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Generalized Bayes Prediction Study Based on Joint Type-II Censoring

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Abstract: In this paper, the problem of predicting future failure times based on a jointly type-II censored sample from k exponential populations is considered. The Bayesian prediction intervals and point predictors were then obtained. Generalized Bayes is a Bayesian study based on a learning rate parameter. This study investigated the effects of the learning rate parameters on the prediction results. The loss functions of squared error, Linex, and general entropy were used as point predictors. Monte Carlo simulations were performed to show the effectiveness of the learning rate parameter in improving the results of prediction intervals and point predictors.

Keywords: learning rate parameter; exponential distribution; joint type-II censoring; point predictor; prediction intervals; squared-error loss; Linex loss; general entropy loss

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1. Introduction and Motivations

Generalized Bayes is a Bayesian study based on a learning rate parameter ($\eta > 0$) as a power of the likelihood function $L(\theta;\ data)$. The traditional Bayes framework is obtained for $\eta = 1$, and we demonstrate the effect of the learning rate parameter on the prediction results. That is, if the prior distribution of the parameter θ is $\pi(\theta)$ then the generalized Bayes posterior distribution for θ is

$$\pi^*(\theta \mid data) \propto L^{\eta}(\theta;\ data)\ \pi(\theta), \quad \theta \in \Theta, \quad \eta > 0. \tag{1}$$

For more details on the generalized Bayes method and the choice of the value of the rate parameter, we refer the reader to [1–11]. In a special way, the choice of the learning rate η was studied in [3–6] by a so-called safe Bayes algorithm based on the minimization of a sequential risk measure. Another learning rate selection method considers the two different information-matching strategies proposed in [7,8]. In addition, a generalized Bayes estimation based on a joint censored sample of type II from k exponential populations using different values of the learning rate parameter was studied in [11]. An exact inference method based on maximum likelihood estimates (MLEs) was developed in [12], and its performance was compared with that of approximate, Bayesian, and bootstrap methods. The joint progressive censoring type II and the expected number of failures for two populations under the joint progressive censoring type II were introduced and studied by [13]. In contrast, the exact likelihood inference for two exponential populations under joint progressive censoring of type II was studied in [14], and some precise results were obtained based on the maximum likelihood estimates developed by [15]. Exact likelihood inference for two populations of two-parameter exponential distributions under type II joint censoring was studied by [16].

One might be interested in predicting future failures using a joint type II censored sample. To accomplish this, prediction points or intervals should be determined. Bayesian prediction bounds for future observations based on certain distributions were discussed by several authors. A study of Bayesian estimation and prediction based on a joint censored sample of type II from two exponential populations was presented by [17]. Prediction (various classical and Bayesian point predictors) for future failures in the Weibull distribution under hybrid censoring was studied by [18]. A Bayesian prediction based on generalized order statistics with multiple censoring of type II was developed by [19].

The main objective of this study is to predict future failures based on a joint type-II censoring scheme for k-exponential populations when censoring is performed on k-samples in a combined manner. Suppose that products from k different lines are produced in the same factory, and k independent samples of size $n_h, 1 \leq h \leq k$ are selected from these k lines and simultaneously placed in a lifetime experiment. To reduce the cost and time of the experiment, the experimenter may decide to stop the lifetime test when a certain number (r) of failures occurs. The nature of the problem and the distributions used in our study are presented below.

Suppose $\{\mathbf{X}_h^{n_h}, h = 1, \ldots, k\}$ are k-samples, where $\mathbf{X}_h^{n_h} = \{X_{h1}, X_{h2}, \ldots, X_{hn_h}\}$ are the lifetimes of n_h samples of product line A_h and are assumed to be independent and identically distributed (iid) random variables from a population with a probability density function (pdf) $f_h(x)$ and a cumulative distribution function (cdf) $F_h(x)$.

Furthermore, let $N = \sum_{i=1}^{k} n_i$ be the total sample size, and let r be the total number of observed failures. Let $W_1 \leq \ldots \leq W_N$ denote the order statistics of N random variables $\{\mathbf{X}_h^{n_h}, h = 1, \ldots, k\}$. Under the joint type-II censoring scheme for the k-samples, the observable data then consist of $(\boldsymbol{\delta}, \mathbf{W})$, where $\mathbf{W} = (W_1, \ldots, W_r)$, $W_i \in \left\{\mathbf{X}_{h_i}^{n_{hi}}, h_i = 1, \ldots, k\right\}$, with $r < N$ being a pre-fixed integer and $\boldsymbol{\delta} = (\delta_1(h), \ldots, \delta_r(h))$ associated to $(h_1, \ldots, h_r)$ is defined by

$$\delta_i(h) = \begin{cases} 1, & \text{if} \quad h = h_i \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Letting $M_r(h) = \sum_{i=1}^{r} \delta_i(h)$ denote the number of X_h-failures in $\mathbf{W}$ and $r = \sum_{h=1}^{k} M_r(h)$, the joint density function of $(\boldsymbol{\delta}, \mathbf{w})$ is given by

$$f(\boldsymbol{\delta}, \mathbf{w}) = \prod_{h=1}^{k} c_r (\overline{F}_h(w_r))^{n_h - M_r(h)} . \prod_{i=1}^{r} \prod_{h=1}^{k} (f_h(w_i))^{\delta_i(h)} \tag{3}$$

where $\overline{F}_h = 1 - F_h$ is the survival functions of hth population and $c_r = \frac{n_h!}{(n_h - M_r(h))!}$.

For any continuous variables $Y_1 \leq \ldots \leq Y_n$, the joint density function of $Y_1, \ldots, Y_r, Y_s$, $r < s \leq n$ is given by.

$$f(y_1, \ldots, y_r, y_s) = \frac{n!}{(s-r-1)!(n-s)!} [\overline{F}(y_r) - \overline{F}(y_s)]^{s-r-1} (\overline{F}(y_s))^{n-s} f(y_s) \prod_{i=1}^{r} f(y_i).$$

Here, $(W_1, \ldots, W_r, W_s) \equiv (\mathbf{W}, W_s)$ linked with the discrete variables $(\delta_1, \ldots, \delta_r, \delta_s) \equiv (\boldsymbol{\delta}, \delta_s)$. Then the joint density function of $(\boldsymbol{\delta}, \delta_s, \mathbf{W}, W_s), r < s \leq N$ is given by

$$\begin{aligned} f(\boldsymbol{\delta}, \delta_s, \mathbf{w}, w_s) = \quad & \sum_{Q_{s-1}} \prod_{h=1}^{k} c_{1h} \left\{\overline{F}_h(w_r) - \overline{F}_h(w_s)\right\}^{M_{rs}(h)} \\ & \times (\overline{F}_h(w_s))^{\overline{n}_{hs}} (f_h(w_s))^{\delta_s(h)} \prod_{i=1}^{r} \prod_{h=1}^{k} (f_h(w_i))^{\delta_i(h)} \end{aligned} \tag{4}$$

where

$w_r < w_s \leq w_N$, $M_{rs}(h) = M_{s-1}(h) - M_r(h), \overline{n}_{hs} = n_h - M_s(h), c_{1h} = \frac{n_h!}{M_{rs}(h)!\overline{n}_{hs}!}$, $\sum_{\mathbf{Q_s}}$ $= \sum_{\delta_{r+1}=0}^{1} \cdots \sum_{\delta_s=0}^{1}$ with $Q_s = \{\boldsymbol{\delta}(\mathbf{h}) = (\delta_{r+1}, \ldots, \delta_s),\ \text{for}\ 1 \leq h \leq k\}$.

The conditional density function of W_s given $(\boldsymbol{\delta}, \mathbf{W}) = (\boldsymbol{\delta}, \mathbf{w})$, is given by

$$\begin{aligned} f(w_s \mid \boldsymbol{\delta}, \mathbf{w}) &= \sum_{Q_s} \prod_{h=1}^{k} c_h (f_h(w_s))^{\delta_s(h)} \{\overline{F}_h(w_r - \overline{F}_h(w_s)\}^{M_{rs}(h)} \frac{(\overline{F}_h(w_s))^{\overline{n}_{hs}}}{(\overline{F}_h(w_r))^{\overline{n}_{hr}}} \\ &= \sum_{Q_s} \prod_{h=1}^{k} c_h (f_h(w_s))^{\delta_s(h)} \sum_{l_h=0}^{M_{rs}(h)} a_{l_h} \frac{(\overline{F}_h(w_s))^{\overline{n}_{hs}+l_h}}{(\overline{F}_h(w_r))^{\overline{n}_{h(s-1)}+l_h}} \end{aligned} \quad (5)$$

where

$$c_h = \frac{\overline{n}_{hr}!}{M_{rs}(h)!\overline{n}_{hs}!},\ a_{l_h} = (-1)^{l_h} \binom{M_{rs}(h)}{l_h}.$$

In addition, when the k populations are exponential, the pdf is given by

$$f_h(w) = \theta_h \exp(-\theta_h w), \text{and cdf } F_h(w) = 1 - \exp(-\theta_h w), \quad (6)$$

where $w > 0,\ \theta_h > 0;\ 1 \le h \le k.$

Then, the likelihood function in (3) becomes

$$\begin{aligned} f(\boldsymbol{\Theta}, \boldsymbol{\delta}, \mathbf{w}) &= \prod_{h=1}^{k} c_r \{\exp(-\theta_h w_r)\}^{\overline{n}_{hr}} \prod_{i=1}^{r} \prod_{h=1}^{k} \{\theta_h \exp(-\theta_h w_i)\}^{\delta_i(h)} \\ &= \prod_{h=1}^{k} c_r \theta_h^{M_r(h)} \exp\{-\theta_h u_h\} \end{aligned} \quad (7)$$

where $\Theta = (\theta_1, \ldots, \theta_k)$ and $u_h = \sum_{i=1}^{r} w_i \delta_i(h) + w_r \overline{n}_{hr}$.

Substituting (6) into (5), we obtain the conditional density function of W_s, given $(\boldsymbol{\delta}, \mathbf{W}) = (\boldsymbol{\delta}, \mathbf{w})$,

$$f(w_s \mid \boldsymbol{\delta}, \mathbf{w}) = \sum_{Q_s} \prod_{h=1}^{k} \theta_h^{\delta_s(h)} \sum_{l_h=0}^{M_{rs}(h)} C_h \exp\{-\theta_h D_h (w_s - w_r)\} \quad (8)$$

where $C_h = c_h a_{l_h}$, $D_h = \overline{n}_{h(s-1)} + l_h$ and $\overline{n}_{hs} + \delta_s(h) = \overline{n}_{h(s-1)}$.

Some special cases of the conditional density function are described as follows:

Case 1:

Suppose that $k-1$ is the number of samples satisfy $M_r(h) = n_h$, but only one sample from the k samples say $\left\{\mathbf{X}_q^{n_q}\right\}$ satisfies $M_r(q) < n_q$ or $\{w_r,\ \ldots,\ w_N\} \in \left\{\mathbf{X}_q^{n_q}\right\}$.

Under Case 1, the conditional density function of W_s given $(\boldsymbol{\delta}, \mathbf{W}) = (\boldsymbol{\delta}, \mathbf{w})$ becomes

$$f_1(w_s \mid \boldsymbol{\delta}, \mathbf{w}) = \frac{\overline{n}_{qr}!}{(\overline{n}_{qr} - s + r)!} \sum_{l_q=0}^{s-r-1} a_{l_q} \theta_q \exp\{-\theta_q D_q (w_s - w_r)\} \quad (9)$$

where $a_{l_q} = (-1)^{l_q} \binom{s-r-1}{l_q}$, $D_q = \overline{n}_{qr} - s + r + l_q + 1$, $w_r < w_s \le x_{qn_q}$.

Case 2:

Suppose that $k - R$ is the number of samples satisfy $M_r(h_i) = n_{h_i}$ for $h_i = 1, \ldots, k$; and $R < k$ is the number of samples satisfy $M_r(h_j) < n_{h_j}$ for $h_j = 1, \ldots, k;\ h_i \ne h_j$, equivalently, $W_r > \max\left\{\mathbf{X}_{h_i}^{n_{h_i}}, h_i = 1, \ldots, k\right\}$ but $W_r \in \left\{\mathbf{X}_{h_j}^{n_{h_j}}, h_j = 1, \ldots, k; h_i \ne h_j\right\}$.

Under Case 2, let us just consider the R samples, where $q = 1, \ldots, R$, then the conditional density function of W_s given $(\boldsymbol{\delta}, \mathbf{W}) = (\boldsymbol{\delta}, \mathbf{w})$ becomes

$$f_2(w_s|\boldsymbol{\delta}, \mathbf{w}) = \sum_{Q'_s} \prod_{q=1}^{R} \theta_q^{\delta_s(q)} \sum_{l_q=0}^{M_{rs}(q)} C_q \exp\{-\theta_q D_q (w_s - w_r)\}$$

where $Q'_s = \{\boldsymbol{\delta}(\mathbf{q}) = (\delta_{r+1}, \ldots, \delta_s) \text{ for } 1 \le q \le R\}$, $C_q = c_q a_{l_q}$, $D_q = \overline{n}_{q(s-1)} + l_q$ and $\overline{n}_{qs} + \delta_s(q) = \overline{n}_{q(s-1)}$.

The remainder of this article is organized as follows: Section 2 presents the generalized Bayesian and Bayesian prediction points and intervals using squared error, Linex, and general entropy loss functions in the point predictor. A numerical study of the results from Section 2 is presented in Section 3. Finally, we conclude the paper in Section 4.

2. Generalized Bayes Prediction

In this section, we introduce the concept of generalized Bayesian prediction, which is an investigation of Bayesian prediction under the influence of a learning rate parameter $\eta > 0$. To apply the concept of generalized Bayesian prediction to a prediction study, we give a brief description of generalized Bayesian prediction based on a learning rate parameter $\eta > 0$. A scheme for predicting a sample based on joint censoring of type II samples from k exponential distributions is presented. The main goal is to obtain the point predictors and prediction intervals given at the end of this section.

2.1. Generalized Bayes

The parameters Θ are assumed to be unknown, we may consider the conjugate prior distributions of Θ as independent gamma prior distributions, i.e., $\theta_h \sim Gam(a_h, b_h)$. Hence, the joint prior distribution of Θ is given by

$$\pi(\Theta) = \prod_{h=1}^{k} \pi_h(\theta_h), \tag{10}$$

where

$$\pi_h(\theta_h) = \frac{b_h^{a_h}}{\Gamma(a_h)}\, \theta_h^{a_h-1} \exp\{-b_h\theta_h\}, \tag{11}$$

and $\Gamma(\cdot)$ denotes the complete gamma function.

Combining (7) and (11) after raising (7) to the power η, the posterior joint density function of Θ is then

$$\begin{aligned} \pi^*(\Theta \mid \boldsymbol{\delta}, \mathbf{w}) &= \prod_{h=1}^{k} \frac{(u_h\eta+b_h)^{\eta M_r(h)+a_h}\theta_h^{\eta M_r(h)+a_h-1}}{\Gamma(\eta M_r(h)+a_h)} \exp\{-\theta_h(u_h\eta + b_h)\}, \\ &= \prod_{h=1}^{k} \frac{\xi_h^{\mu_h}\theta_h^{\mu_h-1}}{\Gamma(\mu_h)} \exp\{-\theta_h\xi_h\}, \end{aligned} \tag{12}$$

where $\xi_h = u_h\eta + b_h$, $\mu_h = \eta M_r(h) + a_h$.

Since π_h is a conjugate prior, where $\theta_h \sim Gam(a_h, b_h)$, then it follows that the posterior density function of $(\theta_h|\boldsymbol{\delta}, \mathbf{w})$ is $Gam(\eta M_r(h) + a_h, u_h\eta + b_h)$.

2.2. One Sample Prediction

A sample prediction scheme for the case of the joint censoring of samples from two exponential distributions was studied in [17], and then three cases for the future failures were derived; where in the first case, the future predicted failure surly belongs to $\boldsymbol{X}_1^{n_1}$ failures if $M_r(1) < n_1, M_r(2) = n_2$, in the second case, the future predicted failure surly belongs to $\boldsymbol{X}_2^{n_2}$ failures if $M_r(2) < n_2, M_r(1) = n_1$, and in the third case, it is unknown to which sample the future predicted failure belongs. Here, we generalize the results reported in [17] and examine two special cases in addition to the general case.

In the general case, the size of any sample is greater than the number of observed failures; that is, $M_r(h) < n_h \equiv w_r < x_{hn_h}$ for $h = 1, \ldots, k$. The first special case arises when all future values (predictors) belong to only one sample and the observations of the remaining $k-1$ samples are less than w_r. The second special case arises when all future values (predictors) belong to some samples and all observations of the other samples are less than w_r. The forms of all functions related to the second special case are similar to those related to the general case; therefore, we will introduce only the general case and the first special case.

For the general case, to predict w_r for $r < s \leq N$ based on the observed data $(\boldsymbol{\delta}, \mathbf{w})$, we use the conditional density function (9). Let us define the following integral:

$$I_h^{\delta_s(h)} = \frac{\xi_h^{\mu_h}}{\Gamma(\mu_h)} \int_0^\infty \theta_h^{\mu_h - 1 + \delta_s(h)} \exp\{-\theta_h[\xi_h + D_h(w_s - w_r)]\} \mathrm{d}\theta_h = \begin{cases} \frac{\mu_h}{\xi_h}\left(1 + \frac{D_h(w_s - w_r)}{\xi_h}\right)^{-(\mu_h+1)}, & \text{for } \delta_s(h) = 1 \\ \left(1 + \frac{D_h(w_s - w_r)}{\xi_h}\right)^{-\mu_h}, & \text{for } \delta_s(h) = 0 \end{cases}. \tag{13}$$

Since $\frac{a^b}{\Gamma(b)} \int_0^\infty x^b \, exp\{-x(a+c)\}dx = \frac{a^b}{\Gamma(b)} \, \frac{\Gamma(b+1)}{(a+c)^{b+1}} = \frac{b}{a} \, (1 + \frac{c}{a})^{-(b+1)}$,

$$\frac{a^b}{\Gamma(b)} \int_0^\infty x^{b-1} \, exp\{-x(a+c)\}dx = \frac{a^b}{\Gamma(b)} \, \frac{\Gamma(b)}{(a+c)^b} = \left(1 + \frac{c}{a}\right)^{-b}.$$

Using (8), (13), and (14), the Bayesian predictive density function of W_s given $(\boldsymbol{\delta}, \mathbf{W}) = (\boldsymbol{\delta}, \mathbf{w})$ becomes

$$\begin{aligned} f_B(w_s \mid \boldsymbol{\delta}, \mathbf{w}) &= \int_\Theta f(w_s \,|\, \delta, \mathrm{w}) \pi^*(\Theta \mid \delta, \mathrm{w}) d\Theta \\ &= \sum_{Q_s} \prod_{h=1}^k \sum_{l_h=0}^{M_{rs}} \frac{C_h \xi_h^{\mu_h}}{\Gamma(\mu_h)} \int_0^\infty \theta_h^{\mu_h - 1 + \delta_s(h)} \exp\{-\theta_h[\xi_h + D_h(w_s - w_r)]\} \mathrm{d}\theta_h \\ &= \sum_{Q_s} \prod_{h=1}^k C_h \left\{ \sum_{v=1}^k \left(I_v^1 \prod_{q=1, q \neq v}^k I_q^0 \right) \right\}, \end{aligned} \tag{14}$$

where

$$\prod_{h=1}^k \int_0^\infty f(\theta_h) d\theta_h = \int_0^\infty \ldots \int_0^\infty f(\theta_1) \ldots f(\theta_k) d\theta_1 \ldots d\theta_k \, .$$

Under Case 1, the Bayesian predictive density function of W_s given $(\boldsymbol{\delta}, \mathbf{W}) = (\boldsymbol{\delta}, \mathbf{w})$ becomes

$$f_{1B}(w_s | \boldsymbol{\delta}, \mathbf{w}) = \frac{\overline{n}_{qr}! \mu_q}{(\overline{n}_{qr} - s + r)! \xi_q} \sum_{l_q=0}^{s-r-1} a_{l_q} \left(1 + \frac{D_q(w_s - w_r)}{\xi_q}\right)^{-(\mu_q+1)}, \tag{15}$$

where $\xi_q = u_q \eta + b_q$, $\mu_q = \eta M_r(q) + a_q$, $w_r < w_s \leq x_{q n_q}$.

2.3. Bayesian Point Predictors

For the point predictor, we considered three types of loss functions:

(i). The squared error loss function (SE), which is classified as a symmetric function, is given by

$$L_{SE}(\varphi^*, \varphi) \propto (\varphi^* - \varphi)^2,$$

where φ^* is an estimate of φ.

(ii). The Linex loss function, which is asymmetric, is given by

$$L_L(\varphi^*, \varphi) \propto e^{\tau(\varphi^* - \varphi)} - \tau(\varphi^* - \varphi) - 1, \ \ \tau \neq 0.$$

(iii). The generalization of the entropy (GE) loss function is

$$L_{GE}(\varphi^*, \varphi) \propto \left(\frac{\varphi^*}{\varphi}\right)^c - c \ln\left(\frac{\varphi^*}{\varphi}\right) - 1, \ \ c \neq 0.$$

It is worth noting that the Bayes estimates under the GE loss function coincide with those under the SE loss function when $c = -1$. However, when $c = 1, -2$, the Bayes estimates under GE become those under the weighted squared error loss function and the precautionary loss function, respectively.

Now, the Bayesian point predictors $W_s,\ r < s \leq N$, under different loss functions (SE, Linex, and GE) can be obtained using the predictive density function (15), which are denoted, respectively, by W_{SP}, W_{LP}, W_{EP} and given as follows:

$$\begin{aligned} W_{SP} &= E(W_s \mid \boldsymbol{\delta}, \mathbf{w}) \\ &= \int_0^\infty w_s f_B(w_s|\boldsymbol{\delta}, \mathbf{w}) dw_s \\ &= \sum_{Q_{s-1}} \prod_{h=1}^k \sum_{l_h=0}^{M_{rs}} C_h \left\{ \sum_{v=1}^k \int_0^\infty w_s \left(I_v^1 \ \prod_{q=1, q\neq v}^k I_q^0 \right) dw_s \right\} \end{aligned} \tag{16}$$

$$\begin{aligned} W_{LP} &= -\frac{1}{\tau} \ln\left[E\left(e^{-\tau W_s} \middle| \boldsymbol{\delta}, \mathbf{w}\right)\right] \\ &= -\frac{1}{\tau} \ln\left[\int_0^\infty e^{-\tau w_s} f_B(w_s|\boldsymbol{\delta}, \mathbf{w}) dw_s\right] \\ &= -\frac{1}{\tau} \ln\left[\sum_{Q_{s-1}} \prod_{h=1}^k \sum_{l_h=0}^{M_{rs}} C_h \left\{ \sum_{v=1}^k \int_0^\infty e^{-\tau w_s} \left(I_v^1 \prod_{q=1, q\neq v}^k I_q^0 \right) dw_s \right\} \right] \end{aligned} \tag{17}$$

$$\begin{aligned} W_{EP} &= \left[E(W_s^{-c}|\boldsymbol{\delta}, \mathbf{w})\right]^{-\frac{1}{c}} \\ &= \left[\int_0^\infty w_s^{-c} f_B(w_s|\boldsymbol{\delta}, \mathbf{w}) dw_s\right]^{-\frac{1}{c}} \\ &= \left[\sum_{Q_{s-1}} \prod_{h=1}^k \sum_{l_h=0}^{M_{rs}} C_h \left\{ \sum_{v=1}^k \int_0^\infty w_s^{-c} \left(I_v^1 \ \prod_{q=1, q\neq v}^k I_q^0 \right) dw_s \right\} \right]^{-\frac{1}{c}} \end{aligned} \tag{18}$$

Under Case 1, $W_{SP}, W_{LP},$ and W_{EP} are, respectively, given by

$$W_{SP} = \frac{\overline{n}_{qr}!\mu_q}{(\overline{n}_{qr} - s + r)!\xi_q} \sum_{l_q=0}^{s-r-1} a_{l_q} \int_0^\infty w_s (1 + \frac{D_q(w_s - w_r)}{\xi_q})^{-(\mu_q+1)} dw_s \tag{19}$$

$$W_{LP} = -\frac{1}{\tau} \ln\left[\frac{\overline{n}_{qr}!\mu_q}{(\overline{n}_{qr} - s + r)!\xi_q} \sum_{l_q=0}^{s-r-1} a_{l_q} \int_0^\infty e^{-\tau w_s} (1 + \frac{D_q(w_s - w_r)}{\xi_q})^{-(\mu_q+1)} dw_s \right] \tag{20}$$

$$W_{EP} = \left[\frac{\overline{n}_{qr}!\mu_q}{(\overline{n}_{qr} - s + r)!\xi_q} \sum_{l_q=0}^{s-r-1} a_{l_q} \int_0^\infty w_s^{-c} (1 + \frac{D_q(w_s - w_r)}{\xi_q})^{-(\mu_q+1)} dw_s \right]^{-\frac{1}{c}}. \tag{21}$$

The above equations are solved numerically to obtain the predictors W_{SP}, W_{LP}, and W_{EP}.

2.4. Prediction Interval

The predictive survival function of W_s is given by

$$\begin{aligned} \overline{F}_B(t) &= P(W_s > t \mid \boldsymbol{\delta}, \mathbf{w}) = \int_t^\infty f_B(w_s|\delta, \mathrm{w}) dw_s \\ &= \sum_{Q_{s-1}} \prod_{h=1}^k \sum_{l_h=0}^{M_{rs}} C_h \left[\sum_{v=1}^k \int_t^\infty \left(I_v^1 \ \prod_{q=1, q\neq v}^k I_q^0 \right) dw_s \right] \end{aligned} \tag{22}$$

Numerical integration is required to obtain the predictive survival function in Equation (23).

In Case 1, the predictive survival function of W_s is given by

$$\begin{aligned} \overline{F}_{1B}(t) &= P(W_s > t|\delta, \mathrm{w}) = \int_t^\infty f_{1B}(w_s|\delta, \mathrm{w}) dw_s \\ &= \frac{\overline{n}_{qr}!}{(\overline{n}_{qr} - s + r)!} \sum_{l_q=0}^{s-r-1} \frac{a_{l_q}}{D_q} \left(1 + \frac{D_q(t - w_r)}{\xi_q}\right)^{-\mu_q}. \end{aligned} \tag{23}$$

The Bayesian predictive bounds of a two-sided equi-tailed $100(1-\gamma)\%$ interval for $W_s,\ r < s \leq N$, can be obtained by solving the following two equations numerically,

$$\overline{F}(L|\delta, \mathrm{w}) = 1 - \frac{\gamma}{2}, \quad \overline{F}(U|\delta, \mathrm{w}) = \frac{\gamma}{2}. \tag{24}$$

3. Numerical Study

In this section, the results of the Monte Carlo simulation study are conducted to evaluate the performance of the prediction study derived in the previous section, and an example is presented to illustrate the prediction methods discussed here.

3.1. Simulation Study

We considered three samples from three populations with (n_1, n_2, n_3, r) for choices $(10,\ 10,\ 10,\ 25)$ and $(15,\ 15,\ 15,\ 40)$. In Case 1, we choose the exponential parameters $(\theta_1, \theta_2, \theta_3)$ as (2, 1, 0.1) based on the hyperparameters represented by $\Delta = (a_1, b_1, a_2, b_2, a_3, b_3)$, where $\Delta = \Delta_1 = (2, 1, 2, 2, 1, 10)$.

In the general case, we choose the exponential parameters $(\theta_1,\ \theta_2, \theta_3)$ as (2, 2.5, 3) based on the hyperparameters $\Delta = \Delta_2 = (2,\ 1,\ 5,\ 2,\ 3,\ 1)$.

For the generalized Bayesian study, three values are chosen for the learning rate parameter $\eta = 1,\ 2,\ 5,$ and 10,000 repetitions are used for the Monte Carlo simulations.

The mean observations values of the three generated samples $X_1, X_2,$ and $X_{3,}$ and their joint sample W, using 10,000 repetitions, are presented in Tables 1–4, where the underlined values are greater than w_r.

Table 1. The mean observations values for $(n_1, n_2, n_3, r) = (10,\ 10,\ 10,\ 25)$, $\Delta = \Delta_1$ (Case 1).

Sample	Data
X_1	0.0498, 0.1058, 0.1674, 0.2391, 0.3225, 0.4226, 0.5477, 0.7159, 0.9664, 1.4685.
X_2	0.1005, 0.2130, 0.3363, 0.4780, 0.6444, 0.8434, 1.0905, 1.4248, 1.9253, 2.9195.
X_3	1.01284, 2.1473, 3.3872, 4.8117, 6.4598, 8.4947, 10.9570, 14.3184, 19.3712, 29.4384.
	Ordered data (w, h_i), $r = 25$.
	(0.0498, 1), (0.1005, 2), (0.1058, 1), (0.1674, 1), (0.2130, 2), (0.3225, 1), (0.3225, 1), (0.3363, 2), (0.4226, 1), (0.4780, 2), (0.5477, 1), (0.6444, 2), (0.7159, 1), (0.8434, 2), (0.9664, 1), (1.01284, 3), (1.0905, 2), (1.4248, 2), (1.4685, 1), (1.9253, 2), (2.1473, 3), (2.9195, 2), (3.3872, 3), (4.8117, 3), (6.4598, 3).

Table 2. The mean observations values for $(n_1, n_2, n_3, r) = (15,\ 15,\ 15,\ 40)$, $\Delta = \Delta_1$ (Case 1).

Sample	Data
X_1	0.0333, 0.0693, 0.1078, 0.1487, 0.1944, 0.2449, 0.3004, 0.3646, 0.4363, 0.5206, 0.6208, 0.7479, 0.9146, 1.1641, 1.6557.
X_2	0.0663, 0.1388, 0.2164, 0.2991, 0.3916, 0.4914, 0.6029, 0.7289, 0.8690, 1.0349, 1.2367, 1.4884, 1.8181, 2.316, 3.3304.
X_3	0.6610, 1.3818, 2.1570, 2.9962, 3.9136, 4.9102, 6.0317, 7.2834, 8.7249, 10.3833, 12.3990, 14.9019, 18.2287, 23.2551, 33.3370.
	Ordered data (w, h_i), $r = 40$
	(0.0333,1), (0.0663,2), (0.0693,1), (0.1078,1), (0.1388,2), (0.1487,1), (0.1944,1), (0.2164,2), (0.2449,1), (0.2991,2), (0.3004,1), (0.3646,1), (0.3916,2), (0.4363,1), (0.4914,2), (0.5206,1), (0.6029,2), (0.6208,1), (0.6610,3), (0.7289,2), (0.7479,1), (0.8690,2), (0.9146,1), (1.0349,2), (1.1641,1), (1.2367,2), (1.3818,3), (1.4884,2), (1.6557,1), (1.8181,2), (2.1570,3), (2.3160,2), (2.9962,3), (3.3304,2), (3.9136,3), (4.9102,3), (6.0317,3), (7.2834,3), (8.7249,3), (10.3833,3).

We notice from Tables 1 and 2 that, the future values come only from sample X_3.

Table 3. The mean observations values for $(n_1, n_2, n_3, r) = (10,\ 10,\ 10,\ 25)$, $\Delta = \Delta_2$, (general case).

Sample	Data
X_1	0.0508, 0.1062, 0.1693, 0.2417, 0.3241, 0.4219, 0.5451, 0.7110, 0.9583, 1.4663.
X_2	0.0401, 0.0844, 0.1338, 0.1907, 0.2575, 0.3384, 0.4395, 0.5727, 0.7732, 1.1655.
X_3	0.0334, 0.0703, 0.1127, 0.1604, 0.2167, 0.2839, 0.3671, 0.4778, 0.6474, 0.9823.
	Ordered data (w, h_i), $r = 25$
	(0.0334, 3), (0.0401, 2), (0.0508, 1), (0.0703, 3), (0.0844, 2), (0.1062, 1), (0.1127, 3), (0.1338, 2), (0.1604, 3), (0.1693, 1), (0.1907,2), (0.2167, 3), (0.2417, 1), (0.2575, 2), (0.2839, 3), (0.3241, 1), (0.3384, 2), (0.3671, 3), (0.4219, 1), (0.4395, 2), (0.4778, 3), (0.5451, 1), (0.5727, 2), (0.6474, 3), (0.7110, 3).

Table 4. The mean observations values for $(n_1, n_2, n_3, r) = (15,\ 15,\ 15,\ 40)$, $\Delta = \Delta_2$, (general case).

Sample	Data
X_1	0.0326, 0.0681, 0.1068, 0.1487, 0.1944, 0.2449, 0.3003, 0.3632, 0.4344, 0.5186, 0.6195, 0.7433, 0.9125, 1.1605, 1.6591.
X_2	0.0269, 0.0556, 0.0864, 0.1193, 0.1557, 0.1965, 0.2414, 0.2913, 0.3484, 0.4137, 0.4943, 0.5941, 0.7264, 0.9289, 1.3301.
X_3	0.0220, 0.0459, 0.0713, 0.0988, 0.1295, 0.1624, 0.1989, 0.2399, 0.2872, 0.3424, 0.4084, 0.4915, 0.6032, 0.7706, 1.1045.
	Ordered data (w, h_i), $r = 40$
	(0.0220,3), (0.0269,2), (0.0326,1), (0.0459,3), (0.0556,2), (0.0681,1), (0.0713,3), (0.0864,2), (0.0988,3), (0.1068,1), (0.1193,2), (0.1295,3), (0.1487,1), (0.1557,2), (0.1624,3), (0.1944,1), (0.1965,2), (0.1989,3), (0.2399,3), (0.2414,2), (0.2449,1), (0.2872,3), (0.2913,2), (0.3003,1), (0.3424,3), (0.3484,2), (0.3632,1), (0.4084,3), (0.4137,2), (0.4344,1), (0.4915,3), (0.4943,2), (0.5186,1), (0.5941,2), (0.6032,3), (0.6195,1), (0.7264,2), (0.7433,1), (0.7706,3), (0.9125,1).

We notice from Tables 3 and 4 that the future values come from the three samples.

For $(n_1, n_2, n_3, r) = (10,\ 10,\ 10,\ 25)$, under Case 1, we use (20), (21), and (22) to calculate the mean squared prediction errors (MSPEs) of the point predictors ($W_{SP},\ W_{LP},$ and W_{EP}) for $s = 26, \ldots, 30$, where $\tau = 0.1,\ 0.5$; $c = 0.1,\ 0.5$, and the results are presented in Table 5.

Table 5. MSPE of point predictions for $\eta = 1, 2, 5$; $\Delta = \Delta_1$ in Case 1.

			η=1			
(n_1,n_2,n_3,r)	s	SP	LP		EP	
			τ=0.1	τ=0.5	c=0.1	c=0.5
$(10, 10, 10, 25)$	26	0.0238	0.0215	0.0211	0.0193	0.0175
	27	0.2623	0.2425	0.2383	0.2179	0.1916
	28	0.8924	0.8914	0.8767	0.8620	0.8125
	29	1.4967	1.4687	1.3263	1.2579	1.2191
	30	2.1610	2.0341	1.9864	1.7451	1.5218
			$\eta = 2$			
	26	0.0213	0.0203	0.0198	0.0191	0.0171
	27	0.2620	0.2319	0.2303	0.2094	0.1815
	28	0.8781	0.8723	0.8627	0.8205	0.7685
	29	1.4687	1.4514	1.3041	1.2256	1.1873
	30	2.0245	1.9341	1.8561	1.6381	1.2552
			$\eta = 5$			
	26	0.0183	0.0174	0.0171	0.0154	0.0144
	27	0.2423	0.2253	0.2230	0.2201	0.1796
	28	0.8165	0.8064	0.8137	0.7905	0.7125
	29	1.3987	1.3782	1.3585	1.2682	1.0671
	30	2.0201	1.9152	1.8610	1.4373	1.1782

The results of the MSPEs in the general case are calculated using (17), (18), and (19), and shown in Table 6.

Table 6. MSPEs of point predictions for $\eta = 1, 2, 5;\ \Delta = \Delta_2$.

			η=1			
(n_1,n_2,n_3,r)	***s***	***SP***	***LP***		***EP***	
			τ=0.1	**τ=0.5**	***c*=0.1**	***c*=0.5**
	26	0.0112	0.0115	0.0113	0.0110	0.0097
	27	0.1342	0.1233	0.1132	0.0872	0.0821
	28	0.2752	0.2571	0.2168	0.2205	0.2064
	29	1.3567	1.3125	1.3061	1.2143	1.2083
	30	1.5984	1.6213	1.5783	1.4437	1.2981
			$\eta = 2$			
	26	0.0101	0.0112	0.0086	0.0088	0.0078
	27	0.1245	0.1156	0.1012	0.0789	0.0689
	28	0.2234	0.2233	0.2087	0.2015	0.2001
	29	1.4010	1.4111	1.3021	1.2182	1.1892
	30	1.6125	1.5987	1.5654	1.4127	1.2678
			$\eta = 5$			
	26	0.0097	0.0072	0.0027	0.0025	0.0012
	27	0.1024	0.1003	0.0998	0.0775	0.0567
	28	0.1892	0.1566	0.1026	0.0876	0.0278
	29	1.2346	1.3271	1.2987	1.1765	1.0482
	30	1.8762	1.5987	1.5654	1.2354	1.1567

For $(n_1, n_2, n_3, r) = (10,\ 10,\ 10,\ 25)$ and $(n_1, n_2, n_3, r) = (15, 15, 15, 40)$, the results of the prediction bounds of $W_s,\ s = 26, \ldots, 30$ and $s = 41, \ldots, 45$, respectively, are calculated using (24) and (25) in Case 1, then are presented in Table 7.

Table 7. Lower and upper 95% prediction bounds for W_s in Case 1, for different choices of n_1, n_2, n_3, r and $\Delta = \Delta_1$.

(n_1,n_2,n_3,r)	***s***	**η=1**		**η=2**		**η=5**	
		L	***U***	***L***	***U***	***L***	***U***
$(10, 10, 10, 25)$	26	7.1426	9.3742	7.1517	9.3268	7.1546	9.2985
	27	9.8765	13.6496	9.9164	13.6412	9.9843	13.6191
	28	12.6289	21.3289	12.6714	21.2692	12.8921	21.21482
	29	16.7643	29.1496	16.8653	28.9896	16.9225	29.9641
	30	25.7658	49.5654	25.8585	49.5154	25.9765	48.3654
$(15, 15, 15, 40)$	41	11.6539	13.8764	11.9152	13.8472	11.9584	13.8174
	42	12.9876	17.6824	13.0256	17.1934	13.2876	17.1264
	43	15.2879	23.1859	15.5429	23.0674	15.7698	22.5429
	44	20.3289	33.9126	20.9721	33.3126	21.9968	32.8952
	45	29.1289	51.1610	29.5289	51.0326	29.7853	50.4761

Table 8 presents the prediction bounds using (23), and (25) to show the results of the general case.

Table 8. Lower and upper 95% prediction bounds for W_s, for different choices of n_1, n_2, n_3, r and $\Delta = \Delta_2$.

(n_1,n_2,n_3,r)	s	η=1		η=2		η=5	
		L	U	L	U	L	U
$(10, 10, 10, 25)$	26	0.6528	0.8889	0.6814	0.8592	0.6920	0.8342
	27	0.7589	1.1096	0.7768	1.2033	0.7879	1.2191
	28	0.8934	1.9583	0.9214	1.9462	0.9520	1.9321
	29	0.9789	3.5696	0.9901	3.3733	1.0127	3.29341
	30	1.1289	5.5610	1.2541	5.3451	1.2964	5.2218
$(15, 15, 15, 40)$	41	0.7902	1.8846	0.7968	1.8732	0.8079	1.8841
	42	0.8674	2.2354	0.8776	2.2125	0.8841	2.1254
	43	0.9282	3.7583	0.9245	3.7483	0.9582	3.6783
	44	0.9949	4.5610	1.0237	4.5516	1.0263	4.4712
	45	1.2367	5.9810	1.2568	5.9736	1.31664	5.9523

3.2. Illustrative Example

To illustrate the usefulness of the results developed in the previous sections, we consider three samples of size $n_1 = n_2 = n_3 = 10$ from Nelson's data (groups 1, 4, and 5) corresponding to the breakdown of an insulating fluid subjected to a high-stress load (see [20] p. 462). These breakdown times, referred to here as samples X_i, $i = 1, 2, 3$, are jointly type-II censored data in the form of (w, h_i) obtained from these three samples with $r = 24$ and are shown in Table 9.

Table 9. The failure time data for X_1, X_2, and X_3, and their order (w, h_i), where $\delta_{h_i} = 1$.

Sample	Data
X_1	1.89, 4.03, 1.54, 0.31, 0.66, 1.7, 2.17, 1.82, 9.99, 2.24
X_2	1.17, 3.87, 2.8, 0.7, 3.82, 0.02, 0.5, 3.72, 0.06, 3.57
X_3	8.11, 3.17, 5.55, 0.80, 0.20, 1.13, 6.63, 1.08, 2.44, 0.78
	Ordered data (w, h_i)
	(0.02, 2), (0.06, 2), (0.20, 3), (0.31, 1), (0.50, 2), (0.66, 1), (0.70, 2), (0.78, 3), (0.80, 3), (1.08, 3) (1.13, 3), (1.17, 2), (1.54, 1), (1.70, 1), (1.82, 1), (1.89, 1), (2.17, 1), (2.24, 1), (2.44, 3), (2.80, 2) (3.17, 3), (3.57, 2), (3.72, 2), (3.82, 2).

Using (17), (18), (19), (23), and (25), the MSPEs of the point predictors and prediction intervals of $w_s, s = 25, \ldots, 30$ are calculated and presented in Table 10 using $\eta = 1,\ 2,\ 5$ and $\Delta = \Delta_3 = (1, 2.6, 1, 2, 1, 3)$; and $\tau = 0.1,\ 0.5$ and $c = 0.1,\ 0.5$.

Table 10. The values of point predictors and 95% prediction interval of W_s, for $\eta = 1,\ 2,\ 5;\ \Delta = \Delta_3$.

				η=1			
s	(w_s,h_i)	W_{SP}	W_{LP}		W_{EP}		(L,U)
	Exactvalue		τ=0.1	τ=0.5	c=0.1	c=0.5	
25	(3.87,2)	3.5678	3.6728	3.7876	3.7789	3.8102	(3.8476,4.7658)
26	(4.03,1)	4.6453	4.5364	4.4573	4.4653	4.1653	(3.8645,5.7653)
27	(5.55,3)	5.8972	5.4676	5.3864	5.4943	5.5127	(3.9785,7.8946)
28	(6.63,3)	7.1236	6.9456	6.8757	6.9764	6.7685	(4.5632,12.5467)
29	(8.11,3)	10.2738	9.9594	9.5734	9.4876	9.2236	(5.8762,18.3765)
30	(9.99,1)	13.6758	12.8654	12.1765	12.5638	12.1128	(6.4657,30.4687)
				$\eta = 2$			
25	(3.87,2)	3.6132	3.6527	3.7964	3.8967	3.8662	(3.8499,4.5473)
26	(4.03,1)	4.3567	4.3384	4.2582	4.2765	4.1234	(3.8764,5.7564)
27	(5.55,3)	5.7653	5.4765	5.3876	5.5123	5.5742	(3.9967,7.8125)
28	(6.63,3)	7.1168	6.9174	6.8542	6.8789	6.7125	(4.7842,12.5165)
29	(8.11,3)	10.1375	9.92875	9.6213	9.2134	8.9984	(5.8923,17.8964)
30	(9.99,1)	13.8234	12.6753	12.3476	12.12273	11.8657	(6.8973,30.1374)
				$\eta = 5$			
25	(3.87,2)	3.6954	3.7135	3.8217	3.8378	3.8675	(3.8564,4.4623)
26	(4.03,1)	4.2765	4.2187	4.2071	4.2135	4.1098	(3.8976,5.7245)
27	(5.55,3)	5.7321	5.4876	5.452	5.5216	5.5731	(3.9986,7.8087)
28	(6.63,3)	7.1065	6.9276	6.8628	6.7522	6.7081	(4.8569,12.5097)
29	(8.11,3)	10.0879	9.9134	9.5675	9.1561	8.9786	(5.9872,17.2876)
30	(9.99,1)	13.5437	12.5674	12.3652	12.1135	11.8543	(6.9965,29.4567)

4. Conclusions

In this study, we examined the effects of learning rate parameters on prediction results. We used Monte Carlo simulations to show the effectiveness of the learning rate parameter in improving the results of prediction intervals and point predictors. Formally, we considered a joint type-II censoring scheme in which the lifetimes of the three populations have exponential distributions. We determined the MSPEs of the point predictors and prediction intervals using different values for the learning rate parameter η and different values for the parameters of the losses in both the simulation study and the illustrative example. From all tables in this prediction study, it can be seen that the results improve with increasing the loss parameters c, τ, and learning rate parameter η. In the simulation study, a comparison of the results in Tables 5 and 6 shows that the results in Table 5 are better, and the length of the prediction intervals in Table 8 is smaller than those in Table 7 because the observed values used in Table 7 are larger than those used in Table 8. The results of the illustrative example improve with larger values of loss parameters and learning rate parameter. So we conclude that the results of the prediction study became better as learning rate parameter increased. However, in both studies, the lengths of the prediction intervals increased for the larger future lifetimes. It may be interesting to examine this work using a different type of censoring.

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