

Article

Overlapping of Lévai's and Milson's e -Tangent-Polynomial Potentials along Symmetric Curves

Gregory Natanson

AI-Solutions Inc., Lanham, MD 20706, USA; greg_natanson@yahoo.com

Abstract: The paper examines common elements between Lévai's and Milson's potentials obtained by Liouville transformations of two rational canonical Sturm–Liouville equations (RCSLEs) with even density functions which are exactly solvable via Jacobi polynomials in a real or accordingly imaginary argument. We refer to the polynomial numerators of the given rational density function as 'tangent polynomial' (TP) and thereby term the aforementioned potentials as ' e -TP'. Special attention is given to the overlap between the two potentials along symmetric curves which represent two different rational forms of the Ginocchio potential exactly quantized via Gegenbauer and Masjed-Jamei polynomials, respectively. Our analysis reveals that the actual interconnection between Lévai's parameters for these two rational realizations of the Ginocchio potential is much more complicated than one could expect based on the striking resemblance between two quartic equations derived by Lévai for 'averaged' Jacobi indexes.

Keywords: rational Sturm–Liouville equation; Liouville transformation; complex Jacobi polynomials; classical Jacobi polynomials; Romanovski–Routh polynomials; Masjed-Jamei polynomials; quasi-rational solutions; almost-everywhere holomorphic solutions

MSC: 34B24



Citation: Natanson, G. Overlapping of Lévai's and Milson's e -Tangent-Polynomial Potentials along Symmetric Curves. *Axioms* **2023**, *12*, 584. <https://doi.org/10.3390/axioms12060584>

Academic Editor: Clemente Cesarano

Received: 2 May 2023

Revised: 5 June 2023

Accepted: 6 June 2023

Published: 12 June 2023



Copyright: © 2023 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

Nearly 60 years ago, Bose [1,2] made a revolutionary discovery that the most solvable potentials known at that time can be obtained by the 'Liouville transformation' [3,4] of some special cases of a rational Sturm–Liouville equation [4] written in its canonical (no first derivative) form and simply referred to below as 'RCSLE'. Several years later, the author [5] made use of Bose's discovery to construct the family of rational potentials *exactly* solvable in terms of superpositions of two hypergeometric functions of a *real* variable as well as its confluent counterpart exactly solvable in terms of superpositions of confluent hypergeometric series and Whittaker function (see, e.g., Ch. 5 and Ch. 6 in [3]). It was shown that the corresponding eigenfunctions are expressible in terms of *classical* Jacobi and *classical* Laguerre polynomials with *degree-dependent* indexes in general. Thereby, we refer to these two families of solvable rational potentials as 'Jacobi-reference' (JRef) and 'Laguerre-reference' (LRef) potentials.

The Bose approach was further advanced by Milson [6], who demonstrated that there is another family of solvable potentials obtained by the Liouville transformation of the real Fuschian SLE with two poles at complex-conjugated points. It was proven by us later [7] under the influence of Alvarez-Castillo and Kirchbach's breakthrough paper [8] that the eigenfunctions of this RCSLE are expressible in terms of Romanovski–Routh (R–Routh) polynomials with degree-dependent indexes. (For the reasons explained in [9,10], we prefer to refer to pseudo-Jacobi polynomials [11–13] as Routh polynomials [14], so the epithet 'R–Routh polynomials' is fully consistent with the term 'Romanovski/pseudo-Jacobi polynomials' in Leski's [15,16] classification scheme of Romanovski polynomials [17,18]).

While making some inspirational remarks on the original draft of [19], Kirchbach drew the author's attention to the reference to Milson's paper [6] in the brand-new review

article [20]. It became clear that Milson has already addressed this problem to a large extent. However, our study revealed some new important elements not discussed in [6]. First, it was shown that hypergeometric polynomials constructed in [6] are nothing but R–Routh polynomials. Secondly, the author took advantage of Stevenson’s idea [21] (also pointed to the author by Kirchbach) to express an analytically continued solution in terms of hypergeometric polynomials in a complex argument. It was just confirmed that the latter *formally complex* polynomials can be converted to real R–Routh polynomials in the real argument and that the Sturm–Liouville problem in question is indeed exactly solvable.

Our current analysis explicitly takes advantage of the interconnection [8] between the real R–Routh polynomials and Jacobi polynomials with complex-conjugated indexes, provided that the argument of the complex polynomials is restricted to the imaginary axis [22,23]. To our knowledge, it was Bagrov et al. [24] who brought the latter polynomials into the quantum mechanics using them as polynomial components of the eigenfunctions for the *very unique trigonometric* limit of the potentials solvable in terms of the R–Routh polynomials—the so-called ‘Rosen–Morse II’ potential in the Cooper–Khare–Sukhatme (CKS) [25,26] classification scheme of solvable rational potentials. In following [27], we refer to this trigonometric modification of the Rosen–Morse potential [28] simply as ‘T-RM’. The cited textbook [24] made no mathematical references and did not provide any arguments in support of the conjecture that the constructed eigenfunctions are real.

A few years later, Dabrowska et al. [29] ran into these polynomials while examining the eigenfunctions for the Gendenshtein potential [30] (the Scarf II potential in the CKS classification scheme [25,26]). The cited authors explicitly computed the first three eigenfunctions and did confirm that they are real and mutually orthogonal. Under the evident influence of these authors, Lévai [31], in his search for ‘shape-invariant’ [30] rational potentials, extended to *complex* Jacobi polynomials the systematic method suggested by Bhattacharjie and Sudarshan [32] for constructing potentials exactly solvable via classical Jacobi polynomials. He found that the list of shape-invariant potentials compiled in [29] missed the aforementioned T-RM potential.

Though the fact that the eigenvalues of the JRef CSLE coincide with one of real roots of a quartic equation was originally recognized by Grosche [33], it was Lévai [34] who proved that the average of the Jacobi indexes of the polynomial forming a ‘quasi-rational’ [35] eigenfunction is determined by a root of a similar quartic equation in the particular case of an even density function. It was originally overlooked by the author [7,10] that Lévai (apparently unaware of Milson’s paper [6]) implicitly discussed both JRef and Routh-reference (‘RRef’) CSLEs with even density functions. (We use the epithet ‘RRef’ to stress that the CSLE in question have ‘quasi-rational’ [35] solutions (q-RSs) expressible in terms of Routh polynomials.) In particular, the quartic equation derived by us [7] for complex-conjugated indexes of the Jacobi polynomials forming eigenfunctions of the RRef CSLE happened to be another representation for the equation obtained by Lévai [34] in a more general context. In following the terminology suggested in our papers [7,36,37], we refer to the numerator of the rational density function in the RCSLEs of our interest as the ‘tangent polynomial’ (TP), so we term the Liouville potential associated with the even density function as the ‘e-TP’ potential for brevity.

In [38], Lévai presented a more thorough analysis comparing his approach [31,34,39] and our original technique [5] used for constructing the general 1D Schrödinger equation exactly solvable via a superposition of hypergeometric functions. An examination of his arguments illuminated in the next section revealed that, while starting from the complex Jacobi equation, Lévai implicitly converts it to the CSLE initially introduced by us [5] under the real field, with the only (though very important for his analysis) difference being that the new CSLE was expressed in terms of the variable $2z(x) - 1$ instead of the variable $z(x)$ used in [5] to convert the 1D Schrödinger equation to the hypergeometric equation in z . Obviously, the conversion of the CSLE by a linear transformation of the variable may not affect the results, so there is no surprise that introducing the solved-by-polynomials equation [31] and then converting it to its canonical form leads to exactly the same results

as the Bose technique [1,2] directly starting from the resultant CSLE. Contrary to Lévai’s assertion in [40], it is the new variable (not the difference in the approach) ‘suits better the formulation of solvable PT-symmetric potentials than that in [5]’. The author has already taken advantage of this variable introduced in Lévai’s earlier works [31,39] while discussing the exactly solvable JRef and RRef problems in parallel in [19]. Regrettably, this re-examination of our original study [5] (as well as its extension by Milson [6]) was disregarded in Lévai’s papers [38,40,41] (see Section 2 for more details).

Though the primary focus of this paper is the real rational Sturm–Liouville problems solvable by polynomials, we (under the influence of Lévai’s cited works [31,39]) start our analysis in Section 2 from a complex (non-self-adjoint) RCSLE such that its two real-field reductions result in the *self-adjoint* RCSLEs with JRef and RRef Liouville potentials. It is worth stressing in this context that our persistent references to the Schrödinger equation with solvable rational potentials are made solely because the RCSLEs represented in their Liouville form have broad applications in quantum mechanics and are much better known to scientists. Also, note that Everitt’s [4] catalogue of Sturm–Liouville differential equations includes some renowned examples of the solvable JRef and LRef Liouville potentials, while disregarding their generalizations presented in our paper [5] as well as the extension of our technique to the Fuschian RCSLE with two poles on the imaginary axis [6].

In Section 3, we present a unified approach to the Liouville potentials of the JRef and RRef CSLEs with even density functions by treating them as two real branches of the generally complex potential of the ‘Lévai class’. We term these two branches as Lévai’s and Milson’s *e*-TP potentials. The new element of our analysis of Lévai’s *e*-TP potential, compared with [34], is the proof [36] that the $(n + 1)$ -th eigenfunction (specified by the label **c** in our notation) for ‘nearly symmetric’ potentials is accompanied by a triplet of q-RSs composed of Jacobi polynomials of the same degree n which belong to three different types **a**, **b**, and **d**. Since any solution vanishing at one of the quantization ends (types **a** and **b**) and lying below the ground-energy level are necessarily nodeless [42], the q-RSs of these two types can be used as the factorization functions (FFs) for the so-called [43] ‘rational Darboux transformations’ (RDTs) giving rise to new exactly solvable rational potentials [37]. We also point to some important details in both Milson’s [6] and our [7,10] papers which are absent in Lévai’s sketch of this little-known problem of exactly solvable in terms of R–Routh polynomials with degree-dependent indexes.

As has been already asserted by Lévai [34], the symmetric reduction of the RRef potential is nothing but an alternative representation for the Ginocchio potential on the line [44]. Though the assertion itself happened to be correct [45], the interrelationship between the two alternative parametrizations of the Ginocchio potential examined in Section 4 turned out to be much more complicated than one would expect based on Lévai’s observation that the quartic equations for the averaged Jacobi indexes in both quantization schemes may be rewritten in the unified fashion.

The most important consequence of the presented proof is that, in addition to the renowned quantization scheme by classical Gegenbauer polynomials [44], the cited symmetric potential can be alternatively quantized by R–Routh polynomials of a definite parity termed by us [7,45] as ‘Masjed-Jamei polynomials’ to give credit to the scrupulous analysis of these polynomials in [46]. In other words, the RCSLEs with even JRef [44] and even RRef [7] Bose invariants constitute the same spectral problem unambiguously defined by the common Liouville form of its two rational realizations—the main result of this study proven in Section 4.

2. Quartic Equation for the Average of Indexes of Jacobi Polynomial Forming a q-RS of Complex Fuschian CSLE with Three Singular Points

Let us start our analysis from the *complex* (non-self-adjoint) Fuschian CSLE with three singular points

$$\left\{ \frac{d^2}{d\eta^2} + I[\eta; \bar{\Lambda}; T_K; \varepsilon] \right\} \Phi[\eta; \bar{\Lambda}; T_K; \varepsilon] = 0, \tag{1}$$

representing its Bose invariant [1,2,6] as

$$I[\eta; \bar{\Lambda}; T_K; \varepsilon] = I^o[\eta; \bar{\Lambda}] + \rho[\eta; T_K] \varepsilon, \tag{2}$$

with both reference polynomial fraction (RefPF)

$$I^o[\eta; \bar{\Lambda}] = -\frac{h_{o;-}}{4(\eta + 1)^2} - \frac{h_{o;+}}{4(1 - \eta)^2} - \frac{O_0^o}{4(1 - \eta^2)} \tag{3}$$

and the density function

$$\rho[\eta; T_K] = \frac{T_K[\eta]}{(1 - \eta^2)^2} \quad (K \leq 2) \tag{4}$$

having second-order poles at $\eta = \pm 1$. Note that, in contrast with JRefPF (3), Bose invariant (2) also depends on the coefficients of the TP of degree $K \leq 2$:

$$T_K[\eta; a, c_{\pm}] = 1/2 [c_+(1 + \eta) + c_-(1 - \eta)] - a(1 - \eta^2) \tag{5}$$

$$= 1/4 [c_+(1 + \eta)^2 + c_-(1 - \eta)^2 - d(1 - \eta^2)], \tag{6}$$

where

$$d \equiv 4a - c_+ - c_- \tag{7}$$

The real-field self-adjoint reduction of JRef CSLE (1) was initially introduced by us in [19] to treat JRef and RRef CSLEs in parallel, to a large extent under the influence of Lévai’s renowned papers [31,39]. Examination of (4) in [38] shows that

$$I[\eta; \bar{\Lambda}; T_K; \varepsilon] = R[\eta; \bar{\Lambda}; T_K; \varepsilon] - 1/2 \dot{Q}[\eta; \bar{\Lambda}; T_K; \varepsilon] - 1/4 Q[\eta; \bar{\Lambda}; T_K; \varepsilon], \tag{8}$$

while Lévai’s second-order differential Equation (2) can be rewritten as

$$\left\{ (1 - \eta^2) \frac{d^2}{d\eta^2} + \tau[\eta; \bar{\Lambda}; T_K; \varepsilon] \frac{d}{d\eta} + \varepsilon \right\} F[\eta; \bar{\Lambda}; T_K; \varepsilon] = 0 \tag{9}$$

with

$$\tau[\eta; \bar{\Lambda}; T_K; \varepsilon] \equiv (1 - \eta^2) Q[\eta; \bar{\Lambda}; T_K; \varepsilon] \tag{10}$$

and

$$(1 - \eta^2) R[\eta; \bar{\Lambda}; T_K; \varepsilon] = \varepsilon. \tag{11}$$

Note that the parameters p_I , p_{II} , and p_{III} in (9) in [38] are related to the coefficients a, c_{\pm} of TP (5) in the trivial fashion:

$$p_I \equiv -a, \quad p_{II} \equiv 1/2(c_+ + c_-), \quad p_{III} \equiv 1/2(c_+ - c_-). \tag{12}$$

Lévai’s starting Formula (13) in [38], with

$$\phi[\eta] \equiv T_K[\eta] = (1 - \eta^2)^2 \rho[\eta; a, c_{\pm}] \tag{13}$$

in our notation, is nothing but the conventional representation of the given Liouville potential rewritten as

$$V[\eta; h_{o;\pm}, O_0^o; a, c_{\pm}] + 1/2\{\eta, x\} = -\rho^{-1}[\eta; a, c_{\pm}] I^o[\eta; h_{o;\pm}, O_0^o; a, c_{\pm}], \tag{14}$$

where the Schwarzian derivative $\{\eta, x\}$ is expressed in terms of η (cf. (2.6) in [19]), provided that the change of variable $\eta(x; T_K)$ satisfies the ordinary differential equation (ODE)

$$\frac{dx}{d\eta} = \frac{1 - \eta^2}{\sqrt{T_K[\eta]}} \tag{15}$$

with prime denoting the derivative with respect to x . The parameters s_I , s_{II} , and s_{III} in the mentioned formula (or similarly in (1.29) in [41]) are thus related to the parameters $\bar{\Lambda}$ of JRefPF (3) above as follows:

$$\begin{aligned} s_I &\equiv 1/4 (O_0^o - h_{o;-} - h_{o;+}), \\ s_{II} &\equiv 1/2(h_{o;+} - h_{o;-}), \\ s_{III} &\equiv 1/2(h_{o;-} + h_{o;+}). \end{aligned} \tag{16}$$

Disregarding the enhancements suggested by us in [19], Lévai simply compared his approach with our initial scheme [5] using the variable $z(x)$ satisfying the ODE

$$\frac{dx}{dz} = \frac{2z(1 - z)}{\sqrt{{}_1T_K[z]}}. \tag{17}$$

This transformation converts the Schrödinger equation into the JRef CSLE with the Bose invariant

$${}_1I[z; \bar{\Lambda}; {}_1T_K; \varepsilon] \equiv 4 I[2z - 1; \bar{\Lambda}; T_K; \varepsilon], \tag{18}$$

where ${}_1T_K$ are the coefficients of the TP

$${}_1T_K[z] \equiv T_K[2z - 1]. \tag{19}$$

Though our initial technique [5] dealing with hypergeometric functions of the variable $z \in (0, 1)$ is necessary to prove that the given potential is indeed exactly solvable (as well as to derive close-form expressions for the scattering amplitudes [47,48]), the use of the variable $\eta = 2z - 1 \in (-1, 1)$ allows one to treat JRef and RRef potentials in a symmetric fashion and also makes it easier to examine PT-symmetric reductions of the complex JRefPF potential as it has been done by Lévai [34,38,40,41]. Obviously, the conversion of the CSLE by a linear transformation of the variable may not affect the results, so there is no surprise that starting from the solved-by-polynomials equation [31] and then converting it to the canonical form leads to exactly the same results as the Bose technique [1,2] directly starting from the resultant CSLE. Lévai’s assertion in [40] that the discussion of this problem in [38] ‘revealed that his approach suits better the formulation of solvable PT-symmetric potentials than that in [5]’ is not precisely accurate—it is the change of variable

$$\eta(x; T_K) = 2z(x; {}_1T_K) - 1 \tag{20}$$

that suits better his analysis [38] compared with the variable $z(x; T_K)$. Namely, if the reflection $\eta \rightarrow -\eta$ keeps the TP unchanged—the case of our current interest—then one can choose $\eta(x; T_K)$ to be an odd function of x . If all the TP coefficients are required to be real, then the Schwarzian derivative $\{\eta, x\}$ becomes an even real function of x and the PT-transformation is equivalent to the complex conjugation of RefPF (3) followed by the reflection of the argument η .

As another novel development inspired by Lévai’s works [31,34,38,39], the author can point to our recent idea [9,49,50] to introduce complex (non-self-adjoint) CSLE (1) and then examine its so-called [36,51,52] ‘almost-everywhere holomorphic’ (AEH) solutions

$$\Phi_{k,m}[\eta; \bar{\Lambda}; T_K] = \prod_{\aleph = \pm} (1 + \aleph\eta)^{\rho_{\aleph;k,m}} \Pi_m[\eta; \bar{\eta}^{(m)}] \tag{21}$$

which exist at some energies $\epsilon_{k,m}$ inside the vertical band $|Re \eta| < 1$ in the complex plane. It will be proven below that the monomial product

$$\Pi_m[\eta; \bar{\eta}^{(m)}] \equiv \prod_{l=1}^m (\eta - \eta_l^{(m)}) \tag{22}$$

coincides with the monic Jacobi polynomial with generally complex indexes

$$\Pi_m[\eta; \bar{\eta}^{(m)}] = \hat{P}_m^{(\lambda_{\pm;k,m}, \lambda_{-;k,m})}(\eta). \tag{23}$$

(It would be more accurate to use the notation $\rho_{\pm;k,m}(\bar{\Lambda}; T_K), \lambda_{\pm;k,m}(\bar{\Lambda}; T_K)$, and $\epsilon_{k,m}(\bar{\Lambda}; T_K)$, but we disregard the dependence of the latter quantities on the parameters $\bar{\Lambda}$ and T_K for brevity.) Note that we use the term ‘AEH solutions’, instead of the equivalent epithet ‘q-RSs’ appearing in the section title, to stress that we deal with complex functions analytically continued from the real axis into the complex plane.

We then took advantage of the fact that the so-called [53,54] ‘differential polynomial system’ (DPS) composed of complex Jacobi polynomials [55] allows the second real-field reduction formed by Routh polynomials [14], in addition to the one formed by conventional (real) Jacobi polynomials.

To derive the necessary and sufficient conditions for CSLE (1) to have an AEH solution (21), first note that characteristic exponents (ChExps) of these solutions for the poles of CSLE (1) at ± 1 satisfy the indicial equations

$$(2\rho_{\pm;k,m} - 1)^2 = h_{o;\pm} + 1 - c_{\pm} \epsilon_{k,m}. \tag{24}$$

Introducing the complex exponent differences (ExpDiffs)

$$\lambda_{\pm;k,m} \equiv 2\rho_{\pm;k,m} - 1, \tag{25}$$

we come to the equation

$$\lambda_{\pm;k,m}^2 = h_{o;\pm} + 1 - c_{\pm} \epsilon_{k,m}. \tag{26}$$

Representing RefPF (3) as

$$I^o[\eta; h_{o;\pm}, \mu_o] = - \frac{h_{o;-} + h_{o;+} + (h_{o;+} - h_{o;-})\eta}{2(1 - \eta^2)^2} + \frac{\mu_o^2 - 1}{4(1 - \eta^2)}, \tag{27}$$

where the parameter

$$\mu_o \equiv \sqrt{h_{o;+} + h_{o;-} - O_0^o + 1} \tag{28}$$

is chosen to coincide with the ExpDiff for the pole of the JRef CSLE at infinity, and examining asymptotic behavior of AEH solutions near this pole, we can write the closing equation for the given system of algebraic equations in $\lambda_{\pm;k,m}$ and $\epsilon_{k,m}$ as follows:

$$a \epsilon_{k,m} = 1/4 \mu_o^2 - \rho_{\infty;k,m}(\rho_{\infty;k,m} + 1), \tag{29}$$

where

$$\rho_{\infty;k,m} = -\omega_{k,m} - m - 1 \tag{30}$$

is the ChExp for the pole of CSLE (1) at infinity and

$$\omega_{k,m} \equiv 1/2(\lambda_{-;k,m} + \lambda_{+;k,m}). \tag{31}$$

It can be shown that the derived system of algebraic Equations (26), (29)–(31) is simply another representation for coupled Equations (14)–(16) in [34].

Squaring (31) gives

$$2\lambda_{+;k,m} \lambda_{-;k,m} = 4\omega_{k,m}^2 - \lambda_{+;k,m}^2 - \lambda_{-;k,m}^2 \tag{32}$$

while substituting (30) into (29) brings us to the following energy dispersion formula:

$$a \epsilon_{k,m} = 1/4 (\mu_0^2 + 1) - (\omega_{k,m} + m + 1/2)^2 (K = 2, a \neq 0). \tag{33}$$

Rewriting (26) as

$$\lambda_{\pm; k,m}^2 = \hbar_{o;\pm} + 1 + c_{\pm} (\omega_{k,m} + m + 1/2)^2 / a, \tag{34}$$

where

$$\hbar_{o;\pm} \equiv h_{o;\pm} - 1/4 c_{\pm} \mu_0^2 / a, \tag{35}$$

squaring (32) and making use of (34), we come to the following quartic equation

$$\begin{aligned} & 1/4 a^2 [4\omega_{k,m}^2 - \hbar_{o;+} - \hbar_{o;-} - 2 - (c_+ + c_-) (\omega_{k,m} + m + 1/2)^2 / a]^2 - \\ & [a (\hbar_{o;+} + 1) - c_+ (\omega_{k,m} + m + 1/2)^2] \times [a (\hbar_{o;-} + 1) - c_- (\omega_{k,m} + m + 1/2)^2] = 0 \end{aligned} \tag{36}$$

in $\omega_{k,m}$. Note that the leading coefficient of quartic Equation (36) coincides with the TP discriminant

$$\Delta_T(a, c_{\pm}) = 1/4(c_+ - c_-)^2 - 2a(c_+ + c_-) + 4a^2. \tag{37}$$

For each of *four* generally complex roots of this equation (or for each of three roots if the TP discriminant vanishes [56]), the corresponding pair of the ExpDiffs $\lambda_{\pm; k,m}$ coincides with two roots of the quadratic equation

$$\begin{aligned} & \lambda_{\pm; k,m}^2 - 2\omega_{k,m} \lambda_{\pm; k,m} + 1/2\omega_{k,m}^2 - 1/2(\hbar_{o;-} + \hbar_{o;+}) - 1 \\ & - 1/2(c_- + c_+) (\omega_{k,m} + m + 1/2)^2 / a = 0. \end{aligned} \tag{38}$$

Expressing again $\lambda_{\pm; k,m}^2$ in terms of $\omega_{k,m}^2$ via (34), one finds

$$\lambda_{\mp; k,m} = \frac{\hbar_{o;\mp} - \hbar_{o;\pm} - (c_{\pm} - c_{\mp}) (\omega_{k,m} + m + 1/2)^2 / a_2}{\omega_{k,m}} - \omega_{k,m}. \tag{39}$$

Keeping in mind that AEH solutions (21) obey the ODE

$$\ddot{\Phi}_{k,m}[\eta; h_{o;\pm}, \mu_0; T_2] + I[\eta; h_{o;\pm}; T_2; \epsilon_{k,m}] \Phi_{k,m}[\eta; h_{o;\pm}, \mu_0; T_2] = 0 \tag{40}$$

with the Bose invariant

$$\begin{aligned} I[\eta; h_{o;\pm}, \mu_0; T_2; \epsilon_{k,m}] = & - \frac{h_{o;-} - c_- \epsilon_{k,m}}{4(\eta+1)^2} - \frac{h_{o;+} - c_+ \epsilon_{k,m}}{4(1-\eta)^2} \\ & + \frac{h_{o;+} + h_{o;-} + 1 - \mu_0^2 + d \epsilon_{k,m}}{4(\eta^2 - 1)} \end{aligned} \tag{41}$$

and substituting (21) and (23) into (40) then brings us to the ODE

$$\begin{aligned}
 & (1 - \eta^2) \ddot{P}_m^{(\lambda_{+;k,m}, \lambda_{-;k,m})}(\eta) - \\
 & 2P_1^{(\lambda_{+;k,m}, \lambda_{-;k,m})}(\eta) \dot{P}_m^{(\lambda_{+;k,m}, \lambda_{-;k,m})}(\eta) + \\
 & m(m + \lambda_{+;k,m} + \lambda_{-;k,m} + 1) P_m^{(\lambda_{+;k,m}, \lambda_{-;k,m})}(\eta) = 0
 \end{aligned} \tag{42}$$

if we set

$$2\rho_{+;k,m} \rho_{-;k,m} + 1/4(h_{o;+} + h_{o;-} - \mu_o^2 + 1 + d\varepsilon_{k,m}) = -m(2\omega_{k,m} + m + 1) \tag{43}$$

One can easily verify the latter relation by substituting (7), (25), (26), and (33) into the left-hand side of (43).

We thus proved that q-RSs (21) have the form

$$\phi_{k,m}[\eta; \bar{\Lambda}; a, c_{\pm}] \propto \prod_{\aleph = \pm} (1 + \aleph \eta) \rho_{\aleph; k, m} P_m^{(\lambda_{+;k,m}, \lambda_{-;k,m})}(\eta) \tag{44}$$

and thereby arrived at the starting point of Lévai’s approach [34,38,40,41] while moving in the opposite direction.

Setting

$$\delta \equiv -1/2(c_+ + c_-) / a, \tag{45}$$

$$\Sigma \equiv 1/2(h_{o;+} + h_{o;-} + 2) - \delta - 1/4, \tag{46}$$

and also taking into account that, as a direct consequence of (34) and (45),

$$1/2(\lambda_{+;k,m}^2 + \lambda_{-;k,m}^2) = 1/2(h_{o;+} + h_{o;-} + 2) - \delta(\omega_{k,m} + m + 1/2)^2, \tag{47}$$

we come to (10) in [34] with the Jacobi indexes α and β dependent on the polynomial degree; namely,

$$\delta(\omega_{k,m} + m + 1/2)^2 - \delta + 1/2(\lambda_{+;k,m}^2 + \lambda_{-;k,m}^2) - 1/4 = \Sigma \tag{48}$$

in our notation.

The simplification utilized in Lévai’s aforementioned papers takes place in the particular case:

$$c_+ = c_- = \delta / C \tag{49}$$

when the parameter

$$\Lambda_{k,m} \equiv 1/4(\lambda_{-;k,m}^2 - \lambda_{+;k,m}^2) \tag{50}$$

becomes independent of the polynomial degree:

$$\Lambda_{k,m} \equiv \Lambda = 1/4(h_{o;-} - h_{o;+}). \tag{51}$$

Subtracting one of Equation (34) from another, we come to the crucial relation [34]

$$\lambda_{-;k,m}^2 - \lambda_{+;k,m}^2 = h_{o;-} - h_{o;+} = 4\Lambda \tag{52}$$

which allows one to combine two separate quartic equations in $\lambda_{\pm; k, m}$ into a single quartic equation in $\omega_{k,m}$. Namely, making use of (31), one finds

$$\lambda_{-;k,m} - \lambda_{+;k,m} = 2\Lambda / \omega_{k,m}, \tag{53}$$

so [34]

$$\lambda_{\pm;k,n} = \omega_{k,n} \mp \Lambda/\omega_{k,n} \tag{54}$$

and

$$1/2(\lambda_{+;k,m}^2 + \lambda_{-;k,m}^2) = \omega_{k,m}^2 + \Lambda^2/\omega_{k,m}^2. \tag{55}$$

We can thus rewrite (47) as

$$\omega_{k,m}^2 + \Lambda^2/\omega_{k,m}^2 = 1/2(\mathfrak{h}_{o;+} + \mathfrak{h}_{o;-} + 2) - \delta(\omega_{k,m} + m + 1/2)^2. \tag{56}$$

Substituting (46) into the right-hand side of this equation gives

$$\delta(\omega_{k,m} + m + 1/2)^2 + \omega_{k,m}^2 - \Sigma - \delta - 1/4 + \Lambda^2/\omega_{k,m}^2 = 0, \tag{57}$$

which brings us to quartic Equation (13) in [34]:

$$(1 + \delta)\omega_{k,m}^4 + \delta(2m + 1)\omega_{k,m}^3 + [\delta(m + 1/2)^2 - \delta - \Xi - 1/4]\omega_{k,m}^2 + \Lambda^2 = 0 \tag{58}$$

The trivial real-field reduction of CSLE (1) obtained by choosing all six parameters $\bar{\Lambda}, T_K$ to be real leads to the very special representative of the family of the JRef potentials [5,36] referred to by us as ‘Lévai’s e -TP potential’. However, as it has been noticed by Lévai himself [34], there is another family of real CSLEs discovered by Milson [6] and thereby referred to by us [45] as ‘Milson’s e -TP potential’. It is obtained by requiring the parameters $\mathfrak{h}_{o;+}$ and $\mathfrak{h}_{o;-}$ to be complex-conjugated while keeping real the parameter O_0^o . Both RefPF and density function thus become real when expressed in terms of the new variable $\eta_I = -i\eta$. Lévai’s formalism outlined above made it possible to treat both potentials in the unified matter outlined in Section 3.

3. Two Real-Field Reductions of Complex JRef SLE

The important common feature of the JRef and RRef potentials *on the line* is that the corresponding density functions

$$\rho[\eta; T_2] = \frac{T_2[\eta]}{(1 - \eta^2)^2} \quad (T_2[\pm 1] \neq 0) \tag{59}$$

and

$${}_i\rho[\eta_I; {}_i T_2] = \frac{{}_i T_2[\eta_I]}{(1 + \eta_I^2)^2} \tag{60}$$

have second-order poles at the endpoints ± 1 and $\pm\infty$ accordingly. As a direct consequence of this observation, the algebraic Schrödinger equation

$$\left\{ \frac{d}{d\xi} {}_i\rho^{-1/2}[\xi; {}_i T_2] \frac{d}{d\xi} - ({}_i V[\xi; {}_i \bar{\Lambda}; {}_i T_2] - \varepsilon) {}_i\rho^{1/2}[\xi; {}_i T_2] \right\} {}_i\Psi[\xi; {}_i \bar{\Lambda}; {}_i T_2; \varepsilon] \tag{61}$$

(with $\xi \equiv \eta$ or η_I for $\iota = ''$ or i , respectively) has the so-called [37] ‘prime’ form; namely, the sum of two ChExps equal to 0, and as a result, the principal Frobenius solution is unambiguously determined by the Dirichlet boundary conditions (DBC). In [37], we have proved the cited below theorem which allows one to formulate the given spectral problem as the Dirichlet problem for the prime SLE.

Theorem 1. *If the density function ${}_i\rho[\xi; {}_i T_2]$ of the given RCSLE has second-order poles at both endpoints, then the corresponding Dirichlet problem formulated for the prime SLE is fully equivalent to the requirement that the eigenfunctions of RCSLE are square-integrable with the weight ${}_i\rho[\xi; {}_i T_2]$.*

One of the advantages of formulating the given spectral problem as the Dirichlet problem for the prime SLE is that we can automatically adopt the rigorous theorems proven in [42] for SLEs solved under the DBCs. As explained in [37], the requirement for the density function to have *second-order* poles at the endpoints automatically assures that the corresponding Liouville transformation leads to the 1D Schrödinger equation on the line. Again Gesztesy, Simon, and Teschl’s meticulous proofs made it possible to extend the conventional results of the regular Sturm–Liouville theory [57] to singular Sturm–Liouville problems including the 1D Schrödinger equation on the line.

It is worth stressing in this context that the theorem does not hold if the density function has a simple pole at one of the endpoints and thereby is invalid for the radial Schrödinger equation [37,58].

In this paper, we focus solely on the JRef and RRef CSLEs with the numerators of density functions (59) and (60) formed by the even second-degree polynomials

$$T_e[\eta; \delta, C] = C^{-1}(1 + \delta - \eta^2)(\delta/C > 0) \tag{62}$$

and

$${}_i T_e[\eta; {}_i a, \kappa_+] = {}_i a(\eta^2 + \kappa_+) \quad ({}_i a, \kappa_+ \equiv 1 + \delta_+ > 0). \tag{63}$$

3.1. Lévai’s Real e-TP Potential

As initially noticed in [59], the variable $\eta_{-;s}(x; \delta, C)$ obtained by solving the ODE

$$\eta'_{-;s}(x; \delta, C) = \rho_L^{-1/2} [\eta_{-;s}(x; \delta, C); \delta, C], \tag{64}$$

under the boundary condition

$$\eta_{-;s}(0; \delta, C) = 0 \tag{65}$$

with

$$\rho_L [\eta; \delta, C] \equiv \frac{1 + \delta - \eta^2}{C(1 - \eta^2)^2}, \tag{66}$$

coincides with variable (60) in [44] if one sets

$$\delta = (\lambda^2 - 1)^{-1}, C = \delta\lambda^4 \tag{67}$$

in Ginocchio’s notation. Namely,

$$\eta_{-;s}[y; \lambda] = \frac{\lambda y}{\sqrt{Y[y^2; \lambda^2]}}, \tag{68}$$

where the function $y(x; \lambda)$ is a solution of the ODE [60,61]

$$y'(x, \lambda) = [1 - y^2(x, \lambda)] Y[y^2(x, \lambda); \lambda^2] \text{ for } -1 < y < +1 \tag{69}$$

and

$$Y[y^2; \lambda^2] \equiv 1 + (\lambda^2 - 1) y^2. \tag{70}$$

To prove Lévai’s assertion [59], one can follow Wu’s pioneering arguments [60,61] in support of the claim that the radial Ginocchio potential [62] is nothing but another representation of the implicit radial potential exactly solvable in terms of hypergeometric functions [47,48]. First multiplying the square of (68) by its reverse

$$y^2 = \frac{\eta_{-;s}^2}{N[\eta_{-;s}^2; \lambda^2]}, \tag{71}$$

where

$$N[\eta^2; \lambda^2] \equiv \lambda^2 + (1 - \lambda^2)\eta^2 \tag{72}$$

gives

$$N[\eta^2; \lambda^2] = \frac{\lambda^2}{Y[y^2; \lambda^2]}. \tag{73}$$

Setting

$$\eta_{-;s}(x; \lambda) \equiv \eta_{-;s}[y(x, \lambda); \lambda] = \eta_{-;s}(x; \delta, \delta\lambda^4) \tag{74}$$

with $\delta = (\lambda^2 - 1)^{-1}$ and also taking into account that

$$N[\eta^2; 1 + 1/\delta] = (1 + \delta - \eta^2)/\delta, \tag{75}$$

we can rewrite (64) as

$$\eta'_{-;s}(x; \lambda) = \frac{\lambda^2 [1 - \eta^2_{-;s}(x; \lambda)]}{\sqrt{N[\eta^2_{-;s}(x; \lambda); \lambda^2]}} \tag{76}$$

or, which is equivalent,

$$\eta'_{-;s}(x; \lambda) = [1 - y^2(x; \lambda)] \sqrt{N[\eta^2_{-;s}(x; \lambda); \lambda^2]}. \tag{77}$$

On the other hand, we come to the same relation multiplying (70) by the derivative

$$\frac{d\eta_{-;s}}{dy} = \lambda^{-2} N^{3/2}[\eta^2_{-;s}; \lambda^2] \tag{78}$$

and making use of (73).

Algebraic Schrödinger Equation (61) thus takes the form

$$\left\{ \frac{d}{d\eta} \rho_L^{-1/2} [\eta; \delta, C] \frac{d}{d\eta} - \rho_L^{1/2} [\eta; \delta, C] (V_L [\eta; \bar{\Lambda}; \delta, C] - \varepsilon) \right\} \times \Psi_L [\eta; \bar{\Lambda}; \delta, C; \varepsilon] = 0, \tag{79}$$

where $V_L [\eta_{-;s}; \bar{\Lambda}; C, \delta]$ is the Liouville potential converted back to the variable $\eta_{-;s}(x; \delta, C)$, i.e.,

$$V_L [\eta_{-;s}(x; \lambda); \bar{\Lambda}; \delta, \delta\lambda^4] = - [\eta'_{-;s}(x; \lambda)]^2 I^0 [\eta_{-;s}(x; \lambda); \bar{\Lambda}] - 1/2 \{ \eta_{-;s}(x; \lambda), x \}, \tag{80}$$

where again $\delta = (\lambda^2 - 1)^{-1}$. It has been shown in [45] that the Schwarzian derivative expressed in terms of the variable $\eta_{-;s}$ can be written as

$$\{ \eta_{-;s}, x \} = - \frac{2C}{1 + \delta - \eta^2_{-;s}} + \frac{C (1 - \eta^2_{-;s})^2}{2(1 + \delta - \eta^2_{-;s})^2} \left\{ 1 - \frac{4}{1 - \eta^2_{-;s}} + \frac{5(\delta + 1)}{1 + \delta - \eta^2_{-;s}} \right\}. \tag{81}$$

Substituting (3), (64), (66), and (81) into potential function (80) expressed in terms of the variable $\eta_{-;s}$, one finds

$$V_L [\eta; \bar{\Lambda}; \delta, C] / C = \frac{h_{0;-}(1-\eta)^2 + h_{0;+}(1+\eta)^2 + O_0^0(1-\eta^2) + 4}{4(1+\delta-\eta^2)} + \frac{1-\eta^2}{(1+\delta-\eta^2)^2} - \frac{(1-\eta^2)^2}{4(1+\delta-\eta^2)^2} \left\{ 1 + \frac{5(\delta+1)}{1+\delta-\eta^2} \right\} \tag{82}$$

or, alternatively,

$$V_L [\eta; \bar{\Lambda}; \delta, C] / C = 1/4(\mu_0^2 - 2) + \frac{\Xi}{1+\delta-\eta^2} + \frac{3\delta(3\delta+2)}{4(1+\delta-\eta^2)^2} - \frac{5\delta^2(\delta+1)}{4(1+\delta-\eta^2)^3} + \frac{(h_{0;+}-h_{0;-})\eta}{2(1+\delta-\eta^2)}, \tag{83}$$

where

$$\Sigma \equiv 1/2(h_{0;+} + h_{0;-} + 2) + 1/4[\delta(\mu_0^2 - 4) - 1] \tag{84}$$

is simply another representation for parameter (46) taking into account (28) and (35).

It is essential that solutions of prime SLE (79) are related to solutions of the real JRef CSLE with density function (66) via the simple gauge transformation

$$\Psi_L[\eta; \bar{\Lambda}; \delta, C; \varepsilon] \propto \rho_L^{1/4}[\eta; \delta, C] \Phi_L[\eta; \bar{\Lambda}; \delta, C; \varepsilon], \tag{85}$$

and therefore the eigenfunctions of the prime SLE and the eigenfunctions of the JRef CSLE,

$$\phi_{cn}[\eta; \Xi, \Lambda; \delta, C] = \prod_{\aleph = \pm} (1 + \aleph\eta)^{1/2\lambda_{\aleph;cn}} P_n^{(\lambda_{+,cn}, \lambda_{-,cn})}(\eta), \tag{86}$$

are interrelated via the elementary formula

$$\psi_{cn}[\eta; \Xi, \Lambda; \delta, C] \propto \rho_L^{1/4}[\eta; C, \delta] \phi_{cn}[\eta; \Xi, \Lambda; \delta, C]. \tag{87}$$

Examination of the asymptotic behavior of eigenfunctions (86) near poles of the JRef CSLE at ± 1 reveals that solutions (87) obey the DBCs

$$\lim_{\eta \rightarrow \pm 1} \psi_{cn}[\eta; \Xi, \Lambda; \delta, C; \varepsilon] = 0 \tag{88}$$

if

$$\lambda_{\pm;cn} > 0. \tag{89}$$

One can then directly verify that constraints (89) are the necessary and sufficient conditions for q-RSs (86) to be square-integrable with nonnegative weight (66):

$$\int_{-1}^{+1} \phi_{cn}^2[\eta; \Xi, \Lambda; \delta, C] \rho_L[\eta; \delta, C] d\eta < \infty, \tag{90}$$

as prescribed by Theorem 1. It directly follows from definition (31) of the sought-for roots, coupled with constraints (89), that the root ω_{cn} associated with the $(n + 1)$ -th eigenfunction must be positive.

Examination of asymptotic behavior of potential (82) near the singular endpoints shows that

$$V_L[\pm 1; \bar{\Lambda}; \delta, C]/C = (h_{o;\pm} + 1)/\delta. \tag{91}$$

We choose the energy reference point from the requirement that the potential vanishes at $\eta = +1$ setting $h_{o;+} = -1$. We also require that the potential takes a nonnegative value at the lower endpoint by choosing

$$h_{o;-} + 1 \equiv \lambda_o^2 \equiv 4\Lambda \geq 0, \tag{92}$$

where the parameter λ_o coincides with the zero-energy ExpDiff for the pole of JRef CSLE (1) at $\eta = -1$. Taking into account (28), we can thus represent JRef PF (3) as

$$I^o[\eta; \lambda_o, \mu_o] = \frac{1 - \lambda_o^2}{4(\eta + 1)^2} + \frac{1}{4(1 - \eta)^2} + \frac{\mu_o^2 + 1 - \lambda_o^2}{4(1 - \eta^2)}. \tag{93}$$

With such a choice of the energy reference point, two positive roots $\lambda_{\pm;cn}$ of quadratic Equation (38) specifying the $(n + 1)$ -th eigenfunction satisfy the inequality

$$\lambda_{-,cn} \geq \lambda_{+,cn} > 0, \tag{94}$$

so the sought-for roots take the form

$$\lambda_{\pm;cn} = \omega_{cn} \mp \Lambda/\omega_{cn}. \tag{95}$$

Comparing (95) with (12) in [34], we conclude that Lévai’s definition of the ϵ -TP potential differs from ours by a reflection of the argument, i.e., $z = -\eta_{-;s}$ and $\alpha = \lambda_{-;cn}, \beta = \lambda_{+;cn}$ in our terms.

In the limiting case of the Ginocchio potential on the line ($\lambda_o = \Lambda = 0$), quartic Equation (58) takes the form:

$$\tilde{\omega}_{\dagger n}^2 [\delta (\tilde{\omega}_{\dagger n} + n + 1/2)^2 + \tilde{\omega}_{\dagger n}^2 - \Sigma - \delta - 1/4] = 0 \tag{96}$$

with the double-zero root

$$\tilde{\omega}_{\dagger -n} = 0 \tag{97}$$

and two other roots determined by the quadratic equation

$$\tilde{\omega}_{\dagger +n}^2 + \delta (\tilde{\omega}_{\dagger +n} + n + 1/2)^2 - \delta - \Xi - 1/4 = 0. \tag{98}$$

The average of the indexes of the Jacobi polynomials in the right-hand side of (86) thus coincides with the positive root $\tilde{\omega}_{cn}$ of quadratic Equation (98). Since the TP may not have zeros between -1 and $+1$, the parameter δ must be either positive or smaller than -1 , i.e., the linear coefficient of the quadratic equation

$$\tilde{\omega}_{\dagger +n}^2 + \frac{\delta}{\delta + 1} (n + 1/2) \tilde{\omega}_{\dagger +n} + [\delta (n + 1/2)^2 - \delta - \Xi - 1/4] / (\delta + 1) = 0 \tag{99}$$

is necessarily positive, and therefore the quadratic equation has a positive root if its free term is negative, i.e., if

$$(n + 1/2)^2 < (\Xi + 1/4) / \delta + 1 \tag{100}$$

in agreement with (23) in [34]. Setting $\Lambda = 0$ in (95) confirms that

$$\tilde{\lambda}_{+;cn}(\Xi; \delta) = \tilde{\lambda}_{-;cn}(\Xi; \delta) = \tilde{\omega}_{cn}, \tag{101}$$

as expected. The negative root $\tilde{\omega}_{dn}$ of quadratic Equation (99) specifies the q-RS composed of the Jacobi polynomial with two equal negative indexes:

$$\tilde{\lambda}_{+;dn}(\Xi; \delta) = \tilde{\lambda}_{-;dn}(\Xi; \delta) = \tilde{\omega}_{dn} < 0. \tag{102}$$

Examination of quartic Equation (58) reveals that

$$\lim_{\Lambda \rightarrow 0} (\Lambda / \tilde{\omega}_{\dagger -n}) = \sqrt{\delta + \Xi + 1/4 - \delta (n + 1/2)^2} \tag{103}$$

and therefore

$$\tilde{\lambda}_{\pm; \dagger -n}(\Xi; \delta) \equiv \lim_{\Lambda \rightarrow 0} \lambda_{\pm; \dagger -n}(\Xi, \Lambda; \delta) = \mp \sqrt{\delta + \Xi + 1/4 - \delta (n + 1/2)^2}. \tag{104}$$

So,

$$\tilde{\lambda}_{+; \dagger -n}(\Xi; \delta) \tilde{\lambda}_{-; \dagger -n}(\Xi; \delta) < 0 \tag{105}$$

as far as condition (100) holds, and therefore both Jacobi indexes are real. We thus conclude that

$$\dagger - = \mathbf{a} \text{ or } \mathbf{b}. \tag{106}$$

We thus confirmed that the $(n + 1)$ -th eigenfunction is indeed accompanied by a triplet of q-RSs \mathbf{an} , \mathbf{bn} , and \mathbf{dn} , in agreement with our more general results [36] for asymptotically levelled (AL) potentials ($\lambda_o = 0, c_+ \neq c_-$).

As the asymmetry parameter $\Lambda > 0$ increases, the e -TP JRef potential has at least $n + 1$ bound energy levels as far as the positive root of quartic Equation (58) remains larger than $\sqrt{\Lambda}$, and thereby both Jacobi indexes stay positive, as prescribed by (89). Indeed, the latter equation may not have a positive double root larger than $\sqrt{\Lambda}$, because this would imply the existence of two eigenfunctions with exactly the same number of nodes inside the quantization interval. Therefore, two positive roots may not merge giving rise to a complex-conjugated pair as far as $\tilde{\omega}_{cn}^2 > \Lambda$. The $(n + 1)$ -th bound energy level may disappear if Λ reaches the positive root of the quadratic equation

$$(\delta + 2)\Lambda + (2n + 1)\sqrt{\Lambda} + (n + 1/2)^2 - (\Xi + 1/4)/\delta - 1 = 0 \tag{107}$$

in $\sqrt{\Lambda}$.

3.2. Milson’s e -TP Potential

As stressed above, choosing all the parameters of RefPF (3) to be real and making the Liouville transformation on the real axis [1,2] brings us directly to the conventional JRef potential [5]. However, as mentioned in Section 2, there is another less obvious option to formulate the real Sturm–Liouville problem utilized by Milson [6]. Namely, keeping real only O_0^o and a_2 while allowing the parameters in each pair $h_{o;\pm}$ and c_{\pm} to be complex and also restricting CSLE (1) to the imaginary axis $\eta = i\eta_I$, we come to the real CSLE [6]:

$$\left\{ \frac{d^2}{d\eta_I^2} + I[\eta_I; h_{o,i} O_0^o; i T_K] + i\rho [\eta_I; i T_K; \tilde{\epsilon}] \right\} \Phi[\eta_I; h_{o,i} O_0^o; i T_K; \tilde{\epsilon}] = 0 \tag{108}$$

with the RRef PF

$${}_i I^o[\eta_I; h_{o,i} O_0^o] = \frac{h_o}{4(1 + i\eta_I)^2} + \frac{h_o^*}{4(1 - i\eta_I)^2} + \frac{{}_i O_0^o}{4(1 + \eta_I^2)} \tag{109}$$

and the *positive* density function

$${}_i \rho_M [\eta_I; i T_K] = \frac{{}_i T_K[\eta_I]}{(1 + \eta_I^2)^2} \quad (K = 0 \text{ or } 2) \tag{110}$$

(assuming that the TP has a positive leading coefficient and a negative discriminant ${}_i \Delta_T$ for $K = 2$).

The conventional Liouville transformation of CSLE (108) then leads to the *real* RRef potential in our classification scheme. Though the full credit for the discovery of the latter potential should be certainly given to Milson [6], two particular cases of CSLE (108) with a constant TP ($K = 0$) and the second-degree TP

$${}_i T_2[\eta_I] = {}_i a(1 + \eta_I^2) \tag{111}$$

associated with the translationally shape-invariant (TSI) RRef potentials (t-RM and Gendenshtein potentials, respectively) have already implicitly appeared in Lévai’s pioneering paper [31], which originally drew the author’s attention to this extraneous class of rational potentials solvable in terms of Jacobi polynomials with complex-conjugated indexes and imaginary argument.

Let us now discuss the energy spectrum of RRef CSLE (108). Since the corresponding Liouville potential is asymptotically levelled, we can choose the energy reference point via the requirement that the potential vanishes at both quantization endpoints $\pm\infty$. This can be achieved by setting [7,10]

$${}_i O_0^o = 2h_{o;R} + 1, \tag{112}$$

where $h_{o;R}$ is the real part of the complex parameter

$$h_{o;-} \equiv h_o \equiv h_{o;R} + i h_{o;I}. \tag{113}$$

In other words, we make parameter (28) equal to zero so $h_{o;\pm} \equiv h_{o;\pm}$ and RRef CSLE (108) takes the form [7,10]

$$\left\{ \frac{d^2}{d\eta_{+;s}^2} + {}_iI^o[\eta_{+;s}; h_o] + {}_i\rho_M[\eta_{+;s}; \kappa_+] \xi \right\} {}_i\Phi[\eta_{+;s}; h_o; \kappa; \xi] = 0, \tag{114}$$

where

$${}_iI^o[\eta_{+;s}; h_{o;R} + i h_{o;I}] = \frac{h_{o;R} + h_{o;I}\eta_{+;s}}{(\eta_{+;s}^2 + 1)^2} + \frac{1}{4(\eta_{+;s}^2 + 1)}, \tag{115}$$

$\xi \equiv -\varepsilon$, and

$$\rho_M[\eta_{+;s}; \kappa_+, a_+] = \frac{a_+(\eta_{+;s}^2 + \kappa_+)}{(\eta_{+;s}^2 + 1)^2} \quad (a_+ > 0). \tag{116}$$

(The density function with a constant TP, giving rise to the translationally form-invariant (TFI) CSLE of Group B [63] with the trigonometric Liouville potential [27], requires a special consideration.) In Lévai’s notation [34],

$$\kappa_+ - 1 = \delta_+ > -1, \quad a_+ = C_+^{-1} > 0, \tag{117}$$

and

$$h_{o;R} - \delta_+ + \frac{3}{4} \equiv \Sigma_+, \tag{118}$$

while parameter (51) becomes imaginary:

$$\Lambda_+ \equiv 1/4(\lambda_{-;k,m}^2 - \lambda_{+;k,m}^2) = 1/2 {}_i h_{o;I} i. \tag{119}$$

The change of variable defined by the ODE

$$\eta'_{+;s}(x; \kappa_+, a_+) = \rho_M^{-1/2}[\eta_{+;s}(x; \kappa_+, a_+); \kappa_+, a_+] \tag{120}$$

solved under the boundary condition

$$\eta_{+;s}(0; \kappa_+, a_+) = 0 \tag{121}$$

converts CSLE (114) into the Schrödinger equation with Milson’s [6] potential

$$\begin{aligned} V_M(x; h_o; \kappa_+, a_+) &\equiv V_M[\eta_{+;s}(x; \kappa_+, a_+); h_o; \kappa_+, a_+] \\ &= (\eta'_{+;s})^2 {}_iI^o[\eta_{+;s}(x; \kappa_+, a_+); h_o] - 1/2\{\eta_{+;s}, x\}, \end{aligned} \tag{122}$$

where [45]

$$\begin{aligned} \{\eta_{+;s}, x\} &= \frac{2}{a_+(\eta_{+;s}^2 + \kappa_+)} + \frac{2(1 + \eta_{+;s}^2)}{a_+(\eta_{+;s}^2 + \kappa_+)^2} - \\ &\frac{(1 + \eta_{+;s}^2)^2}{2a_+(\eta_{+;s}^2 + \kappa_+)^2} \left(1 + \frac{5\kappa_+}{\eta_{+;s}^2 + \kappa_+} \right). \end{aligned} \tag{123}$$

Setting

$${}_i\omega_{k,n} \equiv \text{Re } \lambda_{\pm;k,n} \tag{124}$$

and again taking into account that parameter (119) is independent of the polynomial degree, we come to the quartic equation

$$(1 + \delta_+) {}_i\omega_{k,n}^4 + \delta_+(2n + 1) {}_i\omega_{k,n}^3 + [\delta_+(n + 1/2)^2 - \delta_+ - \Xi_+ - 1/4] {}_i\omega_{k,n}^2 + \Lambda_+^2 = 0, \tag{125}$$

while combining (54) with (119) gives

$$\lambda_{\pm;k,n} = {}_i\omega_{k,n} \mp i \frac{h_{o;l}}{2{}_i\omega_{k,n}}. \tag{126}$$

Though quartic Equation (125) seems identical to (58), there is a catch: the roots specifying the eigenvalues of the corresponding real CSLEs with density functions (66) and (110) have opposite signs. Namely, while ω_{cn} is positive, its counterpart ${}_i\omega_{cn}$ specifying the eigenvalue of the Schrödinger equation with Milson’s e -TP potential is restricted by the negative upper bound specified by (139) below. Though Lévai’s assertion that the symmetric RRef potential coincides with the Ginocchio potential on the line [44] is formally correct [45], its proof relies on rather sophisticated arguments thoroughly illuminated in Section 4.

The quartic equation similar to (125) was independently derived by us in [19] under the influence of Alvarez-Castillo and Kirchbach’s breakthrough study [8] on the quantization of the Gendenshtein (‘Scarf II) potential [30] in terms of Romanovski polynomials [18]. The author realized that the corresponding Schrödinger equation can be obtained by the Liouville transformation of RRef CSLE (114) with the density function

$${}_i\rho_M[\eta_I; 1] = \frac{1}{\eta_I^2 + 1}. \tag{127}$$

By searching for eigenfunctions of the given CSLE in the ‘quasi-rational’ [35] form

$${}_i\Phi_{cn}[\eta_I; h_o; \kappa_+] \propto (\eta_I^2 + 1)^{i\omega_{cn}+1} \left(\frac{1 - i\eta_I}{1 + i\eta_I} \right)^{\lambda_{cn;l}} \Pi_n[\eta_I; h_o; \kappa_+] \tag{128}$$

and examining the ExpDiff for the pole of CSLE (114) at infinity, we found that n -degree polynomials in the right-hand side of (128) satisfy the Routh equation [14]

$$\left[(\eta_I^2 + 1) \frac{d^2}{d\eta_I^2} + 2[(i\omega_{cn} + 1)\eta_I - \lambda_{cn;l}] \frac{d}{d\eta_I} - n(n + 2\omega_{cn} - 1) \right] \times \mathfrak{R}_n^{(i\omega_{cn}+1, \lambda_{cn;l})}[\eta_I] = 0, \tag{129}$$

such that its polynomial solutions $\mathfrak{R}_n^{(i, \lambda_{cn})}[\eta_I]$ after being rewritten in the monic form coincide with the pseudo-Jacobi polynomials

$$\hat{\mathfrak{R}}_n^{(i, \lambda_{cn})}[\eta] = P_n(\eta; -\lambda_{cn;l}, -i\omega_{cn} - 1) \tag{130}$$

defined via (9.9.1) in [11]. It was found that the eigenvalues of RRef CSLE (114) are unambiguously determined by the real part ${}_i\omega_{cn}$ of the complex index ${}_i\lambda_{cn}$ of the Routh polynomial:

$${}_i\varepsilon_{cn}(h_o; \kappa_+) = -{}_i\rho_{\infty;cn}^2 < 0, \tag{131}$$

with

$${}_i\rho_{\infty;cn} = -{}_i\omega_{cn} - n - 1/2 > 0, \tag{132}$$

while examination of the ExpDiffs for the poles of CSLE (114) brought us to the algebraic equation

$${}_i\lambda_{cn}^2 = h_{o;R} + 1 + ih_{o;l} + (\kappa_+ - 1) {}_i\varepsilon_{cn}(h_o; \kappa_+), \tag{133}$$

similar to (26) except that the coefficients c_+ and c_- are real and coincide with each other. It then directly follows from (133) that

$$\text{Im } {}_i\lambda_{cn}^2 = 2{}_i\omega_{cn} \lambda_{cn;l} = h_{o;l}, \tag{134}$$

bringing us back to Lévai’s formula (126).

Substituting (131) and (132) into the right-hand side of (133) and eliminating $\lambda_{cn,I}$ from the resultant formula

$${}_i\omega_{cn}^2 - \lambda_{cn,I}^2 = h_{o;R} + 1 + (1 - \kappa_+) ({}_i\omega_{cn} + n + 1/2)^2, \tag{135}$$

we [19] came to the quartic equation

$${}_i\omega_{cn}^2 [{}_i\omega_{cn}^2 - h_{o;R} - 1 + (\kappa_+ - 1) ({}_i\omega_{cn} + n + 1/2)^2] - 1/4h_{o;I}^2 = 0, \tag{136}$$

which is nothing but an alternative form of (125) with $\kappa_+ = 1 + \delta_+$ and Lévai’s parameters Ξ_+ and Λ_+ defined via (118) and (119) accordingly.

By requiring eigenfunctions (128) to be square-integrable with weight (116):

$$\int_{-\infty}^{+\infty} {}_i\phi_{cn}^2[\eta_I; h_o; \kappa_+] {}_i\rho_M[\eta_I; \kappa_+] d\eta_I < \infty, \tag{137}$$

we come to the constraint

$${}_i\rho_{\infty;cn} = -{}_i\omega_{cn} - n - 1/2 > 0, \tag{138}$$

which implies that the real part of the Routh index ${}_i\lambda_{cn}$ must be restricted by the upper bound

$${}_i\omega_{cn} < -n - 1/2. \tag{139}$$

We define R–Routh polynomials (Romanovski/pseudo-Jacobi polynomials in Leski’s terms [15,16]) via (3.5) and (3.6) in [64]:

$$\begin{aligned} R_n^{(2\alpha_I, \alpha_R+1)}(x) &= (-i)^n P_n^{(\alpha_R+1\alpha_I, \alpha_R-i\alpha_I)}(ix) \\ &\equiv \mathfrak{R}_n^{(\alpha_R+i\alpha_I)}(x) \end{aligned} \tag{140}$$

for

$$n \leq N_0 \equiv \lfloor -\alpha_R - 1/2 \rfloor. \tag{141}$$

(Remember that we use the term ‘pseudo-Jacobi polynomials’ as a synonym for Routh polynomials and do not require the parameter N_0 [12] to be a nonnegative integer, so our use of this term is similar to the definition of these polynomials in [13].)

The requirement for quartic equation

$$({}_i\rho_{\infty;cn} + n + 1/2)^2 [({}_i\rho_{\infty;cn} + n + 1/2)^2 - h_{o;R} - 1 + (\kappa_+ - 1) {}_i\rho_{\infty;cn}^2] - 1/4h_{o;I}^2 = 0 \tag{142}$$

to have an odd number of negative roots [7] results in Milson’s formula for the number n_o of discrete energy levels [6]

$$n_o = \left\lfloor \sqrt{1/2(h_{o;R} + 1 + |h_o + 1|)} + 1/2 \right\rfloor. \tag{143}$$

Finally, let us point to the fact that each eigenfunction is accompanied by another q-RS associated with the second real root of the quartic equation. If the Routh polynomial forming the second q-RS has no real zeros, then the q-RS in question can be used as the FF for the RDT, giving rise to a new solvable rational potential [10] (as originally suggested by Quesne [64] for the Scarf II potential).

It is also worth mentioning that our argumentation in [7] was restricted solely to the real (self-adjoint) CSLE, and within this approach, we were unable to prove that the real

part of sought-for complex ExpDiff $i\lambda_{en}$ coincides with a negative real root of a quartic equation in the general case of the density function [9]

$$i\rho [\eta; \kappa_+ + i \kappa_I] = \frac{\eta^2 + \kappa_I \eta + \kappa_+}{(\eta^2 + 1)^2} \tag{144}$$

with $|\kappa_I| > 0$, assuming that the TP has a negative discriminant

$$\Delta_T = 1/4\kappa_I^2 - \kappa_+ < 0. \tag{145}$$

To cover the general case [9], we (under the influence of Lévai’s works [31,34,38,39]) had to start from complex (non-self-adjoint) CSLE (1) and then consider the second real-field reduction of Bochner’s [55] complex Jacobi DPS to the DPS formed by Routh polynomials [14].

4. Ginocchio Potential as an Overlap of Lévai’s and Milson’s e -TP Potentials

The main purpose of this section is to prove that Lévai’s [34] and Milson’s [6] e -TP reductions of the JRef and RRef potentials defined via (80) and (122) overlap along the symmetric potential curves

$$V_L[\eta_{-;s}(x; \delta, C); 0, \mu_0; \delta, C] = V_M[\eta_{+;s}(x; \kappa_+, a_+); h_0; R; \kappa_+, a_+] \tag{146}$$

obtained by setting corresponding asymmetry parameters (92) and (119) to zero:

$$\Lambda = \Lambda_+ = 0 \tag{147}$$

(assuming that the JRefPF in the right-hand side of (80) is parametrized by the parameters λ_0 and μ_0 according to (93).) As explained below, this assertion can be considered as the corollary of the following theorem:

Theorem 2. *The RCSLEs with evenJRef and evenRRef Bose invariants constitute two rational realizations of the same Sturm–Liouville problem interrelated via an algebraic change of variable.*

Proof of Theorem 2. Let us first prove that the solutions of ODEs (64) and (120) under the boundary conditions (65) and (121), respectively, are interrelated via the following algebraic formulas:

$$\eta_{+;s} = \frac{\eta_{-;s}}{\sqrt{1 - \eta_{-;s}^2}} \tag{148}$$

or

$$\eta_{-;s} = \frac{\eta_{+;s}}{\sqrt{\eta_{+;s}^2 + 1}} \tag{149}$$

if we choose

$$1 - \kappa_+ = (\kappa_- + 1)^{-1} \tag{150}$$

and

$$a_+ = a_-(\kappa_- + 1) \tag{151}$$

or, in Lévai’s notation,

$$\delta_+ \delta_- = 1 \tag{152}$$

and

$$C_- = \delta_- C_+, \tag{153}$$

so

$$\delta_{\pm} + 1 \equiv \pm \kappa_{\pm} \tag{154}$$

and

$$a_{\pm} = \pm C_{\pm}^{-1} \tag{155}$$

with $\delta_{-} \equiv \delta$, $C_{-} \equiv C$ in the notation of Section 3.1.

If we, following [45], introduce two nonnegative variables

$$z_{\pm} \equiv \eta_{\pm,s}^2 \tag{156}$$

satisfying the ODEs

$$z'_{\pm}(x; \kappa_{\pm}, a_{\pm}) = \frac{2\sqrt{z_{\pm}}(1 \pm z_{\pm})}{\sqrt{a_{\pm}(z_{\pm} + \kappa_{\pm})}}, \tag{157}$$

then algebraic relation (148) is equivalent to the linear fractional transformation

$$z_{+} = \frac{z_{-}}{1 - z_{-}}, \tag{158}$$

assuming that the parameters κ_{\pm} , a_{\pm} are interrelated as specified above.

To prove (158), let us introduce the auxiliary PF

$$\zeta_{-}[z_{-}] \equiv \frac{z_{-}}{1 - z_{-}} \tag{159}$$

and demonstrate that the function

$$\zeta_{-}(x; \kappa_{-}, a_{-}) \equiv \zeta_{-}[z_{-}(x; \kappa_{-}, a_{-})] \tag{160}$$

satisfies ODE (157) for the variable $z_{+}(x; \kappa_{+}, a_{+})$ with

$$\zeta_{-}(0; \kappa_{-}, a_{-}) = z_{\pm}(0; \kappa_{+}, a_{+}) = 0. \tag{161}$$

Indeed, differentiating (160) with respect to x and making use of (157) with the lower subscript, one finds

$$\zeta'_{-}(x; \kappa_{-}, a_{-}) = 2[\zeta_{-}(x; \kappa_{-}, a_{-}) + 1] \sqrt{\frac{z_{-}(x; \kappa_{-}, a_{-})}{a_{-}[z_{-}(x; \kappa_{-}, a_{-}) + \kappa_{-}]}} \tag{162}$$

where we also took into account that

$$\zeta_{-}[z_{-}] + 1 = \frac{1}{1 - z_{-}}. \tag{163}$$

Under constraints (150) and (151), the denominator of the PF in the right-hand side of (162) can be rewritten as follows:

$$a_{-}[z_{-}(x; \kappa_{-}, a_{-}) + \kappa_{-}] = a_{+}[1 - z_{-}(x; \kappa_{-}, a_{-})] \times [\zeta_{-}(x; \kappa_{-}, a_{-}) + \kappa_{+}] \tag{164}$$

which gives

$$\zeta'_{-}(x; \kappa_{-}, a_{-}) = 2[\zeta_{-}(x; \kappa_{-}, a_{-}) + 1] \sqrt{\frac{\zeta_{-}(x; \kappa_{-}, a_{-})}{a_{+}[\zeta_{-}(x; \kappa_{-}, a_{-}) + \kappa_{+}]}}. \tag{165}$$

This confirms that

$$z_{+}(x; \kappa_{+}, a_{+}) \equiv \zeta_{-}(x; \kappa_{-}, a_{-}) \tag{166}$$

as asserted.

Our next step is to prove that the parameters μ_o and $h_{o;R}$ can be interrelated in such a way that

$$I^o[\eta_{-;s}; 0, \mu_o] = \left(\frac{d\eta_{+;s}}{d\eta_{-;s}}\right)^2 {}_iI^o[\eta_{+;s}; h_{o;R}] + 1/2\{\eta_{+;s}, \eta_{-;s}\}, \tag{167}$$

provided that the coefficients of e -TPs (62) and (63) obey constraints (150)–(153) and therefore, as proven above, the variables z_{\pm} are related via linear fractional transformation (158), so [65]

$$\{z_+, z_-\} = 0. \tag{168}$$

This implies that interrelation Formula (167) is equivalent to the requirement that the RefPFs

$${}_1I^o_-[z_-; \mu_o] = \left(\frac{d\eta_{-;s}}{dz_-}\right)^2 \left(I^o[\sqrt{z_-}; 0, \mu_o] - 1/2\{\eta^2_{+;s}, \eta_{-;s}\}\right) \tag{169}$$

and

$${}_1I^o_+[z_+; h_{o;R}] \equiv \left(\frac{d\eta_{+;s}}{dz_+}\right)^2 \left({}_iI^o[\sqrt{z_+}; h_{o;R}] - 1/2\{\eta^2_{+;s}, \eta_{+;s}\}\right), \tag{170}$$

where

$$\{\eta^2, \eta\} = -3/2\eta^{-2}, \tag{171}$$

are interrelated as follows:

$${}_1I^o_-[z_-; \mu_o] \equiv \left(\frac{dz_+}{dz_-}\right)^2 {}_1I^o_+[z_+; h_{o;R}]. \tag{172}$$

Substituting (93) with $\lambda_o = 0$ and (115) with $h_{o;I} = 0$, into the right-hand sides of (169) and (170), accordingly, we can represent the RefPFs in question as

$${}_1I^o_-[z_-; \mu_o] = \frac{3}{16z_-^2} + \frac{1}{4z_-(1-z_-)^2} + \frac{\mu_o^2 - 1}{16z_-(1-z_-)} \tag{173}$$

and

$${}_1I^o_+[z_+; h_{o;R}] = \frac{3}{16z_+^2} + \frac{h_{o;R}}{4z_+(z_+ + 1)^2} + \frac{1}{16z_+(z_+ + 1)}. \tag{174}$$

Expressing the right-hand side of (173) in terms of the variable z_- , we then rewrite (172) as

$$(1-z_-)^4 {}_1I^o_-[z_-; \mu_o] = \frac{1}{16z_+^2(z_+ + 1)^2} \left[3 + z_+(\mu_o^2 + 3) + 4z_+^2\right] \tag{175}$$

and thereby confirm that the right-hand side of the latter formula turns into (174) if we choose [45]

$$h_{o;R} + 1 = 1/4 \mu_o^2, \tag{176}$$

which concludes the proof. \square

It seems beneficiary to also reformulate Theorem 2 as the following proposition:

Corollary 1. *The Liouville transformations of the RCSLEs with even JRef and even RRef Bose invariants on the intervals $(-1, +1)$ and $(-\infty, +\infty)$ accordingly results in the Schrödinger equation with exactly the same potential symmetric under the reflection of its argument x .*

This constitutes the main result of this paper.

As illuminated in detail in [45], the Liouville transformation of the CSLE

$$\left\{ \frac{d^2}{dz_-^2} + {}_1I^o_-[z_-; \mu_o] - \frac{(z_- - \delta_- - 1) \varepsilon}{4C_- z_- (1-z_-)^2} \right\} {}_1\Phi_-[z_-; \mu_o; \delta_-, C_-; \varepsilon] = 0 \tag{177}$$

on the finite interval (0, 1) converts it to the Schrödinger equation with the non-singular radial JRef potential. By reflecting the latter potential around the origin, the author [5] originally constructed the symmetric potential later rediscovered by Ginocchio [44] in the form

$$V_G[y] = -(1 - y^2) \left\{ \lambda^2 \nu(\nu + 1) - 1/4(1 - \lambda^2)[5(1 - \lambda^2)y^4 - (7 - \lambda^2)y^2 + 2] \right\}, \tag{178}$$

where the variable y is related to the variable z_- as follows:

$$z_- = \frac{\lambda^2 y^2}{1 + (\lambda^2 - 1) y^2}. \tag{179}$$

It was Wu [60] who recognized the equivalence of two representations as soon as Ginocchio presented his results.

Later, Lévai [59] pointed to the fact that the Ginocchio potential on the line turns into the symmetric JRef potential associated with e -TP (62) if we choose

$$\mu_o = 2\nu + 1, \delta_- = (\lambda^2 - 1)^{-1}, C_- = \delta_- \lambda^4 \tag{180}$$

and then express (178) in terms of variable (68). Setting $h_{o,+} = h_{o,-} = -1$ in (84) and also making use of constraint (49) to replace $\lambda_{\pm; k, m}$ in (26) for

$$\lambda_{-; cn} = \lambda_{+; cn} \equiv \tilde{\omega}_{cn} \tag{181}$$

thus gives

$$\Sigma_- + \delta_- + 1/4 = 1/4 \delta_- \mu_o^2 \tag{182}$$

and

$$\tilde{\varepsilon}_{cn} = -\lambda^4 \tilde{\omega}_{cn}^2, \tag{183}$$

respectively. As expected, energy dispersion formula (183) and quadratic equation

$$(\lambda^2 - 1) \tilde{\omega}_{cn}^2 + (\tilde{\omega}_{cn} + n + 1/2)^2 - 1/4 \mu_o^2 = 0 \tag{184}$$

match, respectively, (3.9) and (3.10) in [44], with μ standing for $\tilde{\omega}_{cn}$ in our notation. Taking into account that both leading and linear coefficients of the quadratic equation are positive, the latter may have a positive root

$$\tilde{\omega}_{cn} = \lambda^{-2} \left[\sqrt{1/4 \lambda^2 \mu_o^2 - (\lambda^2 - 1)(n + 1/2)^2} - n - 1/2 \right] > 0 \tag{185}$$

iff its free term is negative, i.e., iff

$$0 \leq n < 1/2(\mu_o - 1). \tag{186}$$

On the other hand, substituting (176) into (118) shows that

$$\Sigma_+ + \delta_+ + 1/4 = 1/4 \mu_o^2, \tag{187}$$

so the quadratic polynomial in the brackets in quartic Equation (136) takes the form

$$\begin{aligned} \tilde{\omega}_{cn}^2 + (\lambda^2 - 1)(\tilde{\omega}_{cn} + n + 1/2)^2 - 1/4 \mu_o^2 &= 0 \\ (\tilde{\omega}_{cn} < -n - 1/2). \end{aligned} \tag{188}$$

Setting side by side quadratic Equations (184) and (188), we conclude that the positive root of the former equation specifies ChExp (132) of the $(n + 1)$ -th eigenfunction for the pole of CSLE (114) at infinity:

$${}_i\check{\rho}_{\infty;cn} = \check{\omega}_{cn} = -{}_i\check{\omega}_{cn} - n - 1/2 > 0, \tag{189}$$

provided that

$$0 \leq n < \sqrt{h_o + 1} - 1/2, \tag{190}$$

as prescribed by (143) with $h_o = h_{o;R}$.

Finally, comparing (187) with (182), we find that Lévai’s parameters (84) and (118) are related via a nontrivial formula

$$\Sigma_+ = \delta_+(\Sigma_- - 3/4) + 3, \tag{191}$$

which confirms our assertion that, despite the *formal similarity* between quartic Equations (58) and (125), the two equations are related in a rather complicated fashion in the symmetric limit $\Lambda = \Lambda_{\pm} = 0$.

As already pointed to in the Introduction, the fact that the symmetric reduction of the RRef potential leads to a certain subclass of the JRef potentials has been already recognized by Milson [6], who cited in this connection the quadratic transformation of the hypergeometric function using the substitution

$$1/2(1 - i\eta_{+,s}) \rightarrow \eta_{+,s}^2 + 1, \tag{192}$$

which is reminiscent of (156) for the variable z_+ . However, the author was unable to figure out all the details necessary for the suggested (possibly alternative) representation of the Schrödinger equation with the given symmetric potential.

As originally discovered by Ginocchio [44], the eigenfunctions of the Schrödinger equation with potential (178) can be expressed in terms of classical Gegenbauer polynomials with degree-dependent indexes after being converted to the variable $\eta_{-,s}$:

$$\check{\Phi}_{cn}[\eta_{-,s}; \mu_o; \lambda] = (1 - \eta_{-,s}^2)^{1/2(\check{\omega}_{cn}+1)} C_n^{(\check{\omega}_{cn}+1/2)}(\eta_{-,s}). \tag{193}$$

On the other hand, as demonstrated in [7,45], the eigenfunctions of RRef CSLE (114) with real h_o are expressible in terms of orthogonal Masjed-Jamei polynomials [46]

$${}_i\check{\Phi}_{cn}[\eta_{+,s}; \mu_o; \lambda] \propto (\eta_{+,s}^2 + 1)^{1/2{}_i\check{\omega}_{cn}+1/2} I_n^{(p_n)}(\eta_{+,s}) \tag{194}$$

with degree-dependent indexes $p_n = 1/2 - {}_i\check{\omega}_{cn}$ larger than $n + 1$.

5. Example: Sech-Squared Potential

Lévai’s [34] and Milson’s [6] e -TP potentials for $a_- = 0$ ($a_- \kappa_- = 1$) and $\kappa_+ = a_+ = 1$ turn into the Rosen–Morse [28] and, respectively, Gendenshtein [30] TSI potentials:

$$V_{RM}[\tanh(2x); \lambda_o, \mu_o] = -1/4(\mu_o^2 - 1)\text{sech}^2(2x) + 1/2\lambda_o^2[1 - \tanh(2x)] \tag{195}$$

and

$$V_G[\sinh(2x); h_{o;R} + i h_{o;I}] = -\text{sech}^2(2x)[h_{o;R} + \frac{3}{4} + h_{o;I} \sinh(2x)], \tag{196}$$

where

$$V_{RM}[\eta_{-,s}; \lambda_o, \mu_o] = -1/2(1 - \eta_{-,s})[1/2(\mu_o^2 - 1)(1 + \eta_{-,s}) - \lambda_o^2] \tag{197}$$

and

$$V_G[\eta_{+,s}; h_{o;R} + i h_{o;I}] = -\frac{4 h_{o;R} + 3 + 4 h_{o;I} \eta_{+,s}}{4(\eta_{+,s}^2 + 1)}. \tag{198}$$

Our choice of the TP coefficients $a_- = 0$, $\kappa_- = +\infty$, and $\kappa_+ = a_+ = 1$ assures that conditions (150) and (151) automatically hold and, as a result, the variables

$$z_-(x) = \tanh^2(2x) \tag{199}$$

and

$$z_+(x) = \sinh^2(2x) \tag{200}$$

are related via linear fractional Formula (158).

The corresponding prime SLEs take the form

$$\left\{ (1 - \eta^2) \frac{d}{d\eta} (1 - \eta^2) \frac{d}{d\eta} - V_{RM}[\eta; \lambda_o, \mu_o] - \varepsilon \right\} \Psi[\eta; \lambda_o, \mu_o; \varepsilon] = 0 \tag{201}$$

and

$$\left\{ (1 + \eta^2) \frac{d}{d\eta} \sqrt{1 + \eta^2} \frac{d}{d\eta} - V_G[\eta; h_o] - \varepsilon \right\} {}_i\Psi[\eta; h_o; \varepsilon] = 0, \tag{202}$$

which are solved under the DBCs at ± 1 and $\pm\infty$, respectively. The representation of the RCSLEs of our interest in the prime form [37] makes the ChExps of two Frobenius solutions differ only by sign, and therefore the DBC automatically selects the principal solution.

Setting $\kappa_+ = 1$ in (136) brings us to the following quadratic equation

$${}_i\omega_{\dagger m}^2 ({}_i\omega_{\dagger m}^2 - h_{o;R}) - 1/4h_{o;I}^2 = 0 \tag{203}$$

in ${}_i\omega_{\dagger m}^2$, where we changed cn for $\dagger m$ to indicate that the given equation holds for the averaged indexes of the Jacobi polynomials forming not only the eigenfunctions of CSLE (114), but also any other q-RSs. The crucial feature of this quadratic equation is that its coefficients are independent of the polynomial degree, and therefore, as a direct consequence of (126), this is also true for the complex-conjugated pairs of the Jacobi indexes in question. So, in contrast with RM potential (195), Gendenshtein potential (196) belongs to Group A (not B!) in Odake and Sasaki’s [66] classification scheme of the rational TSI potentials. This is the direct consequence of our observation [51] that the ExpDiffs for the poles of SLE (202) at $\pm\infty$ are energy-independent, so the TFI CSLE under consideration belongs to Group A [63]. Since all the q-RSs in this case are specified by a single series of Maya diagrams [63], all the solvable rational Darboux–Crum transforms of SLE (202) with a complex parameter h_o can be obtained using admissible Wronskians of Routh polynomials [67] with the common complex index. It is convenient to choose the sequence which starts from a finite orthogonal set of R–Routh polynomials forming n_o eigenfunctions of SLE (202). As a result [66,67], the corresponding Wronskian transforms of R–Routh polynomials form finite sequences of exceptional orthogonal polynomials (EOSs) in Quesne’s terms [64].

The symmetric reductions $\lambda_o = 0$ and $h_{o;I} = 0$ of potentials (195) and (196) represent exactly the same *sech*-squared potential if the parameters μ_o and $h_{o;R}$ are related via (176). This potential can be thus quantized either via classical Gegenbauer polynomials with *degree-dependent* indexes [44] or via Masjed-Jamei polynomials [46] with *degree-independent* indexes [45]. As pointed to in [63], the A or B grouping suggested by Odake and Sasaki [66] for rational TSI potentials is actually an attribute of the particular rational realization of the given potential rather than the potential itself. The h-PT and Morse potentials represent two other instances of such an atypical dualism [63,68].

The latter quantization scheme using R–Routh polynomials of a definite parity is obviously preferable for constructing the symmetric Darboux–Crum transforms of the *sech*-squared potential [66]. Since it is the symmetric limit of the Gendenshtein potential, the q-RSs of a definite parity in this limiting case are also specified by a single series of Maya diagrams, and therefore all the rational Darboux–Crum transforms (RDCT) of SLE (202) with a real parameter h_o can be obtained using admissible Wronskians of Routh polynomials of a definite parity with the common real index such that the first n_o polynomials in the given infinite polynomial sequence constitute a finite orthogonal set of Masjed-Jamei polynomials.

As a result, the corresponding Wronskian transforms of Masjed-Jamei polynomials must form finite sequences of EOPs. A detailed study on this remarkable family of finite EOP sequences is currently under way.

6. Discussion

The presented analysis scrutinizes the striking resemblance between quartic Equations (58) and (125) suggested by Lévai [34] for computing the energy spectrum of real Fuschian CSLEs (1) and (114) with two poles located symmetrically around the origin on either the real or imaginary axis. In both cases, even density functions (66) and, accordingly, (110) remain positive within the corresponding quantization intervals, so the change of variable $\eta_{-,s}(x; \delta, C)$ or subsequently $\eta_{+,s}(x; \delta_+, C_+)$ converting the given CSLE into the Schrödinger equation is an odd real function of x . It has been shown that the two variables are interrelated via simple algebraic Formula (148) or its reverse (149).

It was also proven that the corresponding Liouville potentials referred to in the paper as Lévai's [34] and Milson's [6] e -TP potentials overlap along the symmetric curves, which are nothing but two alternative rational representations of Ginocchio's [44] potential function (178).

Since the Liouville transformation converts RRef CSLE (114) with real h_0 into the Schrödinger equation with the Ginocchio potential on the line, the eigenfunctions of the latter equations can be expressed in terms of Masjed-Jamei polynomials [46] with degree-dependent index [45]. In the TSI limit represented by sech-squared potential, the index of the Masjed-Jamei polynomials forming the corresponding eigenfunctions becomes degree-independent and, as a result, the Wronskians of the eigenfunctions turn into weighted Wronskians of the *orthogonal* Masjed-Jamei polynomials with the common index. This finite polynomial sequence starts an infinite sequence of Routh polynomials of a definite parity with the same *real* index. If the Wronskian formed by a subset of the latter polynomials does not have real roots, then the corresponding Wronskian transforms of the Masjed-Jamei polynomials form a finite EOP sequence. The EOP sequences constructed in such a way represent the symmetric reduction of the EOP sequences generated by us in [67] for Gendenshtein potential (196) using the admissible Wronskians of Routh polynomials with a common complex index.

7. Conclusions

Though the main purpose of the paper was to reveal some remarkable overlapping features of two real-field reductions of the complex (non-self-adjoint) JRef CSLE with an even density function, the starting point for our analysis has a much broader area of application. Namely, the density function does not have to be even—we only need to require that it remains positive inside the finite real interval or on the imaginary axis. The Liouville transformation on these intervals results in two families of complex Liouville potentials referred to by us as complex potentials of the Lévai class.

As suggested by Lévai [34], one can significantly simplify the analysis of the resultant Schrödinger equation by choosing the even density function. Such a choice of the density function assures that the variable $\eta(x)$ used to convert the CSLE to the Schrödinger equation in x is an odd function of x . The direct consequence of this constraint is that the corresponding Liouville potential is PT-symmetric if this is true for the JRefPF in question.

The important advantage of the suggested complexification of the JRef CSLE comes from the fact that it has a quartet of AEH solutions which can be used as FFs for RDTs whether the Liouville transformation is done on the real or imaginary axis. In particular, it will be shown in a separate publication that Lévai's PT-symmetric potential [34,38,40] obtained via the Liouville transformation of the PT-symmetric JRef CSLE on the real axis has AEH solutions composed of R-Routh polynomials in an imaginary argument η_I . Since these functions may not have zeros on the real axis $-\infty < \eta_I(x) < +\infty$, each of them can be used as the FF to construct a new solvable PT-symmetric potential. The sketched complexification of the JRef CSLE thus provides a new mechanism for constructing

PT-symmetric potentials with real energy spectra specified by positive roots of quartic Equation (58).

The presented analysis of the AEH solutions of the complex JRef CSLE with the even density function thus opens new horizons for Lévai's unified approach [34] to two families of generally complex Liouville potentials obtained via the Liouville transformations of the mentioned CSLE, either inside the interval $(-1, +1)$ or on the imaginary axis.

Funding: This research received no external funding.

Data Availability Statement: No new data were created.

Acknowledgments: I wish to use this opportunity to thank Mariana Kirchbach for her inspiring advice, which greatly influenced the early stage of my studies on the quantization of the RRef potential by Romanovski–Routh polynomials. The author is also grateful to the referees for their stimulating critical suggestions to tune the paper results.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Bose, A.K. Solvable potentials. *Phys. Lett.* **1963**, *7*, 245–246. [CrossRef]
2. Bose, A.K. A class of solvable potentials. *Nuovo Cim.* **1964**, *32*, 679–688. [CrossRef]
3. Erdelyi, A.; Bateman, H. *Transcendental Functions*; McGraw Hill: New York, NY, USA, 1953; Volume 1.
4. Everitt, W.N. A Catalogue of Sturm-Liouville Differential Equations. In *Sturm-Liouville Theory, Past and Present*; Amrein, W.O., Hinz, A.M., Pearson, D.B., Eds.; Birkhäuser Verlag: Basel, Switzerland, 2005; pp. 271–331.
5. Natanson, G.A. Study of the one-dimensional Schrödinger equation generated from the hypergeometric equation. *Vestn. Leningr. Univ.* **1971**, *10*, 22–28. Available online: arxiv.org/PS_cache/physics/pdf/9907/9907032v1.pdf (accessed on 20 July 1999).
6. Milson, R. Liouville transformation and exactly solvable Schrödinger equations. *Int. J. Theor. Phys.* **1998**, *37*, 1735–1752. [CrossRef]
7. Natanson, G. Exact quantization of the Milson potential via Romanovski-Routh polynomials. *arXiv* **2015**, arXiv:1310.0796v3.
8. Avarez-Castillo, D.E.; Kirchbach, M. Exact spectrum and wave functions of the hyperbolic Scarf potential in terms of finite Romanovski polynomials. *Rev. Mex. Fis. E* **2007**, *53*, 143–154.
9. Natanson, G. Routh Polynomials: Hundred Years in Obscurity. 2022. pp. 1–61. Available online: researchgate.net/publication/326522529 (accessed on 1 November 2022).
10. Natanson, G. Rediscovery of Routh Polynomials after Hundred Years in Obscurity. In *Recent Research in Polynomials*; Özger, F., Ed.; IntechOpen: London, UK, 2023; 27p.
11. Koekoek, R.; Lesky, P.A.; Swarttouw, R.F. *Hypergeometric Orthogonal Polynomials and Their q-Analogues*; Springer: Berlin/Heidelberg, Germany, 2010.
12. Koornwinder, T. Additions to the Formula Lists in ‘Hypergeometric Orthogonal Polynomials and Their q-Analogues’ by Koekoek, Lesky and Swarttouw. 2022. pp. 1–45. Available online: [Staff.fnwi.uva.nl/t.h.koornwinder/art/informal/KLSadd.pdf](https://staff.fnwi.uva.nl/t.h.koornwinder/art/informal/KLSadd.pdf) (accessed on 4 February 2022).
13. Jordaan, K.; Toókos, F. Orthogonality and asymptotics of Pseudo-Jacobi polynomials for non-classical parameters. *J. Approx. Theory* **2014**, *178*, 1–12. [CrossRef]
14. Routh, E.J. On some properties of certain solutions of a differential equation of second order. *Proc. Lond. Math. Soc.* **1884**, *16*, 245–261. [CrossRef]
15. Lesky, P.A. Vervollständigung der klassischen Orthogonalpolynome durch Ergänzungen zum Askey-Schema der hypergeometrischen orthogonalen Polynome. *Ost. Akad. Wiss.* **1995**, *204*, 151–166.
16. Lesky, P.A. Endliche und unendliche Systeme von kontinuierlichen klassischen Orthogonalpolynomen. *Z. Angew. Math. Mech.* **1996**, *76*, 181–184. [CrossRef]
17. Romanovsky, V. Sur la généralisation des courbes de Pearson. *Atti Congresso Intern. Mat.* **1928**, *6*, 107–110.
18. Romanovski, V.I. Sur quelques classes nouvelles de polynomes orthogonaux. *C. R. Acad. Sci. Paris* **1929**, *188*, 1023–1025.
19. Natanson, G. Exact quantization of the Milson potential via Romanovski polynomials. *arXiv* **2013**, arXiv:1310.0796v1.
20. Dereziński, J.; Wrochna, M. Exactly solvable Schrödinger Operators. *Ann. Henri Poincaré* **2011**, *12*, 397–418. [CrossRef]
21. Stevenson, A.F. Note on the ‘Kepler problem’ in a spherical space, and the factorization method of solving eigenvalue problems. *Phys. Rev.* **1941**, *59*, 842–843. [CrossRef]
22. Cryer, C.W. Rodrigues’ formulas and the classical orthogonal polynomials. *Boll. Unione Mat. Ital.* **1970**, *25*, 1–11.
23. Askey, R. An integral of Ramanujan and orthogonal polynomials. *J. Indian Math. Soc.* **1987**, *51*, 27–36.
24. Bagrov, V.G.; Gitman, D.M.; Ternov, I.M.; Khalilov, V.R.; Shapovalov, V.N. *Tochnye Resheniya Relativistskikh Volnovykh Uravnenii*; Nauka: Novosibirsk, Russia, 1982. (In Russian)
25. Cooper, F.; Khare, A.; Sukhatme, U.P. Supersymmetry and quantum mechanics. *Phys. Rep.* **1995**, *251*, 267–385. [CrossRef]
26. Cooper, F.; Khare, A.; Sukhatme, U.P. *Supersymmetry in Quantum Mechanics*; World Scientific: Denver, CO, USA, 2001.

27. Compean, C.B.; Kirchbach, M. The trigonometric Rosen–Morse potential in the supersymmetric quantum mechanics and its exact solutions. *J. Phys. A* **2006**, *39*, 547–557. [CrossRef]
28. Rosen, N.; Morse, P.M. On the vibrations of polyatomic molecules. *Phys. Rev.* **1932**, *42*, 210–217. [CrossRef]
29. Dabrowska, J.W.; Khare, A.; Sukhatme, U.P. Explicit wavefunctions for shape-invariant potentials by operator techniques. *J. Phys. A* **1988**, *21*, L195–L200. [CrossRef]
30. Gendenshtein, L.E. Derivation of exact spectra of the Schrödinger equation by means of supersymmetry. *JETP Lett.* **1983**, *38*, 356–359.
31. Lévai, G. A search for shape-invariant solvable potentials. *J. Phys. A* **1989**, *22*, 689–702. [CrossRef]
32. Bhattacharjie, A.; Sudarshan, E.C.G. A class of solvable potentials. *Nuovo Cim.* **1962**, *25*, 864–879. [CrossRef]
33. Grosche, C. The general Besselian and Legendrian path integrals. *J. Phys. A* **1996**, *29*, L183–L189. [CrossRef]
34. Lévai, G. Gradual spontaneous breakdown of \mathcal{PT} symmetry in a solvable potential. *J. Phys. A* **2012**, *45*, 444020. [CrossRef]
35. Gibbons, J.; Veselov, A.P. On the rational monodromy-free potentials with sextic growth. *J. Math. Phys.* **2009**, *50*, 013513. [CrossRef]
36. Natanson, G. Survey of nodeless regular almost-everywhere holomorphic solutions for exactly solvable Gauss-reference Liouville potentials on the line I. Subsets of nodeless Jacobi-seed solutions co-existent with discrete energy spectrum. *arXiv* **2016**, arXiv:1606.08758.
37. Natanson, G. Darboux-Crum Nets of Sturm-Liouville Problems Solvable by Quasi-Rational Functions I. General Theory. 2018. pp. 1–99. Available online: researchgate.net/publication/323831953 (accessed on 1 March 2018).
38. Lévai, G. \mathcal{PT} Symmetry in Natanzon-class Potentials. *Int. J. Theor. Phys.* **2015**, *54*, 2724–2736. [CrossRef]
39. Lévai, G. A class of exactly solvable potentials related to the Jacobi polynomials. *J. Phys. A* **1991**, *24*, 131–146. [CrossRef]
40. Lévai, G. Accidental crossing of energy eigenvalues in \mathcal{PT} -symmetric Natanzon-class potentials. *Ann. Phys.* **2017**, *38*, 1–11. [CrossRef]
41. Lévai, G. Exactly Solvable \mathcal{PT} -Symmetric Models. In *\mathcal{PT} Symmetry in Classical and Quantum Physics*; Bender, C., Ed.; World Scientific: London, UK, 2019; pp. 221–260.
42. Gesztesy, F.; Simon, B.; Teschl, G. Zeros of the Wronskian and renormalize oscillation theory. *Am. J. Math.* **1996**, *118*, 571–594.
43. Garcia-Ferrero, M.; Gémez-Ullate, D.; Milson, R. A Bochner type classification theorem for exceptional orthogonal polynomials. *J. Math. Anal. Appl.* **2019**, *472*, 584–626. [CrossRef]
44. Ginocchio, J. A class of exactly solvable potentials: I. One-dimensional Schrödinger equation. *Ann. Phys.* **1984**, *152*, 203–219. [CrossRef]
45. Natanson, G. Quantization of One-Dimensional Ginocchio Potential by Masjed-Jamei Polynomials with Degree-Dependent Indexes. 2022. pp. 1–52. Available online: researchgate.net/publication/360950671 (accessed on 1 May 2022).
46. Masjedjamei, M. Three finite classes of hypergeometric orthogonal polynomials and their application in functions approximation. *Integr. Transf. Spec. Funct.* **2002**, *13*, 169–190. [CrossRef]
47. Natanzon, G.A. Construction of the Jost function and the S-matrix for a general potential allowing solution of the Schrödinger equation in terms of hypergeometric functions. *Sov. Phys. J.* **1978**, *21*, 855–859. [CrossRef]
48. Natanzon, G.A. General properties of potentials for which the Schrödinger equation can be solved by means of hypergeometric functions. *Theor. Math. Phys.* **1979**, *38*, 146–153. [CrossRef]
49. Natanson, G. Single-source nets of Fuschian rational canonical Sturm-Liouville equations with common simple-poles density functions and related sequences of multi-indexed orthogonal Heine eigenpolynomials. In Proceedings of the 14th International Symposium on Orthogonal Polynomials, Special Functions and Applications, Canterbury, UK, 3–7 July 2017. Available online: <https://www.researchgate.net/publication/317643178> (accessed on 1 July 2017).
50. Natanson, G. Use of Wronskians of Jacobi Polynomials with Common Complex Indexes for Constructing X-DPSs and Their Infinite and Finite Orthogonal Subsets. 2019. pp. 1–57. Available online: researchgate.net/publication/331638063 (accessed on 1 March 2019).
51. Natanson, G. Gauss-seed nets of Sturm-Liouville problems with energy-independent characteristic exponents and related sequences of exceptional orthogonal polynomials I. Canonical Darboux transformations using almost-everywhere holomorphic factorization functions. *arXiv* **2013**, arXiv:1305.7453v1.
52. Natanson, G. Single-source nets of algebraically-quantized reflective Liouville potentials on the line I. Almost-everywhere holomorphic solutions of rational canonical Sturm-Liouville equations with second-order poles. *arXiv* **2015**, arXiv:1503.04798v2.
53. Everitt, W.N.; Littlejohn, L.L. Orthogonal polynomials and spectral theory: A survey. In *Orthogonal Polynomials and Their Applications*; IMACS Annals on Computing and Applied, Mathematics; Brezinski, C., Gori, L., Ronveaux, A., Eds.; J. C. Baltzer AG Publishers: Tulsa, OK, USA, 1991; Volume 9, pp. 21–55.
54. Everitt, W.N.; Kwon, K.H.; Littlejohn, L.L.; Wellman, R. Orthogonal polynomial solutions of linear ordinary differential equations. *J. Comp. Appl. Math.* **2001**, *133*, 85–109. [CrossRef]
55. Bochner, S. Über Sturm-Liouvillesche Polynomsysteme. *Math. Z.* **1929**, *29*, 730–736. [CrossRef]
56. Natanson, G. Dutt-Khare-Varshni Potential and Its Quantized-by-Heun-Polynomial SUSY Partners as Nontrivial Examples of Solvable Potentials Explicitly Expressible in Terms of Elementary Functions. *arXiv* **2015**, arXiv:1508.04738v1.
57. Courant, R.; Hilbert, D. *Methods of Mathematical Physics*; Interscience: New York, NY, USA, 1953; Volume 1.
58. Natanson, G. Breakup of SUSY quantum mechanics in the Limit-Circle region of the reflective Kratzer oscillator. *arXiv* **2014**, arXiv:1405.2059v1.

59. Lévai, G. Non-compact groups and solvable potentials. In *Proceedings of the XXI International Colloquium on Group Theoretical Methods in Physics, Goslar, Germany, 15–20 July 1996*; Doebner, H.-D., Nattermann, P., Scherer, W., Eds.; World Scientific: Singapore, 1997; Volume 1, pp. 461–466.
60. Wu, J. Group Theory Approach to Scattering. Ph.D. Thesis, Yale University, New Haven, CT, USA, 1985.
61. Wu, J.; Alhassid, Y.; Gürsey, F. Group theory approach to scattering. IV. Solvable potentials associated with $SO(2,2)$. *Ann. Phys.* **1989**, *196*, 163–181. [[CrossRef](#)]
62. Ginocchio, J. A Class of exactly solvable potentials II. The three-dimensional Schrodinger equation. *Ann. Phys.* **1985**, *159*, 467–480. [[CrossRef](#)]
63. Natanson, G. Equivalence Relations for Darboux-Crum Transforms of Translationally Form-Invariant Sturm-Liouville Equations. 2021. pp. 1–58. Available online: [researchgate.net/publication/353131294](https://www.researchgate.net/publication/353131294) (accessed on 1 August 2021).
64. Quesne, C. Extending Romanovski polynomials in quantum mechanics. *J. Math. Phys.* **2013**, *54*, 122103. [[CrossRef](#)]
65. McIntosh, H.V. *Complex Analysis*; Universidad Autónoma de Puebla: Puebla, Mexico, 2001.
66. Otake, S.; Sasaki, R. Krein–Adler transformations for shape-invariant potentials and pseudo virtual states. *J. Phys. A* **2013**, *46*, 245201. [[CrossRef](#)]
67. Natanson, G. On Finite Exceptional Orthogonal Polynomial Sequences Composed of Wronskian Transforms of Romanovski-Routh Polynomials. 2022. pp. 1–38. Available online: <https://www.researchgate.net/publication/364350392> (accessed on 1 October 2022).
68. Natanson, G. Quantization of rationally deformed Morse potentials by Wronskian transforms of Romanovski-Bessel polynomials. *Acta Polytec.* **2022**, *62*, 100–117. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.