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Inequalities for the Windowed Linear Canonical Transform of Complex Functions

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Abstract: In this paper, we generalize the N-dimensional Heisenberg's inequalities for the windowed linear canonical transform (WLCT) of a complex function. Firstly, the definition for N-dimensional WLCT of a complex function is given. In addition, the N-dimensional Heisenberg's inequality for the linear canonical transform (LCT) is derived. It shows that the lower bound is related to the covariance and can be achieved by a complex chirp function with a Gaussian function. Finally, the N-dimensional Heisenberg's inequality for the WLCT is exploited. In special cases, its corollary can be obtained.

Keywords: Fourier transform; linear canonical transform; inequality; complex function

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1. Introduction

Inequalities for the Fourier transform (FT) are widely used in mathematics, physics and engineering [1–6]. The classical N-dimensional Heisenberg's inequality of the FT is given by the following formula [7]:

$$\int_{\mathbb{R}^{\mathbb{N}}}(\mathbf{t}-\mathbf{t}^{\mathbf{f}})^2 \mid f(\mathbf{t}) \mid^2 \mathrm{d}\mathbf{t} \int_{\mathbb{R}^{\mathbb{N}}}(\mathbf{u}-\mathbf{u}^{\mathbf{f}})^2 \mid \widehat{f}(\mathbf{u}) \mid^2 \mathrm{d}\mathbf{u} \geq \eth\|f\|^4_{L^2(\mathbb{R}^{\mathbb{N}})}, \tag{1}$$

where $\eth = (\frac{N}{4\pi})^2$, $\mathbf{t} = (t_1, t_2, \cdots, t_N)$, $\mathbf{u} = (u_1, u_2, \cdots, u_N)$. $\widehat{f}(\mathbf{u})$ is the FT of any function $f \in L^2(\mathbb{R}^{\mathbb{N}})$,

$$\widehat{f}(\mathbf{u}) = F\{f(\mathbf{t})\}(\mathbf{u}) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^{\mathbb{N}}} f(\mathbf{t})e^{-i\mathbf{t}\mathbf{u}}\mathrm{d}\mathbf{t}, \tag{2}$$

$$\mathbf{t}^{\mathbf{f}} = \int_{\mathbb{R}^{\mathbb{N}}} \mathbf{t} \mid f(\mathbf{t}) \mid^2 \mathrm{d}\mathbf{t}, \tag{3}$$

$$\mathbf{u}^{\mathbf{f}} = \int_{\mathbb{R}^{\mathbb{N}}} \mathbf{u} \mid \widehat{f}(\mathbf{u}) \mid^2 \mathrm{d}\mathbf{u}, \tag{4}$$

$$\|f\|^2_{L^2(\mathbb{R}^{\mathbb{N}})} = \|f\|^2 = \int_{\mathbb{R}^{\mathbb{N}}} |f(\mathbf{t})|^2 \mathrm{d}\mathbf{t}, \tag{5}$$

Based on Formula (1), Zhang obtained the N-dimensional Heisenberg's inequality of the fractional Fourier transform (FRFT) [8].

The windowed linear canonical transform (WLCT) [9–11] is a generalized integral transform of the FT [12] and the FRFT [13]. In recent years, inequality of the WLCT has become a hot topic. Many scholars [14–17] have studied different types of inequalities for the WLCT.

The purpose of this paper is to obtain various kinds of N-dimensional inequalities associated with the WLCT.

2. Preliminary

Let any function $f(\mathbf{t}) = f_1(\mathbf{t})e^{i\phi(\mathbf{t})} \in L^2(\mathbb{R}^{\mathrm{N}})$ and window function $0 \neq g(\mathbf{t}) = g_1(\mathbf{t})e^{i\varphi(\mathbf{t})} \in L^2(\mathbb{R}^{\mathrm{N}})$.

Definition 1 ([18])**.** *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a matrix parameter satisfying* $a, b, c, d \in \mathbb{R}$ *and* $ad - bc = 1$*. For any function* $f(\mathbf{t})$*, the linear canonical transform (LCT) of* $f(\mathbf{t})$ *is defined as*

$$L_A^f(\mathbf{u}) = L_A[f(\mathbf{t})](\mathbf{u}) = \begin{cases} \int_{\mathbb{R}^{\mathrm{N}}} f(\mathbf{t})K_A(\mathbf{t},\mathbf{u})\mathrm{d}\mathbf{t}, & b \neq 0 \\ \sqrt{d}e^{i\frac{cd}{2}\mathbf{u}^2} f(d\mathbf{u})), & b = 0 \end{cases} \tag{6}$$

where

$$K_A(\mathbf{t},\mathbf{u}) = \frac{1}{\sqrt{i2\pi b}} e^{i\frac{a}{2b}\mathbf{t}^2 - i\frac{1}{b}\mathbf{t}\mathbf{u} + i\frac{d}{2b}\mathbf{u}^2}. \tag{7}$$

Additionally, the paper [19] presented the following properties:

$$K_A^*(\mathbf{t},\mathbf{u}) = K_{A^{-1}}(\mathbf{u},\mathbf{t}), \tag{8}$$

$$2\pi\delta(\mathbf{x}) = \int_{\mathbb{R}^{\mathrm{N}}} e^{\pm i\mathbf{u}\mathbf{x}}\mathrm{d}\mathbf{u}, \tag{9}$$

where $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, $\mathbf{x} = (x_1, x_2, \cdots, x_N)$.

If $b = 0$, then the LCT becomes a kind of scaling and chirp multiplication operations [20]. In this paper, we only consider $b \neq 0$.

The inverse formula of the LCT is given by [19]

$$f(\mathbf{t}) = \int_{\mathbb{R}^{\mathrm{N}}} L_A^f(\mathbf{u})K_{A^{-1}}(\mathbf{u},\mathbf{t})\mathrm{d}\mathbf{u}. \tag{10}$$

Definition 2 ([9])**.** *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a matrix parameter satisfying* $a, b, c, d \in \mathbb{R}$ *and* $ad - bc = 1$*. The WLCT of function* f *with respect to* g *is defined by*

$$\begin{aligned} W_g^A f(\mathbf{t},\mathbf{u}) &= \int_{\mathbb{R}^{\mathrm{N}}} f(\mathbf{y})g^*(\mathbf{y}-\mathbf{t})K_A(\mathbf{y},\mathbf{u})\mathrm{d}\mathbf{y} \\ &= \int_{\mathbb{R}^{\mathrm{N}}} f_{\mathbf{t}}(\mathbf{y})K_A(\mathbf{y},\mathbf{u})\mathrm{d}\mathbf{y}, \end{aligned} \tag{11}$$

where $\mathbf{y} = (y_1, y_2, \cdots, y_N)$ *and* $f_{\mathbf{t}}(\mathbf{y}) = f(\mathbf{y})g^*(\mathbf{y}-\mathbf{t}) = f_1(\mathbf{y})g_1^*(\mathbf{y}-\mathbf{t})e^{i(\phi(\mathbf{y})-\varphi(\mathbf{y}-\mathbf{t}))}$.

Next, we will give a lemma.

Lemma 1. *For* $f \in L^2(\mathbb{R}^{\mathrm{N}})$ *and* $g \in L^2(\mathbb{R}^{\mathrm{N}})$*, we have*

$$W_g^A f(\mathbf{t},\mathbf{u}) = \int_{\mathbb{R}^{\mathrm{N}}} L_A^f(\mathbf{k})Q^*(\mathbf{k}|\mathbf{u},\mathbf{t})\mathrm{d}\mathbf{k}, \tag{12}$$

where $A_1 = \begin{bmatrix} 0 & b' \\ -\frac{1}{b'} & d' \end{bmatrix}$, $0 \neq b' = b \in \mathbb{R}$,

$$Q^*(\mathbf{k}|\mathbf{u},\mathbf{t}) = \sqrt{-i2\pi b} e^{i\frac{d'}{2b}(\mathbf{k}-\mathbf{u})^2} L^g_{A_1}(\mathbf{k}-\mathbf{u})^* K_A(\mathbf{t},\mathbf{u}) K^*_A(\mathbf{t},\mathbf{k}).$$

Proof. According to Definition 2 and Formula (10), we obtain

$$\begin{aligned} W^A_g f(\mathbf{t},\mathbf{u}) &= \int_{\mathbb{R}^{\mathbb{N}}} f(\mathbf{y}) \overline{g(\mathbf{y}-\mathbf{t})} K_A(\mathbf{y},\mathbf{u}) \mathrm{d}\mathbf{y} \\ &= \int_{\mathbb{R}^{\mathbb{N}}} L^f_A(\mathbf{k}) \int_{\mathbb{R}^{\mathbb{N}}} K_{A^{-1}}(\mathbf{k},\mathbf{y}) \overline{g(\mathbf{y}-\mathbf{t})} K_A(\mathbf{y},\mathbf{u}) \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{k}. \end{aligned} \tag{13}$$

Assume that $Q^*(\mathbf{k}|\mathbf{u},\mathbf{t}) = \int_{\mathbb{R}^{\mathbb{N}}} K_{A^{-1}}(\mathbf{k},\mathbf{y}) \overline{g(\mathbf{y}-\mathbf{t}} K_A(\mathbf{y},\mathbf{u}) \mathrm{d}\mathbf{y}$ and $\mathbf{y}-\mathbf{t} = \mathbf{p}$, then

$$\begin{aligned} Q^*(\mathbf{k}|\mathbf{u},\mathbf{t}) &= \int_{\mathbb{R}^{\mathbb{N}}} K_{A^{-1}}(\mathbf{k},\mathbf{y}) \overline{g(\mathbf{y}-\mathbf{t}} K_A(\mathbf{y},\mathbf{u}) \mathrm{d}\mathbf{y} \\ &= \int_{\mathbb{R}^{\mathbb{N}}} \overline{g(\mathbf{p})} \frac{1}{\sqrt{-i2\pi b}} \frac{1}{\sqrt{i2\pi b}} e^{-i\frac{(\mathbf{u}-\mathbf{k})}{b}(\mathbf{p}+\mathbf{t}) + i\frac{d}{2b}(\mathbf{u}^2-\mathbf{k}^2)} \mathrm{d}\mathbf{p} \\ &= \frac{1}{\sqrt{i2\pi b}} \int_{\mathbb{R}^{\mathbb{N}}} \frac{1}{\sqrt{-i2\pi b}} \overline{g(\mathbf{p})} e^{i\frac{0}{-2b}\mathbf{p}^2 - i\frac{(\mathbf{k}-\mathbf{u})}{-b}\mathbf{p} + i\frac{d'}{-2b}(\mathbf{k}-\mathbf{u})^2} \mathrm{d}\mathbf{p} \\ &\times e^{i\frac{d'}{2b}(\mathbf{k}-\mathbf{u})^2 + i\frac{d}{2b}(\mathbf{u}^2-\mathbf{k}^2) - i\frac{(\mathbf{u}-\mathbf{k})}{b}\mathbf{t}} \\ &= \frac{1}{\sqrt{i2\pi b}} e^{i\frac{d'}{2b}(\mathbf{k}-\mathbf{u})^2} L^g_{A_1}(\mathbf{k}-\mathbf{u})^* e^{-i\frac{\mathbf{ut}}{b} + i\frac{d}{2b}\mathbf{u}^2} e^{i\frac{\mathbf{kt}}{b} + i\frac{d}{-2b}\mathbf{k}^2} \\ &= \sqrt{-i2\pi b} e^{i\frac{d'}{2b}(\mathbf{k}-\mathbf{u})^2} L^g_{A_1}(\mathbf{k}-\mathbf{u})^* K_A(\mathbf{t},\mathbf{u}) K^*_A(\mathbf{t},\mathbf{k}). \end{aligned} \tag{14}$$

Hence the Formula (13) becomes (12). □

3. Inequalities Associated with the WLCT

The aim of this section is to obtain the new inequalities for the WLCT by the precise mathematical formulation.

Definition 3. *Let $f \in L^2(\mathbb{R}^{\mathbb{N}})$, then we can define [21]*

$$\mathbf{t^f} = \frac{1}{E} \int_{\mathbb{R}^{\mathbb{N}}} \mathbf{t} \mid f(\mathbf{t}) \mid^2 \mathrm{d}\mathbf{t}, \tag{15}$$

$$\mathbf{u^f} = \frac{1}{E} \int_{\mathbb{R}^{\mathbb{N}}} \mathbf{u} \mid \widehat{f}(\mathbf{u}) \mid^2 \mathrm{d}\mathbf{u}, \tag{16}$$

$$\mathbf{u^A_f} = \frac{1}{E} \int_{\mathbb{R}^{\mathbb{N}}} \mathbf{u} \mid L^f_A(\mathbf{u}) \mid^2 \mathrm{d}\mathbf{u}. \tag{17}$$

$$\Delta^2_f = \frac{1}{E} \int_{\mathbb{R}^{\mathbb{N}}} (\mathbf{t}-\mathbf{t^f})^2 \mid f(\mathbf{t}) \mid^2 \mathrm{d}\mathbf{t}, \tag{18}$$

$$\Lambda^2_f = \frac{1}{E} \int_{\mathbb{R}^{\mathbb{N}}} (\mathbf{u}-\mathbf{u^f})^2 \mid \widehat{f}(\mathbf{u}) \mid^2 \mathrm{d}\mathbf{u}, \tag{19}$$

$$\Lambda^2_{A,f} = \frac{1}{E} \int_{\mathbb{R}^{\mathbb{N}}} (\mathbf{u}-\mathbf{u^A_f})^2 \mid L^f_A(\mathbf{u}) \mid^2 \mathrm{d}\mathbf{u}, \tag{20}$$

where

$$E = \int_{\mathbb{R}^{\mathbb{N}}} \mid f(\mathbf{t}) \mid^2 \mathrm{d}\mathbf{t} = \int_{\mathbb{R}^{\mathbb{N}}} \mid \mathrm{L}^f_{\mathrm{A}}(\mathbf{u}) \mid^2 \mathrm{d}\mathbf{u} = \int_{\mathbb{R}^{\mathbb{N}}} \mid \widehat{f}(\mathbf{u}) \mid^2 \mathrm{d}\mathbf{u}, \tag{21}$$

$$\mathbf{t^f} = (t_1^f, t_2^f, \cdots, t_N^f), \tag{22}$$

$$t_k^f = \frac{1}{E} \int_{\mathbb{R^N}} t_k \mid f(\mathbf{t}) \mid^2 \mathrm{d}\mathbf{t}, \tag{23}$$

$$\mathbf{u^f} = (u_1^f, u_2^f, \cdots, u_N^f), \tag{24}$$

$$u_k^f = \frac{1}{E} \int_{\mathbb{R^N}} u_k \mid \widehat{f}(\mathbf{u}) \mid^2 \mathrm{d}\mathbf{u}. \tag{25}$$

Zhang [8] has generalized the N-dimensional Heisenberg's inequality of the FT for complex function. It can be restated as follows:

Lemma 2. *Let $f(\mathbf{t}) = f_1(\mathbf{t})e^{i\phi(\mathbf{t})} \in L^2(\mathbb{R^N})$, for any $1 \leq \varepsilon \leq N$, the classical partial derivatives $\frac{\partial f}{\partial t_\varepsilon}, \frac{\partial f_1}{\partial t_\varepsilon}, \frac{\partial \phi}{\partial t_\varepsilon}$ exist at any point $\mathbf{t} \in \mathbb{R^N}$, then the inequality of the N-dimensional FT can be obtained:*

$$\Delta_f^2 \Lambda_f^2 \geq \frac{N^2}{16\pi^2} \|f\|^2 + COV_f^2, \tag{26}$$

where

$$COV_f = \int_{\mathbb{R^N}} |\mathbf{t} - \mathbf{t^f}||\varpi_{\mathbf{t}}\phi - \mathbf{u^f}|f_1^2(\mathbf{t})\mathrm{d}\mathbf{t}, \tag{27}$$

and $\varpi_{\mathbf{t}}\phi = (\frac{\partial \phi}{\partial t_1}, \frac{\partial \phi}{\partial t_2} \cdots, \frac{\partial \phi}{\partial t_N})$. If $\varpi_{\mathbf{t}}\phi$ is continuous and $f_1 \neq 0$, then the equality holds if and only if $f(\mathbf{t})$ is a chirp function, the function is

$$f(\mathbf{t}) = e^{-\frac{|\mathbf{t}-\mathbf{t^f}|^2}{2\epsilon}} + \iota e^{2\pi i \left[\frac{1}{2\vartheta} \sum_{\kappa=1}^{N} \varrho(t_\kappa)|t_\kappa - t_\kappa^f|^2 + \mathbf{t}\mathbf{u^f} + \iota^{\prod_{\sigma=1}^{N} \varrho(t_\sigma)}\right]}, \tag{28}$$

where $\epsilon, \vartheta > 0$ and $\iota, \iota^{\prod_{\sigma=1}^{N} \varrho(t_\sigma)} \in \mathbb{R}$,

$$\varrho(t_\sigma) = \begin{cases} 1, & \sigma \in \mathbf{z}_{1\tau} \\ -1, & \sigma \in \mathbf{z}_{2\tau} \\ sgn(t_\sigma - t_\sigma^f), & \sigma \in \mathbf{z}_{3\tau} \\ -sgn(t_\sigma - t_\sigma^f), & \sigma \in \mathbf{z}_{4\tau} \end{cases}, \tag{29}$$

$$\mathbf{z}_{1\tau} = \{z_{11}, z_{12}, \cdots, z_{1\tau}\} = \left\{1 \leq s \leq N \mid \frac{\partial \phi}{\partial t_s} = \frac{1}{\vartheta}(t_s - t_s^f) + u_s^f\right\}, \tag{30}$$

$$\mathbf{z}_{2\tau} = \{z_{21}, z_{22}, \cdots, z_{2\tau}\} = \left\{1 \leq s \leq N \mid \frac{\partial \phi}{\partial t_s} = -\frac{1}{\vartheta}(t_s - t_s^f) + u_s^f\right\}, \tag{31}$$

$$\mathbf{z}_{3\tau} = \{z_{31}, z_{32}, \cdots, z_{3\tau}\} = \left\{1 \leq s \leq N \mid \frac{\partial \phi}{\partial t_s} = \begin{cases} \frac{1}{\vartheta}(t_s - t_s^f) + u_s^f, & t_s \geq t_s^f \\ -\frac{1}{\vartheta}(t_s - t_s^f) + u_s^f, & t_s < t_s^f \end{cases}\right\}, \tag{32}$$

$$\mathbf{z}_{4\tau} = \{z_{41}, z_{42}, \cdots, z_{4\tau}\} = \left\{1 \leq s \leq N \mid \frac{\partial \phi}{\partial t_s} = \begin{cases} -\frac{1}{\vartheta}(t_s - t_s^f) + u_s^f, & t_s \geq t_s^f \\ \frac{1}{\vartheta}(t_s - t_s^f) + u_s^f, & t_s < t_s^f \end{cases}\right\}, \tag{33}$$

and $\bigcup_{\rho=1}^{4} \mathbf{z}_{\rho\tau} = \{1, 2, \cdots, N\}$, $\mathbf{z}_{\rho'\tau} \cap \mathbf{z}_{\rho\tau} = \emptyset$ *for* $\rho \neq \rho'$.

Theorem 1. *Let* $f(\mathbf{t}) = f_1(\mathbf{t})e^{i\phi(\mathbf{t})} \in L^2(\mathbb{R}^{\mathbb{N}})$, $\mathbf{t}f(\mathbf{t}) \in L^2(\mathbb{R}^{\mathbb{N}})$, *for any* $1 \leq \varepsilon \leq N$ *the classical partial derivatives* $\frac{\partial f}{\partial t_\varepsilon}, \frac{\partial f_1}{\partial t_\varepsilon}, \frac{\partial \phi}{\partial t_\varepsilon}$ *exist at any point* $\mathbf{t} \in \mathbb{R}^{\mathbb{N}}$, $E = 1$, *then inequality of the N-dimensional LCT can be obtained*

$$\Delta_f^2 \Lambda_{A,f}^2 \geq \frac{(bN)^2}{i16\pi^2}\|f\|^2 + COV_{f,A}^2, \tag{34}$$

where

$$COV_{f,A} = \int_{\mathbb{R}^{\mathbb{N}}} |\mathbf{t} - \mathbf{t}^{\mathbf{f}}||\varpi_{\mathbf{t}}\phi' - \mathbf{u}_{\mathbf{f}}^{\mathbf{A}}| f_1^2(\mathbf{t}) \mathrm{d}\mathbf{t}, \tag{35}$$

$\phi'(\mathbf{t}) = \phi(\mathbf{t}) + \frac{a}{2b}\mathbf{t}^2$ *and* $\varpi_{\mathbf{t}}\phi' = (\frac{\partial \phi'}{\partial t_1}, \frac{\partial \phi'}{\partial t_2} \cdots, \frac{\partial \phi'}{\partial t_N})$, *If* $\varpi_{\mathbf{t}}\phi$ *is continuous and* $f_1 \neq 0$, *then the equality holds if and only if* $f(\mathbf{t})$ *is a chirp function (28).*

Proof. According to the Formulas (2) and (6), we have

$$L_A[f(\mathbf{t})](\mathbf{u}) = \frac{1}{\sqrt{ib}} F\{f(\mathbf{t})e^{i\frac{a}{2b}\mathbf{t}^2}\}\left(\frac{\mathbf{u}}{b}\right)e^{i\frac{d}{2b}\mathbf{u}^2}, \tag{36}$$

let $\mathbf{u}' = \frac{\mathbf{u}}{b}$ and $f'(\mathbf{t}) = f(\mathbf{t})e^{i\frac{a}{2b}\mathbf{t}^2}$, then

$$\begin{aligned} \Delta_f^2 \Lambda_{A,f}^2 &= \int_{\mathbb{R}^{\mathbb{N}}} (\mathbf{t} - \mathbf{t}^{\mathbf{f}})^2 \mid f(\mathbf{t}) \mid^2 \mathrm{d}\mathbf{t} \int_{\mathbb{R}^{\mathbb{N}}} (\mathbf{u} - \mathbf{u}_{\mathbf{f}}^{\mathbf{A}})^2 \mid \mathrm{L}_{\mathrm{A}}^{f}(\mathbf{u}) \mid^2 \mathrm{d}\mathbf{u} \\ &= \frac{1}{ib} \int_{\mathbb{R}^{\mathbb{N}}} (\mathbf{t} - \mathbf{t}^{\mathbf{f}})^2 \mid f'(\mathbf{t}) \mid^2 \mathrm{d}\mathbf{t} \int_{\mathbb{R}^{\mathbb{N}}} (\mathbf{u} - \mathbf{u}_{\mathbf{f}}^{\mathbf{A}})^2 \mid F\{f'(\mathbf{t})\}\left(\frac{\mathbf{u}}{\mathrm{b}}\right) \mid^2 \mathrm{d}\mathbf{u} \\ &= \frac{b^2}{i} \int_{\mathbb{R}^{\mathbb{N}}} (\mathbf{t} - \mathbf{t}^{\mathbf{f}})^2 \mid f'(\mathbf{t}) \mid^2 \mathrm{d}\mathbf{t} \int_{\mathbb{R}^{\mathbb{N}}} (\mathbf{u}' - \mathbf{u}_{\mathbf{f}}'^{\mathbf{A}})^2 \mid F\{f'(\mathbf{t})\}(\mathbf{u}') \mid^2 \mathrm{d}\mathbf{u}'. \end{aligned} \tag{37}$$

By the Formula (26), we have

$$\Delta_f^2 \Lambda_{A,f}^2 \geq \frac{(bN)^2}{i16\pi^2}\|f\|^2 + COV_{f,A}^2. \tag{38}$$

□

Corollary 1. *When* $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, *the above theorem can become the Lemma 2.*

Corollary 2. *When* $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, *the above theorem can reduce the N-dimensional Heisenberg's inequality of the FRFT for complex function [8].*

Definition 4. *Let* $f, g \in L^2(\mathbb{R}^{\mathbb{N}})$, *then we can give the definition [11]*

$$\mathbf{t}_A^W = \frac{1}{\|W_g^A f(\mathbf{t}, \mathbf{u})\|^2} \int_{\mathbb{R}^{\mathbb{N}}} \int_{\mathbb{R}^{\mathbb{N}}} \mathbf{t} \mid W_g^A f(\mathbf{t}, \mathbf{u}) \mid^2 \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{u}, \tag{39}$$

$$\mathbf{u}_A^W = \frac{1}{\|W_g^A f(\mathbf{t}, \mathbf{u})\|^2} \int_{\mathbb{R}^{\mathbb{N}}} \int_{\mathbb{R}^{\mathbb{N}}} \mathbf{u} \mid W_g^A f(\mathbf{t}, \mathbf{u}) \mid^2 \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{u}, \tag{40}$$

$$\Phi^2_{A,W} = \frac{1}{\|W_g^A f(\mathbf{t},\mathbf{u})\|^2}\int_{\mathbb{R}^{\mathbb{N}}}\int_{\mathbb{R}^{\mathbb{N}}}(\mathbf{t}-\mathbf{t}_A^W)^2 \mid W_g^A f(\mathbf{t},\mathbf{u}) \mid^2 \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{u}, \tag{41}$$

$$\Psi^2_{A,W} = \frac{1}{\|W_g^A f(\mathbf{t},\mathbf{u})\|^2}\int_{\mathbb{R}^{\mathbb{N}}}\int_{\mathbb{R}^{\mathbb{N}}}(\mathbf{u}-\mathbf{u}_A^W)^2 \mid W_g^A f(\mathbf{t},\mathbf{u}) \mid^2 \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{u}, \tag{42}$$

Next, the N-dimensional Heisenberg's inequality of the WLCT will be obtained.

Theorem 2. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix parameter satisfying $a,b,c,d \in \mathbb{R}$ and $ad - bc = 1$. For $f(\mathbf{t}) = f_1(\mathbf{t})e^{i\varphi(\mathbf{t})} \in L^2(\mathbb{R}^{\mathbb{N}})$, $g(\mathbf{t}) = g_1(\mathbf{t})e^{i\varphi(\mathbf{t})} \in L^2(\mathbb{R}^{\mathbb{N}})$,$\mathbf{t}f(\mathbf{t}) \in L^2(\mathbb{R}^{\mathbb{N}})$, we have*

$$\Phi^2_{A,W}\Psi^2_{A,W} \geq \frac{(bN)^2}{i16\pi^2}\|f\|^2 + COV^2_{f,A} + \frac{(bN)^2}{i16\pi^2}\|g\|^2 + COV^2_{g,A_1} \tag{43}$$

$$+2\left(\left(\frac{(bN)^2}{i16\pi^2}\|f\|^2 + COV^2_{f,A}\right)\left(\frac{(bN)^2}{i16\pi^2}\|g\|^2 + COV^2_{g,A_1}\right)\right)^{\frac{1}{2}}, \tag{44}$$

where $A_1 = \begin{bmatrix} 0 & b' \\ -\frac{1}{b'} & d' \end{bmatrix}$, $0 \neq b' = b \in \mathbb{R}$, the equality holds if and only if $f(\mathbf{t})$ is a chirp function (28).

Proof. On the one hand, according to Lemma 1 and the Formula (9), we obtain

$$\begin{aligned}
\|W_g^A f(\mathbf{t},\mathbf{u})\|^2 &= \int_{\mathbb{R}^{\mathbb{N}}}\int_{\mathbb{R}^{\mathbb{N}}} |W_g^A f(\mathbf{t},\mathbf{u})|^2 \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{u} \\
&= \int_{\mathbb{R}^{\mathbb{N}}}\int_{\mathbb{R}^{\mathbb{N}}}\Big[\int_{\mathbb{R}^{\mathbb{N}}} L_A^f(\mathbf{m})\sqrt{-i2\pi b}e^{i\frac{d'}{2b}(\mathbf{m}-\mathbf{u})^2} \\
&\times L_{A_1}^g(\mathbf{m}-\mathbf{u})^* K_A(\mathbf{t},\mathbf{u})K_A^*(\mathbf{t},\mathbf{m})\mathrm{d}\mathbf{m}\Big] \\
&\times \Big[\int_{\mathbb{R}^{\mathbb{N}}} L_A^f(\mathbf{n})\sqrt{-i2\pi b}e^{i\frac{d'}{2b}(\mathbf{n}-\mathbf{u})^2} \\
&\times L_{A_1}^g(\mathbf{n}-\mathbf{u})^* K_A(\mathbf{t},\mathbf{u})K_A^*(\mathbf{t},\mathbf{n})\mathrm{d}\mathbf{n}\Big]^* \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{u} \\
&= \int_{\mathbb{R}^{\mathbb{N}}}\int_{\mathbb{R}^{\mathbb{N}}} |L_A^f(\mathbf{m})|^2 |L_{A_1}^g(\mathbf{m}-\mathbf{u})|^2 \mathrm{d}\mathbf{m}\mathrm{d}\mathbf{u}.
\end{aligned} \tag{45}$$

Let $\mathbf{m}-\mathbf{u}=\mathbf{v}$, then

$$\begin{aligned}
\|W_g^A f(\mathbf{t},\mathbf{u})\|^2 &= \int_{\mathbb{R}^{\mathbb{N}}}\int_{\mathbb{R}^{\mathbb{N}}} |L_A^f(\mathbf{m})|^2 |L_{A_1}^g(\mathbf{v})|^2 \mathrm{d}\mathbf{m}\mathrm{d}\mathbf{v} \\
&= \|L_A^f(\mathbf{m})\|^2 \|L_{A_1}^g(\mathbf{v})\|^2.
\end{aligned} \tag{46}$$

Moreover, we obtain

$$\begin{aligned}
\mathbf{t}_A^W &= \frac{1}{\|W_g^A f(\mathbf{t},\mathbf{u})\|^2}\int_{\mathbb{R}^{\mathbb{N}}}\int_{\mathbb{R}^{\mathbb{N}}} \mathbf{t} \mid W_g^A f(\mathbf{t},\mathbf{u}) \mid^2 \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{u} \\
&= \frac{1}{\|L_A^f(\mathbf{m})\|^2\|L_{A_1}^g(\mathbf{v})\|^2}\int_{\mathbb{R}^{\mathbb{N}}}\int_{\mathbb{R}^{\mathbb{N}}} \mathbf{t}\Big[\int_{\mathbb{R}^{\mathbb{N}}} f(\mathbf{m}')\overline{g(\mathbf{m}'-\mathbf{t})}K_A(\mathbf{m}',\mathbf{u})\mathrm{d}\mathbf{m}'\Big] \\
&\times \Big[\int_{\mathbb{R}^{\mathbb{N}}} f(\mathbf{n}')\overline{g(\mathbf{n}'-\mathbf{t})}K_A(\mathbf{n}',\mathbf{u})\mathrm{d}\mathbf{n}'\Big]^* \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{u} \\
&= \frac{1}{\|L_A^f(\mathbf{m})\|^2\|L_{A_1}^g(\mathbf{v})\|^2}\int_{\mathbb{R}^{\mathbb{N}}}\int_{\mathbb{R}^{\mathbb{N}}} \mathbf{t}|f(\mathbf{m}')|^2|g(\mathbf{m}'-\mathbf{t})|^2 \mathrm{d}\mathbf{m}'\mathrm{d}\mathbf{t}.
\end{aligned} \tag{47}$$

Let $\mathbf{m}' - \mathbf{t} = \mathbf{r}$, then

$$\begin{aligned} \mathbf{t}_A^W &= \frac{1}{\|L_A^f(\mathbf{m})\|^2 \|L_{A_1}^g(\mathbf{v})\|^2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (\mathbf{m}' - \mathbf{r}) |f(\mathbf{m}')|^2 |g(\mathbf{r})|^2 \mathrm{d}\mathbf{m}' \mathrm{d}\mathbf{r} \\ &= \frac{1}{\|L_A^f(\mathbf{m}')\|^2} \int_{\mathbb{R}^N} \mathbf{m}' |f(\mathbf{m}')|^2 |\mathrm{d}\mathbf{m}' - \frac{1}{\|L_{A_1}^{\mathrm{g}}(\mathbf{v})\|^2} \int_{\mathbb{R}^N} \mathbf{r} |\mathrm{g}(\mathbf{r})|^2 \mathrm{d}\mathbf{r} \\ &= \mathbf{t}^{\mathbf{f}} - \mathbf{t}^{\mathbf{g}}. \end{aligned} \tag{48}$$

Using the same method, we can obtain

$$\mathbf{u}_A^W = \mathbf{u}_f^A - \mathbf{u}_g^{A_1}. \tag{49}$$

From the Formula (46), then

$$\begin{aligned} \Psi_{A,W}^2 &= \frac{1}{\|W_g^A f(\mathbf{t},\mathbf{u})\|^2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (\mathbf{u} - \mathbf{u}_A^W)^2 \mid W_g^A f(\mathbf{t},\mathbf{u}) \mid^2 \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{u} \\ &= \frac{1}{\|L_A^f(\mathbf{m})\|^2} \int_{\mathbb{R}^N} (\mathbf{m}' - \mathbf{u}_f^A)^2 |L_A^f(\mathbf{m}')|^2 \mathrm{d}\mathbf{m}' + \frac{1}{\|L_{A_1}^g(\mathbf{v})\|^2} \\ &\times \int_{\mathbb{R}^N} (\mathbf{v}' - \mathbf{u}_g^{A_1})^2 |L_{A_1}^g(\mathbf{v}')|^2 \mathrm{d}\mathbf{v}' - 2\frac{1}{\|L_{\mathrm{A}}^f(\mathbf{m})\|^2} \\ &\times \int_{\mathbb{R}^N} (\mathbf{m}' - \mathbf{u}_f^A) |L_A^f(\mathbf{m}')|^2 \mathrm{d}\mathbf{m}' \frac{1}{\|L_{\mathrm{A}_1}^{\mathrm{g}}(\mathbf{v})\|^2} \\ &\times \int_{\mathbb{R}^N} (\mathbf{v}' - \mathbf{u}_g^{A_1}) |L_{A_1}^g(\mathbf{v}')|^2 \mathrm{d}\mathbf{v}' \\ &= \Lambda_{A,f}^2 + \Lambda_{A_1,g}^2. \end{aligned} \tag{50}$$

From the same method, we can obtain

$$\Phi_{A,W}^2 = \Delta_f^2 + \Delta_g^2, \tag{51}$$

On the other hand, using the Formulas (48)–(51), we can obtain

$$\begin{aligned} \Phi_{A,W}^2 \Psi_{A,W}^2 &= (\Delta_f^2 + \Delta_g^2)(\Lambda_{A,f}^2 + \Lambda_{A_1,g}^2) \\ &= \Delta_f^2 \Lambda_{A,f}^2 + \Delta_g^2 \Lambda_{A_1,g}^2 + \Delta_f^2 \Lambda_{A_1,g}^2 + \Delta_g^2 \Lambda_{A,f}^2. \end{aligned} \tag{52}$$

According to the fact: $n^2 + m^2 \geq 2nm$, for $\forall n, m \in \mathbb{R}$, then

$$\begin{aligned} \Phi_{A,W}^2 \Psi_{A,W}^2 &= \Delta_f^2 \Lambda_{A,f}^2 + \Delta_g^2 \Lambda_{A_1,g}^2 + \Delta_f^2 \Lambda_{A_1,g}^2 + \Delta_g^2 \Lambda_{A,f}^2 \\ &\geq \Delta_f^2 \Lambda_{A,f}^2 + \Delta_g^2 \Lambda_{A_1,g}^2 + 2\sqrt{\Delta_f^2 \Lambda_{A,f}^2 \Delta_g^2 \Lambda_{A_1,g}^2}. \end{aligned} \tag{53}$$

From the Formula (34), we can obtain the result. □

Corollary 3. *When* $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, *the N-dimensional Heisenberg's inequality of the windowed fractional Fourier transform (WFRFT) [22] for the complex function can be obtained:*

$$\Phi_{\alpha,W}^2 \Psi_{\alpha,W}^2 \geq \frac{(\sin\alpha N)^2}{i16\pi^2} \|f\|^2 + COV_{f,\alpha}^2 + \frac{(\sin\alpha N)^2}{i16\pi^2} \|g\|^2 + COV_{g,\alpha_1}^2 \tag{54}$$

$$+ 2\left(\left(\frac{(\sin\alpha N)^2}{i16\pi^2} \|f\|^2 + COV_{f,\alpha}^2 \right) \left(\frac{(\sin\alpha N)^2}{i16\pi^2} \|g\|^2 + COV_{g,\alpha_1}^2 \right) \right)^{\frac{1}{2}}, \tag{55}$$

where

$$\Phi^2_{\alpha,W} = \frac{1}{\|W^\alpha_g f(\mathbf{t},\mathbf{u})\|^2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(\mathbf{t}-\mathbf{t}^W_\alpha)^2 \mid W^\alpha_g f(\mathbf{t},\mathbf{u}) \mid^2 \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{u}, \tag{56}$$

$$\Psi^2_{\alpha,W} = \frac{1}{\|W^\alpha_g f(\mathbf{t},\mathbf{u})\|^2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(\mathbf{u}-\mathbf{u}^W_\alpha)^2 \mid W^\alpha_g f(\mathbf{t},\mathbf{u}) \mid^2 \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{u}, \tag{57}$$

$$\mathbf{t}^W_\alpha = \frac{1}{\|W^\alpha_g f(\mathbf{t},\mathbf{u})\|^2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathbf{t} \mid W^\alpha_g f(\mathbf{t},\mathbf{u}) \mid^2 \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{u}, \tag{58}$$

$$\mathbf{u}^W_\alpha = \frac{1}{\|W^\alpha_g f(\mathbf{t},\mathbf{u})\|^2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathbf{u} \mid W^\alpha_g f(\mathbf{t},\mathbf{u}) \mid^2 \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{u}, \tag{59}$$

$$COV_{f,\alpha} = \int_{\mathbb{R}^N}|\mathbf{t}-\mathbf{t}^{\mathbf{f}}||\varpi_{\mathbf{t}}\phi' - \mathbf{u}^\alpha_{f,W}|f_1^2(\mathbf{t})\mathrm{d}\mathbf{t}, \tag{60}$$

$$COV_{g,\alpha'} = \int_{\mathbb{R}^N}|\mathbf{t}-\mathbf{t}^{\mathbf{g}}||\varpi_{\mathbf{t}}\phi' - \mathbf{u}^{\alpha'}_{g,W}|g_1^2(\mathbf{t})\mathrm{d}\mathbf{t}, \tag{61}$$

$$\mathbf{u}^\alpha_{f,W} = \frac{1}{E}\int_{\mathbb{R}^N}\mathbf{u} \mid W^\alpha_g f(\mathbf{t},\mathbf{u}) \mid^2 \mathrm{d}\mathbf{u}, \tag{62}$$

$$\mathbf{u}^{\alpha'}_{g,W} = \frac{1}{E}\int_{\mathbb{R}^N}\mathbf{u} \mid W^\alpha_g f(\mathbf{t},\mathbf{u}) \mid^2 \mathrm{d}\mathbf{u}, \tag{63}$$

and $W^\alpha_g f(\mathbf{t},\mathbf{u})$ *is the WFRFT of complex function*

$$W^\alpha_g f(\mathbf{t},\mathbf{u}) = \begin{cases} \int_{\mathbb{R}^N} f(\mathbf{y})g^*(\mathbf{y}-\mathbf{t})K_\alpha(\mathbf{y},\mathbf{u})\mathrm{d}\mathbf{y}, & \alpha \neq n\pi \\ f(\mathbf{u}), & \alpha = 2n\pi \\ -f(\mathbf{u}), & \alpha = (2n+1)\pi \end{cases}, \tag{64}$$

and $K_\alpha(\mathbf{y},\mathbf{u}) = (1 - i\cot\alpha)^{\frac{N}{2}} e^{\pi i(|\mathbf{y}|^2+|\mathbf{u}|^2)\cot\alpha - 2\pi i\mathbf{y}\mathbf{u}\csc\alpha}$.

Corollary 4. *When* $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$*, the N-dimensional Heisenberg's inequality of the windowed Fourier transform (WFT) [23] for the complex function can be obtained:*

$$\Phi^2_W\Psi^2_W \geq \frac{N^2}{i16\pi^2}(\|f\|^2 + \|g\|^2) + COV_f^2 + COV_g^2 \tag{65}$$

$$+ 2\left(\left(\frac{N^2}{i16\pi^2}\|f\|^2 + COV_f^2\right)\left(\frac{N^2}{i16\pi^2}\|g\|^2 + COV_g^2\right)\right)^{\frac{1}{2}}, \tag{66}$$

where

$$\Phi^2_W = \frac{1}{\|W_g f(\mathbf{t},\mathbf{u})\|^2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(\mathbf{t}-\mathbf{t}^W)^2 \mid W_g f(\mathbf{t},\mathbf{u}) \mid^2 \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{u}, \tag{67}$$

$$\Psi^2_W = \frac{1}{\|W_g f(\mathbf{t},\mathbf{u})\|^2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(\mathbf{u}-\mathbf{u}^W)^2 \mid W_g f(\mathbf{t},\mathbf{u}) \mid^2 \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{u}, \tag{68}$$

$$\mathbf{t}^W = \frac{1}{\|W_g f(\mathbf{t}, \mathbf{u})\|^2} \int_{\mathbb{R}^{\mathbb{N}}} \int_{\mathbb{R}^{\mathbb{N}}} \mathbf{t} \mid W_g f(\mathbf{t}, \mathbf{u}) \mid^2 \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{u}, \tag{69}$$

$$\mathbf{u}^W = \frac{1}{\|W_g f(\mathbf{t}, \mathbf{u})\|^2} \int_{\mathbb{R}^{\mathbb{N}}} \int_{\mathbb{R}^{\mathbb{N}}} \mathbf{u} \mid W_g f(\mathbf{t}, \mathbf{u}) \mid^2 \mathrm{d}\mathbf{t}\mathrm{d}\mathbf{u}, \tag{70}$$

$$COV_f = \int_{\mathbb{R}^{\mathbb{N}}} |\mathbf{t} - \mathbf{t}^{\mathbf{f}}||\varpi_{\mathbf{t}}\phi' - \mathbf{u}_{\mathbf{f},\mathbf{W}}|f_1^2(\mathbf{t})\mathrm{d}\mathbf{t}, \tag{71}$$

$$COV_g = \int_{\mathbb{R}^{\mathbb{N}}} |\mathbf{t} - \mathbf{t}^{\mathbf{g}}||\varpi_{\mathbf{t}}\phi' - \mathbf{u}_{\mathbf{g},\mathbf{W}}|g_1^2(\mathbf{t})\mathrm{d}\mathbf{t}, \tag{72}$$

$$\mathbf{u}_{\mathbf{f},\mathbf{W}} = \frac{1}{E} \int_{\mathbb{R}^{\mathbb{N}}} \mathbf{u} \mid W_g f(\mathbf{t}, \mathbf{u}) \mid^2 \mathrm{d}\mathbf{u}, \tag{73}$$

$$\mathbf{u}_{\mathbf{g},\mathbf{W}} = \frac{1}{E} \int_{\mathbb{R}^{\mathbb{N}}} \mathbf{u} \mid W_g f(\mathbf{t}, \mathbf{u}) \mid^2 \mathrm{d}\mathbf{u}, \tag{74}$$

and $W_g f(\mathbf{t}, \mathbf{u})$ is the WFT of the complex function

$$W_g f(\mathbf{t}, \mathbf{u}) = \begin{cases} \int_{\mathbb{R}^{\mathbb{N}}} f(\mathbf{y}) g^*(\mathbf{y} - \mathbf{t}) e^{-i\mathbf{y}\mathbf{u}} \mathrm{d}\mathbf{y}, & \alpha \neq n\pi \\ f(\mathbf{u}), & \alpha = 2n\pi \\ -f(\mathbf{u}), & \alpha = (2n+1)\pi \end{cases}. \tag{75}$$

4. Conclusions

In this paper, by the N-dimensional Heisenberg's inequality of the FT, the N-dimensional Heisenberg's inequalities for the WLCT of a complex function are generalized. Firstly, the definition for N-dimensional WLCT of a complex function is given. In addition, according to the second-order moment of the LCT, the N-dimensional Heisenberg's inequality for the linear canonical transform (LCT) is derived. It shows that the lower bound is related to the covariance and can be achieved by a complex chirp function with a Gaussian function. Finally, the second-order moment of the WLCT is given, the relationship between the LCT and WLCT is obtained, and the N-dimensional Heisenberg's inequality for the WLCT is exploited. In special cases, its corollaries can be obtained.

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