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On Approximation by an Absolutely Convergent Integral Related to the Mellin Transform

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Abstract: In this paper, we consider the modified Mellin transform of the product of the square of the Riemann zeta function and the exponentially decreasing function, and we discuss its probabilistic and approximation properties. It turns out that this Mellin transform approximates the identical zero in the strip $\{s \in \mathbb{C} : 1/2 < \sigma < 1\}$.

Keywords: Mellin transform; Riemann zeta function; space of analytic functions; weak convergence

MSC: 11M06

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Citation: Laurinčikas, A. On Approximation by an Absolutely Convergent Integral Related to the Mellin Transform. *Axioms* **2023**, *12*, 789. https://doi.org/10.3390/axioms12080789

Academic Editors: Fabio Caldarola and Gianfranco d'Atri

Received: 29 June 2023
Revised: 27 July 2023
Accepted: 8 August 2023
Published: 14 August 2023

1. Introduction

Throughout this paper, we denote by $s = \sigma + it$ the main complex variable. The main object of our investigations is the Riemann zeta function $\zeta(s)$, which, for $\sigma > 1$, is given by Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}.$$

The function $\zeta(s)$ is analytically continuable to the complex plane $\mathbb{C}$, with the exception of one point $s = 1$, which is a simple pole, and $\mathrm{Res}_{s=1}\zeta(s) = 1$. The function $\zeta(s)$ plays an important role in mathematics. The Riemann hypothesis that all non-trivial zeros of $\zeta(s)$ (the zeros of $\zeta(s)$ lying in the strip $0 < \sigma < 1$) are on the line $\sigma = 1/2$ has not been proven or disproven to date, and it occupies an honorable place among the seven Millennium Problems of mathematics (see [1]). The function $\zeta(s)$ is mentioned in cosmology, quantum mechanics, finance mathematics, even in music, etc. [2–6]. In this sense, $\zeta(s)$ is close to the great Greek philosopher and mathematician Pythagoras, who saw mathematics in all fields of life. However, there are problems and conjectures related to the function $\zeta(s)$. Without the mentioned Riemann hypothesis, we know the Lindellöf hypothesis that, for every $\varepsilon > 0$,

$$\zeta\left(\frac{1}{2} + it\right) \ll_\varepsilon t^\varepsilon, \quad t \geqslant t_0.$$

Here and in what follows, the notation $f(s) \ll_\theta g(s)$, $f(s) \in \mathbb{C}$, $g(s) > 0$ for $s \in \mathscr{X}$ indicates that there exists a constant $c = c(\theta)$ such that $|f(s)| \leqslant cg(s)$ for $s \in \mathscr{X}$. One more difficult problem of the value distribution of $\zeta(s)$ is the moment problem, which consists of finding the asymptotics, or precise estimates, of the quantities

$$M_k(\sigma, T) \stackrel{\mathrm{def}}{=} \int_0^T |\zeta(\sigma + it)|^{2k} \, \mathrm{d}t, \quad \sigma \geqslant \frac{1}{2},\ k \geqslant 0,\ T \to \infty.$$

Notice that individual values of $\zeta(s)$ can sometimes be successfully replaced by mean values, i.e., by moments $M_k(\sigma, T)$. For example, this idea works in the approximation of analytic functions by shifts $\zeta(s + i\tau)$, a property of $\zeta(s)$ called universality [7]. For an investigation of the moments $M_k(1/2, T)$, Y. Motohashi proposed [8] the use of the modified Mellin transforms

$$\mathcal{Z}_k(s) = \int_1^\infty \left|\zeta\left(\frac{1}{2} + ix\right)\right|^{2k} x^{-s} \, \mathrm{d}x.$$

They differ from the classical Mellin transforms only by integration over $(0, 1)$. However, they are more convenient because they allow for the avoidance of convergence problems at $x = 0$. Using the inversion formula, we have [9]

$$\left|\zeta\left(\frac{1}{2} + ix\right)\right|^{2k} = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathcal{Z}_k(s) x^{s-1} \, \mathrm{d}x$$

with a certain σ. Thus, the study of $M_k(1/2, T)$ reduces to that of the mean values of a simpler than $\zeta(1/2 + ix)$ function $\mathcal{Z}_k(s)$.

We recall one example of using the function $\mathcal{Z}_2(s)$. Denote

$$E_2(T) = \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^4 \mathrm{d}t - TP_4(\log T),$$

where it is assumed that $P_4(x)$ is a polynomial of degree 4. Using $\mathcal{Z}_2(s)$, the following strong results on $E_2(T)$ were obtained. We recall that $f(x) = \Omega_\pm(g(x))$ means that $\limsup_{x\to\infty} f(x)/g(x)$ is positive, and $\liminf_{x\to\infty} f(x)/g(x)$ is negative. First, in [8], it was proved that $E_2(T) = \Omega_\pm(T^{1/2})$. A. Ivič applied $\mathcal{Z}_2(s)$ to obtain the following bounds [10,11]. There exist positive constants $c_1 > 1$ and c_2 such that, for sufficiently large T, any interval $[T, c_1 T]$ contains points T_1, T_2, T_3 and T_4 such that

$$E_2(T_1) > c_2 T_1^{1/2}, \qquad E_2(T_2) < -c_2 T_2^{1/2},$$

and

$$\int_0^{T_3} E_2(t) \, \mathrm{d}t > c_2 T_3^{3/2}, \qquad \int_0^{T_4} E_2(t) \, \mathrm{d}t < -c_2 T_4^{3/2}.$$

More results of applications of the modified Mellin transform in the theory of the Riemann zeta function can be found in [12–16].

In Reference [17], we were interested in the approximation of analytic functions by shifts $\mathcal{Z}(s + i\tau) \overset{\text{def}}{=} \mathcal{Z}_1(s + i\tau)$, $\tau \in \mathbb{R}$. Denote by D the strip $1/2 < \sigma < 1$ on the complex plane, and let $H(D)$ stand for the space of analytic functions on D with the topology of uniform convergence on compact sets. Let $\mu(A)$ be the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then, the main result of Reference [17] is the following statement:

Theorem 1 ([17])**.** *There exists a closed non-empty set $F \subset H(D)$ such that, for any compact set $K \subset D$, $f(s) \in F$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mu\left\{\tau \in [0, T] : \sup_{s \in K} |\mathcal{Z}(s + i\tau) - f(s)| < \varepsilon\right\} > 0.$$

Moreover, the lower limit in the above inequality can be replaced by the limit, except for the at most countable set of values of $\varepsilon > 0$.

Theorem 1 was inspired by the Voronin universality theorem [7] for the function $\zeta(s)$ on the approximation of analytic functions from $H(D)$ by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$.

The aim of this paper is the approximation of analytic functions from a certain class by shifts of an absolutely convergent integral related to the Mellin transform $\mathcal{Z}(s)$. Suppose that $\theta > 1/2$ is a fixed number. For $x, y \geqslant 1$, let $v_y(x) = \exp\{-(x/y)^\theta\}$, where $\exp\{a\} = \mathrm{e}^a$. We consider the approximation of analytic functions by shifts of the modified Mellin transform

$$\widehat{\mathcal{Z}}_y(s) = \int_1^\infty \left|\zeta\left(\frac{1}{2} + ix\right)\right|^2 v_y(x) x^{-s}\,\mathrm{d}x.$$

Since $\zeta(1/2 + ix) \ll x^{1/6}$ for large $x > 0$, and the function $v_y(x)$ decreases exponentially with respect to x, the integral for $\widehat{\mathcal{Z}}_y(s)$ is absolutely convergent in the half-plane $\sigma > \sigma_0$ with any finite σ_0 and defines there an analytic function. Our aim is to replace the function $\mathcal{Z}(s)$ in Theorem 1 by $\widehat{\mathcal{Z}}_y(s)$ with a certain y. One motivation for this is the extension of a class of approximating functions. Moreover, since $\widehat{\mathcal{Z}}_y(s)$ is given by an absolutely convergent integral, its use is simpler than that of $\mathcal{Z}(s)$.

In this paper, we prove the following theorem:

Theorem 2. *Suppose that $y_T \to \infty$ and $y_T \ll T^2$ as $T \to \infty$. Then, there exists a closed non-empty set $F_\theta \subset H(D)$ such that, for every compact set $K \subset D$ and $f(s) \in F_\theta$, the limit*

$$\lim_{T\to\infty} \frac{1}{T}\mu\left\{\tau \in [0, T] : \sup_{s\in K}\left|\widehat{\mathcal{Z}}_{y_T}(s + i\tau) - f(s)\right| < \varepsilon\right\}$$

exists and is positive, except for the at most countable set of values of $\varepsilon > 0$.

Theorem 2 shows that, for all but at most countably many $\varepsilon > 0$, and $f \in F_\theta$, there exists $T_0 = T_0(\varepsilon, f, K) > 0$ such that, for $T > T_0$,

$$\mu\left\{\tau \in [0, T] : \sup_{s\in K}\left|\widehat{\mathcal{Z}}_{y_T}(s + i\tau) - f(s)\right| < \varepsilon\right\} \geqslant cT$$

with a certain positive c. Hence, the set of shifts $\widehat{\mathcal{Z}}_{y_T}(s + i\tau)$ is infinite.

Theorem 2, as Theorem 1, is theoretical; however, in virtue of the definition of $\widehat{\mathcal{Z}}_{y_T}(s)$ by an absolutely convergent integral, Theorem 2 has a certain advantage over Theorem 1 involving the function $\mathcal{Z}(s)$ given by analytic continuation.

We derive Theorem 2 by using a certain probabilistic model in the space of analytic functions.

2. Estimate for a Metric

We start by recalling some results on the function $\mathcal{Z}(s)$ obtained in [16].

Let

$$E(T) = \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 \mathrm{d}t - T\log\frac{T}{2\pi} - (2\gamma_0 - 1)T,$$

where γ_0 is the Euler constant. Set

$$g(T) = \int_1^T E(t)\,\mathrm{d}t - \pi T, \qquad g_1(T) = \int_1^T g(t)\,\mathrm{d}t.$$

Lemma 1 ([16])**.** *The function $\mathcal{Z}(s)$ has analytic continuation to the half-plane $\sigma > -3/4$, except for a point $s = 1$, which is a double pole, and it has simple poles at the points $-(2m-1)$, $m \in \mathbb{N}$. Moreover, by setting $b_0 = 2\gamma_0 - \log 2\pi$, for $\sigma > -3/4$, we have*

$$\mathcal{Z}(s) = \frac{1}{(s-1)^2} + \frac{b_0}{s-1} + \pi(s-1) + s(s+1)(s+2)\int_1^\infty g_1(t)t^{-s-3}\,\mathrm{d}t - E(1).$$

Lemma 2 ([16])**.** *For $\sigma \in [1/2, 1]$ and any $\eta > 0$, the estimate*

$$\int_0^T |\mathcal{Z}(\sigma+it)|^2\,\mathrm{d}t \ll_\eta T^{2-2\sigma+\eta}$$

is valid.

Lemma 3. *Suppose that $y_T \to \infty$ and $y_T \ll T^2$ as $T \to \infty$. Then, for every compact set $K \subset D$,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \sup_{s\in K} |\mathcal{Z}(s+i\tau) - \widehat{\mathcal{Z}}_{y_T}(s+i\tau)|\,\mathrm{d}\tau = 0.$$

Proof. As usual, let $\Gamma(s)$ stand for the Euler gamma function, and set

$$a_{y_T}(s) = \theta^{-1}\Gamma(\theta^{-1}s)y_T^s,$$

where θ is from the definition of $v_y(x)$. Then, in [17], Lemma 7, the representation for $s \in D$,

$$\widehat{\mathcal{Z}}_{y_T}(s) = \frac{1}{2\pi i}\int_{\theta-i\infty}^{\theta+i\infty} \mathcal{Z}(s+z)a_{y_T}(z)\,\mathrm{d}z \tag{1}$$

is obtained.

Fix a compact set $K \subset D$. Since set K is closed, there exists a number $\delta > 0$ such that $1/2 + 2\delta \leqslant \sigma \leqslant 1 - \delta$ for $s = \sigma + it$ lying in K. For brevity, let $\theta_1 = 1/2 + \delta - \sigma$. Moreover, we take $\theta = 1/2 + \delta$. Then, in view of Lemma 1, the integrand in Equation (1) has a simple pole $z = 0$ of $\Gamma(\theta^{-1}z)$ and a double pole $z = 1 - s$ of $\mathcal{Z}(s+z)$ in the strip $\theta_1 \leqslant \operatorname{Re} z \leqslant \theta$. Therefore, Equation (1), together with the residue theorem, for $s = \sigma + it \in K$, gives

$$\widehat{\mathcal{Z}}_{y_T}(s) - \mathcal{Z}(s) = \frac{1}{2\pi i}\int_{\theta_1-i\infty}^{\theta_1+i\infty} \mathcal{Z}(s+z)a_{y_T}(z)\,\mathrm{d}z + r_{y_T}(s) \tag{2}$$

with

$$r_{y_T}(s) = \operatorname*{Res}_{z=1-s} \mathcal{Z}(s+z)a_{y_T}(z). \tag{3}$$

Hence, we have

$$\begin{aligned}
\widehat{\mathcal{Z}}_{y_T}(s+i\tau) - \mathcal{Z}(s+i\tau) =& \frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathcal{Z}\left(\frac{1}{2} + \delta + it + i\tau + iu\right) a_{y_T}\left(\frac{1}{2} + \delta - \sigma + iu\right)\mathrm{d}u \\
&+ r_{y_T}(s+i\tau) \\
\ll& \int_{-\infty}^{\infty} \left|\mathcal{Z}\left(\frac{1}{2} + \delta + i\tau + iu\right)\right| \sup_{s\in K}\left|a_{y_T}\left(\frac{1}{2} + \delta - s + iu\right)\right| \mathrm{d}u \\
&+ \sup_{s\in K}\left|r_{y_T}(s+i\tau)\right|.
\end{aligned}$$

Therefore, the mean value of the lemma is estimated as

$$\begin{aligned}
&\frac{1}{T}\int_0^T \sup_{s\in K}\left|\mathcal{Z}(s+i\tau)+\widehat{\mathcal{Z}}(s+i\tau)\right| \ll \\
&\qquad \int_{-\infty}^{\infty}\left(\frac{1}{T}\int_0^T\left|\mathcal{Z}\left(\frac{1}{2}+\delta+i\tau+iu\right)\right|\mathrm{d}\tau\right)\sup_{s\in K}\left|a_{y_T}\left(\frac{1}{2}+\delta-s+i\tau\right)\right|\mathrm{d}u \\
&\qquad + \frac{1}{T}\int_0^T \sup_{s\in K}\left|r_{y_T}(s+i\tau)\right|\mathrm{d}\tau \overset{\text{def}}{=} J_1+J_2. \qquad (4)
\end{aligned}$$

Using the classical estimate for the function $\Gamma(s)$,

$$\Gamma(\sigma+it) \ll \exp\{-c|t|\}, \quad c>0, \qquad (5)$$

we find

$$\begin{aligned}
a_{y_T}\left(\frac{1}{2}+\delta-s+iu\right) &\ll_\theta y_T^{1/2+\delta-\sigma}\left|\Gamma\left(\frac{1}{\theta}\left(\frac{1}{2}+\delta-\sigma-it+iu\right)\right)\right| \\
&\ll_\theta y_T^{-\varepsilon_1}\exp\left\{-\frac{c}{\theta}|t-u|\right\} \ll_{\theta,K} y_T^{-\delta}\exp\{-c_1|u|\}, \quad c_1>0. \qquad (6)
\end{aligned}$$

Moreover, using Lemma 2,

$$\begin{aligned}
\int_0^T\left|\mathcal{Z}\left(\frac{1}{2}+\delta+i\tau+iu\right)\right|\mathrm{d}\tau &\ll \left(\int_{T-|u|}^{T+|u|}\left|\mathcal{Z}\left(\frac{1}{2}+\delta+i\tau\right)\right|^2\mathrm{d}\tau\right)^{1/2} \\
&\ll_\eta \left(T(T+|u|)^{2-2(1/2+\delta)-\eta}\right)^{1/2} \\
&\ll_\eta T^{1-\delta+\eta/2}+|u|^{1/2-\delta+\eta/2} \ll_\eta T\left(1+|u|^{1/2}\right)
\end{aligned}$$

if we choose $0<\eta=2\delta$. Therefore, taking into account Equation (6), we obtain the estimate

$$J_1 \ll_{\theta,K,\varepsilon} y_T^{-\delta}\int_{-\infty}^{\infty}\left(1+|u|^{1/2}\right)\exp\{-c_1|u|\}\,\mathrm{d}u \ll_{\theta,K,\varepsilon} y_T^{-\delta}. \qquad (7)$$

For the estimation of J_2, we also apply the bound Equation (5). However, first, we have to estimate $r_{y_T}(s)$. Since the function $\mathcal{Z}(s)$, in view of Lemma 1, has a double pole at the point $s=1$, by using Equation (3), we have

$$r_{y_T}(s) = (a_{y_T}(z))'_{z=1-s}+b_0a_{y_T}(1-s). \qquad (8)$$

By the definition of $a_{y_T}(z)$,

$$a'_{y_T}(z) = \frac{y_T^z}{\theta}\Gamma'\left(\frac{z}{\theta}\right)\frac{1}{\theta}+\frac{y_T^z}{\theta}\log y_T\Gamma\left(\frac{z}{\theta}\right) = \frac{y_T^z}{\theta}\Gamma\left(\frac{z}{\theta}\right)\left(\frac{\Gamma'(z/\theta)}{\theta\Gamma(z/\theta)}+\log y_T\right).$$

Hence, Equation (8) gives

$$r_{y_T}(s) = \frac{y_T^s}{\theta}\Gamma\left(\frac{1}{\theta}s\right)\left(\frac{1}{\theta}\left(\log\Gamma\left(\frac{1}{\theta}s\right)\right)\right)'+\log y_T+b_0\frac{y_T^s}{\theta}\Gamma\left(\frac{1}{\theta}s\right).$$

Using Equation (5), from this, we obtain

$$r_{y_T}(s+i\tau) \ll_\theta y_T^{1-\sigma} \exp\left\{-\frac{c}{\theta}|t+\theta|\left(\log\left|\frac{t+\tau}{\theta}\right| + \log y_T + 1\right)\right\}$$
$$\ll_{\theta,K,\delta} y_T^{1/2-2\delta} \exp\left\{-\frac{c}{\theta}|\tau|(\log 2|\tau| + 2) + \log y_T\right\}$$
$$\ll_{\theta,K,\delta} y_T^{1/2-2\delta} \exp\{-c_2|\tau|\}, \quad c_2 > 0.$$

Therefore, J_2 has the estimate

$$J_2 \ll_{\theta,K,\delta} y_T^{1/2-2\delta} \frac{1}{T} \int_0^T e^{-c_2\tau}\, d\tau \ll_{\theta,K,\delta} \frac{y_T^{1/2-2\delta}}{T}.$$

This and Equations (7) and (4) show that

$$\frac{1}{T}\int_0^T \sup_{s\in K}\left|\mathcal{Z}(s+i\tau) - \widehat{\mathcal{Z}}(s+i\tau)\right| d\tau \ll_{\theta,K,\delta} y_T^{-\delta} + \frac{y_T^{1/2-2\delta}}{T}.$$

Since $y_T \ll T^2$, from this, we obtain the assertion of the lemma. □

3. Limit Lemma

In this section, we show the weak convergence for

$$\widehat{P}_T(A) \stackrel{\text{def}}{=} \mu\left\{\tau \in [0,T] : \widehat{\mathcal{Z}}_{y_T}(s+i\tau) \in A\right\}, \quad A \in \mathscr{B}(H(D)),$$

as $T \to \infty$. First, we recall a limit theorem for

$$P_T(A) \stackrel{\text{def}}{=} \mu\{\tau \in [0,T] : \mathcal{Z}(s+i\tau) \in A\}, \quad A \in \mathscr{B}(H(D)),$$

from Reference [17], Theorem 2.

Lemma 4 ([17])**.** *On $(H(D), \mathscr{B}(H(D)))$, there is a probability measure P such that P_T weakly converges to P as $T \to \infty$.*

For $h_1, h_2 \in H(D)$, let

$$d(h_1,h_2) = \sum_{m=1}^{\infty} 2^{-m} \frac{\sup_{s\in K_m}|h_1(s) - h_2(s)|}{1 + \sup_{s\in K_m}|h_1(s) - h_2(s)|}.$$

Here, $\{K_m : m \in \mathbb{N}\} \subset D$ is a sequence of compact sets, $K_m \subset K_{m+1}$, and $K \subset K_m$ for a compact set $K \subset D$ with some $m \in \mathbb{N}$. Then, d is a metric on $H(D)$, which induces its topology.

Lemma 5. *Suppose that P is the same as in Lemma 4, and $y_T \to \infty$ and $y_T \ll T^2$ as $T \to \infty$. Then, $\widehat{P}_T$ also weakly converges to P as $T \to \infty$.*

Proof. Suppose that ξ_T is a random variable on a certain probability space $(\Omega, \mathscr{B}(\Omega), Q)$ and that it is uniformly distributed in $[0,T]$. Consider two $H(D)$-valued random elements

$$Y_T = Y_T(s) = \mathcal{Z}(s+i\xi_T)$$

and

$$\widehat{Y}_T = \widehat{Y}_T(s) = \widehat{\mathcal{Z}}_{y_T}(s+i\xi_T).$$

Denote by $\xrightarrow{\mathcal{D}}$ the convergence in the distribution. Then, Lemma 4 implies the relation

$$Y_T \xrightarrow[T\to\infty]{\mathcal{D}} P. \tag{9}$$

Let F be an arbitrary closed set in the space $H(D)$. Fix $\delta > 0$, and define the set

$$F_\delta = \{h \in H(D) : d(h, F) \leqslant \delta\},$$

where $d(h, f) = \inf_{g\in F} d(h, g)$. Then, set F_δ is also closed. Using the equivalent of weak convergence in terms of closed sets, in virtue of Lemma 4, we have

$$\limsup_{T\to\infty} P_T(F) \leqslant P(F). \tag{10}$$

By using Lemma 3 and the definition of the metric d,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T d\Big(\mathcal{Z}(s + i\tau), \widehat{\mathcal{Z}}_{y_T}(s + i\tau)\Big)\, \mathrm{d}\tau = 0. \tag{11}$$

Moreover, the inclusion

$$\Big\{\widehat{Y}_T \in F_\delta\Big\} \subset \{Y_T \in F\} \cup \Big\{d(Y_T, \widehat{Y}_T) \geqslant \delta\Big\}$$

holds. Therefore,

$$Q\Big\{\widehat{Y}_T \in F_\delta\Big\} \leqslant Q\{Y_T \in F\} + Q\Big\{d(Y_T, \widehat{Y}_T) \geqslant \delta\Big\}. \tag{12}$$

By using the definition of the random variable ξ_T, and Equation (11),

$$\begin{aligned} Q\Big\{d(Y_T, \widehat{Y}_T) \geqslant \delta\Big\} &= \tfrac{1}{T}\mu\Big\{\tau \in [0, T] : d\Big(\mathcal{Z}(s + i\tau), \widehat{\mathcal{Z}}_{y_T}(s + i\tau)\Big) \geqslant \delta\Big\} \\ &\leqslant \tfrac{1}{\delta T} \int_0^T d\Big(\mathcal{Z}(s + i\tau), \widehat{\mathcal{Z}}_{y_T}(s + i\tau)\Big)\, \mathrm{d}\tau = o(1) \end{aligned} \tag{13}$$

as $T \to \infty$. Similarly, in virtue of Equation (10),

$$\limsup_{T\to\infty} Q\{Y_T \in F\} = \limsup_{T\to\infty} P_T(F) \leqslant P(F).$$

Thus, Inequality (12), together with Equation (13), yields

$$\limsup_{T\to\infty} \widehat{P}_T(F_\delta) = \limsup_{T\to\infty}\Big\{\widehat{Y}_T \in F_\delta\Big\} \leqslant P(F).$$

Now, by letting $\delta \to +0$, we have $F_\delta \to F$, and

$$\limsup_{T\to\infty} \widehat{P}_T(F) \leqslant P(F).$$

This and the equivalent of weak convergence in terms of closed sets show that $\widehat{P}_T$ weakly converges to P as $T \to \infty$. $\square$

4. Proof of Theorem 2

We derive Theorem 2 from Lemma 5.

Proof of Theorem 2. Suppose that F_θ is a support of the probability measure P in Lemma 5. Thus, we find that $F_\theta \neq \varnothing$ since $P(F_\theta) = 1$, and F_θ is a closed set. For $f(s) \in F_\theta$, set

$$\mathscr{G}_\varepsilon = \left\{ h \in H(D) : \sup_{s \in K} |h(s) - f(s)| < \varepsilon \right\}.$$

Then, $\mathscr{G}_\varepsilon$ is an open neighborhood of the element $f(s)$ of the support of the measure P. Therefore, by using the support property,

$$P(\mathscr{G}_\varepsilon) > 0. \tag{14}$$

Denoting by $\partial\mathscr{G}_\varepsilon$ the boundary of set $\mathscr{G}_\varepsilon$, we find that $\partial\mathscr{G}_\varepsilon$ lies in the set

$$\left\{ h \in H(D) : \sup_{s \in K} |h(s) - f(s)| = \varepsilon \right\}.$$

This remark implies that $\partial\mathscr{G}_{\varepsilon_1} \cap \mathscr{G}_{\varepsilon_2} \neq \varnothing$ for different positive ε_1 and ε_2. Therefore, set $\mathscr{G}_\varepsilon$ is a continuity set of the measure P for all $\varepsilon > 0$, except for at most countably many values. Thus, the equivalent of weak convergence with continuity sets, Lemma 5 and inequality (14) show that

$$\lim_{T \to \infty} \widehat{P}(\mathscr{G}_\varepsilon) = P(\mathscr{G}_\varepsilon) > 0$$

for all $\varepsilon > 0$, except for the at most countable set of values. By the definition of $\widehat{P}$, for the above values of $\varepsilon > 0$, the inequality

$$\lim_{T \to \infty} \frac{1}{T} \mu \left\{ \tau \in [0, T] : \sup_{s \in K} \left| \widehat{\mathscr{Z}}_{y_T}(s + i\tau) - f(s) \right| < \varepsilon \right\} > 0$$

is valid. □

There is an another proof of Theorem 2.

Proof of Theorem 2. Let mapping $u : H(D) \to \mathbb{R}$ be given by the formula

$$u(h) = \sup_{s \in K} |h(s) - f(s)|.$$

Then mapping u is continuous. Actually, let $h_n(s) \xrightarrow[n \to \infty]{} h(s)$ in the space $H(D)$; i.e., for every compact set $K \subset D$,

$$\lim_{n \to \infty} \sup_{s \in K} |h_n(s) - h(s)| = 0. \tag{15}$$

Using a triangle inequality, we have

$$\begin{aligned} |u(h_n) - u(h)| &= \left| \sup_{s \in K} |h_n(s) - f(s)| - \sup_{s \in K} |h(s) - f(s)| \right| \\ &\leqslant \sup_{s \in K} |h_n(s) - f(s) - h(s) + f(s)| = \sup_{s \in K} |h_n(s) - h(s)| \to 0 \end{aligned}$$

as $n \to \infty$ by Equation (15). This proves the continuity of u.

The continuity of u and the preservation of weak convergence under continuous mappings (see, for example, Theorem 5.1 in Reference [18]) show that $\widehat{P}_T u^{-1}$ converges weakly to Pu^{-1} in space $\mathbb{R}$. Here, the measures $P_T u^{-1}$ and Pu^{-1} are defined, for $A \in \mathscr{B}(\mathbb{R})$, by $\widehat{P}_T(u^{-1}A)$ and $P(u^{-1}A)$, respectively.

It is well known that the weak convergence of the probability measures in $\mathbb{R}$ is equivalent to that of distribution functions. Thus, we find that the distribution function

$$\mathfrak{G}_T(\varepsilon) = \frac{1}{T}\mu\left\{\tau \in [0,T] : \sup_{s\in K}\left|\widehat{\mathcal{Z}}_{y_T}(s+i\tau) - f(s)\right| < \varepsilon\right\}$$

converges weakly to the distribution function

$$\mathfrak{G}(\varepsilon) \overset{\text{def}}{=} P\left\{h \in H(D) : \sup_{s\in K}|h(s) - f(s)| < \varepsilon\right\} > 0$$

as $T \to \infty$. The weak convergence of distribution functions is understood as convergence at all continuity points of the limit function $\mathfrak{G}(\varepsilon)$. However, each distribution function has an at most countable set of discontinuity points. This shows that the equality

$$\lim_{T\to\infty} \mathfrak{G}_T(\varepsilon) = \mathfrak{G}(\varepsilon)$$

holds for all $\varepsilon > 0$, except for the at most countable set of values. $\square$

5. Identification of Set F_θ

It is well known (see, for example, References [19,20]) that, in the case of the universality theorem for the Riemann zeta function, the class of approximated functions by shifts $\zeta(s+i\tau)$ coincides with the set of all non-vanishing analytic functions on D complemented by the function $h_0(s) \equiv 0$, $s \in D$. More precise estimates show, unfortunately, that, in the case of Theorem 2, the approximated class reduces to the function $h_0(s)$ only.

Let $K \subset D$ be an arbitrary compact set. Then, there is $\delta > 0$ such that $1/2 + \delta \leqslant \sigma \leqslant 1 - \delta$ for $s = \sigma + it \in K$. We estimate the mean value

$$M_T(K) \overset{\text{def}}{=} \frac{1}{T}\int_0^T \sup_{s\in K}|\mathcal{Z}(s+i\tau)|\,\mathrm{d}\tau.$$

By using Lemma 1, the function $\mathcal{Z}(s)$ is analytic in strip D. Therefore, we can use the integral Cauchy formula, which yields

$$\mathcal{Z}(s+i\tau) = \frac{1}{2\pi i}\int_{\mathscr{L}} \frac{\mathcal{Z}(z+i\tau)}{z-s}\,\mathrm{d}z,$$

where $\mathscr{L}$ is a certain closed simple contour enclosing set K, completely lying in strip D and satisfying the inequality

$$\inf_{s\in K}\inf_{z\in\mathscr{L}}|z-s| \geqslant C(K,\mathscr{L}) > 0.$$

Hence, we find

$$\sup_{s\in K}|\mathcal{Z}(s+i\tau)| \leqslant \frac{1}{2\pi C}\int_{\mathscr{L}}|\mathcal{Z}(z+i\tau)|\,|\mathrm{d}z| \ll_{K,\mathscr{L}} \int_{\mathscr{L}}|\mathcal{Z}(z+i\tau)|\,|\mathrm{d}z|.$$

Therefore, in view of the Cauchy–Schwarz inequality and Lemma 2, we obtain

$$\begin{aligned}
M_T(K) &\ll_{K,\mathscr{L}} \int\limits_{\mathscr{L}} \left(\frac{1}{T} \int\limits_0^T |\mathcal{Z}(z+i\tau)|\,\mathrm{d}\tau \right) |\mathrm{d}z| \ll_{K,\mathscr{L}} \int\limits_{\mathscr{L}} \left(\frac{1}{T} \int\limits_0^T |\mathcal{Z}(z+i\tau)|^2\,\mathrm{d}\tau \right)^{1/2} |\mathrm{d}z| \\
&\ll_{K,\mathscr{L}} \int\limits_{\mathscr{L}} \left(\frac{1}{T} \int\limits_0^T |\mathcal{Z}(u+iv+i\tau)|^2\,\mathrm{d}\tau \right)^{1/2} |\mathrm{d}z| \\
&\ll_{K,\mathscr{L}} \int\limits_{\mathscr{L}} \left(\frac{1}{T} \int\limits_{-|v|}^{T+|v|} |\mathcal{Z}(u+i\tau)|^2\,\mathrm{d}\tau \right)^{1/2} |\mathrm{d}z| \\
&\ll_{K,\mathscr{L},\eta} \int\limits_{\mathscr{L}} \left(\frac{1}{T}(T+|v|)^{2-2u+\eta} \right)^{1/2} |\mathrm{d}z| \ll_{K,\mathscr{L},\delta} T^{1/2-u+\eta/2} \ll_{K,\mathscr{L},\eta} T^{-\delta/4},
\end{aligned}$$

after a choice

$$\inf_{z\in\mathscr{L}} \mathrm{Re} z \geqslant 1/2+\delta/2 \quad \text{and} \quad \eta = \delta/4.$$

The latter estimate, together with Chebyshev's type inequality, implies that, for every $\varepsilon > 0$,

$$\frac{1}{T}\mu\left\{ \tau \in [0,T] : \sup_{s\in K} |\mathcal{Z}(s+i\tau)| \geqslant \varepsilon \right\} \leqslant \frac{1}{\varepsilon T} \int\limits_0^T \sup_{s\in K} |\mathcal{Z}(s+i\tau)|\,\mathrm{d}\tau \ll_{K,\mathscr{L},\delta,\varepsilon} T^{-\delta/4}.$$

Thus,

$$\lim_{T\to\infty} \frac{1}{T}\mu\left\{ \tau \in [0,T] : \sup_{s\in K} |\mathcal{Z}(s+i\tau)| \geqslant \varepsilon \right\} = 0. \tag{16}$$

Let ξ_T, Y_T and Q be the same as in the proof of Lemma 5 and

$$X_T(s) = h_0(s+i\xi_T)$$

for all $T > 0$. Then, by using Equation (16),

$$\lim_{T\to\infty} Q\{d(Y_T, X_T) \geqslant \varepsilon\} = 0.$$

By using this relation and repeating the proof of Lemma 5, we find that P_T converges weakly to P, where

$$P(A) = \begin{cases} 1 & \text{if } h_0(s) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we find that $\mathcal{Z}(s)$ and $\widehat{\mathcal{Z}}_{y_T}(s)$ satisfy the law of large numbers in the space $H(D)$, and $F_\theta = \{h \in H(D) : h(s) \equiv 0\}$.

6. Conclusions

Let $\zeta(s)$ denote the Riemann zeta function, $y_T \to \infty$ and $y_T \ll T^2$ as $T \to \infty$, $v_{y_T}(x) = \exp\{-(x/y_T)^\theta\}$ with fixed $\theta > 1/2$, $V_{y_T}(x) = |\zeta(1/2+ix)|^2 v_{y_T}(x)$ and

$$\widehat{\mathcal{Z}}_{y_T}(s) = \int\limits_1^\infty V_{y_T}(x) x^{-s}\,\mathrm{d}x$$

the modified Mellin transform of $V_{y_T}(x)$. In view of the definition of $v_{y_T}(x)$, the latter integral is absolutely convergent in every half-plane. By using a probabilistic method, we find that there exists a set F_θ of analytic functions that is approximated by shifts $\widehat{\mathcal{Z}}_{y_T}(s+i\tau)$.

The set of such shifts has a positive density. Unfortunately, the used method implies that F_θ consists only of the identical zero. However, this does not eliminate the possibility of the approximation of other analytic functions. Actually, suppose that the set

$$A_T \stackrel{\text{def}}{=} \left\{ \tau \in [0,T] : \sup_{s\in K} \left| \widehat{\mathcal{Z}}(s+i\tau) \right| < \varepsilon \right\}$$

has a positive density and that $\sup_{s\in K} |f(s)| \leqslant \varepsilon$ with a certain compact set K. Define

$$B_T = \left\{ \tau \in [0,T] : \sup_{s\in K} \left| \widehat{\mathcal{Z}}(s+i\tau) - f(s) \right| < 2\varepsilon \right\}.$$

Since, for $\tau \in A_T$,

$$\sup_{s\in K} \left| \widehat{\mathcal{Z}}(s+i\tau) - f(s) \right| \leqslant \sup_{s\in K} \left| \widehat{\mathcal{Z}}(s+i\tau) \right| + \sup_{s\in K} |f(s)| < 2\varepsilon,$$

we have the inclusion $A_T \subset B_T$. Therefore,

$$\liminf_{T\to\infty} \frac{1}{T} \mu\{\tau \in [0,T] : \tau \in B_T\} \geqslant \liminf_{T\to\infty} \frac{1}{T} \mu\{\tau \in [0,T] : \tau \in A_T\} > 0.$$

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable

Conflicts of Interest: The authors declare no conflict of interest.

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