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Classes of Harmonic Functions Related to Mittag-Leffler Function

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Abstract: The purpose of this paper is to find new inclusion relations of the harmonic class $\mathcal{HF}(\varrho, \gamma)$ with the subclasses $\mathcal{S}^*_{\mathcal{HF}}$, $\mathcal{K}_{\mathcal{HF}}$ and $\mathcal{T}\mathfrak{N}_{\mathcal{HF}}(\tau)$ of harmonic functions by applying the convolution operator $\Theta(\Im)$ associated with the Mittag-Leffler function. Further for $\varrho = 0$, several special cases of the main results are also obtained.

Keywords: harmonic; univalent functions; harmonic starlike; harmonic convex; Mittag-Leffler function

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1. Introduction

Harmonic functions play important roles in many problem in applied mathematics and they are also famous for their use in the study of minimal surfaces. Several differential geometers such as Choquest [1], Kneser [2], Lewy [3] and Rado [4] studied the harmonic functions. In 1984, Clunie and Sheil-Small [5] developed the basic theory of complex harmonic univalent functions $\Im$ defined in the open unit disk $\Xi = \{\xi : |\xi| < 1\}$ for which $\Im(0) = \Im_\xi(0) - 1 = 0$.

Let $\mathcal{HF}$ be the family of all harmonic functions of the form $\Im = \phi + \overline{\psi}$, where

$$\phi(\xi) = \xi + \sum_{\nu=2}^{\infty} a_\nu \xi^\nu, \qquad \psi(\xi) = \sum_{\nu=1}^{\infty} b_\nu \xi^\nu, \quad |b_1| < 1. \tag{1}$$

are analytic in the open unit disk Ξ. Furthermore, let $\mathcal{S}_{\mathcal{HF}}$ denote the family of functions $\Im = \phi + \overline{\psi}$ that are harmonic univalent and sense preserving in Ξ. Note that the family $\mathcal{S}_{\mathcal{HF}} = \mathcal{S}$ if ψ is zero.

We also let the subclass $\mathcal{S}^0_{\mathcal{HF}}$ of $\mathcal{S}_{\mathcal{HF}}$ as

$$\mathcal{S}^0_{\mathcal{HF}} = \{\Im = \phi + \overline{\psi} \in \mathcal{S}_{\mathcal{HF}} : \psi'(0) = b_1 = 0\}.$$

The classes $\mathcal{S}^0_{\mathcal{HF}}$ and $\mathcal{S}_{\mathcal{HF}}$ were first studied in [5].

A sense-preserving harmonic mapping $\Im \in \mathcal{S}^0_{\mathcal{HF}}$ is in the class $\mathcal{S}_{\mathcal{HF}}$ if the range $\Im(\Xi)$ is starlike with respect to the origin. The function $\Im \in \mathcal{S}^*_{\mathcal{HF}}$ is called a harmonic starlike mapping in Ξ. Also, the function $\Im$ defined in Ξ belongs to the class $\mathcal{K}_{\mathcal{HF}}$ if $\Im \in \mathcal{S}^0_{\mathcal{HF}}$ and if $\Im(\Xi)$ is a convex domain. The function $\Im \in \mathcal{K}_{\mathcal{HF}}$ is called harmonic convex in Ξ. Analytically, we have

$$\Im \in \mathcal{S}^*_{\mathcal{HF}} \text{ iff } arg\left(\frac{\partial}{\partial\theta}\Im\left(re^{i\theta}\right)\right) \geq 0,$$

and

$$\Im \in \mathcal{K}_{\mathcal{HF}} \text{ iff } \frac{\partial}{\partial\theta}\left\{arg\left(arg\left(\frac{\partial}{\partial\theta}\Im\left(re^{i\theta}\right)\right)\right)\right\} \geq 0,$$
$$\xi = re^{i\theta} \in \Xi,\ 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1.$$

For definitions and properties of these classes, one may refer to [6] and for other subclasses of harmonic functions one can see [7–17].

Let $\mathcal{T}_{\mathcal{HF}}$ be the class of functions in $\mathcal{S}_{\mathcal{HF}}$ that may be expressed as $\Im = \phi + \overline{\psi}$, where

$$\begin{aligned} \phi(\xi) &= \xi - \sum_{\nu=2}^{\infty}|a_\nu|\xi^\nu, \\ \psi(\xi) &= \sum_{\nu=1}^{\infty}|b_\nu|\xi^\nu \quad |b_1| < 1. \end{aligned} \tag{2}$$

For $0 \leq \tau < 1$, let

$$\mathfrak{N}_{\mathcal{HF}}(\tau) = \left\{\Im \in \mathcal{HF} : \mathrm{Re}\left(\frac{\Im'(\xi)}{\xi'}\right) \geq \tau,\ \xi = re^{i\theta} \in \Xi\right\},$$

and

$$\mathfrak{R}_{\mathcal{HF}}(\tau) = \left\{\Im \in \mathcal{HF} : \mathrm{Re}\left(\frac{\Im''(\xi)}{\xi''}\right) \geq \tau,\ \xi = re^{i\theta} \in \Xi\right\}$$

where

$$\xi' = \frac{\partial}{\partial\theta}\left(\xi = re^{i\theta}\right), \xi'' = \frac{\partial}{\partial\theta}(\xi'), \Im'(\xi) = \frac{\partial}{\partial\theta}\Im\left(re^{i\theta}\right), \Im'' = \frac{\partial}{\partial\theta}(\Im'(\xi)).$$

Define

$$\mathcal{T}\mathfrak{N}_{\mathcal{HF}}(\tau) = \mathfrak{N}_{\mathcal{HF}}(\tau) \cap \mathcal{T}_{\mathcal{HF}} \quad \text{and} \quad \mathcal{T}\mathfrak{R}_{\mathcal{HF}}(\tau) = \mathfrak{R}_{\mathcal{HF}}(\tau) \cap \mathcal{T}_{\mathcal{HF}}.$$

For more details about the classes $\mathcal{T}_{\mathcal{HF}}$, $\mathfrak{N}_{\mathcal{HF}}(\tau)$, $\mathcal{T}\mathfrak{N}_{\mathcal{HF}}(\tau)$, $\mathfrak{R}_{\mathcal{HF}}(\tau)$ and $\mathcal{T}\mathfrak{R}_{\mathcal{HF}}(\tau)$ see [13,18].

In [19] Sokòl et al., introduced the class $\mathcal{HF}(\varrho,\gamma)$ of functions $\Im \in \mathcal{HF}$ that satisfy

$$\mathrm{Re}\left\{\phi'(\xi) + \psi'(\xi) + 3\varrho\xi(\phi''(\xi) + \psi''(\xi)) + \varrho\xi^3(\phi'''(\xi) + \psi'''(\xi))\right\} > \gamma,$$

for some $\varrho \geq 0$ and $0 \leq \gamma < 1$. For $\varrho = 0$, we obtain the class $\mathcal{HF}(\gamma)$ which satisfy

$$\mathrm{Re}\{\phi'(\xi) + \psi'(\xi)\} > \gamma.$$

2. Mittag-Leffler Function

The two-parameter Mittag-Leffler $\mathbb{E}_{\rho,\epsilon}(\xi)$ (also known as the Wiman function [20]) was given by

$$\mathbb{E}_{\rho,\epsilon}(\xi) = \sum_{\nu=0}^{\infty}\frac{\xi^\nu}{\Gamma(\rho\nu+\epsilon)}, \quad (\xi,\rho,\epsilon \in \mathbb{C},\ \text{with } \mathrm{Re}\rho > 0, \mathrm{Re}\epsilon > 0), \tag{3}$$

while in 1903, the one-parameter Mittag-Leffler $\mathbb{E}_\rho(\xi)$ was introduced for $\epsilon = 1$, and given by

$$\mathbb{E}_\rho(\xi) = \sum_{\nu=0}^{\infty}\frac{\xi^\nu}{\Gamma(\rho\nu+1)}, \quad (\xi,\rho \in \mathbb{C},\ \text{with } \mathrm{Re}\rho > 0).$$

As its special case, the function $\mathbb{E}_{\rho,\epsilon}(\xi)$ has many well known functions for example, $\mathbb{E}_{\mathbf{0,0}}(\xi) = \sum_{\nu=0}^{\infty}\xi^\nu$, $\mathbb{E}_{\mathbf{1,1}}(\xi) = e^\xi$, $\mathbb{E}_{\mathbf{1,2}}(\xi) = \frac{e^\xi-1}{\xi}$, $\mathbb{E}_{\mathbf{2,1}}(\xi^2) = cosh\xi$, $\mathbb{E}_{\mathbf{2,1}}(-\xi^2) = cos\xi$,

$\mathbb{E}_{2,2}(\xi^2) = \frac{sinh\xi}{\xi}$, $\mathbb{E}_{2,2}(-\xi^2) = \frac{sin\xi}{\xi}$, $\mathbb{E}_4(\xi) = \frac{1}{2}[cos\xi^{\frac{1}{4}} + cosh\xi^{\frac{1}{4}}]$ and $\mathbb{E}_3(\xi) = \frac{1}{2}[e^{\xi^{\frac{1}{3}}} + 2e^{-\frac{1}{2}\xi^{\frac{1}{3}}} cos(\frac{\sqrt{3}}{2}\xi^{\frac{1}{3}})]$.

Putting $\rho = \frac{1}{2}$ and $\epsilon = 1$, we get

$$\mathbb{E}_{\frac{1}{2},1}(\xi) = e^{\xi^2}.erfc(-\xi) = e^{\xi^2}\left(1 + \frac{2}{\sqrt{\pi}}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{\nu!(2\nu+1)}\xi^{2\nu+1}\right).$$

Numerous properties of the one-parameter Mittag-Leffler $\mathbb{E}_{\rho}(\xi)$ and the two-parameter Mittag-Leffler $\mathbb{E}_{\rho,\epsilon}(\xi)$ can be found e.g., in [21–24].

It is clear that the two-parameter Mittag-Leffler function $\mathbb{E}_{\rho,\epsilon}(\xi) \notin \mathcal{A}$. Thus, we have the following normalization due to Bansal and Prajapat [22]:

$$\chi_{\rho,\epsilon}(\xi) = \xi\Gamma(\epsilon)\mathbb{E}_{\rho,\epsilon}(\xi) = \xi + \sum_{\nu=2}^{\infty}\frac{\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)}\xi^{\nu},$$

where $\rho,\epsilon,\xi \in \mathbb{C}$, with Re$\rho > 0$ and Re$\epsilon > 0$. In this study, we let ρ, ϵ to be real numbers and $\xi \in \Xi$.

The study of operators plays an important role in the geometric function theory. Many differential and integral operators can be written in terms of convolution of certain analytic functions, (see [25–29]).

Very recently, and for the functions

$$\chi_{\rho,\epsilon}(\xi) = \xi + \sum_{\nu=2}^{\infty}\frac{\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)}\xi^{\nu}, \text{ and } \quad \chi_{\eta,\delta}(\xi) = \sum_{\nu=1}^{\infty}\frac{\Gamma(\delta)}{\Gamma(\eta(\nu-1)+\delta)}\xi^{\nu}. \tag{4}$$

Murugusundaramoorthy et al. [30] defined the following convolution operator $\Theta(\Im)$ given by

$$\begin{aligned}\mathfrak{F}(\xi) = \Theta\Im(\xi) &= \phi(\xi) * \chi_{\rho,\epsilon}(\xi) + \overline{\psi(\xi) * \chi_{\eta,\delta}(\xi)} \\ &= \xi + \sum_{\nu=2}^{\infty}\frac{\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)}a_{\nu}\xi^{\nu} + \overline{\sum_{\nu=1}^{\infty}\frac{\Gamma(\delta)}{\Gamma(\eta(\nu-1)+\delta)}b_{\nu}\xi^{\nu}},\end{aligned} \tag{5}$$

where $\rho, \eta, \epsilon, \delta$ are real with $\rho, \eta, \epsilon, \delta \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\} \cup \{0\}$.

Inclusion relations between different subclasses of analytic and univalent functions by using hypergeometric functions [10,31], generalized Bessel function [32–34] and by the recent investigations related with distribution series [35–41], were studied in the literature. Very recently, several authors have investigated mapping properties and inclusion results for the families of harmonic univalent functions, including various linear and nonlinear operators (see [42–48]).

The paper is organized as follows. In Section 3, we recall some lemmas, which will be useful to prove the main results. Section 4 is devoted to establishing some inclusion relations of the harmonic class $\mathcal{HF}(\varrho,\gamma)$ the classes $\mathcal{S}^*_{\mathcal{HF}}$, $\mathcal{K}_{\mathcal{HF}}$, $\mathfrak{N}_{\mathcal{HF}}(\tau)$, and $\mathfrak{R}_{\mathcal{HF}}(\tau)$ by applying the convolution operator Θ related with Mittag-Leffler function following the work performed in [30]. Finally, in Section 5, several special cases of the main results are also obtained when $\varrho = 0$.

3. Preliminary Lemmas

We shall use the following lemmas in our proofs.

Lemma 1 ([19]). *Let $\Im = \phi + \overline{\psi}$ where ϕ and ψ are given by (1) and suppose that $\varrho \geq 0$, $0 \leq \gamma < 1$ and*

$$\sum_{\nu=2}^{\infty} \nu[1+\varrho\left(\nu^2-1\right)]|a_\nu| + \sum_{\nu=1}^{\infty} \nu[1+\varrho\left(\nu^2-1\right)]|b_\nu| \leq 1-\gamma. \tag{6}$$

then $\Im$ is harmonic, sense-preserving univalent functions in Ξ and $\Im \in \mathcal{HF}(\varrho,\gamma)$.

Moreover, if $\Im \in \mathcal{HF}(\varrho,\gamma)$, then

$$|a_\nu| \leq \frac{1-\gamma}{\nu[1+\varrho(\nu^2-1)]},\ \nu \geq 2, \tag{7}$$

and

$$|b_\nu| \leq \frac{1-\gamma}{\nu[1+\varrho(\nu^2-1)]},\ \nu \geq 1. \tag{8}$$

Lemma 2 ([6]). *Let $\Im = \phi + \overline{\psi}$ where ϕ and ψ are given by (2) and suppose that $0 \leq \tau < 1$. Then $\Im \in \mathcal{T}\mathfrak{N}_{\mathcal{HF}}(\tau)$ if and only if*

$$\sum_{\nu=2}^{\infty} \nu|a_\nu| + \sum_{\nu=1}^{\infty} \nu|b_\nu| \leq 1-\tau. \tag{9}$$

Moreover, if $\Im \in \mathcal{T}\mathfrak{N}_{\mathcal{HF}}(\tau)$, then

$$|a_\nu| \leq \frac{1-\tau}{\nu}, \nu \geq 2, \tag{10}$$

and

$$|b_\nu| \leq \frac{1-\tau}{\nu}, \nu \geq 1. \tag{11}$$

Lemma 3 ([18]). *Let $\Im = \phi + \overline{\psi}$ where ϕ and ψ are given by (2), and suppose that $0 \leq \tau < 1$. Then $\Im \in \mathcal{T}\mathfrak{R}_{\mathcal{HF}}(\tau)$ if and only if*

$$\sum_{\nu=2}^{\infty} \nu^2|a_\nu| + \sum_{\nu=1}^{\infty} \nu^2|b_\nu| \leq 1-\tau. \tag{12}$$

Moreover, if $\Im \in \mathcal{T}\mathfrak{R}_{\mathcal{HF}}(\tau)$, then

$$|a_\nu| \leq \frac{1-\tau}{\nu^2},\ \nu \geq 2 \tag{13}$$

and

$$|b_\nu| \leq \frac{1-\tau}{\nu^2},\ \nu \geq 1. \tag{14}$$

Lemma 4 ([5]). *If $\Im = \phi + \overline{\psi} \in \mathcal{S}^*_{\mathcal{HF}}$ where ϕ and ψ are given by (1) with $b_1 = 0$, then*

$$|a_\nu| \leq \frac{(2\nu+1)(\nu+1)}{6} \text{ and } |b_\nu| \leq \frac{(2\nu-1)(\nu-1)}{6}. \tag{15}$$

Lemma 5 ([5]). *If $\Im = \phi + \overline{\psi} \in \mathcal{K}_{\mathcal{HF}}$ where ϕ and ψ are given by (1) with $b_1 = 0$, then*

$$|a_\nu| \leq \frac{\nu+1}{2} \text{ and } |b_\nu| \leq \frac{\nu-1}{2}. \tag{16}$$

Throughout the sequence, we use the following:

$$\chi_{\rho,\epsilon}(\xi) = \xi + \sum_{\nu=2}^{\infty} \frac{\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)} \xi^{\nu}; \quad \chi_{\rho,\epsilon}(1) = 1 + \sum_{\nu=2}^{\infty} \frac{\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)}, \tag{17}$$

$$\chi'_{\rho,\epsilon}(\xi) = 1 + \sum_{\nu=2}^{\infty} \frac{\nu\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)} \xi^{\nu-1}; \quad \chi'_{\rho,\epsilon}(1) - 1 = \sum_{\nu=2}^{\infty} \frac{\nu\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)}, \tag{18}$$

$$\chi''_{\rho,\epsilon}(1) = \sum_{\nu=2}^{\infty} \frac{\nu(\nu-1)\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)}, \tag{19}$$

$$\chi'''_{\rho,\epsilon}(1) = \sum_{\nu=2}^{\infty} \frac{\nu(\nu-1)(\nu-2)\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)}, \tag{20}$$

and in general, we have

$$\chi^{(j)}_{\rho,\epsilon}(1) = \sum_{\nu=2}^{\infty} \frac{\nu(\nu-1)(\nu-2)\cdots(\nu-(j-1))\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)}, j = 1, 2, \ldots \tag{21}$$

4. Inclusion Relations of the Class $\mathcal{HF}(\varrho, \gamma)$

In this section we shall prove that $\Theta(\mathcal{S}^*_{\mathcal{HF}}) \subset \mathcal{HF}(\varrho, \gamma)$ and $\Theta(\mathcal{K}_{\mathcal{HF}}) \subset \mathcal{HF}(\varrho, \gamma)$.

Theorem 1. *Let $\varrho \geq 0$, $\gamma \in [0,1)$ and $\rho, \epsilon, \eta, \delta \notin \mathbb{Z}_0^-$. If*

$$\begin{aligned}
&\Big[2\varrho\Big(\chi^{(5)}_{\rho,\epsilon}(1) + \chi^{(5)}_{\eta,\epsilon}(1)\Big) + 23\varrho\chi^{(4)}_{\rho,\epsilon}(1) + (67\varrho+2)\chi^{(3)}_{\rho,\epsilon}(1) \\
&+ (45\varrho+9)\chi^{(2)}_{\rho,\epsilon}(1) + 6\chi'_{\rho,\epsilon}(1) \\
&+ 17\varrho\chi^{(4)}_{\eta,\epsilon}(1) + (31\varrho+2)\chi^{(3)}_{\eta,\epsilon}(1) + (9\varrho+3)\chi^{(2)}_{\eta,\epsilon}(1)\Big] \\
&\leq 6(1-\gamma),
\end{aligned} \tag{22}$$

then

$$\Theta(\mathcal{S}^*_{\mathcal{HF}}) \subset \mathcal{HF}(\varrho, \gamma).$$

Proof. Let $\Im = \phi + \overline{\psi} \in \mathcal{S}^*_{\mathcal{HF}}$ where ϕ and ψ are of the form (1) with $b_1 = 0$. We need to show that $\Theta(\Im) = \mathfrak{F}(\xi) \in \mathcal{HF}(\varrho, \gamma)$, which given by (5) with $b_1 = 0$. In view of Lemma 1, we need to prove that

$$Q(\varrho, \epsilon, \delta, \eta) \leq 1 - \gamma,$$

where

$$\begin{aligned}
Q(\varrho, \epsilon, \delta, \eta) = &\sum_{\nu=2}^{\infty} \nu\Big(1 + \varrho\Big(\nu^2 - 1\Big)\Big)\left|\frac{\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)} a_\nu\right| \\
&+ \sum_{\nu=2}^{\infty} \nu\Big(1 + \varrho\Big(\nu^2 - 1\Big)\Big)\left|\frac{\Gamma(\delta)}{\Gamma(\eta(\nu-1)+\delta)} b_\nu\right|.
\end{aligned} \tag{23}$$

Using the inequalities (15) of Lemma 4, we get

$$\begin{aligned}
&Q(\varrho,\epsilon,\delta,\eta)\\
&\leq \frac{1}{6}\left[\sum_{\nu=2}^{\infty}(2\nu+1)(\nu+1)\left(\nu+\varrho\nu\left(\nu^2-1\right)\right)\frac{\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)}\right.\\
&\left.+\sum_{\nu=2}^{\infty}(2\nu-1)(\nu-1)\left(\nu+\varrho\nu\left(\nu^2-1\right)\right)\frac{\Gamma(\delta)}{\Gamma(\eta(\nu-1)+\delta)}\right]\\
&=\frac{1}{6}\left[\sum_{\nu=2}^{\infty}\left[2\varrho\nu^5+3\varrho\nu^4+(2-\varrho)\nu^3+(3-3\varrho)\nu^2+(1-\varrho)\nu\right]\frac{\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)}\right.\\
&\left.+\sum_{\nu=2}^{\infty}\left[2\varrho\nu^5-3\varrho\nu^4+(2-\varrho)\nu^3+(3\varrho-3)\nu^2+(1-\varrho)\nu\right]\frac{\Gamma(\delta)}{\Gamma(\eta(\nu-1)+\delta)}\right]
\end{aligned} \tag{24}$$

Writing

$$\nu^2=\nu(\nu-1)+\nu, \tag{25}$$

$$\nu^3=\nu(\nu-1)(\nu-2)+3\nu(\nu-1)+\nu, \tag{26}$$

$$\nu^4=\nu(\nu-1)(\nu-2)(\nu-3)+6\nu(\nu-1)(\nu-2)+7\nu(\nu-1)+\nu, \tag{27}$$

and

$$\begin{aligned}
\nu^5=&\,\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)+10\nu(\nu-1)(\nu-2)(\nu-3)+25\nu(\nu-1)(\nu-2)\\
&+15\nu(\nu-1)+\nu,
\end{aligned} \tag{28}$$

in (24), we have

$$\begin{aligned}
&Q(\varrho,\epsilon,\delta,\eta)\\
&\leq \frac{1}{6}\left[\sum_{\nu=2}^{\infty}[2\varrho\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)+23\varrho\nu(\nu-1)(\nu-2)(\nu-3)\right.\\
&+(67\varrho+2)\nu(\nu-1)(\nu-2)+(45\varrho+9)\nu(\nu-1)\\
&+6\nu]\frac{\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)}\\
&+\sum_{\nu=2}^{\infty}[2\varrho\nu(\nu-1)(\nu-2)(\nu-3)(\nu-4)+17\varrho\nu(\nu-1)(\nu-2)(\nu-3)\\
&\left.+(31\varrho+2)\nu(\nu-1)(\nu-2)+(9\varrho+3)\nu(\nu-1)]\frac{\Gamma(\delta)}{\Gamma(\eta(\nu-1)+\delta)}\right]\\
&=\frac{1}{6}\Big[2\varrho\chi^{(5)}_{\rho,\epsilon}(1)+23\varrho\chi^{(4)}_{\rho,\epsilon}(1)+(67\varrho+2)\chi^{(3)}_{\rho,\epsilon}(1)\\
&+(45\varrho+9)\chi^{(2)}_{\rho,\epsilon}(1)+6\chi'_{\rho,\epsilon}(1)\\
&+2\varrho\chi^{(5)}_{\eta,\epsilon}(1)+17\varrho\chi^{(4)}_{\eta,\epsilon}(1)+(31\varrho+2)\chi^{(3)}_{\eta,\epsilon}(1)+(9\varrho+3)\chi^{(2)}_{\eta,\epsilon}(1)\Big].
\end{aligned}$$

Now $Q(\varrho,\epsilon,\delta,\eta)\leq 1-\gamma$ if (22) holds. $\square$

Theorem 2. *Let $\varrho \geq 0$, $\gamma \in [0,1)$ and $\rho, \epsilon, \eta, \delta \notin \mathbb{Z}_0^-$. If*

$$\begin{aligned}&[\varrho\chi^{(4)}_{\rho,\epsilon}(1)+7\varrho\chi^{(3)}_{\rho,\epsilon}(1)+(9\varrho+1)\chi^{(2)}_{\rho,\epsilon}(1)+2\chi'_{\rho,\epsilon}(1)+\varrho\chi^{(4)}_{\eta,\epsilon}+5\varrho\chi^{(3)}_{\eta,\epsilon}(1)\\&+(5\varrho-1)\chi^{(2)}_{\eta,\epsilon}+2(\varrho-1)\chi'_{\eta,\epsilon}(1)].\\&\leq 2(1-\gamma),\end{aligned} \tag{29}$$

then

$$\Theta(\mathcal{K}_{\mathcal{HF}}) \subset \mathcal{HF}(\varrho,\gamma).$$

Proof. Let $\Im = \phi + \overline{\psi} \in \mathcal{K}_{\mathcal{HF}}$ where ϕ and ψ are of the form (2) with $b_1 = 0$. We need to show that $\Theta(\Im) = \mathfrak{F}(\xi) \in \mathcal{HF}(\varrho,\gamma)$ which given by (5) with $b_1 = 0$. In view of Lemma 1, we need to prove that$Q(\varrho,\epsilon,\delta,\eta)$

$$Q(\varrho,\epsilon,\delta,\eta) \leq 1-\gamma,$$

where $Q(\varrho,\epsilon,\delta,\eta)$ as given in (23). Using the inequalities (16) of Lemma 5, we get

$$\begin{aligned}Q(\varrho,\epsilon,\delta,\eta) &\leq \frac{1}{2}\left[\sum_{\nu=2}^{\infty}(\nu+1)\left(\nu+\varrho\nu\left(\nu^2-1\right)\right)\frac{\Gamma\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)}\right.\\&\left.+\sum_{\nu=2}^{\infty}(\nu-1)\left(\nu+\varrho\nu\left(\nu^2-1\right)\right)\frac{\Gamma(\delta)}{\Gamma(\eta(\nu-1)+\delta)}\right]\\&=\frac{1}{2}\left[\sum_{\nu=2}^{\infty}\left[\varrho\nu^4+\varrho\nu^3+(1-\varrho)\nu^2+(1-\varrho)\nu\right]\frac{\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)}\right.\\&\left.+\sum_{\nu=2}^{\infty}\left[\varrho\nu^4-\varrho\nu^3+(1-\varrho)\nu^2+(\varrho-1)\nu\right]\frac{\Gamma(\delta)}{\Gamma(\eta(\nu-1)+\delta)}\right].\end{aligned}$$

Using the Equations (25)–(27), we have

$$\begin{aligned}&Q(\varrho,\epsilon,\delta,\eta) \leq \frac{1}{2}\left[\sum_{\nu=2}^{\infty}[\varrho\nu(\nu-1)(\nu-2)(\nu-3)+7\varrho\nu(\nu-1)(\nu-2)\right.\\&\left.+(9\varrho+1)\nu(\nu-1)+2\nu]\frac{\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)}\right]\\&+\frac{1}{2}\sum_{\nu=2}^{\infty}[\varrho\nu(\nu-1)(\nu-2)(\nu-3)+5\varrho\nu(\nu-1)(\nu-2)+(5\varrho-1)\nu(\nu-1)\\&2(\varrho-1)\nu]\frac{\Gamma(\delta)}{\Gamma(\eta(\nu-1)+\delta)}\\&=\frac{1}{2}[\varrho\chi^{(4)}_{\rho,\epsilon}(1)+7\varrho\chi^{(3)}_{\rho,\epsilon}(1)+(9\varrho+1)\chi^{(2)}_{\rho,\epsilon}(1)+2\chi'_{\rho,\epsilon}(1)+\varrho\chi^{(4)}_{\eta,\epsilon}+5\varrho\chi^{(3)}_{\eta,\epsilon}(1)+(5\varrho-1)\chi^{(2)}_{\eta,\epsilon}\\&+2(\varrho-1)\chi'_{\eta,\epsilon}(1)].\end{aligned}$$

Now $Q(\varrho,\epsilon,\delta,\eta) \leq 1-\gamma$ if (29) holds. □

The connection between $\mathcal{TN}_{\mathcal{HF}}(\tau)$ and $\mathcal{HF}(\varrho,\gamma)$ is given below in the next theorem.

Theorem 3. *Let $\varrho \geq 0$, $\gamma, \tau \in [0,1)$ and $\rho, \epsilon, \eta, \delta \notin \mathbb{Z}_0^-$. If*

$$\begin{aligned}&(1-\tau)\left[\varrho\left(\chi^{(2)}_{\rho,\epsilon}(1)+\chi^{(2)}_{\eta,\epsilon}(1)\right)+\varrho\left(\chi'_{\rho,\epsilon}(1)+\chi'_{\eta,\epsilon}(1)\right)+(1-\varrho)\left(\chi_{\rho,\epsilon}(1)+\chi_{\eta,\epsilon}(1)-2\right)\right]\\&\leq 1-\gamma-|b_1|,\end{aligned}$$

then

$$\Theta(\mathcal{T}\mathfrak{N}_{\mathcal{HF}}(\tau)) \subset \mathcal{HF}(\varrho, \gamma).$$

Proof. Let $\Im = \phi + \overline{\psi} \in \mathcal{T}\mathfrak{N}_{\mathcal{HF}}(\tau)$ where ϕ and ψ are given by (2). In view of Lemma 1, it is enough to show that $P(\varrho, \epsilon, \delta, \eta) \leq 1 - \gamma$, where

$$\begin{aligned} P(\varrho, \epsilon, \delta, \eta) &= \sum_{\nu=2}^{\infty} \left(\nu + \varrho\nu\left(\nu^2 - 1\right)\right) \left| \frac{\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)} a_\nu \right| \\ &\quad + |b_1| + \sum_{\nu=2}^{\infty} \left(\nu + \varrho\nu\left(\nu^2 - 1\right)\right) \left| \frac{\Gamma(\delta)}{\Gamma(\eta(\nu-1)+\delta)} b_\nu \right|. \end{aligned} \tag{30}$$

Using the inequalities (10) and (11) of Lemma 2, it follows that

$$\begin{aligned} P(\varrho, \epsilon, \delta, \eta) &\leq (1-\tau)\left[\sum_{\nu=2}^{\infty} \left(\varrho\nu^2 + 1 - \varrho\right) \frac{\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)} \right. \\ &\quad \left. + \sum_{\nu=2}^{\infty} \left(\varrho\nu^2 + 1 - \varrho\right) \frac{\Gamma(\delta)}{\Gamma(\eta(\nu-1)+\delta)} \right] + |b_1| \\ &= (1-\tau)\left[\sum_{\nu=2}^{\infty} [\varrho\nu(\nu-1) + \varrho\nu + 1 - \varrho] \frac{\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)} \right. \\ &\quad \left. + \sum_{\nu=2}^{\infty} [\varrho\nu(\nu-1) + \varrho\nu + 1 - \varrho] \frac{\Gamma(\delta)}{\Gamma(\eta(\nu-1)+\delta)} \right] + |b_1| \\ &= (1-\tau)\Big[\varrho\chi^{(2)}_{\rho,\epsilon}(1) + \varrho\chi'_{\rho,\epsilon}(1) + (1-\varrho)\left(\chi_{\rho,\epsilon}(1) - 1\right) \\ &\quad + \varrho\chi^{(2)}_{\eta,\epsilon}(1) + \varrho\chi'_{\eta,\epsilon}(1) + (1-\varrho)\left(\chi_{\eta,\epsilon}(1) - 1\right) \Big] + |b_1| \\ &\leq 1 - \gamma, \end{aligned}$$

by the given hypothesis. □

Below we prove that $\Theta(\mathcal{T}\mathfrak{N}_{\mathcal{HF}}(\tau)) \subset \mathcal{HF}(\varrho, \gamma)$.

Theorem 4. *Let $\varrho \geq 0$, $\gamma, \tau \in [0,1)$ and $\rho, \epsilon, \eta, \delta \notin \mathbb{Z}_0^-$. If*

$$\begin{aligned} &(1-\tau)\left[\varrho\left(\chi'_{\rho,\epsilon}(1) + \chi'_{\eta,\epsilon}(1)\right) + \int_0^1 \frac{\chi_{\rho,\epsilon}(s)}{s} ds + \int_0^1 \frac{\chi_{\eta,\epsilon}(s)}{s} ds \right] \\ &\leq 1 - \delta - |b_1|, \end{aligned}$$

then

$$\Theta(\mathcal{T}\mathfrak{N}_{\mathcal{HF}}(\tau)) \subset \mathcal{HF}(\varrho, \gamma).$$

Proof. Making use of Lemma 1, we need only to prove that $P(\varrho,\epsilon,\delta,\eta) \leq 1-\gamma$, where $P(\varrho,\epsilon,\delta,\eta)$ as given in (30). Using the inequalities (13) and (14) of Lemma 3, it follows that

$$\begin{aligned} P(\varrho,\epsilon,\delta,\eta) &= \sum_{\nu=2}^{\infty}\left(\nu+\varrho\nu\left(\nu^2-1\right)\right)\left|\frac{\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)}a_\nu\right| \\ &+ |b_1| + \sum_{\nu=2}^{\infty}\left(\nu+\varrho\nu\left(\nu^2-1\right)\right)\left|\frac{\Gamma(\delta)}{\Gamma(\eta(\nu-1)+\delta)}b_\nu\right| \\ &\leq (1-\tau)\left[\sum_{\nu=2}^{\infty}\left(\varrho\nu+\frac{1-\varrho}{\nu}\right)\frac{\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)}\right. \\ &\left.+\sum_{\nu=2}^{\infty}\left(\varrho\nu+\frac{1-\varrho}{\nu}\right)\frac{\Gamma(\delta)}{\Gamma(\eta(\nu-1)+\delta)}\right] \\ &= (1-\tau)\left[\varrho\chi'_{\rho,\epsilon}(1)+\int_0^1\frac{\chi_{\rho,\epsilon}(s)}{s}dt+\varrho\chi'_{\eta,\epsilon}(1)+\int_0^1\frac{\chi_{\eta,\epsilon}(s)}{s}ds\right]+|b_1| \\ &\leq 1-\gamma, \end{aligned}$$

by given hypothesis. □

Theorem 5. *Let* $\varrho \geq 0,\ \gamma,\tau \in [0,1)$ *and* $\rho,\epsilon,\eta,\delta \notin \mathbb{Z}_0^-$ *. If*

$$\chi_{\rho,\epsilon}(1)+\chi_{\eta,\epsilon}(1) \leq 3-\frac{|b_1|}{1-\gamma}$$

then

$$\Theta(\mathcal{HF}(\varrho,\gamma)) \subset \mathcal{HF}(\varrho,\gamma).$$

Proof. Using Lemma 1 and the inequalities (7) and (8) of Lemma 1, we obtain

$$\begin{aligned} P(\varrho,\epsilon,\delta,\eta) &\leq (1-\gamma)\left[\sum_{\nu=2}^{\infty}\frac{\Gamma(\epsilon)}{\Gamma(\rho(\nu-1)+\epsilon)}+\sum_{\nu=2}^{\infty}\frac{\Gamma(\delta)}{\Gamma(\eta(\nu-1)+\delta)}\right]+|b_1| \\ &= (1-\gamma)\left[(\chi_{\rho,\epsilon}(1)-1)+(\chi_{\eta,\epsilon}(1)-1)\right]+|b_1| \\ &= (1-\gamma)[\chi_{\rho,\epsilon}(1)+\chi_{\eta,\epsilon}(1)-2]+|b_1| \\ &\leq 1-\gamma, \end{aligned}$$

by the given condition and this completes the proof of the theorem. □

5. Special Cases

Putting $\varrho = 0$ in Theorems 1–4, we obtain the following results.

Corollary 1. *Let* $\gamma \in [0,1)$ *and* $\rho,\epsilon,\eta,\delta \notin \mathbb{Z}_0^-$ *. If*

$$2\left(\chi^{(3)}_{\rho,\epsilon}(1)+\chi^{(3)}_{\eta,\epsilon}(1)\right)+9\chi^{(2)}_{\rho,\epsilon}(1)+6\chi'_{\rho,\epsilon}(1)+3\chi^{(2)}_{\eta,\epsilon}(1) \leq 6(1-\gamma),$$

then

$$\Theta(\mathcal{S}^*_{\mathcal{HF}}) \subset \mathcal{HF}(\gamma).$$

Corollary 2. *Let* $\gamma \in [0,1)$ *and* $\rho,\epsilon,\eta,\delta \notin \mathbb{Z}_0^-$ *. If*

$$\begin{aligned} &[\chi^{(2)}_{\rho,\epsilon}(1)-\chi^{(2)}_{\eta,\epsilon}(1)+2\left(\chi'_{\rho,\epsilon}(1)-\chi'_{\eta,\epsilon}(1)\right)] \\ &\leq 2(1-\gamma), \end{aligned} \tag{31}$$

then

$$\Theta(\mathcal{K}_{\mathcal{HF}}) \subset \mathcal{HF}(\gamma).$$

Corollary 3. *Let* $\gamma \in [0,1)$ *and* $\rho, \epsilon, \eta, \delta \notin \mathbb{Z}_0^-$ *. If*

$$(1-\tau)\left[\left(\chi_{\rho,\epsilon}(1)+\chi_{\eta,\epsilon}(1)\right)-2\right] \leq 1-\gamma-|b_1|,$$

then

$$\Theta(\mathcal{T}\mathfrak{N}_{\mathcal{HF}}(\tau)) \subset \mathcal{HF}(\gamma).$$

Corollary 4. *Let* $\gamma \in [0,1)$ *and* $\rho, \epsilon, \eta, \delta \notin \mathbb{Z}_0^-$ *. If*

$$(1-\tau)\left[\int_0^1 \frac{\chi_{\rho,\epsilon}(s)}{s}dt + \int_0^1 \frac{\chi_{\eta,\epsilon}(s)}{s}dt\right] \leq 1-\gamma-|b_1|,$$

then

$$\Theta(\mathcal{T}\mathfrak{R}_{\mathcal{HF}}(\tau)) \subset \mathcal{HF}(\gamma).$$

6. Conclusions

Making use of the of the operator Θ given in (5) related with Mittag-Leffler function, we found some inclusion relations of the harmonic class $\mathcal{HF}(\varrho,\delta)$ with other classes of harmonic analytic function defined in the open disk. Further, and for $\varrho = 0$, several results of the main results are given. Following this study, one can find new inclusion relations for new harmonic classes of analytic functions using the operator Θ.

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