(t) > 0, x'''(t) > 0, \text{ and } x^{(4)}(t) \leq 0,$$

under the condition (A4). Therefore, we can classify the solutions of FDE (1) into the following two cases:

[C1] $x^{(i)}(t) \geq 0$ for $i = 0, 1, 2, 3$, and $x^{(4)} \leq 0$;

[C2] $x^{(i)}(t) \geq 0$ for $i = 0, 1, 3$, $x''(t) < 0$, and $x^{(4)} \leq 0$.

For convenience, we define

$$\mathcal{A}_i(t) := \int_{t_1}^t \mathcal{A}_{i-1}(s) ds, \text{ for } i = 1, 2.$$

2.1. Monotonic Properties of Solutions in [C1]

In the following, we deduce some monotonic properties of the solutions in [C1] and their derivative.

Lemma 1. *Assume that x satisfies [C1], eventually. Then,*

$$\frac{d}{dt} \left[\frac{x^{(i)}(t)}{\mathcal{A}_{2-i}(t)} \right] \leq 0, \quad (8)$$

for $i = 0, 1, 2$.

Proof. Assume that x satisfies [C1] for $t \geq t_1 \in \mathbb{I}$. From FDE (1), we have

$$\frac{d}{dt} \left[\frac{a_0(t)}{\widehat{a}(t)} \phi(x'''(t)) \right] \leq 0.$$

Thus,

$$\begin{aligned} x''(t) &\geq \int_{t_1}^t \left(\frac{\widehat{a}(\zeta)}{a_0(\zeta)} \right)^{\frac{1}{p-1}} \left[\frac{a_0(\zeta)}{\widehat{a}(\zeta)} \phi(x'''(\zeta)) \right]^{\frac{1}{p-1}} d\zeta \\ &\geq \left[\frac{a_0(t)}{\widehat{a}(t)} \phi(x'''(t)) \right]^{\frac{1}{p-1}} \mathcal{A}_0(t) \\ &= \left(\frac{a_0(t)}{\widehat{a}(t)} \right)^{\frac{1}{p-1}} \mathcal{A}_0(t) x'''(t). \end{aligned}$$

This leads to

$$\begin{aligned} \frac{d}{dt} \left[\frac{x''(t)}{\mathcal{A}_0(t)} \right] &= \frac{1}{\mathcal{A}_0^2(t)} \left[\mathcal{A}_0(t) x'''(t) - \left(\frac{\widehat{a}(t)}{a_0(t)} \right)^{\frac{1}{p-1}} x''(t) \right] \\ &\leq 0. \end{aligned}$$

Next, using this fact, we obtain

$$x'(t) \geq \int_{t_1}^t \frac{x''(\zeta)}{\mathcal{A}_0(\zeta)} \mathcal{A}_0(\zeta) d\zeta \geq \frac{x''(t)}{\mathcal{A}_0(t)} \mathcal{A}_1(t),$$

which in turn gives

$$\frac{d}{dt} \left[\frac{x'(t)}{\mathcal{A}_1(t)} \right] \leq 0.$$

Similarly, we obtain

$$\frac{d}{dt} \left[\frac{x(t)}{\mathcal{A}_2(t)} \right] \leq 0.$$

The proof is complete. $\square$

Lemma 2. Assume that x satisfies [C1], eventually. Then,

$$\mathcal{A}_0(t)x(t) \geq \mathcal{A}_2(t)x''(t)$$

and

$$x(t) \geq \mathcal{A}_2(t) \left(\frac{a_0(t)}{\widehat{a}(t)} \right)^{\frac{1}{p-1}} x'''(t).$$

Proof. Assume that x satisfies [C1] for $t \geq t_1 \in \mathbb{I}$. From Lemma 1, we have that (8) holds. Thus,

$$\begin{aligned} x(t) &\geq \frac{\mathcal{A}_2(t)}{\mathcal{A}_1(t)} x'(t) \geq \frac{\mathcal{A}_2(t)}{\mathcal{A}_1(t)} \frac{\mathcal{A}_1(t)}{\mathcal{A}_0(t)} x''(t) \\ &= \frac{\mathcal{A}_2(t)}{\mathcal{A}_0(t)} x''(t) \\ &\geq \frac{\mathcal{A}_2(t)}{\mathcal{A}_0(t)} \left(\frac{a_0(t)}{\widehat{a}(t)} \right)^{\frac{1}{p-1}} \mathcal{A}_0(t) x'''(t) \\ &= \mathcal{A}_2(t) \left(\frac{a_0(t)}{\widehat{a}(t)} \right)^{\frac{1}{p-1}} x'''(t). \end{aligned}$$

The proof is complete. $\square$

2.2. Comparison Theorem

The comparison technique is usually used to benefit from the development of oscillation criteria for solutions to first- and second-order equations. This is achieved by linking

the oscillation of higher-order equations to one or more equations of the first or second order. This technique relies primarily on the relationships between the solution and the derivatives of the second and third orders, so improving these relationships is reflected in turn in improving the results derived from the use of the comparison technique. In the following theorem, we use a comparison approach to relate the oscillation of FDE (1) with a pair of equations of the second order.

Theorem 3. Assume that $p \geq 2$. FDE (1) is oscillatory if the second-order FDEs

$$\frac{d}{dt} \left[\left(\frac{a_0(t)}{\widehat{a}(t)} \right)^{\frac{1}{p-1}} w'(t) \right] + \frac{1}{p-1} \frac{a_2(t)}{\widehat{a}(t)} \frac{\mathcal{A}_2^{p-1}(g(t))}{\mathcal{A}_0(g(t))} w(g(t)) = 0 \quad (9)$$

and

$$x''(t) + x(g(t)) \int_t^\infty \left[\frac{\widehat{a}(\tau)}{a_0(\tau)} \int_\tau^\infty \frac{a_2(\mathfrak{z})}{\widehat{a}(\mathfrak{z})} d\mathfrak{z} \right]^{\frac{1}{p-1}} d\tau = 0 \quad (10)$$

are oscillatory.

Proof. Based on the converse hypothesis, we assume that FDE (1) has a nonoscillatory solution, which in turn inevitably leads to the existence of an eventually positive solution to this equation. Therefore, there is a $t_1 \in \mathbb{I}$ such that x satisfies [C1] or [C2] for $t \geq t_1$.

Suppose first that x satisfies [C1]. Then, we have

$$\begin{aligned} & \frac{d}{dt} \left[\left(\frac{a_0(t)}{\widehat{a}(t)} \right)^{\frac{1}{p-1}} x'''(t) \right] \\ &= \frac{d}{dt} \left[\left(\frac{a_0(t)}{\widehat{a}(t)} \phi(x'''(t)) \right)^{\frac{1}{p-1}} \right] \\ &= \frac{1}{p-1} \left(\left(\frac{a_0(t)}{\widehat{a}(t)} \right)^{\frac{1}{p-1}} x'''(t) \right)^{2-p} \frac{d}{dt} \left[\frac{a_0(t)}{\widehat{a}(t)} \phi(x'''(t)) \right] \\ &= -\frac{1}{p-1} \left(\left(\frac{a_0(t)}{\widehat{a}(t)} \right)^{\frac{1}{p-1}} x'''(t) \right)^{2-p} \frac{a_2(t)}{\widehat{a}(t)} \phi(x(g(t))). \end{aligned} \quad (11)$$

From Lemma 2, we have

$$x(t) \geq \mathcal{A}_2(t) \left(\frac{a_0(t)}{\widehat{a}(t)} \right)^{\frac{1}{p-1}} x'''(t), \quad (12)$$

Since $x/\mathcal{A}_2$ is nonincreasing, we have that

$$\frac{x(g(t))}{\mathcal{A}_2(g(t))} \geq \frac{x(t)}{\mathcal{A}_2(t)},$$

which, with (12), gives

$$\left(\frac{x(g(t))}{\mathcal{A}_2(g(t))} \right)^{2-p} \leq \left(\left(\frac{a_0(t)}{\widehat{a}(t)} \right)^{\frac{1}{p-1}} x'''(t) \right)^{2-p}.$$

Thus, (11) becomes

$$\begin{aligned} \frac{d}{dt} \left[\left(\frac{a_0(t)}{\widehat{a}(t)} \right)^{\frac{1}{p-1}} x'''(t) \right] &\leq -\frac{1}{p-1} \left(\frac{x(g(t))}{\mathcal{A}_2(g(t))} \right)^{2-p} \frac{a_2(t)}{\widehat{a}(t)} \phi(x(g(t))) \\ &= -\frac{1}{p-1} \mathcal{A}_2^{p-2}(g(t)) \frac{a_2(t)}{\widehat{a}(t)} x(g(t)). \end{aligned} \quad (13)$$

From Lemma 2, we obtain

$$x(g(t)) \geq \frac{\mathcal{A}_2(g(t))}{\mathcal{A}_0(g(t))} x''(g(t)). \quad (14)$$

Combining (13) and (14), we arrive at

$$\frac{d}{dt} \left[\left(\frac{a_0(t)}{\widehat{a}(t)} \right)^{\frac{1}{p-1}} x'''(t) \right] + \frac{1}{p-1} \frac{a_2(t)}{\widehat{a}(t)} \frac{\mathcal{A}_2^{p-1}(g(t))}{\mathcal{A}_0(g(t))} x''(g(t)) \leq 0.$$

Now, if we set $w := x'' > 0$, then w is a positive solution of the inequality

$$\frac{d}{dt} \left[\left(\frac{a_0(t)}{\widehat{a}(t)} \right)^{\frac{1}{p-1}} w'(t) \right] + \frac{1}{p-1} \frac{a_2(t)}{\widehat{a}(t)} \frac{\mathcal{A}_2^{p-1}(g(t))}{\mathcal{A}_0(g(t))} w(g(t)) \leq 0.$$

Using Corollary 1 in [26], the corresponding FDE (9) also has a positive solution; this is a contradiction.

Next, suppose first that x satisfies [C2]. Multiplying FDE (1) by $1/\widehat{a}(t)$, we find

$$\frac{d}{dt} \left[\frac{a_0(t)}{\widehat{a}(t)} \phi(x'''(t)) \right] + \frac{a_2(t)}{\widehat{a}(t)} \phi(x(g(t))) = 0. \quad (15)$$

Integrating (15) from t to ∞ , we obtain

$$\begin{aligned} \frac{a_0(t)}{\widehat{a}(t)} \phi(x'''(t)) &\geq \int_t^\infty \frac{a_2(\mathfrak{z})}{\widehat{a}(\mathfrak{z})} \phi(x(g(\mathfrak{z}))) d\mathfrak{z} \\ &\geq \phi(x(g(t))) \int_t^\infty \frac{a_2(\mathfrak{z})}{\widehat{a}(\mathfrak{z})} d\mathfrak{z}, \end{aligned}$$

and then

$$x'''(t) \geq x(g(t)) \left[\frac{\widehat{a}(t)}{a_0(t)} \int_t^\infty \frac{a_2(\mathfrak{z})}{\widehat{a}(\mathfrak{z})} d\mathfrak{z} \right]^{\frac{1}{p-1}}.$$

By integrating from t to ∞ , we obtain

$$\begin{aligned} -x''(t) &\geq \int_t^\infty x(g(\mathfrak{r})) \left[\frac{\widehat{a}(\mathfrak{r})}{a_0(\mathfrak{r})} \int_\mathfrak{r}^\infty \frac{a_2(\mathfrak{z})}{\widehat{a}(\mathfrak{z})} d\mathfrak{z} \right]^{\frac{1}{p-1}} d\mathfrak{r} \\ &\geq x(g(t)) \int_t^\infty \left[\frac{\widehat{a}(\mathfrak{r})}{a_0(\mathfrak{r})} \int_\mathfrak{r}^\infty \frac{a_2(\mathfrak{z})}{\widehat{a}(\mathfrak{z})} d\mathfrak{z} \right]^{\frac{1}{p-1}} d\mathfrak{r}, \end{aligned}$$

or

$$x''(t) + x(g(t)) \int_t^\infty \left[\frac{\widehat{a}(\mathfrak{r})}{a_0(\mathfrak{r})} \int_\mathfrak{r}^\infty \frac{a_2(\mathfrak{z})}{\widehat{a}(\mathfrak{z})} d\mathfrak{z} \right]^{\frac{1}{p-1}} d\mathfrak{r} \leq 0.$$

Then, x is a positive solution of this inequality. Using Corollary 1 in [26], the corresponding FDE (10) also has a positive solution; this is a contradiction.

The proof is complete. $\square$

Corollary 1. Suppose that $p \geq 2$,

$$\alpha_0 := \liminf_{t \rightarrow \infty} \frac{\mathcal{A}_0(t)}{\mathcal{A}_0(g(t))} < \infty,$$

and

$$\alpha_1 := \liminf_{t \rightarrow \infty} \frac{t}{g(t)} < \infty.$$

FDE (1) is oscillatory if

$$\liminf_{t \rightarrow \infty} \left[\frac{a_2(t)}{\widehat{a}(t)} \left(\frac{a_0(t)}{\widehat{a}(t)} \right)^{\frac{1}{p-1}} \mathcal{A}_0(t) \mathcal{A}_2^{p-1}(g(t)) \right] > (p-1)\delta_0 \quad (16)$$

and

$$\liminf_{t \rightarrow \infty} \left[t g(t) \int_t^\infty \left[\frac{\widehat{a}(\tau)}{a_0(\tau)} \int_\tau^\infty \frac{a_2(\mathfrak{z})}{\widehat{a}(\mathfrak{z})} d\mathfrak{z} \right]^{\frac{1}{p-1}} d\tau \right] > \delta_1, \quad (17)$$

where

$$\delta_i = \max \left\{ \frac{\ell(1-\ell)}{\alpha_i^\ell} : \ell \in (0, 1) \right\}, \text{ for } i = 0, 1.$$

Proof. Based on the converse hypothesis, we assume that FDE (1) has a nonoscillatory solution, which in turn inevitably leads to the existence of an eventually positive solution to this equation. Therefore, there is a $t_1 \in \mathbb{I}$ such that x satisfies [C1] or [C2] for $t \geq t_1$. As in the proof of Theorem 3, the second-order FDEs (9) and (10) have positive solutions. However, according to Theorem 1, conditions (16) and (17) confirm the oscillation of FDEs (9) and (10), respectively, which is a contradiction.

The proof is complete. $\square$

The following corollary is obtained directly by setting $p = 2$ and $a_0(t) = 1$. This corollary studies the oscillation of the linear state of FDE (1).

Corollary 2. Suppose that

$$\alpha := \liminf_{t \rightarrow \infty} \frac{t}{g(t)} < \infty.$$

The FDE

$$\frac{d^4}{dt^4} x(t) + a_1(t) \frac{d^3}{dt^3} x(t) + a_2(t) x(g(t)) = 0$$

is oscillatory if

$$\liminf_{t \rightarrow \infty} \left[\frac{a_2(t)}{\widehat{a}^2(t)} \mathcal{A}_0(t) \mathcal{A}_2(g(t)) \right] > \delta$$

and

$$\liminf_{t \rightarrow \infty} \left[t g(t) \int_t^\infty \widehat{a}(\tau) \int_\tau^\infty \frac{a_2(\mathfrak{z})}{\widehat{a}(\mathfrak{z})} d\mathfrak{z} d\tau \right] > \delta,$$

where

$$\delta = \max \left\{ \frac{\ell(1-\ell)}{\alpha^\ell} : \ell \in (0, 1) \right\}.$$

Example 1. Consider the FDE

$$\frac{d}{dt} \left[\frac{1}{t} \frac{d^3}{dt^3} x(t) \right] + \frac{1}{t^2} \frac{d^3}{dt^3} x(t) + \frac{c_0}{t^5} x(\lambda t) = 0, \quad (18)$$

where $t > 0$, $c_0 > 0$ and $\lambda \in (0, 1)$. We note that $p = 2$, $\phi(t) = u$, $a_0(t) = 1/t$, $a_1(t) = 1/t^2$, $a_2(t) = c_0/t^5$, and $g(t) = \lambda t$. Thus, we have

$$\hat{a}(t) = \frac{1}{t}, \mathcal{A}_0(t) = t, \mathcal{A}_1(t) = \frac{1}{2}t^2,$$

and

$$\mathcal{A}_2(t) = \frac{1}{6}t^3.$$

Moreover, from the definition of α_1 and α_2 , we find that $\alpha_1 = \alpha_2 = 1/\lambda$. Now, conditions (16) and (17) reduce to

$$\frac{\lambda^3}{6}c_0 > \delta_0$$

and

$$\lambda \frac{c_0}{6} > \delta_1,$$

where

$$\delta_i = \max \left\{ \ell(1 - \ell)\lambda^\ell : \ell \in (0, 1) \right\}, \text{ for } i = 0, 1.$$

Thus, using Corollary 1, FDE (18) is oscillatory if

$$\lambda^3 c_0 > 6\delta_0. \quad (19)$$

Remark 1. Using Theorem 2, FDE (18) is oscillatory if the second-order FDEs

$$\frac{d}{dt} \left(\frac{1}{t} \frac{d}{dt} w(t) \right) + \frac{\kappa c_0 \lambda^2}{2} \frac{1}{t^3} w(t) = 0 \quad (20)$$

and

$$\frac{d^2}{dt^2} w(t) + \frac{c_0 \lambda^2}{8} \frac{1}{t^2} w(t) = 0 \quad (21)$$

are oscillatory.

Now, From Theorem 1, FDEs (20) and (21) are oscillatory if

$$\frac{1}{8} \lambda^4 c_0 > \delta_2$$

and

$$\frac{1}{8} \lambda^3 c_0 > \delta_0$$

respectively, where

$$\delta_2 = \max \left\{ \ell(1 - \ell)\lambda^{2\ell} : \ell \in (0, 1) \right\}.$$

Therefore, FDE (18) is oscillatory if

$$c_0 > \max \left\{ \frac{8\delta_2}{\lambda^4}, \frac{8\delta_0}{\lambda^3} \right\}. \quad (22)$$

To compare the two criteria (19) and (22), we consider different values of parameter λ and determine the most efficient criterion through the following table.

We notice from Table 1 that Criterion (19) provides wider intervals for the parameter c_0 , and this means that it is more efficient in testing the oscillation.

Table 1. The lower bounds of the parameter c_0 at which conditions (19) and (22) are satisfied.

λ	0.1	0.5	0.7	0.9
Criterion (19)	635.114	8.74015	3.68796	1.95338
Criterion (22)	5159.99	17.9293	6.01641	2.75110

3. Conclusions

Based on the comparison principle with equations of the second order, we established a new criterion of the Kneser type that confirms the oscillation of all solutions of fourth-order half-linear differential equations. After classifying the positive solutions according to their derivatives, we excluded the existence of positive solutions in each case separately. Then, we obtained a criterion that ensures the oscillation of the solutions to DE (1). By applying the new results to some examples and special cases, we clarified the importance of the new results. Extending our results to the neutral case is a suggested research point. Also, improving the monotonic properties of the studied equation can improve the oscillation criteria.

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