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Necessary and Sufficient Conditions for Commutator of the Calderón–Zygmund Operator on Mixed-Norm Herz-Slice Spaces

Lihua Zhang and Jiang Zhou *

College of Mathematics and System Science, Xinjiang University, Urumqi 830046, China; 107552100585@stu.xju.edu.cn
* Correspondence: zhoujiang@xju.edu.cn

Abstract: We obtain the separability of mixed-norm Herz-slice spaces, establish a weak convergence on mixed-norm Herz-slice spaces, and get the boundedness of the Calderón–Zygmund operator T on mixed-norm Herz-slice spaces. Moreover, we get the necessary and sufficient conditions for the boundedness of the commutator $[b, T]$ on mixed-norm Herz-slice spaces, where b is a locally integrable function.

Keywords: mixed-norm Herz-slice spaces; BMO$(\mathbb{R}^n)$; Calderón–Zygmund operator

MSC: 42B35; 46E30; 42B25

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1. Introduction

To understand the Bernstein theorem for absolutely convergent Fourier transforms, Beurling [1] researched the Herz space $A_u(\mathbb{R}^n)$ in 1964. The Herz space $K_u(\mathbb{R}^n)$ is further explored by Herz [2] in 1968. In the 1990's, Lu and Yang [3] introduced the homogeneous Herz space $(\dot{K}_u^{\beta,s})(\mathbb{R}^n)$ and the non-homogeneous Herz space $(K_u^{\beta,s})(\mathbb{R}^n)$. In recent years, Herz spaces has been extensively studied in the fields of harmonic analysis, see [4–8] and so on.

Wiener amalgam is an indispensable tool in time-frequency analysis [9,10] and sampling theory [11]. At first, amalgam spaces is elaborated by Wiener in [12]. Still, the systematic investigation of amalgam spaces should look at the study of Holland [13], which contains the research of dual spaces and multipliers on $\mathbb{R}^n$. Wiener amalgam spaces are generalized by Feichtinger and Weisz from $\mathbb{R}$ or $\mathbb{R}^n$ to Banach function spaces, see, [14–18] and so on. A definition of the amalgam space $(L^v, L^u)(\mathbb{R}^n)$ is defined by:

$$(L^v, L^u)(\mathbb{R}^n) := \left\{ f \in L^v_{\mathrm{loc}}(\mathbb{R}^n) : \left\| f 1_{B(y,1)} \right\|_{L^v(\mathbb{R}^n)} \in L^u(\mathbb{R}^n) \right\},$$

where

$$\|f\|_{(L^v,L^u)(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left\| f 1_{B(y,1)} \right\|^u_{L^v(\mathbb{R}^n)} \mathrm{d}y \right)^{1/u},$$

with the usual modification for $v = \infty$ or $u = \infty$, denote $B(y, 1)$ the open ball with centered at y with the radius 1. $1_{B(y,1)}$ is the characteristic function of the ball $B(y, 1)$.

Very recently, the slice space $(E^u_t)(\mathbb{R}^n)$ was introduced by Auscher and Mourgoglou [19], an exceptional examples of classical amalgam spaces. Auscher and Prisuelos-Arribas [20] researched some classical operators of harmonic analysis for generalized version slice space $(E^u_v)_t(\mathbb{R}^n)$ in 2017, and proved that amalgam spaces and slice spaces are equivalent. In recent years, many authors studied slice-type spaces, such as, in 2019, Zhang, Yang, Yuan, and Wang [21] introduced Orlicz-slice spaces and Orlicz-slice Hardy spaces. In 2022, Lu, Zhou and Wang [22] introduced Herz-slice spaces. In 2023, we defined mixed-norm Herz-slice space $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$ in [23], and the theory of the Hardy–Littlewood maximal operator is given in this space. Given $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the Hardy-Littlewood maximal function $M(f)$ is defined by

$$M(f)(x) := \sup_B \frac{1}{|B|} \int_B |f(y)|dy, \tag{1}$$

where the supremum is taken over all balls $B \in \mathbb{B}$ containing x.

In 1961, the mixed-norm Lebesgue space $L^{\vec{u}}(\mathbb{R}^n)$ was introduced by Benedek and Panzone [24], where $\vec{u} = (u_1, \ldots, u_n) \in (0,\infty]^n$, which can be traced back to [25]. Because mixed-norm Lebesgue spaces has a more refined structure on partial differential equations [26,27], more scholarly authors like to study problems on it. For example, the mixed-norm amalgam space $(L^{\vec{u}}, L^{\vec{v}})(\mathbb{R}^n)$ was introduced by Zhang and Zhou [28] in 2022, and the boundedness of the Calderón–Zygmund operator T and the commutators with $\mathrm{BMO}(\mathbb{R}^n)$ functions on the mixed-norm amalgam space $(L^{\vec{u}}, L^{\vec{s}})(\mathbb{R}^n)$ was established [29]. The properties of the $\mathrm{BMO}(\mathbb{R}^n)$ function b are referred to [30]. A function $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we define the mean of b over Q by $b_Q := \frac{1}{|Q|}\int_Q b(x)dx$. The bounded mean oscillation space, namely $\mathrm{BMO}(\mathbb{R}^n)$ space is defined by

$$\|b\|_{\mathrm{BMO}} := \|b\|_* := \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q|dx < \infty,$$

the supremum is taken over all cubes Q in $\mathbb{R}^n$. For a deeper discussion about mixed-norm spaces, we can see [26,31–33] and so on.

Our motivation is based on our introduction of the mixed-norm Herz-slice space $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$, to consider whether we can establish the boundedness of the Calderón–Zygmund operator T and commutators with $\mathrm{BMO}(\mathbb{R}^n)$ functions on the mixed-norm Herz-slice space $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$. In this paper, we get the boundedness of the Calderón–Zygmund operator T on mixed-norm Herz-slice spaces, and demonstrate the necessary and sufficient conditions for the boundedness of the commutator $[b, T]$ on mixed-norm Herz-slice spaces. Our results are all new when we return to the classical Herz-slice spaces, slice spaces and Lebesgue spaces. The commutator $[b, T]$ is defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

The remainder of this paper is organized as follows. In Section 2, we introduce some necessary space definitions and operator notation, and we also give crucial lemma. In Section 3, we obtain the separable, weak convergence of mixed-norm Herz-slice spaces. To get the boundedness of the Calderón–Zygmund operator T on mixed-norm Herz-slice spaces, we explain whether $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$ has an absolutely continuous quasi-norm and the class $C_c^\infty(\mathbb{R}^n)$ is dense in $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$. $C_c^\infty(\mathbb{R}^n)$ is the space of infinitely differentiable functions with compact support in $\mathbb{R}^n$. Furthermore, we get the boundedness of the Calderón–Zygmund operator T of commutators with $\mathrm{BMO}(\mathbb{R}^n)$ functions on mixed-norm Herz-slice spaces by M_b is bounded on $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. The commutators of the Hardy-Littlewood maximal function with $\mathrm{BMO}(\mathbb{R}^n)$ functions M_b are defined, respectively, by

$$M_b f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |b(x) - b(y)| \cdot |f(y)|dy, \tag{2}$$

where $B(x,r) = \{y \in \mathbb{R}^n : |y - x| < r\}$ the ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with the radius $r \in (0,\infty)$. Finally, we establish the necessary and sufficient conditions for the boundedness of the commutator $[b, T]$ on mixed-norm Herz-slice spaces.

By $\mathcal{K}(\mathbb{R}^n)$ we denote the class of Lebesgue measurable functions on $\mathbb{R}^n$. Let $\mathbb{N} := \{1, 2, \ldots\}$. Denote the characteristic function of a set G by 1_G. Denote the Lebesgue measure of a measurable set by $|G|$. Given $B_k = B\left(0, 2^k\right)$, let $S_k := B_k \setminus B_{k-1}$ with $k \in \mathbb{N}$, $1_k = 1_{S_k}$ when $k \in \mathbb{N}$, and $1_{S_0} = 1_{B_0}$, 1_S is the characteristic function of S. Let $\mathcal{S}(\mathbb{R}^n)$ denote the collection of all Schwartz functions on $\mathbb{R}^n$, equipped with the well-known classical topology determined by a countable family of semi-norms. We denote $C^\infty(\mathbb{R}^n)$ by the space of infinitely differentiable complex-valued functions. Denotes the unit sphere in $\mathbb{R}^n$ by $\left(\mathbb{S}^{n-1}\right) := \mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$. We denote $C_c(\mathbb{R}^n)$ by the space of all continuos, complex-valued functions with compact support. The letters $\vec{u}, \vec{v}, \ldots$ will denote n-tuples of the numbers in $[0, \infty]$, $\vec{u} = (u_1, \ldots, u_m)$, $\vec{v} = (v_1, \ldots, v_m)$, $m \in \mathbb{N}$. $0 < \vec{u} < \infty$ indicates that $0 < u_i < \infty$ for each $i = 1, \cdots, m$. Moreover, for $\vec{u} = (u_1, \ldots, u_m)$ and $\vartheta \in \mathbb{R}$, let $\vec{u'} = \left(u'_1, \ldots, u'_m\right)$ be its conjugate index, that is, and $\vec{u'}$ satisfies $1/\vec{u} + 1/\vec{u'} = 1$. Let

$$\frac{1}{\vec{u}} = \left(\frac{1}{u_1}, \ldots, \frac{1}{u_m}\right), \quad \frac{\vec{u}}{\vartheta} = \left(\frac{u_1}{\vartheta}, \ldots, \frac{u_m}{\vartheta}\right)$$

The letter $D > 0$ is used for various constants, it is independent of the main parameters and maybe change from line to line. We denote a positive constant depending on the indicated parameters $A, B, \cdots$ by $D_{A,B}$. We write $\phi \lesssim \psi$, $\phi \leq D\psi$ mean that for some constant $D > 0$, especially, $\phi \asymp \psi$ indicates that $\phi \lesssim \psi$ and $\psi \lesssim \phi$.

2. Definition and Preliminary Lemmas

In this section, to more clearly show the progress from the problem to the solution, we first do some preliminary preparation. Now, we recall some basic notation.

Definition 1 ([28]). *Let $0 < t < \infty$ and $\vec{v}, \vec{u} \in (1, \infty)^n$. The mixed amalgam spaces $(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)$ is defined as the set of all measurable functions f satisfy $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n) := \left\{ f : \|f\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} = \left\| \frac{\|f 1_{B(\cdot,t)}\|_{L^{\vec{v}}(\mathbb{R}^n)}}{\|1_{B(\cdot,t)}\|_{L^{\vec{v}}(\mathbb{R}^n)}} \right\|_{L^{\vec{u}}(\mathbb{R}^n)} < \infty \right\},$$

where the usual modification when $u_i = \infty, i = 1, \cdots, n$.

Definition 2 ([23]). *Let $0 < t < \infty$, $0 < s \leq \infty$, $\beta \in \mathbb{R}$ and $\vec{v}, \vec{u} \in (1, \infty)^n$.*

(1) The homogeneous mixed-norm Herz-slice space $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$ is defined by

$$(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)} < \infty \right\},$$

and

$$\|f\|_{(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)} := \left[\sum_{k=-\infty}^{\infty} 2^{k\beta s} \|f 1_{S_k}\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} \right]^{\frac{1}{s}}, \tag{3}$$

where the usual modification made for $s = \infty$.

(2) The non-homogeneous mixed-norm Herz-slice space $(KE^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$ is defined by

$$(KE^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{(KE^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)} < \infty \right\},$$

and

$$\|f\|_{(KE^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)} := \left[\sum_{k=0}^{\infty} 2^{k\beta s} \|f 1_{S_k}\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} \right]^{\frac{1}{s}}, \tag{4}$$

where the usual modification made for $s = \infty$.

Now we introduce ball Banach function spaces. Let $y \in \mathbb{R}^n$ and $R \in (0, \infty)$, for $B(y, R) := \{z \in \mathbb{R}^n : |y - z| < R\}$ and

$$\mathbb{B} := \{B(y, R) : y \in \mathbb{R}^n \text{ and } R \in (0, \infty)\}. \tag{5}$$

Definition 3 ([34])**.** *A quasi-Banach space $X \subset \mathcal{K}(\mathbb{R}^n)$ is called a ball quasi-Banach function space if it satisfies*

(1) $\|\varphi\|_X = 0$ *means that* $\varphi = 0$ *almost everywhere;*
(2) $|\phi| \le |\varphi|$ *almost everywhere means that* $\|\phi\|_X \le \|\varphi\|_X$*;*
(3) $0 \le \varphi_k \uparrow \varphi$ *almost everywhere means that* $\|\varphi_k\|_X \uparrow \|\varphi\|_X$*;*
(4) $B \in \mathbb{B}$ *means that* $\mathbf{1}_B \in X$ *with* $\mathbb{B}$ *is as in* (5)*.*
and, if the norm of X satisfies the triangle inequality, then X is called a ball Banach function space, namely
(5) *Given* $\varphi, \phi \in X$

$$\|\varphi + \phi\|_X \le \|\varphi\|_X + \|\phi\|_X, \tag{6}$$

moreover, let $B \in \mathbb{B}$. There exists a positive constant $D_{(B)}$, depending on B, such that,
(6) *Given* $\phi \in X$

$$\int_B |\phi(x)| dx \le D_{(B)} \|\phi\|_X. \tag{7}$$

Definition 4. *A quasi-Banach function space X is said to have an absolutely continuous quasi-norm if $\left\|1_{G_j}\right\|_X \downarrow 0$ as $j \to \infty$ whenever $\{G_j\}_{j=1}^{\infty}$ is a sequence of measurable sets in $\mathbb{R}^n$ satisfying that $G_j \supset G_{j+1}$ for any $j \in \mathbb{N}$ and $\cap_{j=1}^{\infty} G_j = \emptyset$.*

Definition 5 (Mollifiers, [35])**.** *Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a radial, decreasing, nonnegative function pertain to $C_c^\infty(\mathbb{R}^n)$ and having the properties:*

(1) $\psi(z) = 0$ *when* $|z| \ge 1$,
(2) $\int_{\mathbb{R}^n} \psi(z) dz = 1$.
Let $\eta > 0$. Suppose that the function $\psi_\eta(x) = \eta^{-n} \psi(z/\eta)$ is nonnegative, pertain to $C_c^\infty(\mathbb{R}^n)$, and satisfies
(1) $\psi_\eta(x) = 0$ *if* $|z| \ge \eta$ *and*
(2) $\int_{\mathbb{R}^n} \psi_\eta(x) dx = 1$*, then mollifier is defined the following convolution operator:*

$$\Psi_\varepsilon(f)(z) = \int_{\mathbb{R}^n} \psi_\eta(z - y) f(y) \mathrm{d}y.$$

In what follows, we give the notion of the following operators.

Definition 6 (([36]))**.** *Let $0 < \tau, D < \infty$. A function $K(x, y)$ is called the standard kernel if*

(1) $\forall\, x, y \in \mathbb{R}^n$, $x \ne y$,

$$|K(x, y)| \le \frac{D}{|x - y|^n}, \tag{8}$$

(2) *there exist positive constants* $0 < \tau \le 1$,

$$|K(x, y) - K(u, y)| \le \frac{D|x - u|^\tau}{(|x - y| + |u - y|)^{n+\tau}}, \tag{9}$$

for $|x - u| \le \frac{1}{2} \max(|x - y|, |u - y|)$.

$$|K(x, y) - K(x, v)| \le \frac{D|y - v|^\tau}{(|x - y| + |x - v|)^{n+\tau}}, \tag{10}$$

for $|y - v| \le \frac{1}{2} \max(|x - y|, |x - v|)$.

Definition 7 ([36]). *Let $0<\tau, D<\infty$ and K satisfying (8), (9), and (10). A Calderón-Zygmund operator associated with K is a linear operator T defined on $\mathcal{S}(\mathbb{R}^n)$ that admits a bounded extension on $L^2(\mathbb{R}^n)$,*

$$\|T(f)\|_{L^2} \le B\|f\|_{L^2},$$
$$T(f)(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy$$

with $f \in C_c^\infty(\mathbb{R}^n)$ and $x \notin \operatorname{supp}(f)$.

Let us show some crucial lemmas as follow, first, we give Hölder's inequality on $(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)$.

Lemma 1 ([28]). *Let $0<t<\infty$ and $\vec{v},\vec{u} \in (1,\infty)^n$. Suppose that $\phi \in (E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)$ and $\psi \in (E^{\vec{u}'}_{\vec{v}'})_t(\mathbb{R}^n)$, we have $\phi\psi$ is integrable and*

$$\|\phi\psi\|_{L^1(\mathbb{R}^n)} \le \|\phi\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\|\psi\|_{(E^{\vec{u}'}_{\vec{v}'})_t(\mathbb{R}^n)},$$

with $1/\vec{v}+1/\vec{v}' = 1/\vec{u}+1/\vec{u}' = 1$.

Lemma 2 ([23]). *Let $0 \le t \le \infty$ and $\vec{v},\vec{u} \in (1,\infty)^n$. Then the characteristic function on $B(y_0,\lambda)$ satisfies*

$$\left\|1_{B(y_0,\lambda)}\right\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} \lesssim \lambda^{\sum_{i=1}^n \frac{1}{u_i}}. \tag{11}$$

where $y_0 \in \mathbb{R}^n$, $1<\lambda<\infty$.

Lemma 3 ([37], Proposition 2.7). *Let ψ_η be a mollifier and $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We have*

$$\sup_{\varepsilon>0}\left|\psi_\eta * g(x)\right| \le Mg(x).$$

3. Main Result

In this part, we first establish the separability of Herz-slice spaces with a mixed-norm, get a weak convergence on mixed-norm Herz-slice spaces, then, to show the Calderón–Zygmund operator T is bounded on mixed-norm Herz-slice spaces, we need to indicate that Tf is well-defined on $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$. Furthermore, we get the necessary and sufficient conditions for the boundedness of the commutator $[b,T]$ on mixed-norm Herz-slice spaces.

Now, we prove that $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$ is separable space.

Theorem 1. *Let $0<t,s<\infty$, $\beta \in \mathbb{R}$ and $\vec{v},\vec{u} \in (1,\infty)^n$. Then $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$ is separable space.*

Proof. Let $\phi \in (\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$ and $\varpi<\infty$, by using Corollary 2, there exist $\psi \in C_c(\mathbb{R}^n)$ such that

$$\|\phi-\psi\|_{(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)} < \varpi,$$

which implies that ψ is uniformly continuous. Thus, there exists dyadic cube $\{Q_i\}_{i=1}^M$ sequence and rational number $\{a_i\}_{i=1}^M$ sequence such that

$$\left\|\psi - \sum_{i=1}^M a_i 1_{Q_i}\right\|_{(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)} < \varpi.$$

Denote Θ is a set of simple functions $\hbar$ and

$$\hbar(x) = \sum_{i=1}^{M} a_i \chi_{Q_i}$$

with $\{Q_i\}_{i=1}^{M}$ is a dyadic cube's sequence and $\{a_i\}_{i=1}^{M}$ is a rational numbers's sequence. It suffices to show that $\hbar$ is countable and dense in $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$. Thus, $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$ is separable. The proof is complete. □

Theorem 2. *Let $0 < t, s < \infty$, $\beta \in \mathbb{R}$ and $\vec{v}$, $\vec{u} \in (1,\infty)^n$. Assume that there exists a positive constant D such that*

$$\|\phi_l\|_{(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)} < D,$$

then there exists a subsequence $\{\phi_{l_\tau}\}_{\tau=1}^{\infty}$ that is weakly convergent in the space $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$.

Proof. For $1 < s < \infty$, applying ([23], Theorem 3.1), we can see that

$$\Big((\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)\Big)^* = (\dot{K}E^{-\beta,s\prime}_{\vec{u}\prime,\vec{v}\prime})_t(\mathbb{R}^n).$$

Therefore, we only need to explain that there exists a subset $\{\phi_{l_\tau}\}_{\tau=1}^{\infty}$ such that for any $\psi \in (\dot{K}E^{-\beta,s\prime}_{\vec{u}\prime,\vec{v}\prime})_t(\mathbb{R}^n)$,

$$\lim_{\tau\to\infty} \int_{\mathbb{R}^n} \phi_{l_\tau}(x)\psi(x)dx = \int_{\mathbb{R}^n} \phi(x)\psi(x)dx,$$

where $\phi \in (\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$.

By using Theorem 1, suppose that $\{\psi_l\}_{l=1}^{\infty}$ is dense in $(\dot{K}E^{-\beta,s\prime}_{\vec{u}\prime,\vec{v}\prime})_t(\mathbb{R}^n)$. Denote

$$F_l(\psi) := (\phi_l, \psi) := \int_{\mathbb{R}^n} \phi_l(x)\psi(x)dx.$$

Applying the Hölder inequality,

$$|F_l(\psi_l)| \le D\|\psi_l\|_{(\dot{K}E^{-\beta,s\prime}_{\vec{u}\prime,\vec{v}\prime})_t(\mathbb{R}^n)}$$

Then there exist convergent subsequences $\{F_{l,1}(\psi_1)\}_{l=1}^{\infty}$ via the boundedness of $\{F_l(\psi_1)\}_{l=1}^{\infty}$. Repeat this step, we get a subsequence $\{F_{l,2}(\psi_2)\}_{l=1}^{\infty}$ of $\{F_{l,1}(\psi_2)\}_{l=1}^{\infty}$ satisfying $\{F_{l,2}(\psi_2)\}_{l=1}^{\infty}$ is convergence. Thus, for $\psi_{l_0} (l_0 \le i)$ there exists a convergent subsequences $\{F_{l,i}(\psi_{l_0})\}$. After a diagonal process, for any ψ_ι, we know that a subsequence $\{F_{i,i}(\psi_\iota)\}_{i=1}^{\infty}$ is convergence and

$$F_{i,i}(\psi) = \int_{\mathbb{R}^n} \phi_{l,i}(x)\psi(x)dx := \int_{\mathbb{R}^n} \phi_{l_i}(x)\psi(x)dx.$$

For any $\psi \in (\dot{K}E^{-\beta,s\prime}_{\vec{u}\prime,\vec{v}\prime})_t(\mathbb{R}^n)$ and $\vartheta > 0$, there exists ψ_ι have

$$\|\psi - \psi_\iota\|_{(\dot{K}E^{-\beta,s\prime}_{\vec{u}\prime,\vec{v}\prime})_t(\mathbb{R}^n)} \le \vartheta/2D.$$

Thus

$$\begin{aligned} \big|F_{k,k}(\psi) - F_{k',k'}(\psi)\big| &\le \int_{\mathbb{R}^n} \Big|\phi_{l_k}(x) - \phi_{l_{k'}}(x)\|\psi_\iota(x)\Big|dx \\ &+ \int_{\mathbb{R}^n} \mid f_{l_k}(x) - \phi_{l_{k'}}(x)\|(\psi(x) - \psi_\iota(x) \mid dx \\ &\le \int_{\mathbb{R}^n} \Big|\phi_{l_k}(x) - \phi_{l_{k'}}(x)\|\psi_\iota(x)\Big|dx + \vartheta. \end{aligned}$$

If k and k' are large enough, then

$$\left|F_{k,k}(\psi) - F_{k',k'}(\psi)\right| \leq 2\vartheta,$$

therefore, for any $\psi \in (\dot{K}E^{-\beta,s\prime}_{\vec{u}\prime,\vec{v}\prime})_t(\mathbb{R}^n)$, $\{F_{i,i}(\psi)\}_{i=1}^{\infty}$ is a Cauchy sequence. Let $F(\psi) = \lim_{i\to\infty} F_{i,i}(\psi)$ and $F(\psi)$ be a linear bounded functional on $(\dot{K}E^{-\beta,s\prime}_{\vec{u}\prime,\vec{v}\prime})_t(\mathbb{R}^n)$. By using ([23] Theorem 3.1), we see that there exist $\phi \in (\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \phi(x)\psi(x)dx = (\phi,\psi) = F(\psi) = \lim_{i\to\infty} F_{i,i}(\psi) = \lim_{i\to\infty} \int_{\mathbb{R}^n} \phi_{l_i}(x)\psi(x)dx,$$

the proof is completed. □

In following that, we investigate the boundedness of the Calderón–Zygmund operator T on $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$.

Theorem 3. *Let $0 < t, s < \infty$, $\beta \in \mathbb{R}$, $\vec{v}$, $\vec{u} \in (1,\infty)^n$ and $-\sum_{i=1}^n 1/u_i < \beta < n - 1/\sum_{i=1}^n 1/u_i$. If the Caldeón-Zygmund operator T satisfies*

(1) Given function f, $\operatorname{supp}(f) \subset S_k$ and $|x| \geq 2^{k+1}$ with $k \in \mathbb{R}$,

$$|Tf(x)| \leq C\|f\|_{L^1(\mathbb{R}^n)}|x|^{-n}; \tag{12}$$

(2) Given function f, $\operatorname{supp}(f) \subset S_k$ and $|x| \leq 2^{k-2}$ with $k \in \mathbb{R}$,

$$|Tf(x)| \leq C2^{-kn}\|f\|_{L^1(\mathbb{R}^n)}. \tag{13}$$

We have T is bounded on $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$.

Before we come to the proof of the above theorem, we should to explain that Tf is well-defined on $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$.

Lemma 4. *Let $0 < t, s < \infty$, $\beta \in \mathbb{R}$ and $\vec{v}$, $\vec{u} \in (1,\infty)^n$. Then $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$ has an absolutely continuous quasi-norm.*

Proof. Let $\{G_j\}_{j=1}^{\infty}$ be a sequence of measurable sets. For any $j \in \mathbb{N}$, $G_j \supset G_{j+1}$ and $\cap_{j=1}^{\infty} G_j = \emptyset$. We see that,

$$\left\|1_{G_j}1_{B(x,t)}\right\|_{L^v(\mathbb{R}^n)} = \left[\int_{B(x,t)} 1_{G_j}(y)dy\right]^{1/v},$$

when $j \to \infty$, we know that $\left\|1_{G_j}1_{B(x,t)}\right\|_{L^v(\mathbb{R}^n)} \to 0$, which, together with Definition 1, we have

$$\lim_{j\to\infty}\left\|1_{G_j}\right\|_{(G^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} = \lim_{j\to\infty}\left\|\frac{\left\|1_{G_j}1_{B(x,t)}\right\|_{L^{\vec{v}}(\mathbb{R}^n)}}{\left\|1_{B(x,t)}\right\|_{L^{\vec{v}}(\mathbb{R}^n)}}\right\|_{L^{\vec{u}}} = 0.$$

By Definition 4, we know that $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$ has an absolutely continuous quasi-norm. This accomplishes the desired result. □

Lemma 5. *Let X be a ball quasi-Banach function space having an absolutely continuous quasi-norm and M is bounded on X. If $g \in X$, then $\psi_\eta * g \to g$ in X as $\eta \to 0^+$.*

Proof. Let $g \in X$. Then the assertion is trivial. Combining Lemma 3, we have

$$\left\|\psi_\eta * g\right\|_X \leq \|Mg\|_X \leq C\|g\|_X,$$

where $\psi_\eta * g \in X$ for all $\eta > 0$. We know that the class $C_c(\mathbb{R}^n)$ is dense in X, see [38], where $C_c(\mathbb{R}^n)$ is continuous functions with compact support. Then there is a function $h \in C_c(\mathbb{R}^n)$ such that

$$\|g - h\|_X < \varepsilon. \tag{14}$$

based on these, when $h \in C_c(\mathbb{R}^n)$, we have $\psi_\eta * h \in C_c^\infty(\mathbb{R}^n)$ for all $\eta > 0$. Namely, $\psi_\eta * h \to h$ uniformly on compact sets as $\eta \to 0^+$. Thus, we get

$$\left|\psi_\eta * h(x) - h(x)\right| \longrightarrow 0,$$

with $\operatorname{supp}\left(\psi_\eta * h\right) \cup \operatorname{supp} h \subset K, K \subset \mathbb{R}^n$ compact. By using Definition 2.1, we get

$$\left\|\psi_\eta * h - h\right\|_X < \eta. \tag{15}$$

Finally by using (14) and (15),

$$\left\|g - \psi_\eta * g\right\|_X \leq \|g - h\|_X + \left\|h - \psi_\eta * h\right\|_X + \left\|\psi_\eta * h - \psi_\eta * g\right\|_X < C\eta,$$

the proof is complete. □

As a consequence of Lemma 5, we deduce the following result.

Corollary 1. *Let X be a ball quasi-Banach function space having an absolutely continuous quasi-norm. Suppose that M is bounded on X. We have the class $C_c^\infty(\mathbb{R}^n)$ is dense in X.*

We know that $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$ is ball quasi-Banach function space (see [23], Proposition 3.2). We immediately get the following result, which means that Tf is well-defined on $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$.

Corollary 2. *Let $0 < t, s < \infty$, $\beta \in \mathbb{R}$ and $\vec{v}$, $\vec{u} \in (1,\infty)^n$. Then the class $C_c^\infty(\mathbb{R}^n)$ is dense in $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$.*

Proof of Theorem 3. Let

$$f(x) = \sum_{m\in\mathbb{Z}} f(x)1_{S_m}(x) := \sum_{m\in\mathbb{Z}} f_m(x).$$

By Corollary 2, we know Tf is well defined on $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$, Observe that

$$\begin{aligned}
\|Tf\|_{(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)} &= \left(\sum_{k\in\mathbb{Z}} 2^{k\beta s}\|Tf1_{S_k}\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\right)^{\frac{1}{s}} \\
&= \left(\sum_{k\in\mathbb{Z}} 2^{k\beta s}\left\|\sum_{m=-\infty}^{\infty} Tf_m 1_{S_k}\right\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\right)^{\frac{1}{s}} \\
&\leq \left(\sum_{k\in\mathbb{Z}} 2^{k\beta s}\left\|\sum_{m=-\infty}^{k-2} Tf_m 1_{S_k}\right\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\right)^{\frac{1}{s}} \\
&\quad + \left(\sum_{k\in\mathbb{Z}} 2^{k\beta s}\left\|\sum_{m=k-1}^{k+1} Tf_m 1_{S_k}\right\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\right)^{\frac{1}{s}}
\end{aligned}$$

$$
+\left(\sum_{k\in\mathbb{Z}}2^{k\beta s}\left\|\sum_{m=k+2}^{\infty}Tf_m1_{S_k}\right\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\right)^{\frac{1}{s}}
$$
$$
:=\mathrm{I}+\mathrm{II}+\mathrm{III}.
$$

First estimate I, using Lemma 1 and (13), we have

$$
\begin{aligned}
\left\|\sum_{m=-\infty}^{k-2}Tf_m1_{S_k}\right\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} &\lesssim \left\|\sum_{m=-\infty}^{k-2}\|f_m\|_{L^1(\mathbb{R}^n)}2^{-kn}1_{S_k}\right\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\\
&\lesssim \left\|\sum_{m=-\infty}^{k-2}\|f_m\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\|1_{S_m}\|_{(E^{\vec{u}'}_{\vec{v}'})_t(\mathbb{R}^n)}2^{-kn}1_{S_k}\right\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}.
\end{aligned}
$$

For $s\in(0,1]$, we use Lemma 2, we have

$$
\begin{aligned}
\mathrm{I}&\lesssim\left(\sum_{k\in\mathbb{Z}}2^{k\beta s}\sum_{m=-\infty}^{k-2}\|f_m\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}2^{(m-k)s(n-\sum_{i=1}^n\frac{1}{u_i})}\right)^{\frac{1}{s}}\\
&\lesssim\left(\sum_{m\in\mathbb{Z}}2^{m\beta s}\|f_m\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\right)^{\frac{1}{s}}\lesssim\|f\|_{(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)},
\end{aligned}
$$

First estimate $s\in(1,\infty)$, we use Lemma 1 and (13),we know that

$$
\begin{aligned}
\left\|\sum_{m=-\infty}^{k-2}Tf_m1_{S_k}\right\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} &\lesssim \left\|\sum_{m=-\infty}^{k-2}\|f_m\|_{L^1(\mathbb{R}^n)}2^{-kn}1_{S_k}\right\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\\
&\lesssim\left(\sum_{m=-\infty}^{k-2}\|f_m\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\right)^s\|1_{S_m}\|^s_{(E^{\vec{u}'}_{\vec{v}'})_t(\mathbb{R}^n)}2^{-kns}\|1_{S_k}\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}.
\end{aligned}
$$

We can estimate I by Lemma 2, we have

$$
\begin{aligned}
\mathrm{I}&\lesssim\left(\sum_{k\in\mathbb{Z}}2^{k\beta s}\left(\sum_{m=-\infty}^{k-2}\|f_m\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\right)^s2^{(k-m)s(\sum_{i=1}^n\frac{1}{u_i}-n)}\right)^{\frac{1}{s}}\\
&\lesssim\left(\sum_{k\in\mathbb{Z}}2^{k\beta s}\left(\sum_{m=-\infty}^{k-2}\|f_m\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}2^{(k-m)(\sum_{i=1}^n\frac{1}{u_i}-n)s/2}\right)\right)^{\frac{1}{s}}\\
&\times\left(\left(\sum_{m=-\infty}^{k-2}2^{(k-m)(\sum_{i=1}^n\frac{1}{u_i}-n)s'/2}\right)^{s/s'}\right)^{\frac{1}{s}}\\
&\lesssim\|f\|_{(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)}.
\end{aligned}
$$

We need to pay attention to T is bounded on $(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)$ as in Definition 1 (see [29], Corollary 4.1), to estimate II,

$$\begin{aligned}
\mathrm{II} &\lesssim \left(\sum_{k\in\mathbb{Z}} 2^{k\beta s}\left\|\sum_{m=k-1}^{k+1} Tf_m\right\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\right)^{\frac{1}{s}} \\
&\lesssim \left(\sum_{k\in\mathbb{Z}}\sum_{m=k-1}^{k+1} 2^{(k-m)\beta s}2^{m\beta s}\|f_m\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\right)^{\frac{1}{s}} \\
&\lesssim \left(\sum_{m\in\mathbb{Z}} 2^{m\beta s}\|f_m\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\right)^{\frac{1}{s}} \lesssim \|f\|_{(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)}.
\end{aligned}$$

Finally, we to estimate III, we use Lemma 1 and (5.1), we conclude that

$$\begin{aligned}
&\left\|\sum_{m=k+2}^{\infty} Tf_m 1_{S_k}\right\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} \\
\lesssim &\left\|\sum_{m=k+2}^{\infty} \|f_m\|_{L^1(\mathbb{R}^n)}2^{-mn}1_{S_k}\right\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} \\
\lesssim &\left\|\sum_{m=k+2}^{\infty} 2^{-mn}1_{S_k}\|f_m\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\|1_{S_m}\|_{(E^{\vec{u}'}_{\vec{v}'})_t(\mathbb{R}^n)}\right\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}.
\end{aligned}$$

For $0 < s \le 1$, using Lemma 2, we have

$$\begin{aligned}
\mathrm{III} &\lesssim \left[\sum_{k\in\mathbb{Z}} 2^{k\beta s}\sum_{m=k+2}^{\infty}\left(2^{-mn}\|f_m\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\|1_{B_m}\|_{(E^{\vec{u}'}_{\vec{v}'})_t(\mathbb{R}^n)}\|1_{B_k}\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\right)^s\right]^{\frac{1}{s}} \\
&\lesssim \left(\sum_{k\in\mathbb{Z}} 2^{k\beta s}\sum_{m=k+2}^{\infty}\|f_m\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}2^{ms(n-\sum_{i=1}^n\frac{1}{u_i})}2^{-msn}2^{ks\sum_{i=1}^n\frac{1}{u_i}}\right)^{\frac{1}{s}} \\
&\lesssim \left(\sum_{k\in\mathbb{Z}} 2^{k\beta s}\sum_{m=k+2}^{\infty}\left(2^{m\beta-k\beta}\|f_m\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\right)^s\right)^{\frac{1}{s}} \\
&\lesssim \|f\|_{(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)}.
\end{aligned}$$

Using Lemma 1 and (12), we know that

$$\begin{aligned}
\left\|\sum_{m=k+2}^{\infty} Tf_m 1_{S_k}\right\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} &\lesssim \left\|\sum_{m=k+2}^{\infty}\|f_m\|_{L^1(\mathbb{R}^n)}2^{-mn}1_{S_k}\right\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} \\
&\lesssim \left\|\sum_{m=k+2}^{\infty}\|f_m\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\|1_{S_m}\|_{(E^{\vec{u}'}_{\vec{u}'})_t(\mathbb{R}^n)}2^{-mn}1_{S_k}\right\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}.
\end{aligned}$$

For $1 < s < \infty$, by Lemma 2 yields

$$\begin{aligned}
\mathrm{III} &\lesssim \left[\sum_{k\in\mathbb{Z}} 2^{k\beta s}\left(2^{-mn}\sum_{m=k+2}^{\infty}\|f_m\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\|1_{B_m}\|_{(E^{\vec{u}'}_{\vec{v}'})_t(\mathbb{R}^n)}\|1_{B_k}\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\right)^s\right]^{\frac{1}{s}} \\
&\lesssim \left(\sum_{k\in\mathbb{Z}} 2^{m\beta s}\|f_m\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\right)^{\frac{1}{s}} \lesssim \|f\|_{(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)},
\end{aligned}$$

we got what we want. □

The following theorem aims is to give the necessary conditions for the boundedness of the commutator $[b, T]$ on mixed-norm Herz-slice spaces.

Theorem 4. *Let $0 < t < \infty$ and $\vec{v}, \vec{u} \in (1, \infty)^n$. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. T be an Calderón–Zygmund operator satisfies the local size condition*

$$|Tf(x)| \le C|x|^{-n} \int_{\mathbb{R}^n} |f(y)|dy \tag{16}$$

where $f \in L^1(\mathbb{R}^n)$, $\operatorname{supp} f \subseteq A_k$, and $|x| \ge 2^{k+1}$ with $k \in \mathbb{Z}$, and the condition

$$|Tf(x)| \le C2^{-kn}\|f\|_{L^1(\mathbb{R}^n)} \tag{17}$$

where $f \in L^1(\mathbb{R}^n)$, supp $f \subseteq A_k$ and $|x| \le 2^{k-2}$ with $k \in \mathbb{Z}$. We have $[b, T]$ is also bounded on $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$, provided that $-\sum_{i=1}^n 1/u_i < \beta < n - \sum_{i=1}^n 1/u_i$ and $0 < s < \infty$.

Before we come to the proof of the above Theorem, we need give the boundedness of M_b on $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$.

Lemma 6. *Let $0 < t, s < \infty$, $\vec{v}, \vec{u} \in (1, \infty)^n$ and $-\sum_{i=1}^n 1/u_i < \beta < n - \sum_{i=1}^n 1/u_i$. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then M_b is bounded on $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$.*

Proof. By [23], Lemma 5.1, we find that $M_b f$ is well defined on $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$. Set $f = \sum_{m=-\infty}^{\infty} f1_m \equiv \sum_{m=-\infty}^{\infty} f_m$. We write

$$\begin{aligned}
\|M_b f\|_{(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)} &= \left\{ \sum_{k=-\infty}^{\infty} 2^{k\beta s} \|(M_b f)1_k\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} \right\}^{1/s} \\
&\le C\left\{ \sum_{k=-\infty}^{\infty} 2^{k\beta s} \left(\sum_{m=-\infty}^{k-3} \|(M_b f_m)1_k\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} \right)^s \right\}^{1/s} \\
&\quad + C\left\{ \sum_{k=-\infty}^{\infty} 2^{k\beta s} \left(\sum_{m=k-2}^{k+2} \|(M_b f_m)1_k\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} \right)^s \right\}^{1/s} \\
&\quad + C\left\{ \sum_{k=-\infty}^{\infty} 2^{k\beta s} \left(\sum_{m=k+3}^{\infty} \|(M_b f_m)1_k\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} \right)^s \right\}^{1/s} \\
&:= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

For the part of II, by using [29], Theorem 2.4, we get

$$\mathrm{II} \le C\left\{ \sum_{k=-\infty}^{\infty} 2^{k\beta s} \left(\sum_{m=k-2}^{k+2} \|f_m\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} \right)^s \right\}^{1/s} \le C\|f\|_{(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)}.$$

For I, denote b_m is the mean value of b on $B(0, 2^m)$. Observe that if $m \le k-3$, from the properties of BMO($\mathbb{R}^n$) functions (see Stein [30]), by Lemma 1 and Remark 1, we have

$$\begin{aligned}
&\|(M_b f_m)1_k\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} \\
&\le C2^{-kn}\left\| \left(\|[(b(x)-b(\cdot))\cdot(f_m(\cdot))]1_{A_m}\|_{L_1(\mathbb{R}^n)} \right) 1_{A_k} \right\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} \\
&\le C2^{-kn}\|f_m\|_{L^1(\mathbb{R}^n)}\|(b(\cdot)-b_m)1_{A_k}\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} \\
&\quad + C2^{k(\sum_{m=1}^n \frac{1}{u_i}-n)}\|f_m\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}\|(b_m-b(\cdot))1_{A_m}\|_{(E^{\vec{u}'}_{\vec{v}'})_t(\mathbb{R}^n)} \\
&\le C\|b\|_* 2^{(m-k)(n-\sum_{m=1}^n \frac{1}{u_i})}(k-m)\|f_m\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, for $s \in (0,1]$,

$$\begin{aligned}
\mathrm{I} &\le C\|b\|_* \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{m=-\infty}^{k-3} 2^{m\beta}\|f_m\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} 2^{(k-m)(\beta-n+\sum_{i=1}^n \frac{1}{u_m})}(k-m) \right)^s \right\}^{1/s} \\
&\le C\|b\|_* \left(\sum_{m=-\infty}^{\infty} 2^{m\beta s}\|f_m\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} \right)^{1/s} \\
&\quad \times \left(\sum_{k=m+3}^{\infty} 2^{(k-m)(\beta-(n-\sum_{i=1}^n \frac{1}{u_i}))s}(k-m)^s \right)^{1/s} \\
&\le C\|b\|_* \left\{ \sum_{m=-\infty}^{\infty} 2^{m\beta s}\|f_m\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} \right\}^{1/s} \\
&= C\|b\|_*\|f\|_{(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)}.
\end{aligned}$$

For $s \in (1, \infty)$, using Hölder's inequality can deduce that

$$\begin{aligned}
\mathrm{I} &\le C\|b\|_* \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{m=-\infty}^{k-3} 2^{m\beta}\|f_m\|_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} 2^{(k-m)(\beta-(n-\sum_{i=1}^n \frac{1}{u_i}))}(k-m) \right)^s \right\}^{1/s} \\
&\le C\|b\|_* \left[\sum_{k=-\infty}^{\infty} \left(\sum_{m=-\infty}^{k-3} 2^{m\beta s}\|f_m\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} 2^{(k-m)(\beta-(n-\sum_{i=1}^n \frac{1}{u_i}))s/2} \right) \right]^{1/s} \\
&\quad \times \left[\left(\sum_{m=-\infty}^{k-3} (k-m)^{s'} \right)^{s/s'} 2^{(k-m)(\beta-(n-\sum_{i=1}^n \frac{1}{u_i}))\frac{s}{2}} \right]^{1/s} \\
&\le C\|b\|_* \left\{ \sum_{m=-\infty}^{\infty} 2^{m\beta s}\|f_m\|^s_{(E^{\vec{u}}_{\vec{v}})_t(\mathbb{R}^n)} \right\}^{1/s} \\
&= C\|b\|_*\|f\|_{(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)}.
\end{aligned}$$

Since $\beta < n - \sum_{i=1}^n 1/u_i$. For part of III, when $m \ge k+3$ and $x \in A_k$, we have

$$\begin{aligned}
M_b f_m(x) &\le C2^{-mn}\int_{A_m} |b(x)-b(y)|\cdot|f_m(y)|dy \\
&\le C2^{-mn}|b(x)-b_k|\cdot\|f_m\|_{L^1(\mathbb{R}^n)} + C2^{-mn}\int_{A_m}|b_k-b(y)|\cdot|f(y)|dy.
\end{aligned}$$

Thus, by Lemmas 1 and 2, we can see that

$$\begin{aligned}\|(M_b f_m)1_k\|_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)} &\leq C\|b\|_* 2^{(k-m)\sum_{i=1}^n \frac{1}{u_i}} \|f_m\|_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)} \\ &\quad + C\|b\|_* 2^{(k-m)\sum_{i=1}^n \frac{1}{u_i}} (m-k)\|f_m\|_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)} \\ &\leq C\|b\|_* 2^{(k-m)\sum_{i=1}^n \frac{1}{u_i}} (m-k)\|f_m\|_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)}.\end{aligned}$$

Hence, when $s \in (0,1]$,

$$\begin{aligned}\mathrm{III} &\leq C\|b\|_* \left[\sum_{k=-\infty}^{\infty}\left(\sum_{m=k+3}^{\infty} 2^{m\beta}\|f_m\|_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)}(m-k)2^{-(m-k)(\beta+\sum_{i=1}^n \frac{1}{u_i})}\right)^s\right]^{1/s} \\ &\leq C\|b\|_* \left[\sum_{m=-\infty}^{\infty} 2^{m\beta s}\|f_m\|^s_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)}\left(\sum_{k=-\infty}^{m-3}(m-k)^s 2^{(k-m)(\beta+\sum_{i=1}^n \frac{1}{u_i})s}\right)\right]^{1/s} \\ &= C\|b\|_*\|f\|_{(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)}.\end{aligned}$$

For $s \in (1,\infty)$, using the Hölder inequality, we have

$$\begin{aligned}\mathrm{III} &\leq C\|b\|_* \left[\sum_{k=-\infty}^{\infty}\left(\sum_{m=k+3}^{\infty} 2^{m\beta}\|f_m\|_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)}(m-k)2^{-(m-k)(\beta+\sum_{i=1}^n \frac{1}{u_i})}\right)^s\right]^{1/s} \\ &\leq C\|b\|_* \left[\sum_{k=-\infty}^{\infty}\left(\sum_{m=k+3}^{\infty} 2^{m\beta s}\|f_m\|^s_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)}\cdot 2^{-(m-k)(\beta+\sum_{i=1}^n \frac{1}{u_i})\frac{s}{2}}\right)\right]^{1/s} \\ &\quad\times\left[\left(\sum_{k=-\infty}^{\infty}\sum_{m=k+3}^{\infty}(m-k)^{s'}2^{-(m-k)(\beta+\sum_{i=1}^n \frac{1}{u_i})s'/2}\right)^{s/s'}\right]^{1/s} \\ &= C\|b\|_*\|f\|_{(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)}.\end{aligned}$$

Because $\beta > -\sum_{i=1}^n 1/u_i$. We got what we want. □

In what follows, we show commutators $[b,T]$ is also bounded on $(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)$.

Proof of Theorem 4. Write $f = \sum_{m=-\infty}^{\infty} f1_m := \sum_{m=-\infty}^{\infty} f_m$. We then have

$$\begin{aligned}\|[b,T]f\|_{(\dot{K}E^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)} &= \left\{\sum_{k=-\infty}^{\infty} 2^{k\beta s}\|[b,T]f1_k\|^s_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)}\right\}^{1/s} \\ &\leq C\left\{\sum_{k=-\infty}^{\infty} 2^{k\beta s}\left(\left\|[b,T]\left(\sum_{m=-\infty}^{k-3} f_m\right)1_k\right\|_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)}\right)^s\right\}^{1/s} \\ &\quad + C\left\{\sum_{k=-\infty}^{\infty} 2^{k\beta s}\left(\left\|[b,T]\left(\sum_{m=k-2}^{k+2} f_m\right)1_k\right\|_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)}\right)^s\right\}^{1/s} \\ &\quad + C\left\{\sum_{k=-\infty}^{\infty} 2^{k\beta s}\left(\left\|[b,T]\left(\sum_{m=k+3}^{\infty} f_m\right)1_k\right\|_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)}\right)^s\right\}^{1/s} \\ &:= \mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

For II, since $[b, T]$ is bounded on $(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)$ in [29], we have

$$\begin{aligned} \mathrm{II} &\leq C\left\{\sum_{k=-\infty}^{\infty} 2^{k\beta s}\left(\sum_{m=k-2}^{k+2}\|f_m\|_{E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n}\right)^s\right\}^{1/s} \\ &\leq C\|f\|_{(\dot{K}E_{\vec{u},\vec{v}}^{\beta,s})_t(\mathbb{R}^n)}. \end{aligned}$$

For I, when $x \in A_k$ and $j \leq k-3$, using (16), we get

$$\left|[b, T]\left(\sum_{m=-\infty}^{k-3} f_m\right)(x)\right| \leq C|x|^{-n}\int |b(x)-b(y)| \cdot \left|\sum_{m=-\infty}^{k-3} f_m(y)\right| dy \leq CM_b f(x).$$

Therefore, by Lemma 6,

$$\mathrm{I} \leq C\|M_b f\|_{(\dot{K}E_{\vec{u},\vec{v}}^{\beta,s})_t(\mathbb{R}^n)} \leq C\|f\|_{(\dot{K}E_{\vec{u},\vec{v}}^{\beta,s})_t(\mathbb{R}^n)}.$$

For part of III, when $x \in A_k, y \in A_m$ and $m \geq k+3$, then $2|x| < |y|$. Therefore,

$$\left|[b, T]\left(\sum_{m=k+3}^{\infty} f_m\right)(x)\right| \leq C\int_{2|x|<|y|} |b(x)-b(y)| \cdot |y|^{-n} \cdot |f(y)| dy := CT_b^1(|f|)(x).$$

Set $1 < s < \infty$ and

$$T_b^0 g(x) = |x|^{-n}\int_{|y|<2|x|} |b(x)-b(y)| g(y) dy.$$

Thus, by Corollary 3.1,

$$\begin{aligned} \mathrm{III} &\leq C\left\|T_b^1(|f|)\right\|_{(\dot{K}E_{\vec{u},\vec{v}}^{\beta,s})_t(\mathbb{R}^n)} \\ &= C \sup_{\|g\|_{(\dot{K}E_{\vec{u}',\vec{v}'}^{-\beta,s'})_t(\mathbb{R}^n)}\leq 1} \left|\left(T_b^1(|f|), g\right)\right| \\ &= C \sup_{\|g\|_{(\dot{K}E_{\vec{u}',\vec{v}'}^{-\beta,s'})_t(\mathbb{R}^n)}\leq 1} \left|\left(|f|, T_b^0 g\right)\right| \\ &\leq C\|f\|_{(\dot{K}E_{\vec{u},\vec{v}}^{\beta,s})_t(\mathbb{R}^n)} \sup_{\|g\|_{(\dot{K}E_{\vec{u}',\vec{v}'}^{-\beta,s'})_t(\mathbb{R}^n)}.\leq 1} \|M_b g\|_{(\dot{K}E_{\vec{u}',\vec{v}'}^{-\beta,s'})_t(\mathbb{R}^n)}. \end{aligned}$$

By Lemma 6, we have

$$\|M_b g\|_{(\dot{K}E_{\vec{u}',\vec{v}'}^{\beta,s'})_t(\mathbb{R}^n)} \leq C\|g\|_{(\dot{K}E_{\vec{u}',\vec{v}'}^{-\beta,s'})_t(\mathbb{R}^n)}.$$

Thus,

$$\mathrm{III} \leq C\|f\|_{(\dot{K}E_{\vec{u},\vec{v}}^{\beta,s})_t(\mathbb{R}^n)}.$$

Now let $0 < s \leq 1$. By Lemma 1 and (17), we first deduced that, when $m \geq k+3$,

$$\begin{aligned}
&\|[b,T]f_m 1_k\|_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)} \\
&\leq C\||b(\cdot)-b_k|\cdot|Tf_m(\cdot)|1_{A_k}\|_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)} \\
&\leq C\|b\|_* 2^{(k-m)\sum_{i=1}^n \frac{1}{u_i}}\|f_m\|_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)} \\
&\quad + C2^{(k-m)\sum_{i=1}^n \frac{1}{u_i}}\|f_m\|_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)} \cdot \frac{\left\|(b(\cdot)-b_k)1_{B(0,2^m)}\right\|_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)}}{\left\|1_{B(0,2^m)}\right\|_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)}} \\
&\leq C\|b\|_* 2^{(k-m)\sum_{i=1}^n \frac{1}{u_i}}(m-k)\|f_m\|_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)}.
\end{aligned}$$

By this and (1.1), we have

$$\begin{aligned}
\mathrm{III} &\leq C\|b\|_*\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{m=k+3}^{\infty} 2^{m\beta}\|f_m\|_{(E_{\vec{v}}^{\vec{u}})_t(\mathbb{R}^n)}(m-k)2^{-(m-k)(\beta+\sum_{i=1}^n \frac{1}{u_i})}\right)^s\right\}^{1/s} \\
&\leq C\|b\|_*\left\{\sum_{k=-\infty}^{\infty} 2^{k\beta s}\|f_k\|^s_{(E_{\vec{r}}^{\vec{u}})_t(\mathbb{R}^n)}\right\}^{1/s} \\
&= C\|b\|_*\|f\|_{(\dot{K}E_{\vec{u},\vec{v}}^{\beta,s})_t(\mathbb{R}^n)}.
\end{aligned}$$

because $\beta > -\sum_{i=1}^n 1/u_i$. This accomplishes the desired result. □

Finally, we prove the other side, namely, the sufficient conditions for the boundedness of the commutator $[b,T]$ on mixed-norm Herz-slice spaces.

Theorem 5. *Let $-\sum_{i=1}^n 1/u_i < \beta < n - \sum_{i=1}^n 1/u_i$, $0 < t, s < \infty$ and $\vec{v}, \vec{u} \in (1,\infty)^n$. Let T be an Calderón–Zygmund operator with $K(x)$ satisfying (9), (10), and $K(x) \in C^\infty(\mathbb{S}^{n-1})$. Let b be a locally integrable function. If $[b,T]$ is bounded on $(\dot{K}E_{\vec{u},\vec{v}}^{\beta,s})_t(\mathbb{R}^n)$, then $b \in \mathrm{BMO}(\mathbb{R}^n)$.*

Using a similar way of [23], Proposition 3.2, we immediately have the following lemma.

Lemma 7. *Let $0 < t, s < \infty$ and $\vec{v}, \vec{u} \in (1,\infty)^n$. If $-\sum_{i=1}^n 1/u_i < \beta < n - \sum_{i=1}^n 1/u_i$, then the characteristic function on $B(y_0, R_0)$ satisfies*

$$\left\|1_{B(y_0,R_0)}\right\|_{(\dot{K}E_{\vec{u},\vec{v}}^{\beta,s})_t(\mathbb{R}^n)}\left\|1_{B(y_0,R_0)}\right\|_{(\dot{K}E_{\vec{u}',\vec{v}'}^{-\beta,s'})_t(\mathbb{R}^n)} \lesssim R_0^n, \tag{18}$$

where $y_0 \in \mathbb{R}^n$, $1 < R_0 < \infty$.

Proof of Theorem 5. Suppose that boundedness of commutator $[b,T]$ on $(\dot{K}E_{\vec{u},\vec{v}}^{\beta,s})_t(\mathbb{R}^n)$. We use the same method as Janson [39]. Let us take $0 \neq z_0 \in \mathbb{R}^n$ and $\rho > 0$ such that $0 \neq B(z_0, \sqrt{n}\rho)$. Then for $x \in B(z_0, \sqrt{n}\rho)$, $K(x) \in C^\infty(B(z_0, \sqrt{n}\rho))$ such that $(K(x))^{-1}$ can be written as the absolutely convergent Fourier series,

$$(K(x))^{-1} = \sum_{m\in\mathbb{Z}^n} a_m e^{i<v_m,x>}.$$

with $\sum|a_m| < \infty$, where the exact form of the vectors v_m is unrelated, then we have the expansion

$$(K(x))^{-1} = \frac{\rho^{-n}}{K(\rho x)} = \rho^{-n}\sum a_m e^{i\langle v_m,\rho x\rangle} \text{ for } \left|x - z_0\rho^{-1}\right| < \sqrt{n}.$$

Given cubes $Q = Q(x_0, r)$ and $Q' = Q(x_0 - rz_0/\rho, r)$, if $x \in Q$ and $y \in Q'$, then

$$\left|\frac{x-y}{r} - \frac{z_0}{\rho}\right| \le \left|\frac{x-x_0}{r}\right| + \left|\frac{y - \left(x_0 - \frac{rz_0}{\rho}\right)}{r}\right| < \sqrt{n}.$$

Let $T(x) = \overline{\operatorname{sgn} B}$ and $B = \left(\int_{Q'}(b(x) - b(y))dy\right)$. Then

$$\begin{aligned}
\int_Q |b(x) - b_{Q'}|dx &= \frac{1}{|Q'|}\int_Q |B|dx \\
&= \frac{1}{|Q|}\int_Q \int_{Q'} T(x)(b(x) - b(y))dydx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(b(x) - b(y))K(x-y)\sum a_m e^{i\langle v_m, \frac{x-y}{r}\rangle} s(x) 1_Q(x) 1_{Q'}(y)dydx.
\end{aligned}$$

Setting $g_m(y) = e^{-i\langle v_m, \frac{y}{r}\rangle} 1_{Q'}(y)$ and $h_m(x) = e^{i\langle v_m, \frac{x}{r}\rangle} s(x) 1_Q(x)$, we have

$$\begin{aligned}
\int_Q |b(x) - b_{Q'}|dx &= \sum a_m \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(b(x) - b(y))K(x-y)g_m(y)h_m(x)dydx \\
&= \sum a_m \int_{\mathbb{R}^n}[b, T](g_m)(x)h_m(x)dx.
\end{aligned}$$

Applying Hölder's inequality and Lemma 7, we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |b(x) - b_Q|dx \le \frac{2}{|Q|}\int_Q |b(x) - b_{Q'}|dx \\
&\lesssim \frac{2}{|Q|}\sum_{m\in\mathbb{Z}^n}|a_m| \|[b,T]f_m\|_{(KE^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)} \|g_m\|_{(KE^{-\beta,s'}_{\vec{u}',\vec{v}'})_t(\mathbb{R}^n)} \\
&\lesssim \frac{2}{|Q|}\sum_{m\in\mathbb{Z}^n}|a_m| \|f_m\|_{(KE^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n)} \|g_m\|_{(KE^{-\beta,s'}_{\vec{u}',\vec{v}'})_t(\mathbb{R}^n)} \\
&\quad \times \|[b,T]\|_{(KE^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n) \to (KE^{-\beta,s'}_{\vec{u}',\vec{v}'})_t(\mathbb{R}^n)} \\
&\lesssim \|[b,T]\|_{(KE^{\beta,s}_{\vec{u},\vec{v}})_t(\mathbb{R}^n) \to (KE^{-\beta,s'}_{\vec{u}',\vec{v}'})_t(\mathbb{R}^n)'}
\end{aligned}$$

then $b \in \mathrm{BMO}(\mathbb{R}^n)$. This completes the conclusion. □

4. Conclusions

We obtain the separable of on mixed-norm Herz-slice spaces, establish a weak convergence of on mixed-norm Herz-slice spaces, and get the boundedness of the Calderón–Zygmund operator T on mixed-norm Herz-slice spaces. Moreover, we get the necessary and sufficient conditions for the boundedness of the commutator $[b, T]$ on mixed-norm Herz-slice spaces.

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