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Article

Efficient Difference and Ratio-Type Imputation Methods under Ranked Set Sampling

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Abstract: It is well known that ranked set sampling (RSS) is more efficient than simple random sampling (SRS). Furthermore, the presence of missing data vitiates the conventional results. Only a minuscule amount of work has been conducted under RSS with missing data. This paper makes a modest attempt to provide some efficient difference- and ratio-type imputation methods in the presence of missing values under RSS. The envisaged imputation methods are demonstrated to provide better results than the existing imputation methods. The theoretical results are enhanced by a computational analysis using real and hypothetically generated symmetric (Normal) and asymmetric (Gamma and Weibull) populations. The computational results show that the proposed imputation method outperforms the existing imputation methods in terms of its higher percent relative efficiency. Additionally, the impact of skewness and kurtosis on the efficiency of the suggested imputation methods has also been calculated.

Keywords: bias; mean square error; missing data; imputation; ranked set sampling

MSC: 2020: 62D04

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1. Introduction

The most common problem reported by a survey statistician in their daily life is making inferences from data containing missing values. Such problems of missing values in survey sampling may be tackled through the technique of imputation. A wide range of imputation methods have been suggested by various authors. The authors of [1] discussed three noteworthy concepts on missing values as missing at random (MAR), observed at random (OAR), and parameter distribution (PD). Subsequently, [2–6] suggested different types of imputation methods. The authors of [7] showed that missing at random and missing completely at random ($MCAR$) are totally different approaches. Many renowned authors [8–12] assumed the $MCAR$ approach in their studies for the imputation of missing values. The authors of [13] introduced some imputation methods which outperformed the imputation methods suggested by [14]. The authors of [15] developed logarithmic-type imputation methods under SRS. The authors of [16] utilized robust measures and suggested compromised imputation-based mean estimators.

In real-life applications, situations may arise where the measurement of the study variable is not easy or expensive to do so but can be ranked visually or by a cost-free measure. In this situation, ref. [17] envisaged the concept of ranked set sampling (RSS) but did not provide any rigorous mathematical support. The authors of [18] explored the idea of [17] and furnished the essential mathematical foundation to the theory of RSS. In sample surveys, when each group has very few observations, each observation then becomes essential to make an effective prediction. Furthermore, the utilization of these types of datasets based on missing values may alter the final conclusion and decrease the efficiency of the estimation procedure. To deal with such problems, refs. [19–22] introduced an analytical comparison of imputation methods under RSS.

This paper conducted a search for efficient imputation procedures. We adapted some efficient difference- and ratio-type imputation methods under RSS based on [11,12], which are more efficient compared to the mean imputation method and the imputation methods suggested by [21,22] under RSS.

The paper is organised as follows: Section 2 describes the detailed methodology of RSS as well as the notations utilized throughout the paper. In Section 3, we consider a concise recap of some imputation methods under RSS, whereas in Section 4, we consider the proposed methods of imputation. In Section 5, we provide the efficiency conditions. Section 6 is devoted to the computational analysis and finally, Section 7 considers the conclusions of this study.

2. Sampling Methodology and Notations

The methodology of RSS was initiated by [17], based on drawing m simple random samples of size m from the parent population. These m units are now ranked inside each set regarding the auxiliary variable. The *rank* 1 unit is chosen from the first set for the measurement of the auxiliary variables along with the associated study variable. The *rank* 2 unit is chosen from the second ranked set for the measurement of auxiliary variable X along with the associated study variable Y, and the process is proceeded until the *rank* m unit is chosen from the last set. The above process is referred to as a cycle. This whole procedure is repeated k times, providing $n = mk$ ranked set samples.

In the presence of missing values in a dataset, an alteration in the aforesaid methodology is proposed for the estimation of the population mean of the study variables under the consideration of usable auxiliary information. To facilitate ranking, m bivariate random samples, each consisting of m units, are quantified from the parent population. These m units are ranked within each set regarding the auxiliary variable as it is hypothesized that the study variable has some missing values. Now, from the first sample, the smallest ranked unit of X along with the correlated Y is selected. From the second sample, the second smallest ranked unit of X along with the correlated Y is selected. The above procedure is continued in the same mode until the mth sample from the highest ranked unit of X along with the correlated Y is selected. Compatible to the study variable from the first cycle, m' units can provide a response for the measurement of the element out of the selected m units such that $m > m'$. The whole procedure is repeated k times until responses from n' units out of n selected units is obtained, where $n > n'$.

Notations

Let $\mu_y = N^{-1}\sum_{i=1}^{N} Y_i$ be the mean of the finite population Ω of N identifiable units with values Y_i, $i \in \Omega$. Let a ranked set sample s of size $n = mk$ be quantified from Ω to estimate the population mean μ_y. Let m' be the number of responding units out of the sampled m units. Let P be the probability that the ith respondent belongs to the responding group A and $(1 - P)$ be the probability that the ith respondent belongs to the non-responding group $\bar{A}$ such that $s = A \cup \bar{A}$. The value Y_i, $i \in A$ is observed for every unit, but for the units $i \in \bar{A}$ the values are missing and need imputation to build the complete structure of the data to draw a valid conclusion. The auxiliary variable X assists in the execution of imputation of missing values. Let X_i be the value of X for the unit i which is positive and known $\forall\ i \in s$ such that $X_s = X_i$; $i \in s$ are known. Let $\bar{X}_{r,rss} = \sum_{i=1}^{m'}\sum_{j=1}^{k} X_{(i:i)j}/mkP$ and $\bar{Y}_{r,rss} = \sum_{i=1}^{m'}\sum_{j=1}^{k} Y_{[i:i]j}/mkP$ possess the unbiased estimator of population means μ_x and μ_y, respectively. Here, $X_{(i:i)j}$ and $Y_{[i:i]j}$ are the ith order statistics and ith judgement order in the ith sample, respectively, of size m in cycle j for variable X and Y. For the sake of simplicity, we denote $X_{(i:i)j}$ and $Y_{[i:i]j}$ by $X_{(i)}$ and $Y_{[i]}$, respectively. Let P be the probability of determining the response, then $E(r^{-j}) = \{E(r)\}^{-j}$, which provides the variance as

$$E\{V(\bar{Y}_{r,rss})\} = \left(\frac{\sigma_y^2}{mkP} - \frac{1}{m^2kP}\sum_{i=1}^{m}\tau_{y_{[i]}}^2\right) \tag{1}$$

then $$E(j^{-1}) = \{E(j)\}^{-1} = nP \tag{2}$$

The proof of (1) and (2) can be viewed in [20].

To tabulate the bias and mean square error (MSE), the following notations and results are used throughout this paper. Let $\bar{Y}_{r,rss} = \mu_y(1+\epsilon_0)$, $\bar{X}_{r,rss} = \mu_x(1+\epsilon_1)$, and $\bar{X}_{n,rss} = \mu_x(1+\epsilon_2)$, where ϵ_0, ϵ_1, and ϵ_2 are the error terms, such that $E(\epsilon_0) = E(\epsilon_1) = E(\epsilon_2) = 0$ and

$$E(\epsilon_0^2) = \left(\frac{C_y^2}{mkP} - \frac{1}{m^2kP}\sum_{i=1}^{m}\frac{\tau_{y_{[i]}}^2}{\mu_y^2}\right) = \left(\gamma^* C_y^2 - W_y^{2^*}\right)$$

$$E(\epsilon_1^2) = \left(\frac{C_x^2}{mkP} - \frac{1}{m^2kP}\sum_{i=1}^{m}\frac{\tau_{x_{(i)}}^2}{\mu_x^2}\right) = \left(\gamma^* C_x^2 - W_x^{2^*}\right)$$

$$E(\epsilon_2^2) = \left(\frac{C_x^2}{mk} - \frac{1}{m^2k}\sum_{i=1}^{m}\frac{\tau_{x_{(i)}}^2}{\mu_x^2}\right) = \left(\gamma C_x^2 - W_x^2\right)$$

$$E(\epsilon_0, \epsilon_1) = \left(\frac{\rho_{xy}C_xC_y}{mkP} - \frac{1}{m^2kP}\sum_{i=1}^{m}\frac{\tau_{xy_{[i]}}}{\mu_x\mu_y}\right) = \left(\gamma^*\rho_{xy}C_xC_y - W_{xy}^*\right)$$

$$E(\epsilon_0, \epsilon_2) = \left(\frac{\rho_{xy}C_xC_y}{mk} - \frac{1}{m^2k}\sum_{i=1}^{m}\frac{\tau_{xy_{[i]}}}{\mu_x\mu_y}\right) = \left(\gamma\rho_{xy}C_xC_y - W_{xy}\right)$$

$$E(\epsilon_1, \epsilon_2) = \left(\frac{C_x^2}{mk} - \frac{1}{m^2k}\sum_{i=1}^{m}\frac{\tau_{x_{(i)}}^2}{\mu_x^2}\right) = \left(\gamma C_x^2 - W_x^2\right)$$

where $\gamma^* = 1/mkP$, $\gamma = 1/mk$, $W_y^2 = 1\sum_{i=1}^{m}\tau_{y_{[i]}}^2/m^2kP\mu_y^2$, $W_x^2 = 1\sum_{i=1}^{m}\tau_{x_{(i)}}^2/m^2k\mu_x^2$, $W_x^{2^*} = 1\sum_{i=1}^{m}\tau_{x_{(i)}}^2/m^2kP\mu_x^2$, $W_{xy} = 1\sum_{i=1}^{m}\tau_{xy_{[i]}}/m^2k\mu_x\mu_y$, $W_{xy}^* = 1\sum_{i=1}^{m}\tau_{xy_{[i]}}/m^2kp\mu_x\mu_y$, $\tau_{y_{[i]}} = (\mu_{y_{[i]}} - \mu_y)$, $\tau_{x_{(i)}} = (\mu_{x_{(i)}} - \mu_x)$, $\tau_{xy} = (\mu_{x_{(i)}} - \mu_x)(\mu_{y_{[i]}} - \mu_y)$, $C_x = S_x/\mu_x$, $C_y = S_y/\mu_y$, $\mu_y = E(Y)$, $\mu_x = E(X)$, $\mu_{y_{[i]}} = E(Y_{[i]})$, and $\mu_{x_{(i)}} = E(X_{(i)})$.

Here, S_x and S_y are the population standard deviations due to the auxiliary variable X and study variable Y, respectively, C_x and C_y are the population coefficients of variation due to the auxiliary variable X and study variable Y, respectively, and ρ_{xy} is the population correlation coefficient between the auxiliary variable X and study variable Y. Moreover, we would also like to annotate that the quantities $\mu_{x_{(i)}}$ and $\mu_{y_{[i]}}$ consist of order statistics from some particular distributions and can be easily determined from [23].

3. Review of Imputation Methods under RSS

3.1. Mean Imputation Method

The method of imputation is

$$y_{.i} = \begin{cases} Y_i & \text{for } i \in A \\ \bar{Y}_{r,rss} & \text{for } i \in \bar{A} \end{cases}$$

The consequent estimator is

$$t_m = \bar{Y}_{r,rss} \tag{3}$$

The imputation methods are categorized into three strategies under the consideration of the availability of auxiliary information.

Strategy I: When μ_x is known and $\bar{X}_{n,rss}$ is used.
Strategy II: When μ_x is known and $\bar{X}_{r,rss}$ is used.
Strategy III: When μ_x is unknown and $\bar{X}_{n,rss}$ and $\bar{X}_{r,rss}$ are used.

3.2. The Al-Omari and Bouza Imputation Method

To improve the efficiency of the estimators in the presence of missing data, [9,21] suggested some regression-cum-ratio-type estimators under RSS as
Strategy I

$$\bar{y}_{KC_1} = \frac{\bar{Y}_{r,rss} + b(\mu_x - \bar{X}_{n,rss})\mu_x}{\bar{X}_{n,rss}} \tag{4}$$

Strategy II

$$\bar{y}_{KC_2} = \frac{\bar{Y}_{r,rss} + b(\mu_x - \bar{X}_{r,rss})\mu_x}{\bar{X}_{r,rss}} \tag{5}$$

Strategy III

$$\bar{y}_{KC_3} = \frac{\bar{Y}_{r,rss} + b(\bar{X}_{n,rss} - \bar{X}_{r,rss})\bar{X}_{n,rss}}{\bar{X}_{r,rss}} \tag{6}$$

where $b = S_{xy}/S_x^2$ is the regression coefficient of Y on X.

3.3. The Sohail, Shabbir and Ahmed Imputation Methods

Following [20–22], we examined the ratio-type estimators of [8] using RSS for the imputation of missing values. These imputation methods are
Strategy I

$$y_{.is_1} = \begin{cases} Y_i & \text{for } i \in A \\ \frac{1}{n-r}\left\{n\bar{Y}_{r,rss}\left(\frac{\mu_x}{\bar{X}_{n,rss}}\right)^{\beta_1} - r\bar{Y}_{r,rss}\right\} & \text{for } i \in \bar{A} \end{cases}$$

$$y_{.is_4} = \begin{cases} Y_i & \text{for } i \in A \\ \frac{1}{n-r}\left[n\bar{Y}_{r,rss}\left\{\frac{\mu_x}{\beta_4\bar{X}_{n,rss}+(1-\beta_4)\mu_x}\right\} - r\bar{Y}_{r,rss}\right] & \text{for } i \in \bar{A} \end{cases}$$

Strategy II

$$y_{.is_2} = \begin{cases} Y_i & \text{for } i \in A \\ \frac{1}{n-r}\left\{n\bar{Y}_{r,rss}\left(\frac{\mu_x}{\bar{X}_{r,rss}}\right)^{\beta_2} - r\bar{Y}_{r,rss}\right\} & \text{for } i \in \bar{A} \end{cases}$$

$$y_{.is_5} = \begin{cases} Y_i & \text{for } i \in A \\ \frac{1}{n-r}\left[n\bar{Y}_{r,rss}\left\{\frac{\mu_x}{\beta_5\bar{X}_{r,rss}+(1-\beta_5)\mu_x}\right\} - r\bar{Y}_{r,rss}\right] & \text{for } i \in \bar{A} \end{cases}$$

Strategy III

$$y_{.is_3} = \begin{cases} Y_i & \text{for } i \in A \\ \frac{1}{n-r}\left\{n\bar{Y}_{r,rss}\left(\frac{\bar{X}_{n,rss}}{\bar{X}_{r,rss}}\right)^{\beta_3} - r\bar{Y}_{r,rss}\right\} & \text{for } i \in \bar{A} \end{cases}$$

$$y_{.is_6} = \begin{cases} Y_i & \text{for } i \in A \\ \frac{1}{n-r}\left[n\bar{Y}_{r,rss}\left\{\frac{\bar{X}_{n,rss}}{\beta_6\bar{X}_{r,rss}+(1-\beta_6)\bar{X}_{n,rss}}\right\} - r\bar{Y}_{r,rss}\right] & \text{for } i \in \bar{A} \end{cases}$$

The consequent estimators are

$$t_{s_1} = \bar{y}_{r,rss}\left(\frac{\mu_x}{\bar{X}_{n,rss}}\right)^{\beta_1} \tag{7}$$

$$t_{s_2} = \bar{y}_{r,rss}\left(\frac{\mu_x}{\bar{X}_{r,rss}}\right)^{\beta_2} \tag{8}$$

$$t_{s_3} = \bar{y}_{r,rss}\left(\frac{\bar{X}_{n,rss}}{\bar{X}_{r,rss}}\right)^{\beta_3} \tag{9}$$

$$t_{s_4} = \bar{y}_{r,rss}\left\{\frac{\mu_x}{\beta_4\bar{X}_{n,rss} + (1-\beta_4)\mu_x}\right\} \tag{10}$$

$$t_{s_5} = \bar{y}_{r,rss}\left\{\frac{\mu_x}{\beta_5\bar{X}_{r,rss} + (1-\beta_5)\mu_x}\right\} \tag{11}$$

$$t_{s_6} = \bar{y}_{r,rss}\left\{\frac{\bar{X}_{n,rss}}{\beta_6\bar{X}_{r,rss} + (1-\beta_6)\bar{X}_{n,rss}}\right\} \tag{12}$$

where $\beta_i;\ i = 1,2,\ldots,6$ are appropriately chosen optimizing scalars.

The *MSE* values of the consequent estimators consisting of different imputation methods are given in Appendix A for quick reference and further analytical comparison.

4. The Proposed Imputation Methods

The crux of the present article is:

1. To provide efficient imputation methods for mean estimation.
2. To access the impact of the skewness and kurtosis coefficients on the choice of imputation procedures.

Motivated by the works of [11,12], we propose nine new imputation methods under the three strategies discussed earlier, defined as

Strategy I

$$y_{.i_1} = \begin{cases} \alpha_1 Y_i & \text{for } i \in A \\ \alpha_1\bar{Y}_{r,rss} + \frac{n\theta_1}{n-r}(\bar{X}_{n,rss} - \mu_x) & \text{for } i \in \bar{A} \end{cases}$$

$$y_{.i_4} = \begin{cases} Y_i & \text{for } i \in A \\ \frac{1}{n-r}\left\{n\alpha_4\bar{Y}_{r,rss}\left(\frac{\mu_x}{\bar{X}_{n,rss}}\right)^{\theta_4} - r\bar{Y}_{r,rss}\right\} & \text{for } i \in \bar{A} \end{cases}$$

$$y_{.i_7} = \begin{cases} Y_i & \text{for } i \in A \\ \frac{1}{n-r}\left\{n\alpha_7\bar{Y}_{r,rss}\left(\frac{\mu_x}{\mu_x+\theta_7(\bar{X}_{n,rss}-\mu_x)}\right) - r\bar{Y}_{r,rss}\right\} & \text{for } i \in \bar{A} \end{cases}$$

Strategy II

$$y_{.i_2} = \begin{cases} \alpha_2 Y_i & \text{for } i \in A \\ \alpha_2\bar{Y}_{r,rss} + \frac{n\theta_2}{n-r}(\bar{X}_{r,rss} - \mu_x) & \text{for } i \in \bar{A} \end{cases}$$

$$y_{.i_5} = \begin{cases} Y_i & \text{for } i \in A \\ \frac{1}{n-r}\left\{n\alpha_5\bar{Y}_{r,rss}\left(\frac{\mu_x}{\bar{X}_{r,rss}}\right)^{\theta_5} - r\bar{Y}_{r,rss}\right\} & \text{for } i \in \bar{A} \end{cases}$$

$$y_{.i_8} = \begin{cases} Y_i & \text{for } i \in A \\ \frac{1}{n-r}\left[n\alpha_8\bar{Y}_{r,rss}\left\{\frac{\mu_x}{\mu_x+\theta_8(\bar{X}_{r,rss}-\mu_x)}\right\} - r\bar{Y}_{r,rss}\right] & \text{for } i \in \bar{A} \end{cases}$$

Strategy III

$$y_{.i_3} = \begin{cases} \alpha_3 Y_i & \text{for } i \in A \\ \alpha_3 \bar{Y}_{r,rss} + \frac{n\theta_3}{n-r}(\bar{X}_{r,rss} - \bar{X}_{n,rss}) & \text{for } i \in \bar{A} \end{cases}$$

$$y_{.i_6} = \begin{cases} Y_i & \text{for } i \in A \\ \frac{1}{n-r}\left\{ n\alpha_6 \bar{Y}_{r,rss}\left(\frac{\bar{X}_{n,rss}}{\bar{X}_{r,rss}}\right)^{\theta_6} - r\bar{Y}_{r,rss} \right\} & \text{for } i \in \bar{A} \end{cases}$$

$$y_{.i_9} = \begin{cases} Y_i & \text{for } i \in A \\ \frac{1}{n-r}\left[n\alpha_9 \bar{Y}_{r,rss}\left\{ \frac{\bar{X}_{n,rss}}{\bar{X}_{n,rss}+\theta_9(\bar{X}_{r,rss}-\bar{X}_{n,rss})} \right\} - r\bar{Y}_{r,rss} \right] & \text{for } i \in \bar{A} \end{cases}$$

Under the above strategies, the consequent estimators are

$$T_1 = \alpha_1 \bar{Y}_{r,rss} + \theta_1(\bar{X}_{n,rss} - \mu_x) \tag{13}$$

$$T_2 = \alpha_2 \bar{Y}_{r,rss} + \theta_2(\bar{X}_{r,rss} - \mu_x) \tag{14}$$

$$T_3 = \alpha_3 \bar{Y}_{r,rss} + \theta_3(\bar{X}_{r,rss} - \bar{X}_{n,rss}) \tag{15}$$

$$T_4 = \alpha_4 \bar{Y}_{r,rss}\left(\frac{\mu_x}{\bar{X}_{n,rss}}\right)^{\theta_4} \tag{16}$$

$$T_5 = \alpha_5 \bar{Y}_{r,rss}\left(\frac{\mu_x}{\bar{X}_{r,rss}}\right)^{\theta_5} \tag{17}$$

$$T_6 = \alpha_6 \bar{Y}_{r,rss}\left(\frac{\bar{X}_{n,rss}}{\bar{X}_{r,rss}}\right)^{\theta_6} \tag{18}$$

$$T_7 = \alpha_7 \bar{Y}_{r,rss}\left\{\frac{\mu_x}{\mu_x + \theta_7(\bar{X}_{n,rss} - \mu_x)}\right\} \tag{19}$$

$$T_8 = \alpha_8 \bar{Y}_{r,rss}\left\{\frac{\mu_x}{\mu_x + \theta_8(\bar{X}_{r,rss} - \mu_x)}\right\} \tag{20}$$

$$T_9 = \alpha_9 \bar{Y}_{r,rss}\left\{\frac{\bar{X}_{n,rss}}{\bar{X}_{n,rss} + \theta_9(\bar{X}_{r,rss} - \bar{X}_{n,rss})}\right\} \tag{21}$$

where α_i and θ_i; $i = 1, 2, \ldots, 9$ are suitably chosen scalars.

Theorem 1. *The MSE of the consequent estimators comprising the suggested imputation methods are*

$$MSE(T_1) = \left\{ \begin{array}{l} (\alpha_1 - 1)^2\mu_y^2 + \alpha_1^2\mu_y^2(\gamma^* C_y^2 - W_y^{2^*}) + \theta_1^2\mu_x^2(\gamma C_x^2 - W_x^2) \\ +2\alpha_1\theta_1\mu_x\mu_y(\gamma\rho_{xy}C_xC_y - W_{xy}) \end{array} \right\} \tag{22}$$

$$MSE(T_2) = \left\{ \begin{array}{l} (\alpha_2 - 1)^2\mu_y^2 + \alpha_2^2\mu_y^2(\gamma^* C_y^2 - W_y^{2^*}) + \theta_2^2\mu_x^2(\gamma^* C_x^2 - W_x^{2^*}) \\ +2\alpha_2\theta_2\mu_x\mu_y(\gamma^*\rho_{xy}C_xC_y - W_{xy}^*) \end{array} \right\} \tag{23}$$

$$MSE(T_3) = \left\{ \begin{array}{l} (\alpha_3 - 1)^2\mu_y^2 + \alpha_3^2\mu_y^2(\gamma^* C_y^2 - W_y^{2^*}) + \theta_3^2\mu_x^2(\gamma^* C_x^2 - W_x^{2^*} - \gamma C_x^2 + W_x^2) \\ +2\alpha_3\theta_3\mu_x\mu_y(\gamma^*\rho_{xy}C_xC_y - W_{xy}^* - \gamma\rho_{xy}C_xC_y + W_{xy}) \end{array} \right\} \tag{24}$$

$$MSE(T_4) = \mu_y^2 \left[\begin{array}{l} 1 + \alpha_4^2\left\{ \begin{array}{l} 1 + \gamma^* C_y^2 - W_y^{2^*} + \theta_4(2\theta_4 + 1)(\gamma C_x^2 - W_x^2) \\ -4\theta_4(\gamma\rho_{xy}C_xC_y - W_{xy}) \end{array} \right\} \\ -2\alpha_4\left\{ 1 - \theta_4(\gamma\rho_{xy}C_xC_y - W_{xy}) + \frac{\theta_4(\theta_4+1)}{2}(\gamma C_x^2 - W_x^2) \right\} \end{array} \right] \tag{25}$$

$$MSE(T_5) = \mu_y^2 \left[\begin{array}{l} 1 + \alpha_5^2\left\{ \begin{array}{l} 1 + \gamma^* C_y^2 - W_y^{2^*} + \theta_5(2\theta_5 + 1)(\gamma^* C_x^2 - W_x^{2^*}) \\ -4\theta_5(\gamma^*\rho_{xy}C_xC_y - W_{xy}^*) \end{array} \right\} \\ -2\alpha_5\left\{ 1 - \theta_5(\gamma^*\rho_{xy}C_xC_y - W_{xy}^*) + \frac{\theta_5(\theta_5+1)}{2}(\gamma^* C_x^2 - W_x^{2^*}) \right\} \end{array} \right] \tag{26}$$

$$MSE(T_6) = \mu_y^2 \left[\begin{array}{l} 1 + \alpha_6^2 \left\{ \begin{array}{l} 1 + \gamma^* C_y^2 - W_y^{2^*} + \theta_6(2\theta_6 + 1)(\gamma^* C_x^2 - W_x^{2^*} - \gamma C_x^2 + W_x^2) \\ -4\theta_6(\gamma^* \rho_{xy} C_x C_y - W_{xy}^* - \gamma \rho_{xy} C_x C_y + W_{xy}) \end{array} \right\} \\ -2\alpha_6 \left\{ \begin{array}{l} 1 - \theta_6(\gamma^* \rho_{xy} C_x C_y - W_{xy}^* - \gamma \rho_{xy} C_x C_y + W_{xy}) \\ + \frac{\theta_6(\theta_6+1)}{2}(\gamma^* C_x^2 - W_x^{2^*} - \gamma C_x^2 + W_x^2) \end{array} \right\} \end{array} \right] \tag{27}$$

$$MSE(T_7) = \mu_y^2 \left[\begin{array}{l} 1 + \alpha_7^2 \left\{ 1 + \gamma^* C_y^2 - W_y^{2^*} + 3\theta_7^2(\gamma C_x^2 - W_x^2) - 4\theta_7(\gamma \rho_{xy} C_x C_y - W_{xy}) \right\} \\ -2\alpha_7 \{ 1 + \theta_7^2(\gamma C_x^2 - W_x^2) - \theta_7(\gamma \rho_{xy} C_x C_y - W_{xy}) \} \end{array} \right] \tag{28}$$

$$MSE(T_8) = \mu_y^2 \left[\begin{array}{l} 1 + \alpha_8^2 \left\{ 1 + \gamma^* C_y^2 - W_y^{2^*} + 3\theta_8^2(\gamma^* C_x^2 - W_x^{2^*}) - 4\theta_8(\gamma^* \rho_{xy} C_x C_y - W_{xy}^*) \right\} \\ -2\alpha_8 \left\{ 1 + \theta_8^2(\gamma^* C_x^2 - W_x^{2^*}) - \theta_8(\gamma^* \rho_{xy} C_x C_y - W_{xy}^*) \right\} \end{array} \right] \tag{29}$$

$$MSE(T_9) = \mu_y^2 \left[\begin{array}{l} 1 + \alpha_9^2 \left\{ \begin{array}{l} 1 + \gamma^* C_y^2 - W_y^{2^*} + 3\theta_9^2(\gamma^* C_x^2 - W_x^{2^*} - \gamma C_x^2 + W_x^2) \\ -4\theta_9(\gamma^* \rho_{xy} C_x C_y - W_{xy}^* - \gamma \rho_{xy} C_x C_y + W_{xy}) \end{array} \right\} \\ -2\alpha_9 \left\{ \begin{array}{l} 1 + \theta_9^2 \ (\gamma^* C_x^2 - W_x^{2^*} - \gamma C_x^2 + W_x^2) \\ -\theta_9(\gamma^* \rho_{xy} C_x C_y - W_{xy}^* - \gamma \rho_{xy} C_x C_y + W_{xy}) \end{array} \right\} \end{array} \right] \tag{30}$$

Proof. The precis of the derivations are given in Appendix B for quick reference. □

Corollary 1. *The minimum MSE of the consequent estimators comprising the suggested imputation methods are given by*

$$minMSE(T_i) = \mu_y^2(1 - \alpha_{i(opt)}); \ i = 1,2,3,7,8,9 \tag{31}$$

$$minMSE(T_j) = \mu_y^2\left(1 - \alpha_{j(opt)} A_j\right); \ j = 4,5,6 \tag{32}$$

Proof. A summary of the derivations and the definition of the parametric function A_j are given in Appendix B. □

5. Efficiency Conditions

By successively comparing the *MSEs* of the suggested imputation methods y_{i_1} to y_{i_9} regarding the other existing imputation methods proposed by [21,22], we obtain the following efficiency conditions.

(i). From (31) and (A1)

$$\alpha_{i(opt)} > 1 - \gamma^* C_y^2 + W_y^{2^*}; \ \ i = 1,2,3,7,8,9 \tag{33}$$

(ii). From (32) and (A1)

$$\alpha_{j(opt)} > \frac{1}{A_j}\left(1 - \gamma^* C_y^2 + W_y^{2^*}\right); \ \ j = 4,5,6 \tag{34}$$

(iii). From (31) and (A2)

$$\alpha_{i(opt)} > 1 - \gamma^* C_y^2 + W_y^{2^*} - \left\{1 - \left(\frac{B}{R}\right)^2\right\}(\gamma C_x^2 - W_x^2); \ \ i = 1,2,3,7,8,9 \tag{35}$$

(iv). From (32) and (A2)

$$\alpha_{j(opt)} > \frac{1}{A_j}\left[1 - \gamma^* C_y^2 + W_y^{2^*} - \left\{1 - \left(\frac{B}{R}\right)^2\right\}(\gamma C_x^2 - W_x^2)\right]; \ \ j = 4,5,6 \tag{36}$$

(v). From (31) and (A3)

$$\alpha_{i(opt)} > 1 - \gamma^* C_y^2 + W_y^{2^*} - \gamma^* C_x^2 + W_x^{2^*} + \left(\frac{B}{R}\right)(\gamma^* \rho_{xy} C_x C_y - W_{xy}^*);\ \ i = 1,2,3,7,8,9 \tag{37}$$

(vi). From (32) and (A3)

$$\alpha_{j(opt)} > \frac{1}{A_j}\left\{\ 1 - \gamma^* C_y^2 + W_y^{2^*} - \gamma^* C_x^2 + W_x^{2^*} + \left(\frac{B}{R}\right)(\gamma^* \rho_{xy} C_x C_y - W_{xy}^*)\ \right\};\ \ j = 4,5,6 \tag{38}$$

(vii). From (31) and (A4)

$$\alpha_{i(opt)} > \left[\begin{array}{l} 1 - \gamma^* C_y^2 + W_y^{2^*} - \left\{1 + \left(\frac{B}{R}\right)\right\}^2 (\gamma^* C_x^2 - W_x^{2^*}) \\ +2\left\{1 + \left(\frac{B}{R}\right)\right\}(\gamma^* \rho_{xy} C_x C_y - W_{xy}^*) \end{array}\right];\ \ i = 1,2,3,7,8,9 \tag{39}$$

(viii).From (32) and (A4)

$$\alpha_{j(opt)} > \frac{1}{A_j}\left[\begin{array}{l} 1 - \gamma^* C_y^2 + W_y^{2^*} - \left\{1 + \left(\frac{B}{R}\right)\right\}^2 (\gamma^* C_x^2 - W_x^{2^*}) \\ +2\left\{1 + \left(\frac{B}{R}\right)\right\}(\gamma^* \rho_{xy} C_x C_y - W_{xy}^*) \end{array}\right];\ \ j = 4,5,6 \tag{40}$$

(ix). From (31) and (A8)

$$\alpha_{i(opt)} > 1 - \gamma^* C_y^2 + W_y^{2^*} + \frac{(\gamma \rho_{xy} C_x C_y - W_{xy})^2}{(\gamma C_x^2 - W_x^2)};\ \ i = 1,2,3,7,8,9 \tag{41}$$

(x). From (32) and (A8)

$$\alpha_{j(opt)} > \frac{1}{A_j}\left\{1 - \gamma C_y^2 + W_y^2 + \frac{(\gamma \rho_{xy} C_x C_y - W_{xy})^2}{(\gamma^* C_x^2 - W_x^{2^*})}\right\};\ \ j = 4,5,6 \tag{42}$$

(xi). From (31) and (A9)

$$\alpha_{i(opt)} > 1 - \gamma^* C_y^2 + W_y^{2^*} + \frac{(\gamma^* \rho_{xy} C_x C_y - W_{xy}^* - \gamma \rho_{xy} C_x C_y + W_{xy})^2}{(\gamma^* Cx^2 - W_x^{2^*} - \gamma Cx^2 + W_x^2)};\ \ i = 1,2,3,7,8,9 \tag{43}$$

(xii). From (32) and (A9)

$$\alpha_{j(opt)} > \frac{1}{A_j}\left\{1 - \gamma^* C_y^2 + W_y^{2^*} + \frac{(\gamma^* \rho_{xy} C_x C_y - W_{xy}^* - \gamma \rho_{xy} C_x C_y + W_{xy})^2}{(\gamma^* Cx^2 - W_x^{2^*} - \gamma Cx^2 + W_x^2)}\right\};\ \ j = 4,5,6 \tag{44}$$

It is notable that only under these conditions, we can ensure the efficiency of the suggested imputation methods. Furthermore, we observed that these conditions are usually satisfied in various populations.

6. Computational Study

To enhance the soundness of the efficiency conditions obtained in the previous section, a computational study was designed in three subsections, namely, a numerical analysis based on a real population, a simulation analysis based on an artificially generated population, and a discussion of the computational findings.

6.1. Numerical Study

In this subsection, a numerical study is performed and the performance of the proposed imputation methods is compared with existing imputation methods. The numerical analysis was accomplished on four real datasets. Population 1 was taken from [24], where the level of apple production was taken as the study variable and the number of apple trees taken as the auxiliary variable in 69 villages of the South Anatolia region of Turkey in 1999. Population 2 was taken from [25], where the population (in millions) in 1983 was considered as the study variable and the export (in millions of U.S. dollars) was considered the auxiliary variable. Population 3 was taken from [26], where the amount (in U.S. dollars) of real estate farm loans in different states during 1997 was considered as the study variable and the amount (in U.S. dollars) of non-real estate farm loans in different states during 1997 was considered the auxiliary variable. Population 4 was taken from [25], where the total number of seats in the municipal council in 1982 was considered the study variable and the number of conservative seats in the municipal council in 1982 was considered the auxiliary variable. The necessary values of the parameters for all four populations are reported in Table 1.

Table 1. Description of the population parameters.

Parameters	Population 1	Population 2	Population 3	Population 4
N	69	124	50	284
n	12	12	12	12
m	3	3	3	3
r	4	4	4	4
P	0.5	0.3	0.7	0.6
μ_y	71.34	36.65	555.43	47.50
μ_x	3165.02	14276.03	878.16	9.05
S_y	110.85	116.80	584.82	11.06
S_x	3965.24	31431.81	1084677	4.95
ρ_{xy}	0.91	0.23	0.80	0.66

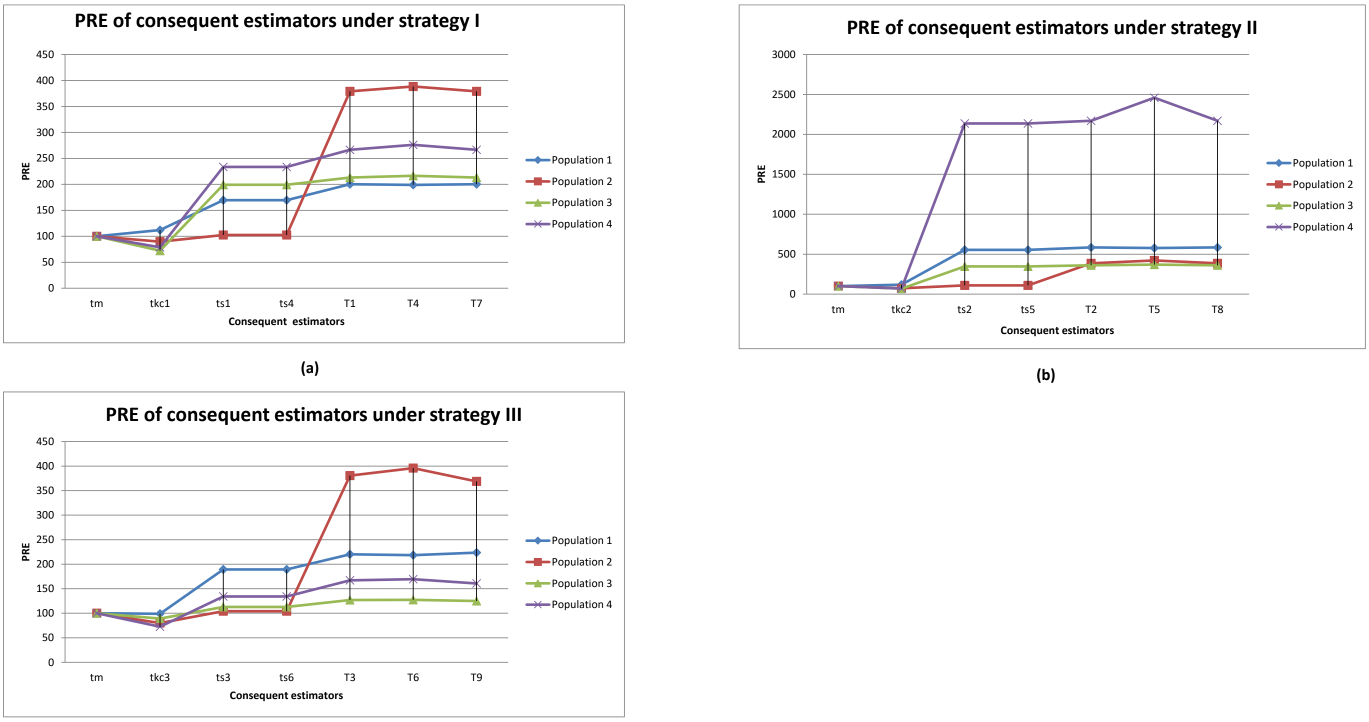
The percent relative efficiency (PRE) of the proposed imputation methods regarding the conventional imputation methods was calculated using the following formula:

$$PRE = \frac{MSE(t_m)}{MSE(T)} \times 100 \tag{45}$$

where $T = t_m, t_{s_i},\ i = 1, 2, \ldots, 6$, and $T_i,\ i = 1, 2, \ldots, 9$. The results of the numerical analysis are summarized in Table 2 and depicted in Figure 1 under strategies I, II, III and III for each population.

Table 2. PRE of the proposed estimators for real populations.

Estimators	Population 1	Population 2	Population 3	Population 4
t_m	100.00	100.00	100.00	100.00
Strategy I				
T_1	200.23	379.07	213.14	266.61
T_4	198.93	388.40	216.47	276.07
T_7	200.23	379.07	213.14	266.61
$t_{s_i}, i = 1, 4$	169.41	102.59	199.16	233.60
$\bar{y}_{kc_1}$	111.93	89.60	72.25	78.77
Strategy II				
T_2	584.83	385.66	360.33	2169.67
T_5	576.85	421.37	369.29	2459.06
T_8	584.83	385.66	360.33	2169.67
$t_{s_i}, i = 2, 5$	554.02	109.18	346.35	2136.65
$\bar{y}_{kc_2}$	118.15	72.99	68.44	69.78
Strategy III				
T_3	220.06	380.63	126.93	167.17
T_6	218.39	395.85	127.34	169.46
T_9	223.56	368.69	125.09	160.86
$t_{s_i}, i = 3, 6$	189.24	104.14	112.94	134.15
$\bar{y}_{kc_3}$	98.91	79.99	89.12	72.58



(c)

Figure 1. PRE results of the consequent estimators under (**a**) strategy I, (**b**) strategy II, and (**c**) strategy III for the real populations reported in Table 2.

6.2. Simulation Analysis

To assess the performance of the suggested imputation methods, following [27], simulation experiments were conducted over three parent populations, namely, Normal, Gamma, and Weibull, of size $N = 1000$ units with variables X and Y, expressed by

$$Y = 2.9 + \sqrt{(1-\rho_{xy}^2)}\, Y^* + \rho_{xy}\left(\frac{S_y}{S_x}\right) X^*$$
$$X = 2.5 + X^*$$

where X^* and Y^* are independent variables of the corresponding parent population. The sampling methodology of Section 2 was used to draw an RSS of size 12 units with set size 3 from each parent population. Using 20,000 iterations, the PRE of the consequent estimators compared to the conventional mean estimator were computed as

$$PRE = \frac{\frac{1}{20{,}000}\sum_{i=1}^{20{,}000}(t_m - \mu_y)^2}{\frac{1}{20{,}000}\sum_{i=1}^{20{,}000}(T - \mu_y)^2} \times 100$$

The outcomes of the simulation experiments are reported in Tables 3–9 by their PRE for each sensibly opted values of response probability $P = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$ with corresponding correlation coefficient $\rho_{xy} = 0.6, 0.7, 0.8, 0.9$.

Table 3. PRE of the proposed estimators at P = 0.20.

ρ_{xy}	**0.6**	**0.7**	**0.8**	**0.9**
t_m	100	100	100	100
$X^* \sim N(20, 25)$				
$Y^* \sim N(30, 35)$				
Strategy I				
T_1	108.2314	110.1882	113.1969	118.9193
$\mathbf{T_4}$	**108.9253**	**110.9860**	**114.1003**	**119.9397**
$t_{s_i}, i = 1, 4$	102.4440	102.7442	102.8370	102.7567
$\bar{y}_{kc_1}$	56.3365	62.5280	70.0558	78.6946
Strategy II				
$T_i, i = 2, 8$	119.3313	122.8567	126.3605	131.6545
$\mathbf{T_5}$	**123.7161**	**127.9222**	**132.0179**	**137.8729**
$t_{s_i}, i = 2, 5$	113.5438	115.4126	116.0005	115.492
$\bar{y}_{kc_2}$	20.5860	25.1720	32.1576	43.0168
Strategy III				
$T_i,\ i = 3, 9$	116.3368	119.4054	122.7634	128.1837
$\mathbf{T_6}$	**119.6366**	**123.2108**	**127.0276**	**132.9048**
$t_{s_i}, i = 3, 6$	110.5493	111.9614	112.4035	112.0212
$\bar{y}_{kc_3}$	21.7699	27.3798	35.9864	49.5031
$X^* \sim Gamma(5.6, 0.9)$				
$Y^* \sim Gamma(6.9, 0.9)$				
Skewness of Y	0.6046	0.6134	0.6571	0.7487
Kurtosis of Y	3.4184	3.4308	3.5393	3.7750
Strategy I				
$\mathbf{T_i}, i = 1, 7$	**104.1361**	**104.0201**	**103.8752**	**103.7148**
T_4	103.9699	103.8591	103.7210	103.5712
$t_{s_i}, i = 1, 4$	102.2484	102.1062	101.8782	101.5178
$\bar{y}_{kc_1}$	84.4838	84.5867	85.0399	86.0938
Strategy II				
$\mathbf{T_i}, i = 2, 8$	**114.2407**	**113.4139**	**112.1508**	**110.2765**
T_5	113.4229	112.6215	111.3919	109.5685
$t_{s_i}, i = 2, 5$	112.3531	111.5000	110.1539	108.0795
$\bar{y}_{kc_2}$	52.5444	52.7365	53.6026	55.6966
Strategy III				
$T_i,\ i = 3, 9$	**111.5319**	**110.9071**	**109.9584**	**108.5579**
T_6	110.8735	110.2693	109.3479	107.9888
$t_{s_i}, i = 3, 6$	109.6442	108.9931	107.9614	106.3608
$\bar{y}_{kc_3}$	50.0729	50.1878	50.9778	52.9983
$X^* \sim Wb(0.945, 1.0)$				
$Y^* \sim Wb(0.953, 0.99)$				
Skewness of Y	1.3561	1.3526	1.4714	1.7607
Kurtosis of Y	5.2276	5.2844	6.0535	7.8079
Strategy I				
$T_i, i = 1, 7$	107.3833	107.2255	107.1537	107.1449
$\mathbf{T_4}$	**107.5074**	**107.3496**	**107.2772**	**107.2674**
$t_{s_i}, i = 1, 4$	105.6587	105.0803	104.5253	103.9469
$\bar{y}_{kc_1}$	80.5933	83.9483	86.6202	88.7951
Strategy II				
$T_i, i = 2, 8$	138.2957	134.0250	130.2555	126.6323
$\mathbf{T_5}$	**139.1752**	**134.8628**	**131.0549**	**127.3948**
$t_{s_i}, i = \text{b}2, 5$	136.5711	131.8799	127.6271	123.4344
$\bar{y}_{kc_2}$	46.4315	52.3644	57.8042	62.7748
Strategy III				
$T_i,\ i = 3, 9$	128.9875	126.1204	123.5728	121.1061
$\mathbf{T_6}$	**129.6256**	**126.7374**	**124.1689**	**121.6812**
$t_{s_i}, i = 3, 6$	127.2628	123.9752	120.9444	117.9082
$\bar{y}_{kc_3}$	55.5027	62.5806	68.7953	74.1058

Bold values indicate maximum PRE.

Table 4. PRE of the proposed estimators at P = 0.30.

ρ_{xy}	**0.6**	**0.7**	**0.8**	**0.9**
t_m	100	100	100	100
$X^* \sim N(20, 25)$				
$Y^* \sim N(30, 35)$				
Strategy I				
$T_i, i = 1, 7$	107.5696	109.1362	111.2233	114.9679
$\mathbf{T_4}$	**108.2824**	**109.9559**	**112.1497**	**116.0109**
$t_{s_i}, i = 1, 4$	103.7113	104.1735	104.3167	104.1929
$\bar{y}_{kc_1}$	46.2412	52.6613	60.9329	71.1186
Strategy II				
$T_i, i = 2, 8$	117.4021	120.3753	122.9071	126.2670
$\mathbf{T_5}$	**120.3133**	**123.7369**	**126.6603**	**130.3914**
$t_{s_i}, i = 2, 5$	113.5438	115.4126	116.0005	115.4920
$\bar{y}_{kc_2}$	20.5860	25.1720	32.1576	43.0168
Strategy III				
$T_i,\ i = 3, 9$	112.9688	115.2747	117.5940	121.1378
$\mathbf{T_6}$	**114.8323**	**117.4214**	**120.0030**	**123.8138**
$t_{s_i}, i = 3, 6$	109.1105	110.3120	110.6873	110.3628
$\bar{y}_{kc_6}$	21.7699	27.3798	35.9864	49.5031
$X^* \sim Gamma(5.6, 0.9)$				
$Y^* \sim Gamma(6.9, 0.9)$				
Skewness of Y	0.6046	0.6134	0.6571	0.7487
Kurtosis of Y	3.4184	3.4308	3.5393	3.7750
Strategy I				
$\mathbf{T_i}, i = 1, 7$	**104.6694**	**104.4689**	**104.1753**	**103.7588**
T_4	104.5033	104.3080	104.0213	103.6153
$t_{s_i}, i = 1, 4$	103.4110	103.1930	102.8440	102.2941
$\bar{y}_{kc_1}$	78.4014	78.5344	79.1216	80.4968
Strategy II				
$\mathbf{T_i}, i = 2, 8$	**113.6115**	**112.7759**	**111.4852**	**109.5442**
T_5	113.0660	112.2474	110.9790	109.0720
$t_{s_i}, i = 2, 5$	112.3531	111.5000	110.1539	108.0795
$\bar{y}_{kc_2}$	52.5444	52.7365	53.6026	55.6966
Strategy III				
$T_i,\ i = 3, 9$	**109.5966**	**109.0575**	**108.2289**	**106.9865**
T_6	109.2114	108.6844	107.8719	106.6538
$t_{s_i}, i = 3, 6$	108.3381	107.7815	106.8976	105.5218
$\bar{y}_{kc_3}$	50.0729	50.1878	50.9778	52.9983
$X^* \sim Wb(0.945, 1.0)$				
$Y^* \sim Wb(0.953, 0.99)$				
Skewness of Y	1.3561	1.3526	1.4714	1.7607
Kurtosis of Y	5.2276	5.2844	6.0535	7.8079
Strategy I				
$T_i, i = 1, 7$	109.8849	109.2492	108.6973	108.1715
$\mathbf{T_4}$	**110.0138**	**109.3775**	**108.8245**	**108.2972**
$t_{s_i}, i = 1, 4$	108.7352	107.8191	106.9450	106.0396
$\bar{y}_{kc_1}$	73.4648	77.7113	81.1888	84.0843
Strategy II				
$T_i, i = 2, 8$	137.7209	133.3100	129.3794	125.5664
$\mathbf{T_5}$	**138.3068**	**133.8682**	**129.9120**	**126.0744**
$t_{s_i}, i = 2, 5$	136.5711	131.8799	127.6271	123.4344
$\bar{y}_{kc_2}$	46.4315	52.3644	57.8042	62.7748
Strategy III				
$T_i,\ i = 3, 9$	124.2186	121.7980	119.6111	117.4585
$\mathbf{T_6}$	**124.574**	**122.1439**	**119.9472**	**117.7845**
$t_{s_i}, i = 3, 6$	123.0688	120.3679	117.8588	115.3266
$\bar{y}_{kc_3}$	55.5027	62.5806	68.7953	74.1058

Bold values indicate maximum PRE.

Table 5. PRE of the proposed estimators at P = 0.40.

ρ_{xy}	0.6	0.7	0.8	0.9
t_m	100	100	100	100
$X^* \sim N(20, 25)$				
$Y^* \sim N(30, 35)$				
Strategy I				
$T_i, i = 1, 7$	107.9041	109.3652	111.0196	113.7510
$\mathbf{T_4}$	**108.6366**	**110.2077**	**111.9699**	**114.8175**
$t_{s_i}, i = 1, 4$	105.0104	105.6432	105.8396	105.6698
$\bar{y}_{kc_1}$	39.2142	45.4841	53.9122	64.8732
Strategy II				
$T_i, i = 2, 8$	116.4375	119.1347	121.1805	123.5733
$\mathbf{T_5}$	**118.6164**	**121.6501**	**123.9885**	**126.6586**
$t_{s_i}, i = 2, 5$	113.5438	115.4126	116.0005	115.4920
$\bar{y}_{kc_2}$	20.5860	25.1720	32.1576	43.0168
Strategy III				
$T_i,\ i = 3, 9$	110.6024	112.4326	114.2028	116.8341
$\mathbf{T_6}$	**111.7649**	**113.7708**	**115.7070**	**118.5106**
$t_{s_i}, i = 3, 6$	107.7087	108.7106	109.0228	108.7528
$\bar{y}_{kc_3}$	21.7699	27.3798	35.9864	49.5031
$X^* \sim Gamma(5.6, 0.9)$				
$Y^* \sim Gamma(6.9, 0.9)$				
Skewness of Y	0.6046	0.6134	0.6571	0.7487
Kurtosis of Y	3.4184	3.4308	3.5393	3.7750
Strategy I				
$\mathbf{T_i}, i = 1, 7$	**105.5441**	**105.2600**	**104.8268**	**104.1809**
T_4	105.3782	105.0993	104.6730	104.0376
$t_{s_i}, i = 1, 4$	104.6003	104.3031	103.8283	103.0824
$\bar{y}_{kc_1}$	73.1359	73.2903	73.9734	75.5831
Strategy II				
$\mathbf{T_i}, i = 2, 8$	**113.2969**	**112.4569**	**111.1524**	**109.1780**
T_5	112.8877	112.0605	110.7727	108.8238
$t_{s_i}, i = 2, 5$	112.3531	111.5000	110.1539	108.0795
$\bar{y}_{kc_2}$	52.5444	52.7365	53.6026	55.6966
Strategy III				
$\mathbf{T_i},\ i = 3, 9$	**108.0067**	**107.5535**	**106.8530**	**105.7945**
T_6	107.7585	107.3131	106.6230	105.5802
$t_{s_i}, i = 3, 6$	107.0628	106.5965	105.8546	104.6959
$\bar{y}_{kc_3}$	50.0729	50.1878	50.9778	52.9983
$X^* \sim Wb(0.945, 1.0)$				
$Y^* \sim Wb(0.953, 0.99)$				
Skewness of Y	1.3561	1.3526	1.4714	1.7607
Kurtosis of Y	5.2276	5.2844	6.0535	7.8079
Strategy I				
$T_i, i = 1, 7$	112.8585	111.7770	110.7937	109.8172
$\mathbf{T_4}$	**112.9926**	**111.9098**	**110.9247**	**109.9461**
$t_{s_i}, i = 1, 4$	111.9962	110.7044	109.4795	108.2182
$\bar{y}_{kc_1}$	67.4948	72.3369	76.3983	79.8482
Strategy II				
$T_i, i = 2, 8$	137.4334	132.9525	128.9413	125.0334
$\mathbf{T_5}$	**137.8727**	**133.3710**	**129.3406**	**125.4143**
$t_{s_i}, i = 2, 5$	136.5711	131.8799	127.6271	123.4344
$\bar{y}_{kc_2}$	46.4315	52.3644	57.8042	62.7748
Strategy III				
$T_i,\ i = 3, 9$	120.0047	118.0372	116.2409	114.4546
$\mathbf{T_6}$	**120.2233**	**118.2512**	**116.4501**	**114.6584**
$t_{s_i}, i = 3, 6$	119.1424	116.9646	114.9267	112.8556
$\bar{y}_{kc_3}$	55.5027	62.5806	68.7953	74.1058

Bold values indicate maximum PRE.

Table 6. PRE of the proposed estimators at $P = 0.50$.

ρ_{xy}	**0.6**	**0.7**	**0.8**	**0.9**
t_m	100	100	100	100
$X^* \sim N(20, 25)$				
$Y^* \sim N(30, 35)$				
Strategy I				
$T_i, i = 1, 7$	108.6574	110.1325	111.5516	113.6541
$\mathbf{T_4}$	**109.4103**	**110.9988**	**112.5270**	**114.7449**
$t_{s_i}, i = 1, 4$	106.3424	107.1549	107.4076	107.1891
$\bar{y}_{kc_1}$	34.0412	40.0286	48.3422	59.6361
Strategy II				
$T_i, i = 2, 8$	115.8588	118.3903	120.1445	121.9570
$\mathbf{T_5}$	**117.5997**	**120.3998**	**122.3876**	**124.4215**
$t_{s_i}, i = 2, 5$	113.5438	115.4126	116.0005	115.4920
$\bar{y}_{kc_2}$	20.5860	25.1720	32.1576	43.0168
Strategy III				
$T_i,\ i = 3, 9$	108.6574	110.1325	111.5516	113.6541
$\mathbf{T_6}$	**109.4103**	**110.9988**	**112.5270**	**114.7449**
$t_{s_i}, i = 3, 6$	106.3424	107.1549	107.4076	107.1891
$\bar{y}_{kc_3}$	21.7699	27.3798	35.9864	49.5031
$X^* \sim Gamma(5.6, 0.9)$				
$Y^* \sim Gamma(6.9, 0.9)$				
Skewness of Y	0.6046	0.6134	0.6571	0.7487
Kurtosis of Y	3.4184	3.4308	3.5393	3.7750
Strategy I				
$\mathbf{T_i}, i = 1, 7$	**106.5723**	**106.2029**	**105.6304**	**104.7617**
T_4	106.4066	106.0424	105.4768	104.6186
$t_{s_i}, i = 1, 4$	105.8172	105.4373	104.8316	103.8829
$\bar{y}_{kc_1}$	68.5332	68.7027	69.4543	71.2347
Strategy II				
$\mathbf{T_i}, i = 2, 8$	**113.1081**	**112.2655**	**110.9527**	**108.9583**
T_5	112.7807	111.9483	110.6489	108.6749
$t_{s_i}, i = 2, 5$	112.3531	111.5000	110.1539	108.0795
$\bar{y}_{kc_2}$	52.5444	52.7365	53.6026	55.6966
Strategy III				
$T_i,\ i = 3, 9$	**106.5723**	**106.2029**	**105.6304**	**104.7617**
T_6	106.4066	106.0424	105.4768	104.6186
$t_{s_i}, i = 3, 6$	105.8172	105.4373	104.8316	103.8829
$\bar{y}_{kc_3}$	50.0729	50.1878	50.9778	52.9983
$X^* \sim Wb(0.945, 1.0)$				
$Y^* \sim Wb(0.953, 0.99)$				
Skewness of Y	1.3561	1.3526	1.4714	1.7607
Kurtosis of Y	5.2276	5.2844	6.0535	7.8079
Strategy I				
$T_i, i = 1, 7$	116.1487	114.6065	113.1884	113.1884
$\mathbf{T_4}$	**116.2883**	**114.7441**	**113.3234**	**113.3234**
$t_{s_i}, i = 2, 4$	115.4588	113.7484	112.1370	112.1370
$\bar{y}_{kc_1}$	62.4222	67.6579	72.1416	72.1416
Strategy II				
$T_i, i = 2, 8$	137.2610	132.7379	128.6785	128.6785
$\mathbf{T_5}$	**137.6123**	**133.0727**	**128.9979**	**128.9979**
$t_{s_i}, i = 2, 5$	136.5711	131.8799	127.6271	127.6271
$\bar{y}_{kc_2}$	46.4315	52.3644	57.8042	57.8042
Strategy III				
$T_i,\ i = 3, 9$	116.1487	114.6065	113.1884	111.7674
$\mathbf{T_6}$	**116.2883**	**114.7441**	**113.3234**	**111.8997**
$t_{s_i}, i = 3, 6$	115.4588	113.7484	112.137	110.4883
$\bar{y}_{kc_3}$	55.5027	62.5806	68.7953	68.7953

Bold values indicate maximum PRE.

Table 7. PRE of the proposed estimators at P = 0.60.

ρ_{xy}	**0.6**	**0.7**	**0.8**	**0.9**
t_m	100	100	100	100
$X^* \sim N(20,25)$				
$Y^* \sim N(30,35)$				
Strategy I				
$T_i, i = 1,7$	109.6378	111.1919	112.4761	114.1403
$\mathbf{T_4}$	**110.4120**	**112.0830**	**113.4776**	**115.2564**
$t_{s_i}, i = 1,4$	107.7087	108.7106	109.0228	108.7528
$\bar{y}_{kc_1}$	30.0739	35.7417	43.8154	55.1814
Strategy II				
$T_i, i = 2,8$	115.4729	117.8940	119.4538	120.8795
$\mathbf{T_5}$	**116.9226**	**119.5671**	**121.3212**	**122.9311**
$t_{s_i}, i = 2,5$	113.5438	115.4126	116.0005	115.4920
$\bar{y}_{kc_2}$	20.5860	25.1720	32.1576	43.0168
Strategy III				
$T_i,\ i = 3,9$	106.9395	108.1245	109.2929	111.0573
$\mathbf{T_6}$	**107.4275**	**108.6858**	**109.926**	**111.7676**
$t_{s_i}, i = 3,6$	105.0104	105.6432	105.8396	105.6698
$\bar{y}_{kc_3}$	21.7699	27.3798	35.9864	49.5031
$X^* \sim Gamma(5.6,0.9)$				
$Y^* \sim Gamma(6.9,0.9)$				
Skewness of Y	0.6046	0.6134	0.6571	0.7487
Kurtosis of Y	3.4184	3.4308	3.5393	3.7750
Strategy I				
$\mathbf{T_i}, i = 1,7$	**107.6921**	**107.2345**	**106.5202**	**105.4283**
T_4	107.5266	107.0743	106.3668	105.2854
$t_{s_i}, i = 1,4$	107.0628	106.5965	105.8546	104.6959
$\bar{y}_{kc_1}$	64.4756	64.6556	65.4555	67.3595
Strategy II				
$\mathbf{T_i}, i = 2,8$	**112.9823**	**112.1380**	**110.8195**	**108.8119**
T_5	112.7094	111.8736	110.5664	108.5756
$t_{s_i}, i = 2,5$	112.3531	111.5000	110.1539	108.0795
$\bar{y}_{kc_2}$	52.5444	52.7365	53.6026	55.6966
Strategy III				
$\mathbf{T_i},\ i = 3,9$	**105.2295**	**104.9411**	**104.4940**	**103.8147**
T_6	105.1188	104.8339	104.3914	103.7192
$t_{s_i}, i = 3,6$	104.6003	104.3031	103.8283	103.0824
$\bar{y}_{kc_3}$	50.0729	50.1878	50.9778	52.9983
$X^* \sim Wb(0.945,1.0)$				
$Y^* \sim Wb(0.953,0.99)$				
Skewness of Y	1.3561	1.3526	1.4714	1.7607
Kurtosis of Y	5.2276	5.2844	6.0535	7.8079
Strategy I				
$T_i, i = 1,7$	119.7173	117.6797	115.8029	113.9216
$\mathbf{T_4}$	**119.8630**	**117.8223**	**115.9423**	**114.0574**
$t_{s_i}, i = 1,4$	119.1424	116.9646	114.9267	112.8556
$\bar{y}_{kc_1}$	58.0588	63.5474	68.3343	72.5392
Strategy II				
$T_i, i = 2,8$	137.1460	132.5949	128.5032	124.5004
$\mathbf{T_5}$	**137.4387**	**132.8739**	**128.7694**	**124.7542**
$t_{s_i}, i = 2,5$	136.5711	131.8799	127.6271	123.4344
$\bar{y}_{kc_2}$	46.4315	52.3644	57.8042	62.7748
Strategy III				
$T_i,\ i = 3,9$	112.571	111.4195	110.3556	109.2842
$\mathbf{T_6}$	**112.6604**	**111.5080**	**110.4430**	**109.3701**
$t_{s_i}, i = 3,6$	111.9962	110.7044	109.4795	108.2182
$\bar{y}_{kc_3}$	55.5027	62.5806	68.7953	74.1058

Bold values indicate maximum PRE.

Table 8. PRE of the proposed estimators at P = 0.70.

ρ_{xy}	**0.6**	**0.7**	**0.8**	**0.9**
t_m	100	100	100	100
$X^* \sim N(20,25)$				
$Y^* \sim N(30,35)$				
Strategy I				
$T_i, i = 1,7$	110.7640	112.4389	113.6473	114.9807
$\mathbf{T_4}$	**111.5604**	**113.3559**	**114.6761**	**116.1231**
$t_{s_i}, i = 1,4$	109.1105	110.3120	110.6873	110.3628
$\bar{y}_{kc_1}$	26.9348	32.2841	40.0638	51.3460
Strategy II				
$T_i, i = 2,8$	115.1973	117.5395	118.9605	120.1099
$\mathbf{T_5}$	**116.4391**	**118.9727**	**120.5600**	**121.8671**
$t_{s_i}, i = 2,5$	113.5438	115.4126	116.0005	115.4929
$\bar{y}_{kc_2}$	20.5860	25.1720	32.1576	43.0168
Strategy III				
$T_i,\ i = 3,9$	105.3648	106.3004	107.2767	108.8108
$\mathbf{T_6}$	**105.6700**	**106.6513**	**107.6731**	**109.2571**
$t_{s_i}, i = 3,6$	103.7113	104.1735	104.3167	104.1929
$\bar{y}_{kc_3}$	21.7699	27.3798	35.9864	49.5031
$X^* \sim Gamma(5.6,0.9)$				
$Y^* \sim Gamma(6.9,0.9)$				
Skewness of Y	0.6046	0.6134	0.6571	0.7487
Kurtosis of Y	3.4184	3.4308	3.5393	3.7750
Strategy I				
$\mathbf{T_i}, i = 1,7$	**108.8775**	**108.3284**	**107.4682**	**106.1495**
T_4	108.7123	108.1684	107.3151	106.0069
$t_{s_i}, i = 1,4$	108.3381	107.7815	106.8976	105.5218
$\bar{y}_{kc_1}$	60.8715	61.0588	61.8921	63.8842
Strategy II				
$\mathbf{T_i}, i = 2,8$	**112.8924**	**112.0468**	**110.7245**	**108.7072**
T_5	112.6585	111.8202	110.5074	108.5048
$t_{s_i}, i = 2,5$	112.3531	111.5000	110.1539	108.0795
$\bar{y}_{kc_2}$	52.5444	52.7365	53.6026	55.6966
Strategy III				
$T_i,\ i = 3,9$	**103.9503**	**103.7398**	**103.4146**	**102.9218**
T_6	103.8791	103.6708	103.3486	102.8603
$t_{s_i}, i = 3,6$	103.4110	103.1930	102.844	102.2941
$\bar{y}_{kc_3}$	50.0729	50.1878	50.9778	52.9983
$X^* \sim Wb(0.945,1.0)$				
$Y^* \sim Wb(0.953,0.99)$				
Skewness of Y	1.3561	1.3526	1.4714	1.7607
Kurtosis of Y	5.2276	5.2844	6.0535	7.8079
Strategy I				
$T_i, i = 1,7$	123.5616	120.9808	118.6098	116.2403
$\mathbf{T_4}$	**123.7138**	**121.1290**	**118.7538**	**116.3799**
$t_{s_i},\ i = 1,4$	123.0688	120.3679	117.8588	115.3266
$\bar{y}_{kc_1}$	54.2655	59.9077	64.9087	69.3645
Strategy II				
$T_i, i = 2,8$	137.0639	132.4928	128.3781	124.3481
$\mathbf{T_5}$	**137.3148**	**132.7318**	**128.6062**	**124.5657**
$t_{s_i}, i = 2,5$	136.5711	131.8799	127.6271	123.4344
$\bar{y}_{kc_2}$	46.4315	52.3644	57.8042	62.7748
Strategy III				
$T_i,\ i = 3,9$	109.2279	108.432	107.696	106.9533
$\mathbf{T_6}$	**109.2831**	**108.4869**	**107.7505**	**107.0071**
$t_{s_i}, i = 3,6$	108.7352	107.8191	106.945	106.0396
$\bar{y}_{kc_3}$	55.5027	62.5806	68.7953	74.1058

Bold values indicate maximum PRE.

Table 9. PRE of the proposed estimators at P = 0.80.

ρ_{xy}	**0.6**	**0.7**	**0.8**	**0.9**
t_m	100	100	100	100
$X^* \sim N(20,25)$				
$Y^* \sim N(30,35)$				
Strategy I				
$T_i, i = 1,7$	111.9961	113.8224	114.9935	116.0618
$\mathbf{T_4}$	**112.8155**	**114.7665**	**116.0507**	**117.2317**
t_{s_i} $i = 1,4$	110.5493	111.9614	112.4035	112.0212
$\bar{y}_{kc_1}$	24.3891	29.4365	36.9040	48.0091
Strategy II				
$T_i, i = 2,8$	114.9907	117.2736	118.5905	119.5327
$\mathbf{T_5}$	**116.0767**	**118.5271**	**119.9893**	**121.0694**
t_{s_i} $i = 2,5$	113.5438	115.4126	116.0005	115.4920
$\bar{y}_{kc_2}$	20.5860	25.1720	32.1576	43.0168
Strategy III				
$T_i,\ i = 3,9$	103.8908	104.6052	105.427	106.7974
$\mathbf{T_6}$	**104.0641**	**104.8043**	**105.6524**	**107.052**
$t_{s_i}, i = 3,6$	102.444	102.7442	102.8370	102.7567
$\bar{y}_{kc_3}$	21.7699	27.3798	35.9864	49.5031
$X^* \sim Gamma(5.6, 0.9)$				
$Y^* \sim Gamma(6.9, 0.9)$				
Skewness of Y	0.6046	0.6134	0.6571	0.7487
Kurtosis of Y	3.4184	3.4308	3.5393	3.7750
Strategy I				
$\mathbf{T_i}, i = 1,7$	**110.1161**	**109.4716**	**108.4607**	**106.9101**
T_4	109.9513	109.3120	108.3079	106.7677
$t_{s_i}, i = 1,4$	109.6442	108.9931	107.9614	106.3608
$\bar{y}_{kc_1}$	57.6491	57.8411	58.6967	60.7498
Strategy II				
$\mathbf{T_i}, i = 2,8$	**112.8250**	**111.9785**	**110.6531**	**108.6288**
T_5	112.6203	111.7802	110.4632	108.4516
$t_{s_i}, i = 2,5$	112.3531	111.5000	110.1539	108.0795
$\bar{y}_{kc_2}$	52.5444	52.7365	53.6026	55.6966
Strategy III				
$T_i,\ i = 3,9$	**102.7203**	**102.5847**	**102.3774**	**102.0671**
T_6	102.6788	102.5444	102.3389	102.0311
$t_{s_i}, i = 3,6$	102.2484	102.1062	101.8782	101.5178
$\bar{y}_{kc_3}$	50.0729	50.1878	50.9778	52.9983
$X^* \sim Wb(0.945, 1.0)$				
$Y^* \sim Wb(0.953, 0.99)$				
Skewness of Y	1.3561	1.3526	1.4714	1.7607
Kurtosis of Y	5.2276	5.2844	6.0535	7.8079
Strategy I				
$T_i, i = 1,7$	127.6940	124.5115	121.6015	118.7077
$\mathbf{T_4}$	**127.8533**	**124.6656**	**121.7504**	**118.8513**
$t_{s_i}, i = 1,4$	127.2628	123.9752	120.9444	117.9082
$\bar{y}_{kc_1}$	50.9375	56.6624	61.8101	66.4561
Strategy II				
$T_i, i = 2,8$	137.0023	132.4162	128.2842	124.2339
$\mathbf{T_5}$	**137.2218**	**132.6253**	**128.4838**	**124.4243**
$t_{s_i}, i = 2,5$	136.5711	131.8799	127.6271	123.4344
$\bar{y}_{kc_2}$	46.4315	52.3644	57.8042	62.7748
Strategy III				
$T_i,\ i = 3,9$	106.0898	105.6166	105.1824	104.7464
$\mathbf{T_6}$	**106.1208**	**105.6476**	**105.2132**	**104.777**
$t_{s_i}, i = 3,6$	105.6587	105.0803	104.5253	103.9469
$\bar{y}_{kc_3}$	55.5027	62.5806	68.7953	74.1058

Bold values indicate maximum PRE.

6.3. Discussion of Computational Findings

After carefully observing the findings reported in Tables 2–9, we discuss the following points:

(i). From the findings of Table 2, the proposed imputation methods $y_{.ij}$, $j = 1, 2, \ldots, 9$ outperform the mean imputation methods, ref. [21] imputation methods and ref. [22] imputation methods in each real population. Furthermore, the proposed imputation methods $y_{.ij}$, $j = 1, 3, 7, 9$ were superior among the proposed imputation methods in population 1, whereas the proposed imputation methods $y_{.ij}$, $j = 4, 5, 6$ were superior among the proposed imputation methods in populations 2–4. This is easily observed in Figure 1.

(ii). From the findings of Tables 3–9, the proposed imputation methods $y_{.ij}$, $j = 1, 2, \ldots, 9$ are also better than the mean imputation, ref. [21] imputation methods and ref. [22] imputation methods under both the symmetric and asymmetric populations for different correlation coefficients ρ_{xy}, coefficients of skewness β_1 and coefficients of kurtosis β_2.

(iii). When the parent population was normal (symmetric) and Weibull (asymmetric), the proposed ratio-type imputation methods $y_{.ij}$, $j = 4, 5, 6$ always performed better than the competitors as well as within the proposed class of imputation methods under strategies I, II and III.

(iv). When the parent population was Gamma (asymmetric), the proposed difference- and ratio-type imputation methods $y_{.ij}$, $j = 1, 2, 3, 7, 8, 9$ were equally efficient and outperformed the conventional methods and performed better in comparison with the proposed imputation methods under strategies I, II and III.

(v). The suggested imputation methods performed better in strategy II compared to strategies I and III in the real and artificially generated populations.

(vi). It can be easily seen that the PRE decreases with the increase in asymmetry and peakedness for asymmetric distributions such as Gamma and Weibull.

(vii). Moreover, the numerical analysis is summarized in Table 2 and Figure 1 under strategies I, II, and III for real populations 1–4. The PRE of the consequent estimators for the remaining simulation results in Tables 3–7 exhibit the same pattern and can be easily presented as line diagrams, if required.

7. Conclusions

In this manuscript, we proposed efficient difference- and ratio-type imputation methods for the estimation of the population mean in the presence of missing data. The efficiency conditions have been derived and sustained with computational analysis on some real and hypothetically generated symmetric and asymmetric populations. The computational and theoretical results show that the proposed imputation methods $y_{.ij}$, $j = 1, 2, \ldots, 9$ outperformed the mean imputation method $y_{.i}$, ref. [21] imputation methods $\bar{y}_{KC_i}$, $i = 1, 2,$ 3, and ref. [22] imputation methods $y_{.is_j}$, $j = 1, 2, \ldots, 6$.

In the simulation analysis, we considered one family of a symmetric population, namely, Normal, and two families of asymmetric populations, namely, Gamma and Weibull, to ascertain the effect of the correlation coefficient for a symmetric population and the effect of skewness and kurtosis for asymmetric populations.

It is worth mentioning that among the asymmetric populations, all imputation methods exhibited a decreasing trend in PRE as the coefficient of skewness and kurtosis increased. Although, in such cases, the proposed estimators fared better than their conventional counterparts. These results are in agreement with the results of [17,28,29], where these authors took skewed distributions and reported that the efficiency of the estimators decreased with an increase in skewness and kurtosis. The same was also true for imputation as well.

Lastly, the proposed imputation methods currently provide the best possible imputation methods for the estimation of a population mean in the presence of the missing data.

Furthermore, the proposed imputation strategies can be defined using multi-auxiliary information, which our future research with investigate.

Appendix A

The expressions of MSE, minimum MSE, and the optimum scalar values of the existing resultant estimators is reported below.

$$V(t_m) = \mu_y^2(\gamma^* C_y^2 - W_y^{2^*}) \tag{A1}$$

$$MSE(\bar{y}_{KC_1}) \cong \left\{ \mu_y^2(\gamma^* C_y^2 - W_y^{2^*}) + (R^2 - B^2)\mu_x^2(\gamma C_x^2 - W_x^2) \right\} \tag{A2}$$

$$MSE(\bar{y}_{KC_2}) = \left\{ \mu_y^2(\gamma^* C_y^2 - W_y^{2^*}) + \mu_x^2(\gamma^* C_x^2 - W_x^{2^*}) - B\mu_x\mu_y(\gamma^*\rho_{xy}C_xC_y - W_{xy}^*) \right\} \tag{A3}$$

$$MSE(\bar{y}_{KC_3}) = \left\{ \begin{array}{l} \mu_y^2(\gamma^* C_y^2 - W_y^{2^*}) + (R+B)^2\mu_x^2(\gamma^* C_x^2 - W_x^{2^*}) \\ -2(R+B)\mu_x\mu_y(\gamma^*\rho_{xy}C_xC_y - W_{xy}^*) \end{array} \right\} \tag{A4}$$

$$MSE(t_{s_i}) = \mu_y^2 \left\{ \gamma^* C_y^2 - W_y^{2^*} + \beta_i^2(\gamma C_x^2 - W_x^2) - 2\beta_i(\gamma\rho_{xy}C_xC_y - W_{xy}) \right\},\ i = 1, 4 \tag{A5}$$

$$MSE(t_{s_i}) = \mu_y^2 \left\{ \gamma^* C_y^2 - W_y^{2^*} + \beta_i^2(\gamma^* C_x^2 - W_x^{2^*}) - 2\beta_i(\gamma^*\rho_{xy}C_xC_y - W_{xy}^*) \right\},\ i = 2, 5 \tag{A6}$$

$$MSE(t_{s_i}) = \mu_y^2 \left\{ \begin{array}{l} \gamma^* C_y^2 - W_y^{2^*} + \beta_i^2(\ \gamma^* C_x^2 - W_x^{2^*} - \gamma C_x^2 + W_x^2\ ) \\ -2\beta_i\left(\gamma^*\rho_{xy}C_xC_y - W_{xy}^* - \gamma\rho_{xy}C_xC_y + W_{xy}\right) \end{array} \right\},\ i = 3, 6 \tag{A7}$$

$$minMSE(t_{s_i}) = \mu_y^2 \left\{ \gamma^* C_y^2 - W_y^{2^*} - \frac{(\gamma\rho_{xy}C_xC_y - W_{xy})^2}{(\gamma C_x^2 - W_x^2)} \right\};\ i = 1, 4 \tag{A8}$$

$$minMSE(t_{s_i}) = \mu_y^2 \left\{ \gamma^* C_y^2 - W_y^{2^*} - \frac{(\gamma^*\rho_{xy}C_xC_y - W_{xy}^*)^2}{(\gamma^* C_x^2 - W_x^{2^*})} \right\};\ i = 2, 5 \tag{A9}$$

$$minMSE(t_{s_i}) = \mu_y^2 \left\{ \ \gamma^* C_y^2 - W_y^{2^*} - \frac{\left(\gamma^*\rho_{xy}C_xC_y - W_{xy}^* - \gamma\rho_{xy}C_xC_y + W_{xy}\right)^2}{\left(\ \gamma^* C_x^2 - W_x^{2^*} - \gamma Cx^2 + W_x^2\ \right)} \ \right\};\ i = 3, 6 \tag{A10}$$

To obtain the minimum $MSEs$, the optimum scalar values associated with the estimators discussed in Section 3 are given below.
$B = \frac{S_{xy}}{S_x^2}$, $\beta_{1(opt)} = \beta_{4(opt)} = \frac{\left(\gamma\rho_{xy}C_xC_y - W_{xy}\right)}{\left(\gamma C_x^2 - W_x^2\right)}$, $\beta_{2(opt)} = \beta_{5(opt)} = \frac{\left(\gamma^*\rho_{xy}C_xC_y - W_{xy}^*\right)}{\left(\gamma^* C_x^2 - W_x^{2^*}\right)}$, $\beta_{3(opt)} = \beta_{6(opt)} = \frac{(\gamma^*\rho_{xy}C_xC_y - W_{xy}^* - \gamma\rho_{xy}C_xC_y + W_{xy})}{(\gamma^* Cx^2 - W_x^{2^*} - \gamma Cx^2 + W_x^2)}$

Appendix B

In this section, we outline the proof of Theorem 1 and Corollary 1.
Under strategy I, consider the estimator

$$T_1 = \alpha_1 \bar{Y}_{r,rss} + \theta_1(\bar{X}_{n,rss} - \mu_x)$$

Using the notations discussed in the earlier section, we obtain

$$T_1 - \mu_y = (\alpha_1 - 1)\mu_y + \alpha_1\mu_y\epsilon_0 + \theta_1\mu_x\epsilon_1 \tag{A11}$$

Squaring both sides of (A11) and taking the expectation, we obtain the MSE of the estimator as

$$MSE(T_1) = \left\{ \begin{array}{l} (\alpha_1 - 1)^2\mu_y^2 + \alpha_1^2\mu_y^2(\gamma^* C_y^2 - W_y^{2^*}) + \theta_1^2\mu_x^2(\gamma C_x^2 - W_x^2) \\ +2\alpha_1\theta_1\mu_x\mu_y(\gamma\rho_{xy}C_xC_y - W_{xy}) \end{array} \right\} \tag{A12}$$

The optimum values of α_1 and θ_1 can be obtained by minimizing (A12) with respect to α_1 and θ_1 as

$$\alpha_{1(opt)} = \frac{1}{\left\{1 + \gamma^* C_y^2 - W_y^{2*} - \frac{(\gamma \rho_{xy} C_x C_y - W_{xy})^2}{(\gamma C_x^2 - W_x^2)}\right\}} = \alpha_{7(opt)} \tag{A13}$$

$$\text{and} \quad \theta_{1(opt)} = -\frac{\mu_y}{\mu_x} \frac{(\gamma \rho_{xy} C_x C_y - W_{xy})}{(\gamma C_x^2 - W_x^2)} \alpha_{1(opt)} \tag{A14}$$

Introducing $\alpha_{1(opt)}$ and $\theta_{1(opt)}$ into (A12), we obtain the minimum MSE as

$$minMSE(T_1) = \mu_y^2 (1 - \alpha_{1(opt)}) \tag{A15}$$

Similarly, we can obtain the optimum values of constants and minimum $MSEs$ of other proposed estimators, which are

$$\alpha_{2(opt)} = \frac{1}{\left\{1 + (\gamma^* C_y^2 - W_y^{2*}) - \frac{(\gamma^* \rho_{xy} C_x C_y - W_{xy}^*)^2}{(\gamma^* C_x^2 - W_x^{2*})}\right\}} = \alpha_{8(opt)} \tag{A16}$$

$$\theta_{2(opt)} = -\frac{\mu_y}{\mu_x} \frac{(\gamma^* \rho_{xy} C_x C_y - W_{xy}^*)}{(\gamma^* C_x^2 - W_x^{2*})} \alpha_{2(opt)} \tag{A17}$$

$$\alpha_{3(opt)} = \frac{1}{\left\{1 + \gamma^* C_y^2 - W_y^{2*} - \frac{(\gamma^* \rho_{xy} C_x C_y - W_{xy}^* - \gamma \rho_{xy} C_x C_y + W_{xy})^2}{(\gamma C_x^2 - W_x^2 - \gamma^* C_x^2 + W_x^{2*})}\right\}} = \alpha_{9(opt)} \tag{A18}$$

$$\theta_{3(opt)} = -\frac{\mu_y}{\mu_x} \left(\frac{\gamma^* \rho_{xy} C_x C_y - W_{xy}^* - \gamma \rho_{xy} C_x C_y + W_{xy}}{\gamma C_x^2 - W_x^2 - \gamma^* C_x^2 + W_x^{2*}}\right) \alpha_{3(opt)} \tag{A19}$$

$$\alpha_{j(opt)} = \frac{A_j}{B_j}; \;\; j = 4, 5, 6 \tag{A20}$$

$$\theta_{j(opt)} = \frac{(\gamma \rho_{xy} C_x C_y - W_{xy})}{(\gamma C_x^2 - W_x^2)}; \;\; j = 4, 7 \tag{A21}$$

$$\theta_{j(opt)} = \frac{(\gamma^* \rho_{xy} C_x C_y - W_{xy}^*)}{(\gamma^* C_x^2 - W_x^{2*})}; \;\; j = 5, 8 \tag{A22}$$

$$\theta_{j(opt)} = \frac{(\gamma^* \rho_{xy} C_x C_y - W_{xy}^* - \gamma \rho_{xy} C_x C_y + W_{xy})}{(\gamma C_x^2 - W_x^2 - \gamma^* C_x^2 + W_x^{2*})}; \;\; j = 6, 9 \tag{A23}$$

where

$$A_4 = 1 + \frac{(\gamma \rho_{xy} C_x C_y - W_{xy})}{2} - \frac{(\gamma \rho_{xy} C_x C_y - W_{xy})^2}{2(\gamma C_x^2 - W_x^2)},$$

$$B_4 = 1 + \gamma^* C_y^2 - W_y^{2*} + \gamma \rho_{xy} C_x C_y - W_{xy} - \frac{2(\gamma \rho_{xy} C_x C_y - W_{xy})^2}{(\gamma C_x^2 - W_x^2)},$$

$$A_5 = 1 + \frac{(\gamma^* \rho_{xy} C_x C_y - W_{xy}^*)}{2} - \frac{(\gamma^* \rho_{xy} C_x C_y - W_{xy}^*)^2}{2(\gamma^* C_x^2 - W_x^{2*})},$$

$$B_5 = 1 + \gamma^* C_y^2 - W_y^{2*} + \gamma^* \rho_{xy} C_x C_y - W_{xy}^* - \frac{2(\gamma^* \rho_{xy} C_x C_y - W_{xy}^*)^2}{(\gamma^* C_x^2 - W_x^{2*})},$$

$$A_6 = 1 - \frac{1}{2} \frac{(\gamma^* \rho_{xy} C_x C_y - W_{xy}^* - \gamma \rho_{xy} C_x C_y + W_{xy})^2}{(\gamma^* C_x^2 - W_x^{2*} - \gamma C_x^2 + W_x^2)} + \frac{1}{2}(\gamma^* \rho_{xy} C_x C_y - W_{xy}^* - \gamma \rho_{xy} C_x C_y + W_{xy}),$$

$$B_6 = \left\{ \begin{array}{l} 1 + \gamma^* C_y^2 - W_y^{2*} + \gamma^* \rho_{xy} C_x C_y - W_{xy}^* - \gamma \rho_{xy} C_x C_y + W_{xy} \\ -2 \frac{(\gamma^* \rho_{xy} C_x C_y - W_{xy}^* - \gamma \rho_{xy} C_x C_y + W_{xy})^2}{(\gamma^* C_x^2 - W_x^{2*} - \gamma C_x^2 + W_x^2)} \end{array} \right\}.$$

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