axioms

MDPI

Article

Solvability of Iterative Classes of Nonlinear Elliptic Equations on an Exterior Domain

Xiaoming Wang [1], Jehad Alzabut [2,3,*], Mahammad Khuddush [4] and Michal Fečkan [5,6]

1 School of Mathematics and Computer Science, Shangrao Normal University, Shangrao 334001, China; wxmsuda03@163.com
2 Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia
3 Department of Industrial Engineering, OSTİM Technical University, Ankara 06374, Türkiye
4 Department of Mathematics, Dr. Lankapalli Bullayya College of Engineering, Visakhapatnam 530013, Andhra Pradesh, India; khuddush89@gmail.com
5 Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Mlynská Dolina, 842 48 Bratislava, Slovakia; michal.feckan@fmph.uniba.sk
6 Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia
* Correspondence: jalzabut@psu.edu.sa

Abstract: This work explores the possibility that iterative classes of elliptic equations have both single and coupled positive radial solutions. Our approach is based on using the well-known Guo–Krasnoselskii and Avery–Henderson fixed-point theorems in a Banach space. Furthermore, we utilize Rus' theorem in a metric space, to prove the uniqueness of solutions for the problem. Examples are constructed for the sake of verification.

Keywords: iterative class; elliptic equations; exterior domain; radial solutions; Banach space; complete metric space; fixed-point theorem

MSC: 35J66; 35J60; 34B18; 47H10

check for updates

Citation: Wang, X.; Alzabut, J.; Khuddush, M.; Fečkan, M. Solvability of Iterative Classes of Nonlinear Elliptic Equations on an Exterior Domain. *Axioms* **2023**, *12*, 474. https://doi.org/10.3390/axioms12050474

Academic Editor: Chris Goodrich

Received: 25 March 2023
Revised: 8 May 2023
Accepted: 10 May 2023
Published: 14 May 2023

1. Introduction

The study of nonlinear elliptic systems has a strong motivation, and important research efforts have been made undertaken recently for these systems, aiming to apply the results of the existence and asymptotic behavior of positive solutions in applied fields (see [1–5]). The investigation of the following system of nonlinear elliptic equations in a bounded domain $\mho \subset \mathbb{R}^N$,

$$\triangle \mathfrak{z}_\beta + \lambda \mathtt{F}_\beta(\mathfrak{z}_{\beta+1}) = 0, \tag{1}$$

where $\mathfrak{z}_\beta = 0$ on $\partial\mho$ and $\mathfrak{z}_1 = \mathfrak{z}_{\eth+1}$, $\beta \in \{1, 2, 3, \cdots, \eth\}$, has an important application in science and technology [6,7]. In [8], Dalmasso discussed the existence of positive solutions to such systems for $\eth = 2$ when the $\mathtt{F}(0)'s$ are non-negative with at least one $\mathtt{F}(0) > 0$ (positone problems). In [7], when $\eth = 2$, Ali–Ramaswamy–Shivaji discussed the existence of multiple positive solutions to such positone problems. In particular, in cases where one of $\frac{\mathfrak{z}}{\mathtt{F}_1(\mathfrak{z})}$ or $\frac{\mathfrak{z}}{\mathtt{F}_2(\mathfrak{z})}$ decreases for some range of $\mathfrak{z}$, they established conditions for the existence of at least three positive solutions for a certain range of λ. In [9], Hai–Shivaji discussed the existence of positive solutions for $\lambda >> 1$ for cases where no sign conditions are assumed on $\mathtt{F}(0)$, $\beta \in \{1, 2\}$ (semipositone problems). In [10], again for $\eth = 2$, Ali–Shivaji discussed the existence of multiple positive solutions for $\lambda >> 1$ when $\mathtt{F}(0) = 0 = \mathtt{F}'(0)$ for $\beta \in \{1, 2\}$. In addition, in [11–20], relevant references to the most recent works on (1) can be found. Next, we quote some recent works on elliptic equations.

In [21], Padhi et al. derived sufficient conditions to the following problem in an annular domain:

$$\triangle\mathfrak{z} = \lambda\mathtt{F}(|\nu|,\mathfrak{z}),\ \mathfrak{z} \in \mho = \{\nu \in \mathbb{R}^{\mathrm{N}} : a_1 < |\nu| < a_2\},$$
$$\mathfrak{z} = 0,\ \mathfrak{z} \in \partial\mho,$$

for the existence of positive radial solutions, by utilizing Gustafson and Schmitt fixed-point theorems. In [22], Chrouda and Hassine established the uniqueness of positive radial solutions to the following Dirichlet boundary value problem for the semilinear elliptic equation in an annulus:

$$\triangle\mathfrak{z} = \mathtt{F}(\mathfrak{z}),\ \mathfrak{z} \in \mho = \{\nu \in \mathbb{R}^{\mathrm{N}} : a_1 < |\nu| < a_2\},$$
$$\mathfrak{z} = 0,\ \mathfrak{z} \in \mathfrak{z} \in \partial\mho,$$

for any dimension $\mathrm{N} \geq 1$. In [23], Dong and Wei established the existence of radial solutions for the following nonlinear elliptic equations with gradient terms in annular domains:

$$\triangle\mathfrak{z} + \mathrm{g}\big(|\nu|,\mathfrak{z},\frac{\nu}{|\nu|}\cdot\nabla\mathfrak{z}\big) = 0 \ \text{ in } \ \Omega_a^b,$$
$$\mathfrak{z} = 0 \ \text{ on } \ \partial\Omega_a^b,$$

by using Schauder's fixed-point theorem and contraction mapping theorem. In [24], R. Kajikiya and E. Ko established the existence of positive radial solutions for a semipositone elliptic equation of the form

$$\triangle\mathfrak{z} + \lambda\mathrm{g}(\mathfrak{z}) = 0 \ \text{ in } \ \Omega,$$
$$\mathfrak{z} = 0 \ \text{ on } \ \partial\Omega,$$

where Ω is a ball or an annulus in $\mathbb{R}^{\mathbf{N}}$. Recently, Son and Wang [25] considered the following system in an exterior ball $\mho_{\mathfrak{X}}$:

$$\triangle\mathfrak{z}_\beta + \lambda\mathrm{K}_\beta(|\nu|)\mathtt{F}_\beta(\mathfrak{z}_{\beta+1}) = 0,$$
$$\mathfrak{z}_\beta \to 0 \text{ as } |\nu| \to +\infty$$
$$\mathfrak{z}_\beta = 0 \text{ on } |\nu| = r_0,$$

where $\beta \in \{1,2,3,\cdots,\mathfrak{d}\}$, $\mathfrak{z}_1 = \mathfrak{z}_{\mathfrak{d}+1}$, and derived sufficient conditions for the existence of positive radial solutions. The above-mentioned works motivated us to study the following iterative classes of nonlinear elliptic equations on an exterior domain:

$$\left.\begin{aligned} \triangle\mathfrak{z}_\beta - \frac{(\mathrm{N}-2)^2 r_0^{2\mathrm{N}-2}}{|\nu|^{2\mathrm{N}-2}}\mathfrak{z}_\beta + \varrho(|\nu|)\mathtt{F}_\beta(\mathfrak{z}_{\beta+1}) = 0,\ \nu \in \mho, \\ \lim_{|\nu|\to\infty}\mathfrak{z}_\beta(\nu) = 0,\ \mathfrak{z}_\beta|_{\partial\mho} = 0, \end{aligned}\right\} \tag{2}$$

where $\beta \in \{1,2,3,\cdots,n\}$, $\mathfrak{z}_1 = \mathfrak{z}_{n+1}$, $\Delta\mathfrak{z} = \mathtt{div}(\nabla\mathfrak{z})$, $\mathrm{N} > 2$, $\mho = \{\mathfrak{z} \in \mathbb{R}^{\mathrm{N}} |\ |\mathfrak{z}| > r_0\}$, $\varrho = \prod_{i=1}^{\kappa}\varrho_i$, each $\varrho_i \in \mathtt{C}((r_0,+\infty),(0,+\infty))$, $r^{\mathrm{N}-1}\varrho$ is integrable. The Guo–Krasnoselskii cone fixed-point theorem is a key tool for obtaining single positive radial solutions, whereas the Avery–Henderson cone fixed-point theorem is utilized to obtain the coupled solutions. We further study the uniqueness of solutions of the problem (2) via Rus' theorem in a metric space.

The study of the positive solutions to the iterative classes of ordinary differential equations with two-point boundary conditions,

$$\left.\begin{aligned} \mathfrak{z}_\beta''(\hat{\mathrm{r}}) - r_0^2\mathfrak{z}_\beta(\hat{\mathrm{r}}) + \varrho(\hat{\mathrm{r}})\mathtt{F}_\beta(\mathfrak{z}_{\beta+1}(\hat{\mathrm{r}})) = 0,\ 0 < \hat{\mathrm{r}} < 1, \\ \mathfrak{z}_\beta(0) = 0,\ \mathfrak{z}_\beta(1) = 0, \end{aligned}\right\} \tag{3}$$

where $\beta \in \{1,2,3,\cdots,\mathfrak{d}\}$, $\mathfrak{z}_1 = \mathfrak{z}_{\mathfrak{d}+1}$, $r_0 > 0$ and $\varrho(\hat{\mathbf{r}}) = \frac{r_0^2}{(\mathrm{N}-2)^2}\hat{\mathbf{r}}^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}\prod_{i=1}^{\kappa}\varrho_i(\hat{\mathbf{r}})$, $\varrho_i(\hat{\mathbf{r}}) = \varrho_i(r_0\hat{\mathbf{r}}^{\frac{1}{2-\mathrm{N}}})$ by a Kelvin-type transformation [26,27] through the change of variables $\mathfrak{m} = |\nu|$ and $\hat{\mathbf{r}} = \left(\frac{\mathfrak{m}}{r_0}\right)^{2-\mathrm{N}}$, facilitates the investigation of the positive radial solutions of (2).

We impose the below-mentioned presumptions whenever necessary:

($\mathcal{J}_1$) $\mathtt{F}_\beta : [0,+\infty) \to [0,+\infty)$ is continuous.

($\mathcal{J}_2$) For $1 \le i \le \mathfrak{d}$, $\varrho_i \in \mathrm{L}^{\mathrm{p}_i}[0,1](1 \le \mathrm{p}_i \le +\infty)$ and $\exists\, \varrho_i^\star > 0 \ni \varrho_i^\star < \varrho_i(\hat{\mathbf{r}}) < \infty$ almost everywhere on the interval $[0,1]$.

The remainder of the paper is structured as follows: The problem (3) is transformed into an analogous integral equation involving the kernel in Section 2. Additionally, we calculate the kernel boundaries that are crucial to our major findings. In Section 3, we employ Guo–Krasnoselskii's cone fixed-point theorem, to provide a criterion for the single positive radial solution. In Section 4, the coupled solutions are established by the Avery–Henderson cone fixed-point theorem. The final portion deals with a unique solution. Meanwhile, some numerical examples are provided.

2. Preliminaries

The essential results are stated here, prior to proceeding to the main results in the subsequent sections.

Lemma 1. *For every $\wp \in \mathtt{C}[0,1]$, the BVP*

$$-\mathfrak{z}_1''(\hat{\mathbf{r}}) + r_0^2\mathfrak{z}_1(\hat{\mathbf{r}}) = \wp(\hat{\mathbf{r}}),\ 0 < \hat{\mathbf{r}} < 1,$$

$$\mathfrak{z}_1(0) = \mathfrak{z}_1(1) = 0,$$

has a unique solution

$$\mathfrak{z}_1(\hat{\mathbf{r}}) = \int_0^1 \mathtt{Q}(\hat{\mathbf{r}},\zeta)\wp(\zeta)\mathtt{d}\zeta,$$

where

$$\mathtt{Q}(\hat{\mathbf{r}},\zeta) = \frac{1}{r_0\sinh(r_0)}\begin{cases}\sinh(r_0\hat{\mathbf{r}})\sinh(r_0(1-\zeta)), & 0 \le \hat{\mathbf{r}} \le \zeta \le 1,\\ \sinh(r_0\zeta)\sinh(r_0(1-\hat{\mathbf{r}})), & 0 \le \zeta \le \hat{\mathbf{r}} \le 1.\end{cases}$$

Lemma 2. *The kernel $\mathtt{Q}(\hat{\mathbf{r}},\zeta)$ has the subsequent characteristics:*

(i) $\mathtt{Q}(\hat{\mathbf{r}},\zeta) \ge 0$ *and continuous on* $[0,1]\times[0,1]$;

(ii) $\mathtt{Q}(\hat{\mathbf{r}},\zeta) \le \mathtt{Q}(\zeta,\zeta)$, $\hat{\mathbf{r}},\zeta \in [0,1]$;

(iii) *there exists* $\xi \in (0,\frac{1}{2})$ *such that* $\sigma(\xi)\mathtt{Q}(\zeta,\zeta) \le \mathtt{Q}(\hat{\mathbf{r}},\zeta)$, $(\hat{\mathbf{r}},\zeta) \in [\xi,1-\xi]\times[0,1]$, *where* $\sigma(\xi) = \frac{\sinh(r_0\xi)}{\sinh(r_0)}$.

Proof. (i) is evident. The following proves (ii):

$$\frac{\mathtt{Q}(\hat{\mathbf{r}},\zeta)}{\mathtt{Q}(\zeta,\zeta)} = \begin{cases}\dfrac{\sinh(r_0\hat{\mathbf{r}})}{\sinh(r_0\zeta)}, & 0 \le \hat{\mathbf{r}} \le \zeta \le 1,\\[2ex] \dfrac{\sinh(r_0(1-\hat{\mathbf{r}}))}{\sinh(r_0(1-\zeta))}, & 0 \le \zeta \le \hat{\mathbf{r}} \le 1,\end{cases}$$
$$\le \begin{cases}1, & 0 \le \hat{\mathbf{r}} \le \zeta \le 1,\\ 1, & 0 \le \zeta \le \hat{\mathbf{r}} \le 1,\end{cases}$$

For (iii), we consider

$$
\begin{aligned}
\frac{\mathtt{Q}(\hat{\mathbf{r}},\zeta)}{\mathtt{Q}(\zeta,\zeta)} &= \begin{cases} \dfrac{\sinh(r_0\hat{\mathbf{r}})}{\sinh(r_0\zeta)}, & 0\le\hat{\mathbf{r}}\le\zeta\le 1,\\ \dfrac{\sinh(r_0(1-\hat{\mathbf{r}}))}{\sinh(r_0(1-\zeta))}, & 0\le\zeta\le\hat{\mathbf{r}}\le 1, \end{cases}\\
&\ge \begin{cases} \dfrac{\sinh(r_0\xi)}{\sinh(r_0)}, & 0\le\hat{\mathbf{r}}\le\zeta\le 1,\ \xi\le\hat{\mathbf{r}}\le 1-\xi,\\ \dfrac{\sinh(r_0\xi)}{\sinh(r_0)}, & 0\le\zeta\le\hat{\mathbf{r}}\le 1,\ \xi\le\hat{\mathbf{r}}\le 1-\xi, \end{cases}\\
&= \sigma.
\end{aligned}
$$

The proof is now completed. □

We observe that a $\mathfrak{d}$-tuple $(\mathfrak{z}_1,\mathfrak{z}_2,\cdots,\mathfrak{z}_{\mathfrak{d}})$ is a solution of BVP (3) from Lemma 1 if and only if

$$
\begin{aligned}
\mathfrak{z}_1(\hat{\mathbf{r}}) = \int_0^1 \mathtt{Q}(\hat{\mathbf{r}},\zeta_1)\varrho(\zeta_1)\mathtt{F}_1\Bigg[\int_0^1 \mathtt{Q}(\zeta_1,\zeta_2)\varrho(\zeta_2)\mathtt{F}_2\Bigg[\int_0^1 \mathtt{Q}(\zeta_2,\zeta_3)\varrho(\zeta_3)\mathtt{F}_4\cdots\\
\mathtt{F}_{\mathfrak{d}-1}\Bigg[\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathrm{d}\zeta_{\mathfrak{d}}\Bigg]\cdots\Bigg]\mathrm{d}\zeta_3\Bigg]\mathrm{d}\zeta_2\Bigg]\mathrm{d}\zeta_1.
\end{aligned}
$$

In general,

$$
\begin{aligned}
\mathfrak{z}_\beta(\hat{\mathbf{r}}) &= \int_0^1 \mathtt{Q}(\hat{\mathbf{r}},\zeta)\varrho(\zeta)\mathtt{F}_\beta\big(\mathfrak{z}_{\beta+1}(\zeta)\big)\mathrm{d}\zeta,\ \beta=1,2,3,\cdots,\mathfrak{d},\\
\mathfrak{z}_1(\hat{\mathbf{r}}) &= \mathfrak{z}_{\mathfrak{d}+1}(\hat{\mathbf{r}}).
\end{aligned}
$$

Let $\aleph := \mathtt{C}((0,1),\mathbb{R})$ be a Banach space equipped with a norm $\|\mathfrak{z}\| = \max\limits_{\hat{\mathbf{r}}\in[0,1]}|\mathfrak{z}(\hat{\mathbf{r}})|$, and

$$
\mathfrak{X}_\xi = \Big\{\mathfrak{z}\in\aleph : \mathfrak{z}(\hat{\mathbf{r}})\ge 0 \text{ on } [0,1],\ \min_{\hat{\mathbf{r}}\in[\xi,1-\xi]}\mathfrak{z}(\hat{\mathbf{r}})\ge\sigma(\xi)\|\mathfrak{z}\|\Big\}
$$

be a cone, for $\xi\in(0,\frac{1}{2})$. For any $\mathfrak{z}_1\in\mathfrak{X}$, define an operator $\pounds:\mathfrak{X}\to\aleph$ by

$$
\begin{aligned}
(\pounds\mathfrak{z}_1)(\hat{\mathbf{r}}) = \int_0^1 \mathtt{Q}(\hat{\mathbf{r}},\zeta_1)\varrho(\zeta_1)\mathtt{F}_1\Bigg[\int_0^1 \mathtt{Q}(\zeta_1,\zeta_2)\varrho(\zeta_2)\mathtt{F}_2\Bigg[\int_0^1 \mathtt{Q}(\zeta_2,\zeta_3)\varrho(\zeta_3)\cdots\\
\mathtt{F}_{\mathfrak{d}-1}\Bigg[\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathrm{d}\zeta_{\mathfrak{d}}\Bigg]\cdots\Bigg]\mathrm{d}\zeta_3\Bigg]\mathrm{d}\zeta_2\Bigg]\mathrm{d}\zeta_1. \qquad (4)
\end{aligned}
$$

Lemma 3. *$\pounds$ is self–mapping on $\mathfrak{X}_\xi$ and $\pounds:\mathfrak{X}_\xi\to\mathfrak{X}_\xi$ is completely continuous.*

Proof. As $\mathtt{F}_\beta(\mathfrak{z}_{\beta+1}(\hat{\mathbf{r}}))\ge 0$ and $\mathtt{Q}(\hat{\mathbf{r}},\zeta)\ge 0$ for $\hat{\mathbf{r}},\zeta\in[0,1]$, we have $\pounds(\mathfrak{z}_1(\hat{\mathbf{r}}))\ge 0$ for $\hat{\mathbf{r}}\in[0,1]$, $\mathfrak{z}_1\in\mathfrak{X}_\xi$. Applying Lemmas 1 and 2, we obtain

$$\begin{aligned}
\min_{\hat{\mathfrak{r}}\in[\xi,1-\xi]}(\pounds\mathfrak{z}_1)(\hat{\mathfrak{r}}) = &\min_{\hat{\mathfrak{r}}\in[\xi,1-\xi]}\Bigg\{\int_0^1 \mathtt{Q}(\hat{\mathfrak{r}},\zeta_1)\varrho(\zeta_1)\mathtt{F}_1\Bigg[\int_0^1 \mathtt{Q}(\zeta_1,\zeta_2)\varrho(\zeta_2)\mathtt{F}_2\Bigg[\int_0^1 \mathtt{Q}(\zeta_2,\zeta_3)\varrho(\zeta_3)\mathtt{F}_4\cdots \\
&\mathtt{F}_{\mathfrak{d}-1}\Bigg[\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathtt{d}\zeta_{\mathfrak{d}}\Bigg]\cdots\Bigg]\mathtt{d}\zeta_3\Bigg]\mathtt{d}\zeta_2\Bigg]\mathtt{d}\zeta_1\Bigg\} \\
\geq\ &\sigma(\xi)\Bigg\{\int_0^1 \mathtt{Q}(\zeta_1,\zeta_1)\varrho(\zeta_1)\mathtt{F}_1\Bigg[\int_0^1 \mathtt{Q}(\zeta_1,\zeta_2)\varrho(\zeta_2)\mathtt{F}_2\Bigg[\int_0^1 \mathtt{Q}(\zeta_2,\zeta_3)\varrho(\zeta_3)\mathtt{F}_4\cdots \\
&\mathtt{F}_{\mathfrak{d}-1}\Bigg[\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathtt{d}\zeta_{\mathfrak{d}}\Bigg]\cdots\Bigg]\mathtt{d}\zeta_3\Bigg]\mathtt{d}\zeta_2\Bigg]\mathtt{d}\zeta_1\Bigg\} \\
\geq\ &\sigma(\xi)\Bigg\{\int_0^1 \mathtt{Q}(\hat{\mathfrak{r}},\zeta_1)\varrho(\zeta_1)\mathtt{F}_1\Bigg[\int_0^1 \mathtt{Q}(\zeta_1,\zeta_2)\varrho(\zeta_2)\mathtt{F}_2\Bigg[\int_0^1 \mathtt{Q}(\zeta_2,\zeta_3)\varrho(\zeta_3)\mathtt{F}_4\cdots \\
&\mathtt{F}_{\mathfrak{d}-1}\Bigg[\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathtt{d}\zeta_{\mathfrak{d}}\Bigg]\cdots\Bigg]\mathtt{d}\zeta_3\Bigg]\mathtt{d}\zeta_2\Bigg]\mathtt{d}\zeta_1\Bigg\} \\
\geq\ &\sigma(\xi)\max_{\hat{\mathfrak{r}}\in[0,1]}|\pounds\mathfrak{z}_1(\hat{\mathfrak{r}})|.
\end{aligned}$$

Thus, $\pounds(\mathfrak{X}_{\xi})\subset\mathfrak{X}_{\xi}$. In light of this, the operator £ is fully continuous according to the Arzela–Ascoli theorem. □

The following theorems are key tools for the existence of positive solutions:

Theorem 1 (Hölder's [28])**.** *For $\ell=1,2,\cdots,\kappa$, and $\mathtt{p}_\ell>1$, let $\hbar\in L^{\mathtt{p}_\ell}[0,1]$ with $\sum_{\ell=1}^{\kappa}\frac{1}{\mathtt{p}_\ell}=1$; then, $\prod_{\ell=1}^{\kappa}\hbar_\ell\in L^1[0,1]$ and $\|\prod_{\ell=1}^{\kappa}\hbar_\ell\|_1\leq\prod_{\ell=1}^{\kappa}\|\hbar_\ell\|_{\mathtt{p}_\ell}$. Furthermore, if $\hbar\in L^1[0,1]$ and $\bar{g}\in L^\infty[0,1]$ then $\hbar\bar{g}\in L^1[0,1]$ and $\|\hbar\bar{g}\|_1\leq\|\hbar\|_1\|\bar{g}\|_\infty$.*

Theorem 2 (Guo–Krasnoselskii [29])**.** *Let $\mathfrak{G}$ be a Banach space, and let $\mathfrak{N}_1,\mathfrak{N}_2$ be bounded open subsets of $\mathfrak{G}$ with $0\in\mathfrak{N}_1\subset\overline{\mathfrak{N}}_1\subset\mathfrak{N}_2$ and $\aleph:\mathfrak{X}\cap(\overline{\mathfrak{N}}_2\backslash\mathfrak{N}_1)\to\mathfrak{X}$ ($\mathfrak{X}\subset\mathfrak{G}$ is a cone) as a completely continuous operator, such that*

(i) $\|\aleph\mathfrak{z}\|\leq\|\mathfrak{z}\|$, $\mathfrak{z}\in\mathfrak{X}\cap\partial\mathfrak{N}_1$, and $\|\aleph\mathfrak{z}\|\geq\|\mathfrak{z}\|$, $\mathfrak{z}\in\mathfrak{X}\cap\partial\mathfrak{N}_2$, or
(ii) $\|\aleph\mathfrak{z}\|\geq\|\mathfrak{z}\|$, $\mathfrak{z}\in\mathfrak{X}\cap\partial\mathfrak{N}_1$, and $\|\aleph\mathfrak{z}\|\leq\|\mathfrak{z}\|$, $\mathfrak{z}\in\mathfrak{X}\cap\partial\mathfrak{N}_2$;

then, $\aleph$ has a fixed point in $\mathfrak{X}\cap(\overline{\mathfrak{N}}_2\backslash\mathfrak{N}_1)$.

Let $\psi\geq0$ be a continuous functional on a cone $\mathfrak{X}$, and let $\mathfrak{f}>0$ and $\mathfrak{h}>0$. Define $\mathfrak{X}(\psi,\mathfrak{h})=\{\mathfrak{z}\in\mathfrak{X}:\psi(\mathfrak{z})<\mathfrak{h}\}$ and $\mathfrak{X}_{\mathfrak{f}}=\{\mathfrak{z}\in\mathfrak{X}:\|\mathfrak{z}\|<\mathfrak{f}\}$.

Theorem 3 (Avery–Henderson [30])**.** *If $\gamma_1\geq0$, $\gamma_2\geq0$, $\gamma_3\geq0$ continuous and increasing functionals on $\mathfrak{X}$, $\gamma_3(0)=0$, such that, for some positive numbers $\mathfrak{h}$ and k, $\gamma_2(\mathfrak{z})\leq\gamma_3(\mathfrak{z})\leq\gamma_1(\mathfrak{z})$ and $\|\mathfrak{z}\|\leq k\gamma_2(\mathfrak{z})$, for all $\mathfrak{z}\in\overline{\mathfrak{X}(\gamma_2,\mathfrak{h})}$, and there exist $\mathfrak{f}>0$ and $\mathfrak{g}>0$ with $\mathfrak{f}<\mathfrak{g}<\mathfrak{h}$, such that $\gamma_3(\lambda\mathfrak{z})\leq\lambda\gamma_3(\mathfrak{z})$, for $0\leq\lambda\leq1$ and $\mathfrak{z}\in\partial\mathfrak{X}(\gamma_3,\mathfrak{g})$. Furthermore, if $\pounds:\overline{\mathfrak{X}(\gamma_2,\mathfrak{h})}\to\mathfrak{X}$ is a completely continuous operator, such that*

(a) $\gamma_2(\pounds\mathfrak{z})>\mathfrak{h}$, for all $\mathfrak{z}\in\partial\mathfrak{X}(\gamma_2,\mathfrak{h})$,
(b) $\gamma_3(\pounds\mathfrak{z})<\mathfrak{g}$, for all $\mathfrak{z}\in\partial\mathfrak{X}(\gamma_3,\mathfrak{g})$,
(c) $\mathfrak{X}(\gamma_1,\mathfrak{f})\neq\emptyset$ and $\gamma_1(\pounds\mathfrak{z})>\mathfrak{f}$, for all $\partial\mathfrak{X}(\gamma_1,\mathfrak{f})$,

then £ has at least two fixed points ${}^1\mathfrak{z},{}^2\mathfrak{z}\in\mathtt{P}(\gamma_2,\mathfrak{h})$, such that $\mathfrak{f}<\gamma_1({}^1\mathfrak{z})$ with $\gamma_3({}^1\mathfrak{z})<\mathfrak{g}$ and $\mathfrak{g}<\gamma_3({}^2\mathfrak{z})$ with $\gamma_2({}^2\mathfrak{z})<\mathfrak{h}$.

Define the non-negative, increasing, continuous functional γ_2,γ_3, and γ_1 by

$$\gamma_2(\mathfrak{z})=\min_{\hat{\mathfrak{r}}\in[\xi,1-\xi]}\mathfrak{z}(\hat{\mathfrak{r}}),\ \gamma_3(\mathfrak{z})=\max_{\hat{\mathfrak{r}}\in[0,1]}\mathfrak{z}(\hat{\mathfrak{r}}),\ \gamma_1(\mathfrak{z})=\max_{\hat{\mathfrak{r}}\in[0,1]}\mathfrak{z}(\hat{\mathfrak{r}}).$$

It is obvious that for each $\mathfrak{z} \in \mathfrak{X}$, $\gamma_2(\mathfrak{z}) \leq \gamma_3(\mathfrak{z}) = \gamma_1(\mathfrak{z})$, and $\gamma_2(\mathfrak{z}) \geq \sigma(\xi)\|\mathfrak{z}\|$. Thus, $\|\mathfrak{z}\| \leq \frac{1}{\sigma(\xi)}\gamma_2(\mathfrak{z})$ for all $\mathfrak{z} \in \mathfrak{X}$. Furthermore, we observe that $\gamma_3(\lambda\mathfrak{z}) = \lambda\gamma_3(\mathfrak{z})$, for $0 \leq \lambda \leq 1$, $\mathfrak{z} \in \mathfrak{X}$.

3. Single Positive Radial Solution

In accordance with Guo–Krasnoselskii's theorem, we demonstrate in this section that problem (3) has a single positive radial solution.

For $\varrho_i \in \mathrm{L}^{\mathrm{p}_i}[0,1]$, we have the following cases:

$$\sum_{i=1}^{\kappa} \frac{1}{\mathrm{p}_i} < 1, \quad \sum_{i=1}^{\kappa} \frac{1}{\mathrm{p}_i} = 1, \quad \sum_{i=1}^{\kappa} \frac{1}{\mathrm{p}_i} > 1.$$

We discuss the positive radial solutions for $\sum_{i=1}^{\kappa} \frac{1}{\mathrm{p}_i} < 1$, in the following theorem:

Theorem 4. *Suppose that* $(\mathcal{J}_1)$–$(\mathcal{J}_2)$ *hold, and there exist positive constants* $a_2 > a_1 > 0$, *such that*

$(\mathcal{J}_3)$ $\mathtt{F}_\beta(\mathfrak{z}(\hat{\mathbf{r}})) \leq \mathfrak{R}_2 a_2$ *for* $0 \leq \hat{\mathbf{r}} \leq 1$, $0 \leq \mathfrak{z} \leq a_2$, *where* $\mathfrak{R}_2 = \left[\frac{r_0^2}{(\mathrm{N}-2)^2}\|\widehat{\mathtt{Q}}\|_{\mathtt{q}} \prod_{i=1}^{\kappa} \|\varrho_i\|_{\mathtt{p}_i}\right]^{-1}$ *and* $\widehat{\mathtt{Q}}(\zeta) = \mathtt{Q}(\zeta,\zeta)\zeta^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}$,

$(\mathcal{J}_4)$ $\mathtt{F}_\beta(\mathfrak{z}(\hat{\mathbf{r}})) \geq \mathfrak{R}_1 a_1$ *for* $\xi \leq \hat{\mathbf{r}} \leq 1-\xi$, $\sigma(\xi)a_1 \leq \mathfrak{z} \leq a_1$, *where*

$$\mathfrak{R}_1 = \left[\frac{\sigma(\xi)r_0^2}{(\mathrm{N}-2)^2} \prod_{i=1}^{\kappa} \varrho_i^\star \int_\xi^{1-\xi} \mathtt{Q}(\zeta,\zeta)\zeta^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}} \mathrm{d}\zeta\right]^{-1},$$

then the BVP (3) *has a solution* $(\mathfrak{z}_1, \mathfrak{z}_2, \cdots, \mathfrak{z}_\mathfrak{d})$, *such that* $\mathfrak{z}_\beta > 0$, $a_1 \leq \|\mathfrak{z}_\beta\| \leq a_2$, $\beta = 1, 2, \cdots, \mathfrak{d}$.

Proof. Let $\mathfrak{N}_1 = \{\mathfrak{z} \in \aleph : \|\mathfrak{z}\| < a_1\}$ and $\mathfrak{N}_2 = \{\mathfrak{z} \in \aleph : \|\mathfrak{z}\| < a_2\}$. For $\mathfrak{z}_1 \in \partial\mathfrak{N}_2$, $0 \leq \mathfrak{z}_1 \leq a_2$ for $\hat{\mathbf{r}} \in [0,1]$. For $\zeta_{\mathfrak{d}-1} \in [0,1]$, and from $(\mathcal{J}_3)$, we obtain

$$\begin{aligned}
\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_\mathfrak{d})\varrho(\zeta_\mathfrak{d})\mathtt{F}_\mathfrak{d}(\mathfrak{z}_1(\zeta_\mathfrak{d}))\mathrm{d}\zeta_\mathfrak{d} &\leq \int_0^1 \mathtt{Q}(\zeta_\mathfrak{d},\zeta_\mathfrak{d})\varrho(\zeta_\mathfrak{d})\mathtt{F}_\mathfrak{d}(\mathfrak{z}_1(\zeta_\mathfrak{d}))\mathrm{d}\zeta_\mathfrak{d} \\
&\leq \mathfrak{R}_2 a_2 \int_0^1 \mathtt{Q}(\zeta_\mathfrak{d},\zeta_\mathfrak{d})\varrho(\zeta_\mathfrak{d})\mathrm{d}\zeta_\mathfrak{d} \\
&\leq \mathfrak{R}_2 a_2 \frac{r_0^2}{(\mathrm{N}-2)^2} \int_0^1 \mathtt{Q}(\zeta_\mathfrak{d},\zeta_\mathfrak{d})\zeta_\mathfrak{d}^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}} \prod_{i=1}^{\kappa} \varrho_i(\zeta_\mathfrak{d})\mathrm{d}\zeta_\mathfrak{d}.
\end{aligned}$$

Now, there exists $\mathtt{q} > 1$, such that $\sum_{i=1}^{\kappa} \frac{1}{\mathrm{p}_i} + \frac{1}{\mathtt{q}} = 1$. From Theorem 1, we have

$$\begin{aligned}
\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_\mathfrak{d})\varrho(\zeta_\mathfrak{d})\mathtt{F}_\mathfrak{d}(\mathfrak{z}_1(\zeta_\mathfrak{d}))\mathrm{d}\zeta_\mathfrak{d} &\leq \mathfrak{R}_2 a_2 \frac{r_0^2}{(\mathrm{N}-2)^2}\|\widehat{\mathtt{Q}}\|_{\mathtt{q}} \prod_{i=1}^{\kappa} \|\varrho_i\|_{\mathtt{p}_i} \\
&\leq a_2.
\end{aligned}$$

Similarly, for $0 < \zeta_{\mathfrak{d}-2} < 1$,

$$\begin{aligned}\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-2},\zeta_{\mathfrak{d}-1})\varrho(\zeta_{\mathfrak{d}-1})\mathtt{F}_{\mathfrak{d}-1}\left[\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathtt{d}\zeta_{\mathfrak{d}}\right]\mathtt{d}\zeta_{\mathfrak{d}-1}&\\ \le \int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}-1})\varrho(\zeta_{\mathfrak{d}-1})\mathtt{F}_{\mathfrak{d}-1}(a_2)\mathtt{d}\zeta_{\mathfrak{d}-1}&\\ \le \mathfrak{R}_2 a_2\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}-1})\varrho(\zeta_{\mathfrak{d}-1})\mathtt{d}\zeta_{\mathfrak{d}-1}&\\ \le \mathfrak{R}_2 a_2\frac{r_0^2}{(\mathrm{N}-2)^2}\|\widehat{\mathtt{Q}}\|_{\mathtt{q}}\prod_{i=1}^{\kappa}\|\varrho_i\|_{\mathtt{p}_i}&\\ \le a_2.&\end{aligned}$$

Following this bootstrapping reasoning, we arrive at

$$\begin{aligned}(\pounds\mathfrak{z}_1)(t) = \int_0^1 \mathtt{Q}(\hat{\mathbf{r}},\zeta_1)\varrho(\zeta_1)\mathtt{F}_1\Bigg[\int_0^1 \mathtt{Q}(\zeta_1,\zeta_2)\varrho(\zeta_2)\mathtt{F}_2\Bigg[\int_0^1 \mathtt{Q}(\zeta_2,\zeta_3)\varrho(\zeta_3)\mathtt{F}_4\cdots&\\ \mathtt{F}_{\mathfrak{d}-1}\left[\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathtt{d}\zeta_{\mathfrak{d}}\right]\cdots\Bigg]\mathtt{d}\zeta_3\Bigg]\mathtt{d}\zeta_2\Bigg]\mathtt{d}\zeta_1&\\ \le a_2.&\end{aligned}$$

As $\mathfrak{N}_2 = \|\mathfrak{z}_1\|$ for $\mathfrak{z}_1 \in \mathfrak{X}\cap\partial\mathfrak{N}_2$, we obtain

$$\|\pounds\mathfrak{z}_1\| \le \|\mathfrak{z}_1\|. \tag{5}$$

Let $\hat{\mathbf{r}} \in [\xi, 1-\xi]$; then, $a_1 = \|\mathfrak{z}_1\| \ge \mathfrak{z}_1(\hat{\mathbf{r}}) \ge \min_{\hat{\mathbf{r}}\in[\xi,1-\xi]} \mathfrak{z}_1(t) \ge \sigma(\xi)\,\|\mathfrak{z}_1\| \ge \sigma(\xi)a_1$. By $(\mathcal{J}_4)$ and for $\zeta_{\mathfrak{d}-1} \in [\xi, 1-\xi]$, we have

$$\begin{aligned}\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathtt{d}\zeta_{\mathfrak{d}} &\ge \int_{\xi}^{1-\xi} \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathtt{d}\zeta_{\mathfrak{d}}\\ &\ge \sigma(\xi)\int_{\xi}^{1-\xi} \mathtt{Q}(\zeta_{\mathfrak{d}},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathtt{d}\zeta_{\mathfrak{d}}\\ &\ge \sigma(\xi)\mathfrak{R}_1 a_1\int_{\xi}^{1-\xi} \mathtt{Q}(\zeta_{\mathfrak{d}},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{d}\zeta_{\mathfrak{d}}\\ &\ge \mathfrak{R}_1 a_1\frac{\sigma(\xi)r_0^2}{(\mathrm{N}-2)^2}\int_{\xi}^{1-\xi} \mathtt{Q}(\zeta_{\mathfrak{d}},\zeta_{\mathfrak{d}})\zeta_{\mathfrak{d}}^{\frac{2(\mathrm{N}-1)}{2-\mathfrak{d}}}\prod_{i=1}^{\kappa}\varrho_i(\zeta_{\mathfrak{d}})\mathtt{d}\zeta_{\mathfrak{d}}\\ &\ge \mathfrak{R}_1 a_1\frac{\sigma(\xi)r_0^2}{(\mathrm{N}-2)^2}\prod_{i=1}^{\kappa}\varrho_i^{\star}\int_{\xi}^{1-\xi} \mathtt{Q}(\zeta_{\mathfrak{d}},\zeta_{\mathfrak{d}})\zeta_{\mathfrak{d}}^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}\mathtt{d}\zeta_{\mathfrak{d}}\\ &\ge a_1.\end{aligned}$$

Similarly, for $0 < \zeta_{\mathfrak{d}-2} < 1$,

$$
\begin{aligned}
\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-2},\zeta_{\mathfrak{d}-1})\varrho(\zeta_{\mathfrak{d}-1})\mathtt{F}_{\mathfrak{d}-1}&\left[\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathrm{d}\zeta_{\mathfrak{d}}\right]\mathrm{d}\zeta_{\mathfrak{d}-1}\\
&\geq \int_{\xi}^{1-\xi} \mathtt{Q}(\zeta_{\mathfrak{d}-2},\zeta_{\mathfrak{d}-1})\varrho(\zeta_{\mathfrak{d}-1})\mathtt{F}_{\mathfrak{d}-1}(a_1)\mathrm{d}\zeta_{\mathfrak{d}-1}\\
&\geq \sigma(\xi)\int_{\xi}^{1-\xi} \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}-1})\varrho(\zeta_{\mathfrak{d}-1})\mathtt{F}_{\mathfrak{d}-1}(a_1)\mathrm{d}\zeta_{\mathfrak{d}-1}\\
&\geq \sigma(\xi)\mathfrak{N}_1 a_1\int_{\xi}^{1-\xi} \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}-1})\varrho(\zeta_{\mathfrak{d}-1})\mathrm{d}\zeta_{\mathfrak{d}-1}\\
&\geq \mathfrak{N}_1 a_1\frac{\sigma(\xi)r_0^2}{(\mathrm{N}-2)^2}\int_{\xi}^{1-\xi} \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}-1})\zeta_{\mathfrak{d}-1}^{\frac{2(\mathrm{N}-1)}{2-\mathfrak{d}}}\prod_{i=1}^{\kappa}\varrho_i(\zeta_{\mathfrak{d}-1})\mathrm{d}\zeta_{\mathfrak{d}-1}\\
&\geq \mathfrak{N}_1 a_1\frac{\sigma(\xi)r_0^2}{(\mathrm{N}-2)^2}\prod_{i=1}^{\kappa}\varrho_i^{\star}\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}-1})\zeta_{\mathfrak{d}-1}^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}\mathrm{d}\zeta_{\mathfrak{d}-1}\\
&\geq a_1.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(\pounds\mathfrak{z}_1)(\hat{\mathbf{r}}) = \int_0^1 \mathtt{Q}(\hat{\mathbf{r}},\zeta_1)\varrho(\zeta_1)\mathtt{F}_1&\left[\int_0^1 \mathtt{Q}(\zeta_1,\zeta_2)\varrho(\zeta_2)\mathtt{F}_2\left[\int_0^1 \mathtt{Q}(\zeta_2,\zeta_3)\varrho(\zeta_3)\mathtt{F}_4\cdots\right.\right.\\
&\left.\left.\mathtt{F}_{\mathfrak{d}-1}\left[\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathrm{d}\zeta_{\mathfrak{d}}\right]\cdots\right]\mathrm{d}\zeta_3\right]\mathrm{d}\zeta_2\Bigg]\mathrm{d}\zeta_1\\
&\geq a_1.
\end{aligned}
$$

Thus, for $\mathfrak{z}_1 \in \mathfrak{X}\cap\partial\mathfrak{N}_1$, we have

$$\|\pounds\mathfrak{z}_1\| \geq \|\mathfrak{z}_1\|. \tag{6}$$

It can be seen that $0 \in \mathfrak{N}_1 \subset \overline{\mathfrak{N}}_1 \subset \mathfrak{N}_2$, and from (5), (6), and Theorem 2, the operator $\pounds$ has a fixed point $\mathfrak{z}_1 \in \mathfrak{X}\cap\big(\overline{\mathfrak{N}}_2\backslash\mathfrak{N}_1\big)$ and $\mathfrak{z}_1(\hat{\mathbf{r}}) \geq 0$ on $(0,1)$. Now, put $\mathfrak{z}_1 = \mathfrak{z}_{\mathfrak{d}+1}$, to obtain an infinite number of solutions:

$$
\begin{aligned}
&\mathfrak{z}_{\beta}(\hat{\mathbf{r}}) = \int_0^1 \mathtt{Q}(\hat{\mathbf{r}},s)\varrho(s)\mathtt{F}_{\beta}(\mathfrak{z}_{\beta+1}(s))ds,\ \beta = 1,2,\cdots,\mathfrak{d}-1,\mathfrak{d},\\
&\mathfrak{z}_{\mathfrak{d}+1}(\hat{\mathbf{r}}) = \mathfrak{z}_1(\hat{\mathbf{r}}),\ \hat{\mathbf{r}}\in(0,1).
\end{aligned}
$$

□

For the cases $\sum_{i=1}^{\kappa}\frac{1}{\mathrm{p}_i} = 1$ and $\sum_{i=1}^{\kappa}\frac{1}{\mathrm{p}_i} > 1$, we have the following theorems:

Theorem 5. *Suppose $(\mathcal{J}_1)$–$(\mathcal{J}_2)$ hold, and there exist constants $b_2 > b_1 > 0$ with $\mathtt{F}_{\beta}$ $(\beta = 1,2,\cdots,\mathfrak{d})$ satisfies $(\mathcal{J}_4)$ and*

$(\mathcal{J}_5)$ $\mathtt{F}_{\beta}(\mathfrak{z}(\hat{\mathbf{r}})) \leq \mathfrak{N}_2 b_2$ for $0 \leq \hat{\mathbf{r}} \leq 1$, $0 \leq \mathfrak{z} \leq b_2$, where $\mathfrak{N}_2 = \left[\frac{r_0^2}{(\mathrm{N}-2)^2}\|\widehat{\mathtt{Q}}\|_{\infty}\prod_{i=1}^{\kappa}\|\varrho_i\|_{\mathrm{p}_i}\right]^{-1}$ and $\widehat{\mathtt{Q}}(\zeta) = \mathtt{Q}(\zeta,\zeta)\zeta^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}$;

then the BVP (3) has a solution $(\mathfrak{z}_1,\mathfrak{z}_2,\cdots,\mathfrak{z}_{\mathfrak{d}})$, such that $\mathfrak{z}_{\beta} > 0$, $b_1 \leq \|\mathfrak{z}_{\beta}\| \leq b_2$, $\beta = 1,2,\cdots,\mathfrak{d}$.

Proof. The proof is similar to the proof of Theorem 4; therefore, we omit the details here. □

Theorem 6. *Suppose $(\mathcal{J}_1)$–$(\mathcal{J}_2)$ hold, and there exist constants $c_2 > c_1 > 0$ with $\mathtt{F}_{\beta}$ $(\beta = 1,2,\cdots,\mathfrak{d})$ satisfying $(\mathcal{J}_4)$ and*

$(\mathcal{J}_6)$ $\mathtt{F}_\beta(\mathfrak{z}(\hat{\mathbf{r}})) \le \mathfrak{M}_2 c_2$ *for all* $0 \le \hat{\mathbf{r}} \le 1$, $0 \le \mathfrak{z} \le c_2$, *where* $\mathfrak{M}_2 = \left[\dfrac{r_0^2}{(\mathrm{N}-2)^2}\|\widehat{\mathtt{Q}}\|_\infty \prod_{i=1}^{\kappa}\|\varrho_i\|_1\right]^{-1}$ *and* $\widehat{\mathtt{Q}}(\zeta) = \mathtt{Q}(\zeta,\zeta)\zeta^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}$,

then the BVP (3) *has a solution* $(\mathfrak{z}_1, \mathfrak{z}_2, \cdots, \mathfrak{z}_\mathfrak{d})$, *such that* $\mathfrak{z}_\beta > 0$, $c_1 \le \|\mathfrak{z}_\beta\| \le c_2$, $\beta = 1, 2, \cdots, \mathfrak{d}$.

Proof. The proof is similar to the proof of Theorem 4; therefore, we omit the details here. □

Example 1. *Consider the problem*

$$\triangle \mathfrak{z}_\beta - \frac{(\mathrm{N}-2)^2 r_0^{2\mathrm{N}-2}}{|\nu|^{2\mathrm{N}-2}}\mathfrak{z}_\beta + \varrho(|\nu|)\mathtt{F}_\beta(\mathfrak{z}_{\beta+1}) = 0,\ 1 < |\nu| < 3, \tag{7}$$

$$\mathfrak{z}_\beta(0) = 0,\ \mathfrak{z}_\beta(1) = 0, \tag{8}$$

where $r_0 = 1$, $\mathrm{N} = 3$, $\beta \in \{1,2\}$, $\mathfrak{z}_3 = \mathfrak{z}_1$, $\varrho(\hat{\mathbf{r}}) = \frac{1}{\hat{\mathbf{r}}^4}\prod_{i=1}^{2}\varrho_i(\hat{\mathbf{r}})$, $\varrho_i(\hat{\mathbf{r}}) = \varrho_i\left(\frac{1}{\hat{\mathbf{r}}}\right)$, *in which* $\varrho_1(t) = \frac{2}{t^2+1}$ *and* $\varrho_2(t) = \frac{1}{\sqrt{t+2}}$, *then* $\varrho_1, \varrho_2 \in L^{\mathrm{p}}[0,1]$ *and* $\prod_{i=1}^{2}\varrho_i^* = \frac{1}{\sqrt{3}}$. *Let* $\xi = \frac{1}{3}$, $\mathtt{F}_1(\mathfrak{z}) = \mathtt{F}_2(\mathfrak{z}) = 1 + \frac{1}{3}|\sin(1+\mathfrak{z})| + \frac{1}{1+\mathfrak{z}}$.

$$\mathtt{Q}(\hat{\mathbf{r}},\zeta) = \frac{1}{\sinh(1)}\begin{cases}\sinh(\hat{\mathbf{r}})\sinh(1-\zeta), & 0 \le \hat{\mathbf{r}} \le \zeta \le 1,\\ \sinh(\zeta)\sinh(1-\hat{\mathbf{r}}), & 0 \le \zeta \le \hat{\mathbf{r}} \le 1,\end{cases}$$

and $\sigma(\xi) = \frac{\sinh(\xi)}{\sinh(1)} = \frac{\sinh(\frac{1}{3})}{\sinh(1)} = 0.2889212153$. *In addition,*

$$\mathfrak{R}_1 = \left[\frac{\sigma(\xi)r_0^2}{(\mathrm{N}-2)^2}\prod_{i=1}^{\kappa}\varrho_i^\star\int_\xi^{1-\xi}\mathtt{Q}(\zeta,\zeta)\zeta^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}\,\mathrm{d}\zeta\right]^{-1} \approx 2.932844681.$$

Let $\mathrm{p}_1 = 2, \mathrm{p}_2 = 3$ *and* $\mathrm{q} = 6$, *then* $\frac{1}{\mathrm{p}_1} + \frac{1}{\mathrm{p}_2} + \frac{1}{\mathrm{q}} = 1$ *and*

$$\mathfrak{R}_2 = \left[\frac{r_0^2}{(\mathrm{N}-2)^2}\|\widehat{\mathtt{Q}}\|_{\mathrm{q}}\prod_{i=1}^{\kappa}\|\varrho_i\|_{\mathrm{p}_i}\right]^{-1} \approx 4.284821634.$$

Choose $a_1 = \frac{1}{2}$ *and* $a_2 = 1$. *Then,*

$$\begin{aligned}\mathtt{F}_1(\mathfrak{z}) = \mathtt{F}_2(\mathfrak{z}) &= 1 + \frac{1}{3}|\sin(1+\mathfrak{z})| + \frac{1}{1+\mathfrak{z}} \le 4.284821634 = \mathfrak{R}_2 a_2,\ 0 \le \mathfrak{z} \le 1,\\ \mathtt{F}_1(\mathfrak{z}) = \mathtt{F}_2(\mathfrak{z}) &= 1 + \frac{1}{3}|\sin(1+\mathfrak{z})| + \frac{1}{1+\mathfrak{z}} \ge 1.466422340 = \mathfrak{R}_1 a_1,\ 0.1444606076 \le \mathfrak{z} \le \frac{1}{2}.\end{aligned}$$

Thus, by Theorem 4, *BVP* (7) *and* (8) *has at least one positive solution* $(\mathfrak{z}_1, \mathfrak{z}_2)$, *such that* $\frac{1}{2} \le \|\mathfrak{z}_\beta\| \le 1$ *for* $\beta = 1, 2$.

4. Existence of Coupled Positive Radial Solutions

By utilizing the Avery–Henderson cone fixed-point theorem, we demonstrate in this section that there are coupled positive solutions for (3). Denote

$$\beta_1 = \frac{\sigma(\xi)r_0^2}{(\mathrm{N}-2)^2}\prod_{i=1}^{\kappa}\varrho_i^\star\int_0^1\mathtt{Q}(\zeta,\zeta)\zeta^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}\,\mathrm{d}\zeta,$$

$$\beta_2 = \frac{r_0^2}{(N-2)^2}\|\widehat{Q}\|_q \prod_{i=1}^{\kappa}\|\varrho_i\|_{p_i},$$

$$\beta_3 = \frac{r_0^2}{(N-2)^2}\|\widehat{Q}\|_\infty \prod_{i=1}^{\kappa}\|\varrho_i\|_{p_i},$$

$$\beta_4 = \frac{r_0^2}{(N-2)^2}\|\widehat{Q}\|_\infty \prod_{i=1}^{\kappa}\|\varrho_i\|_1.$$

Theorem 7. *Suppose that* $(\mathcal{J}_1)$–$(\mathcal{J}_2)$ *hold, and that there exist three positive real numbers* $\mathfrak{f} < \mathfrak{g} < \mathfrak{h}$ *with* $F_\beta(\beta = 1,2,\cdots,\mathfrak{d})$ *satisfying*

$(\mathcal{J}_7)$ $F_\beta(\mathfrak{z}) > \frac{\mathfrak{h}}{\beta_1}$, $\mathfrak{h} \le \mathfrak{z} \le \frac{\mathfrak{h}}{\sigma(\xi)}$,
$(\mathcal{J}_8)$ $F_\beta(\mathfrak{z}) < \frac{\mathfrak{g}}{\beta_2}$, $0 \le \mathfrak{z} \le \frac{\mathfrak{g}}{\sigma(\xi)}$,
$(\mathcal{J}_9)$ $F_\beta(\mathfrak{z}) > \frac{\mathfrak{f}}{\beta_1}$, $\mathfrak{f} \le \mathfrak{z} \le \frac{\mathfrak{f}}{\sigma(\xi)}$,

then the BVP (3) *has coupled positive solutions* $\{({}^1\mathfrak{z}_1, {}^1\mathfrak{z}_2, \cdots, {}^1\mathfrak{z}_\mathfrak{d})\}$ *and* $\{({}^2\mathfrak{z}_1, {}^2\mathfrak{z}_2, \cdots, {}^2\mathfrak{z}_\mathfrak{d})\}$ *satisfying*

$$\mathfrak{f} < \gamma_1({}^1\mathfrak{z}_\beta) \;\; \textit{with} \;\; \gamma_3({}^1\mathfrak{z}_\beta) < \mathfrak{g}, \;\; \beta = 1,2,\cdots,\mathfrak{d}$$

and

$$\mathfrak{g} < \gamma_3({}^2\mathfrak{z}_\beta) \;\; \textit{with} \;\; \gamma_2({}^2\mathfrak{z}_\beta) < \mathfrak{h}, \;\; \beta = 1,2,\cdots,\mathfrak{d}.$$

Proof. It is easy to demonstrate that $\pounds : \overline{\mathfrak{X}(\gamma_2,\mathfrak{h})} \to \mathfrak{X}$ and $\pounds$ are completely continuous from (4): first, we check that the condition (a) of Theorem 3 holds; for this, we choose $\mathfrak{z}_1 \in \partial\mathfrak{X}(\gamma_2,\mathfrak{h})$; then, $\gamma_2(\mathfrak{z}_1) = \min_{\hat{\mathfrak{r}}\in[\xi,1-\xi]} \mathfrak{z}_1(\hat{\mathfrak{r}}) = \mathfrak{h}$, so $\mathfrak{h} \le \mathfrak{z}_1(\hat{\mathfrak{r}})$ for $\hat{\mathfrak{r}} \in [\xi,1-\xi]$. As $\|\mathfrak{z}_1\| \le \frac{1}{\sigma(\xi)}\gamma_2(\mathfrak{z}_1) = \frac{1}{\sigma(\xi)}\mathfrak{h}$, we have $\mathfrak{h} \le \mathfrak{z}_1(\hat{\mathfrak{r}}) \le \frac{\mathfrak{h}}{\sigma(\xi)}$, $\hat{\mathfrak{r}} \in [\xi,1-\xi]$. Let $\zeta_{\mathfrak{d}-1} \in [\xi,1-\xi]$. Then, by $(\mathcal{J}_7)$, we have

$$\begin{aligned}
\int_0^1 Q(\zeta_{\mathfrak{d}-1},\zeta_\mathfrak{d})\varrho(\zeta_\mathfrak{d})F_\mathfrak{d}(\mathfrak{z}_1(\zeta_\mathfrak{d}))d\zeta_\mathfrak{d} &\ge \sigma(\xi)\int_\xi^{1-\xi} Q(\zeta_\mathfrak{d},\zeta_\mathfrak{d})\varrho(\zeta_\mathfrak{d})F_\mathfrak{d}(\mathfrak{z}_1(\zeta_\mathfrak{d}))d\zeta_\mathfrak{d} \\
&\ge \frac{\sigma(\xi)\mathfrak{h}}{\beta_1}\int_\xi^{1-\xi} Q(\zeta_\mathfrak{d},\zeta_\mathfrak{d})\varrho(\zeta_\mathfrak{d})d\zeta_\mathfrak{d} \\
&\ge \frac{\sigma(\xi)\mathfrak{h}r_0^2}{(N-2)^2\beta_1}\int_\xi^{1-\xi} Q(\zeta_\mathfrak{d},\zeta_\mathfrak{d})\zeta_\mathfrak{d}^{\frac{2(N-1)}{2-N}}\prod_{i=1}^{\kappa}\varrho_i(\zeta_\mathfrak{d})d\zeta_\mathfrak{d} \\
&\ge \frac{\sigma(\xi)\mathfrak{h}r_0^2}{(N-2)^2\beta_1}\prod_{i=1}^{\kappa}\varrho_i^\star\int_\xi^{1-\xi} Q(\zeta_\mathfrak{d},\zeta_\mathfrak{d})\zeta_\mathfrak{d}^{\frac{2(N-1)}{2-N}}d\zeta_\mathfrak{d} \\
&\ge \mathfrak{h}.
\end{aligned}$$

Following this, we arrive at

$$\begin{aligned}
\gamma_2(\pounds\mathfrak{z}_1) = \min_{\hat{\mathfrak{r}}\in[0,1]}\int_0^1 Q(\hat{\mathfrak{r}},\zeta_1)\varrho(\zeta_1)F_1\Bigg[\int_0^1 Q(\zeta_1,\zeta_2)\varrho(\zeta_2)F_2\Bigg[\int_0^1 Q(\zeta_2,\zeta_3)\varrho(\zeta_3)F_4\cdots \\
F_{\mathfrak{d}-1}\Bigg[\int_0^1 Q(\zeta_{\mathfrak{d}-1},\zeta_\mathfrak{d})\varrho(\zeta_\mathfrak{d})F_\mathfrak{d}(\mathfrak{z}_1(\zeta_\mathfrak{d}))d\zeta_\mathfrak{d}\Bigg]\cdots\Bigg]d\zeta_3\Bigg]d\zeta_2\Bigg]d\zeta_1 \\
\ge \mathfrak{h}.
\end{aligned}$$

Condition (a) of Theorem 3 is proved. To prove (b), choose $\mathfrak{z}_1 \in \partial\mathfrak{X}(\gamma_3,\mathfrak{g})$. Then, $\gamma_3(\mathfrak{z}_1) = \max_{\hat{\mathfrak{r}}\in[0,1]} \mathfrak{z}_1(\hat{\mathfrak{r}}) = \mathfrak{g}$, so that $0 \le \mathfrak{z}_1(\hat{\mathfrak{r}}) \le \mathfrak{g}$ for $\hat{\mathfrak{r}} \in [0,1]$. As $\|\mathfrak{z}_1\| \le \frac{1}{\sigma(\xi)}\gamma_2(\mathfrak{z}_1) \le$

$\frac{1}{\sigma(\xi)}\gamma_3(\mathfrak{z}_1) = \frac{\mathfrak{g}}{\sigma(\xi)}$, we have $0 \leq \mathfrak{z}_1(\hat{\mathbf{r}}) \leq \sigma(\xi)^2\mathfrak{g}$, $\hat{\mathbf{r}} \in [0,1]$. Let $0 < \zeta_{\mathfrak{d}-1} < 1$. Then, by $(\mathcal{J}_8)$, we have

$$\begin{aligned}\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathrm{d}\zeta_{\mathfrak{d}} &\leq \sigma(\xi)\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathrm{d}\zeta_{\mathfrak{d}} \\ &\leq \frac{\sigma(\xi)\mathfrak{g}}{\beta_2}\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathrm{d}\zeta_{\mathfrak{d}} \\ &\leq \frac{\sigma(\xi)\mathfrak{g}r_0^2}{(\mathrm{N}-2)^2\beta_2}\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}},\zeta_{\mathfrak{d}})\zeta_{\mathfrak{d}}^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}\prod_{i=1}^{\kappa}\varrho_i(\zeta_{\mathfrak{d}})\mathrm{d}\zeta_{\mathfrak{d}}.\end{aligned}$$

For some $\mathtt{q} > 1$, we have $\dfrac{1}{\mathtt{q}} + \sum_{i=1}^{\kappa}\dfrac{1}{\mathtt{p}_i} = 1$. From Theorem 1, we have

$$\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathrm{d}\zeta_{\mathfrak{d}} \leq \frac{\sigma(\xi)\mathfrak{g}r_0^2}{(\mathrm{N}-2)^2\beta_2}\|\widehat{\mathtt{Q}}\|_{\mathtt{q}}\prod_{i=1}^{\kappa}\|\varrho_i\|_{\mathtt{p}_i} \leq \mathfrak{g}.$$

It follows that

$$\begin{aligned}\gamma_3(\pounds\mathfrak{z}_1) = \max_{\hat{\mathbf{r}}\in[0,1]}\int_0^1 \mathtt{Q}(\hat{\mathbf{r}},\zeta_1)\varrho(\zeta_1)\mathtt{F}_1\left[\int_0^1 \mathtt{Q}(\zeta_1,\zeta_2)\varrho(\zeta_2)\mathtt{F}_2\left[\int_0^1 \mathtt{Q}(\zeta_2,\zeta_3)\varrho(\zeta_3)\mathtt{F}_4\cdots\right.\right.\\ \left.\left.\mathtt{F}_{\mathfrak{d}-1}\left[\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathrm{d}\zeta_{\mathfrak{d}}\right]\cdots\right]\mathrm{d}\zeta_3\right]\mathrm{d}\zeta_2\right]\mathrm{d}\zeta_1 \\ \leq \mathfrak{g}.\end{aligned}$$

Thus, (b) holds. Finally, we also check that (c) of Theorem 3 holds. Observe that $\mathfrak{z}_1(\hat{\mathbf{r}}) = \mathfrak{f}/4 \subset \mathfrak{X}(\gamma_1,\mathfrak{f})$ and $\mathfrak{f}/4 < \mathfrak{f}$, so that $\mathfrak{X}(\gamma_1,\mathfrak{f}) \neq \emptyset$. Next, if $\mathfrak{z}_1 \in \mathfrak{X}(\gamma_1,\mathfrak{f})$, then $\mathfrak{f} = \gamma_1(\mathfrak{z}_1) = \max_{\hat{\mathbf{r}}\in[0,1]}\mathfrak{z}_1(\hat{\mathbf{r}}) = \|\mathfrak{z}_1\| = \frac{1}{\sigma(\xi)}\gamma_2(\mathfrak{z}_1) \leq \frac{1}{\sigma(\xi)}\gamma_3(\mathfrak{z}_1) = \frac{1}{\sigma(\xi)}\gamma_1(\mathfrak{z}_1) = \frac{\mathfrak{f}}{\sigma(\xi)}$, i.e., $\mathfrak{f} \leq \mathfrak{z}_1(\hat{\mathbf{r}}) \leq \frac{\mathfrak{f}}{\sigma(\xi)}$ for $\hat{\mathbf{r}} \in [0,1]$. Let $0 < \zeta_{\mathfrak{d}-1} < 1$. Then, by $(\mathcal{J}_9)$, we have

$$\begin{aligned}\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathrm{d}\zeta_{\mathfrak{d}} &\geq \sigma(\xi)\int_{\xi}^{1-\xi} \mathtt{Q}(\zeta_{\mathfrak{d}},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathrm{d}\zeta_{\mathfrak{d}} \\ &\geq \frac{\sigma(\xi)\mathfrak{f}}{\beta_1}\int_{\xi}^{1-\xi} \mathtt{Q}(\zeta_{\mathfrak{d}},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathrm{d}\zeta_{\mathfrak{d}} \\ &\geq \frac{\sigma(\xi)\mathfrak{f}r_0^2}{(\mathrm{N}-2)^2\beta_1}\int_{\xi}^{1-\xi} \mathtt{Q}(\zeta_{\mathfrak{d}},\zeta_{\mathfrak{d}})\zeta_{\mathfrak{d}}^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}\prod_{i=1}^{\kappa}\varrho_i(\zeta_{\mathfrak{d}})\mathrm{d}\zeta_{\mathfrak{d}} \\ &\geq \frac{\sigma(\xi)\mathfrak{f}r_0^2}{(\mathrm{N}-2)^2\beta_1}\prod_{i=1}^{\kappa}\varrho_i^{\star}\int_{\xi}^{1-\xi} \mathtt{Q}(\zeta_{\mathfrak{d}},\zeta_{\mathfrak{d}})\zeta_{\mathfrak{d}}^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}\mathrm{d}\zeta_{\mathfrak{d}} \\ &\geq \mathfrak{f}.\end{aligned}$$

Following this bootstrapping reasoning, we arrive at

$$\begin{aligned}\gamma_1(\pounds\mathfrak{z}_1) &= \max_{\hat{\mathfrak{r}}\in[0,1]}\int_0^1 \mathtt{Q}(\hat{\mathfrak{r}},\zeta_1)\varrho(\zeta_1)\mathtt{F}_1\left[\int_0^1 \mathtt{Q}(\zeta_1,\zeta_2)\varrho(\zeta_2)\mathtt{F}_2\left[\int_0^1 \mathtt{Q}(\zeta_2,\zeta_3)\varrho(\zeta_3)\mathtt{F}_4\cdots\right.\right.\\ &\qquad \left.\left.\mathtt{F}_{\mathfrak{d}-1}\left[\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)d\zeta_{\mathfrak{d}}\right]\cdots\right]d\zeta_3\right]d\zeta_2\right]d\zeta_1\\ &\geq \min_{\hat{\mathfrak{r}}\in[0,1]}\int_0^1 \mathtt{Q}(\hat{\mathfrak{r}},\zeta_1)\varrho(\zeta_1)\mathtt{F}_1\left[\int_0^1 \mathtt{Q}(\zeta_1,\zeta_2)\varrho(\zeta_2)\mathtt{F}_2\left[\int_0^1 \mathtt{Q}(\zeta_2,\zeta_3)\varrho(\zeta_3)\mathtt{F}_4\cdots\right.\right.\\ &\qquad \left.\left.\mathtt{F}_{\mathfrak{d}-1}\left[\int_0^1 \mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)d\zeta_{\mathfrak{d}}\right]\cdots\right]d\zeta_3\right]d\zeta_2\right]d\zeta_1\\ &\geq \mathfrak{f}.\end{aligned}$$

Thus, assumption (c) of Theorem 3 holds. Hence, by Theorem 3, there exist coupled positive solutions as mentioned in the hypothesis. □

The following theorems are for the cases, $\sum_{i=1}^{\kappa}\frac{1}{\mathrm{p}_i}=1$ and $\sum_{i=1}^{\kappa}\frac{1}{\mathrm{p}_i}>1$, respectively:

Theorem 8. *Suppose that $(\mathcal{J}_1)$–$(\mathcal{J}_2)$ hold, and there exist three positive real numbers $\mathfrak{f}<\mathfrak{g}<\mathfrak{h}$ with $\mathtt{F}_\beta(\beta=1,2,\cdots,\mathfrak{d})$ satisfying $(\mathcal{J}_7)$, $(\mathcal{J}_9)$, and*
$(\mathcal{J}_{10})\mathtt{F}_\beta(\mathfrak{z})<\frac{\mathfrak{g}}{\beta_3},\, 0\le\mathfrak{z}\le\frac{\mathfrak{g}}{\sigma(\xi)},$

then the BVP (3) has coupled positive solutions $\{({}^1\mathfrak{z}_1,{}^1\mathfrak{z}_2,\cdots,{}^1\mathfrak{z}_{\mathfrak{d}})\}$ and $\{({}^2\mathfrak{z}_1,{}^2\mathfrak{z}_2,\cdots,{}^2\mathfrak{z}_{\mathfrak{d}})\}$ satisfying

$$\mathfrak{f}<\gamma_1\big({}^1\mathfrak{z}_\beta\big)\ \textit{with}\ \gamma_3\big({}^1\mathfrak{z}_\beta\big)<\mathfrak{g},\ \beta=1,2,\cdots,\mathfrak{d}$$

and

$$\mathfrak{g}<\gamma_3\big({}^2\mathfrak{z}_\beta\big)\ \textit{with}\ \gamma_2\big({}^2\mathfrak{z}_\beta\big)<\mathfrak{h},\ \beta=1,2,\cdots,\mathfrak{d}.$$

Proof. The proof is similar to the proof of Theorem 7; therefore, we omit the details here. □

Theorem 9. *Suppose that $(\mathcal{J}_1)$–$(\mathcal{J}_3)$ hold, and there exist three positive real numbers $0<\mathfrak{f}<\mathfrak{g}<\mathfrak{h}$ with $\mathtt{F}_\beta(\beta=1,2,\cdots,\mathfrak{d})$ satisfying $(\mathcal{J}_7)$, $(\mathcal{J}_9)$ and*
$(\mathcal{J}_{11})\mathtt{F}_\beta(\mathfrak{z})<\frac{\mathfrak{g}}{\beta_4},\, 0\le\mathfrak{z}\le\frac{\mathfrak{g}}{\sigma(\xi)},$

then the BVP (3) has coupled positive solutions $\{({}^1\mathfrak{z}_1,{}^1\mathfrak{z}_2,\cdots,{}^1\mathfrak{z}_{\mathfrak{d}})\}$ and $\{({}^2\mathfrak{z}_1,{}^2\mathfrak{z}_2,\cdots,{}^2\mathfrak{z}_{\mathfrak{d}})\}$ satisfying

$$\mathfrak{f}<\gamma_1\big({}^1\mathfrak{z}_\beta\big)\ \textit{with}\ \gamma_3\big({}^1\mathfrak{z}_\beta\big)<\mathfrak{g},\ \beta=1,2,\cdots,\mathfrak{d}$$

and

$$\mathfrak{g}<\gamma_3\big({}^2\mathfrak{z}_\beta\big)\ \textit{with}\ \gamma_2\big({}^2\mathfrak{z}_\beta\big)<\mathfrak{h},\ \beta=1,2,\cdots,\mathfrak{d}.$$

Proof. The proof is similar to the proof of Theorem 7; therefore, we omit the details here. □

Example 2. *Consider the problem*

$$\triangle\mathfrak{z}_\beta-\frac{(\mathrm{N}-2)^2r_0^{2\mathrm{N}-2}}{|\nu|^{2\mathrm{N}-2}}\mathfrak{z}_\beta+\varrho(|\nu|)\mathtt{F}_\beta(\mathfrak{z}_{\beta+1})=0,\ 1<|\nu|<3, \tag{9}$$

$$\mathfrak{z}_\beta(0)=\mathfrak{z}_\beta(1)=0, \tag{10}$$

where $r_0 = 1$, $\mathrm{N} = 3$, $\beta \in \{1,2\}$, $\mathfrak{z}_3 = \mathfrak{z}_1$, $\varrho(\hat{\mathbf{r}}) = \frac{1}{\hat{\mathbf{r}}^4}\prod_{i=1}^{2}\varrho_i(\hat{\mathbf{r}})$, $\varrho_i(\hat{\mathbf{r}}) = \varrho_i\left(\frac{1}{\hat{\mathbf{r}}}\right)$, *in which* $\varrho_1(\hat{\mathbf{r}}) = \frac{1}{\hat{\mathbf{r}}+2}$ *and* $\varrho_2(\hat{\mathbf{r}}) = \frac{3}{\hat{\mathbf{r}}^2+1}$, *then* $\varrho_1, \varrho_2 \in L^{\mathrm{p}}[0,1]$, $\prod_{i=1}^{2}\varrho_i^{*} = \frac{1}{2}$, *and* $\sigma(\xi) = \frac{\sinh(\xi)}{\sinh(1)} = \frac{\sinh(\frac{1}{3})}{\sinh(1)} = 0.2889212153$. *In addition,*

$$\beta_1 = \frac{\sigma(\xi) r_0^2}{(\mathrm{N}-2)^2}\prod_{i=1}^{\kappa}\varrho_i^{\star}\int_{\xi}^{1-\xi}\mathrm{Q}(\zeta,\zeta)\zeta^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}\mathrm{d}\zeta \approx 0.1704829453.$$

Let $\mathrm{p}_1 = 6, \mathrm{p}_2 = 3$ *and* $\mathrm{q} = 2$, *then* $\frac{1}{\mathrm{p}_1} + \frac{1}{\mathrm{p}_2} + \frac{1}{\mathrm{q}} = 1$ *and*

$$\beta_2 = \frac{r_0^2}{(\mathrm{N}-2)^2}\|\widehat{\mathrm{Q}}\|_{\mathrm{q}}\prod_{i=1}^{\kappa}\|\varrho_i\|_{\mathrm{p}_i} \approx 0.1255931381.$$

Let

$$\mathrm{F}_1(\mathfrak{z}) = \mathrm{F}_2(\mathfrak{z}) = \begin{cases} 3.9, & \mathfrak{z} \leq 1.8, \\ 3.9(\mathfrak{z}-0.8)^2 + \mathfrak{z} - 1.8, & \mathfrak{z} > 1.8. \end{cases}$$

Choose $\mathfrak{f} = \frac{1}{3}$, $\mathfrak{g} = \frac{1}{2}$ *and* $\mathfrak{h} = \frac{3}{5}$. *Then,*

$$\begin{aligned} \mathrm{F}_1(\mathfrak{z}) = \mathrm{F}_2(\mathfrak{z}) &\geq 3.519413622 = \frac{\mathfrak{h}}{\beta_1}, \quad \mathfrak{z} \in \left[\frac{3}{5}, 3.461 \times \frac{3}{5}\right], \\ \mathrm{F}_1(\mathfrak{z}) = \mathrm{F}_2(\mathfrak{z}) &\leq 3.981109220 = \frac{\mathfrak{g}}{\beta_2}, \quad \mathfrak{z} \in \left[0, 3.461 \times \frac{1}{2}\right], \\ \mathrm{F}_1(\mathfrak{z}) = \mathrm{F}_2(\mathfrak{z}) &\geq 1.955229790 = \frac{\mathfrak{f}}{\beta_1}, \quad \mathfrak{z} \in \left[\frac{1}{3}, 3.461 \times \frac{1}{3}\right]. \end{aligned}$$

Hence, by an application of Theorem 4, the BVP (9) and (10) has coupled positive solutions $({}^{\beta}\mathfrak{z}_1, {}^{\beta}\mathfrak{z}_2)$, $\beta = 1,2$, *such that*

$$\frac{1}{3} < \max_{\hat{\mathbf{r}}\in[0,1]} {}^{\beta}\mathfrak{z}_1(\hat{\mathbf{r}}) \;\; \textit{with} \;\; \max_{\hat{\mathbf{r}}\in[0,1]} {}^{\beta}\mathfrak{z}_1(\hat{\mathbf{r}}) < \frac{1}{2}, \;\; \textit{for} \;\; \beta = 1,2,$$

$$\frac{1}{2} < \max_{\hat{\mathbf{r}}\in[0,1]} {}^{\beta}\mathfrak{z}_2(\hat{\mathbf{r}}) \;\; \textit{with} \;\; \min_{\hat{\mathbf{r}}\in[0,1]} {}^{\beta}\mathfrak{z}_2(\hat{\mathbf{r}}) < \frac{3}{5}, \;\; \textit{for} \;\; \beta = 1,2.$$

5. Uniqueness of Positive Radial Solution

We use two metrics, in accordance with Rus' theorem [31,32], in this part, to test if there is a unique positive solution to the BVP (3). Consider the collection of continuous, real-valued functions defined on $[0,1]$: this space is symbolised by the letter X. Take into account the below metrics on X, for functions $\mathfrak{y}, \mathfrak{z} \in \mathrm{X}$:

$$\mathrm{d}(\mathfrak{y},\mathfrak{z}) = \max_{\hat{\mathbf{r}}\in[0,1]} |\mathfrak{y}(\hat{\mathbf{r}}) - \mathfrak{z}(\hat{\mathbf{r}})|; \tag{11}$$

$$\rho(\mathfrak{y},\mathfrak{z}) = \left[\int_0^1 |\mathfrak{y}(\hat{\mathbf{r}}) - \mathfrak{z}(\hat{\mathbf{r}})|^{\mathrm{p}}\mathrm{d}\hat{\mathbf{r}}\right]^{\frac{1}{\mathrm{p}}}, \;\; \mathrm{p} > 1. \tag{12}$$

The combination (X,d) creates a complete metric space for d in (11). Then, (X,ρ) constitutes a metric space for the value of ρ in (12). The equation expressing the connection between the two measures on X is

$$\rho(\mathfrak{y},\mathfrak{z}) \leq \mathrm{d}(\mathfrak{y},\mathfrak{z}) \;\; \text{for all} \;\; \mathfrak{y},\mathfrak{z} \in \mathrm{X}. \tag{13}$$

Theorem 10 (Rus [32]). *Let $\mathtt{F}:\mathtt{X}\to\mathtt{X}$ be a continuous with respect to $\mathtt{d}$ on $\mathtt{X}$ and*

$$\mathtt{d}(\mathtt{F}\mathfrak{y},\mathtt{F}\mathfrak{z})\leq\alpha_1\rho(\mathfrak{y},\mathfrak{z}),\tag{14}$$

for some $\alpha_1>0$ and for all $\mathfrak{y},\mathfrak{z}\in\mathtt{X}$,

$$\rho(\mathtt{F}\mathfrak{y},\mathtt{F}\mathfrak{z})\leq\alpha_2\rho(\mathfrak{y},\mathfrak{z}),\tag{15}$$

for some $0<\alpha_2<1$ for all $\mathfrak{y},\mathfrak{z}\in\mathtt{X}$, then there is a unique $\mathfrak{y}^*\in\mathtt{X}$ such that $\mathtt{F}\mathfrak{y}^*=\mathfrak{y}^*$.

Denote $\mathrm{Y}(\zeta)=\mathtt{Q}(\zeta,\zeta)\zeta^{\frac{2(\mathrm{N}-1)}{2-\mathrm{N}}}\prod_{i=1}^{\kappa}\varrho_i(\zeta)$.

Theorem 11. *Suppose that $(\mathcal{J}_1)$ and $(\mathcal{J}_2)$ and the following*
$(\mathcal{J}_{12})\,|\mathtt{F}_\beta(\mathfrak{z})-\mathtt{F}_\beta(\mathfrak{y})|\leq\mathtt{K}|\mathfrak{z}-\mathtt{y}|$ for $\mathfrak{z},\mathfrak{y}\in\mathtt{X}$, for some $\mathtt{K}>0$
are satisfied. Furthermore, there are two real numbers $\mathtt{p}>1$, $\mathtt{q}>1$ satisfying $\frac{1}{\mathtt{p}}+\frac{1}{\mathtt{q}}=1$, and the following holds:

$$\left[\frac{\sigma(\xi)\mathtt{K}r_0^2}{(\mathrm{N}-2)^2}\right]^{\mathfrak{d}+1}\left[\int_0^1|\mathrm{Y}(\zeta)|\mathtt{d}\zeta\right]^{\mathfrak{d}}\left[\int_0^1|\mathrm{Y}(\zeta)|^{\mathtt{q}}\mathtt{d}\zeta\right]^{\frac{1}{q}}<1;\tag{16}$$

then the BVP (3) has a unique positive solution in $\mathtt{X}$.

Proof. Let $\mathfrak{z}_1,\mathfrak{y}_1\in\mathtt{X}$ and $\zeta_{n-1}\in[0,1]$. The Hölder's inequality gives

$$\begin{aligned}
&\left|\int_0^1\mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathtt{d}\zeta_{\mathfrak{d}}-\int_0^1\mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{y}_1(\zeta_{\mathfrak{d}})\big)\mathtt{d}\zeta_{\mathfrak{d}}\right|\\
&\leq\int_0^1|\mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\,\varrho(\zeta_{\mathfrak{d}})||\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)-\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{y}_1(\zeta_{\mathfrak{d}})\big)|\mathtt{d}\zeta_{\mathfrak{d}}\\
&\leq\int_0^1|\mathtt{Q}(\zeta_{\mathfrak{d}},\zeta_{\mathfrak{d}})\,\varrho(\zeta_{\mathfrak{d}})|\,\mathtt{K}|\mathfrak{z}_1(\zeta_{\mathfrak{d}})-\mathfrak{y}_1(\zeta_{\mathfrak{d}})|\mathtt{d}\zeta_{\mathfrak{d}}\leq\frac{\mathtt{K}r_0^2}{(\mathfrak{d}-2)^2}\int_0^1|\mathrm{Y}(\zeta_{\mathfrak{d}})||\mathfrak{z}_1(\zeta_{\mathfrak{d}})-\mathfrak{y}_1(\zeta_{\mathfrak{d}})|\mathtt{d}\zeta_{\mathfrak{d}}\\
&\leq\frac{\mathtt{K}r_0^2}{(\mathrm{N}-2)^2}\left[\int_0^1|\mathrm{Y}(\zeta_{\mathfrak{d}})|^{\mathtt{q}}\mathtt{d}\zeta_{\mathfrak{d}}\right]^{\frac{1}{\mathtt{q}}}\left[\int_0^1|\mathfrak{z}_1(\zeta_{\mathfrak{d}})-\mathfrak{y}_1(\zeta_{\mathfrak{d}})|^{\mathtt{p}}\mathtt{d}\zeta_{\mathfrak{d}}\right]^{\frac{1}{\mathtt{p}}}\\
&\leq\frac{\mathtt{K}r_0^2}{(\mathrm{N}-2)^2}\left[\int_0^1|\mathrm{Y}(\zeta_{\mathfrak{d}})|^{\mathtt{q}}\mathtt{d}\zeta_{\mathfrak{d}}\right]^{\frac{1}{\mathtt{q}}}\rho(\mathfrak{z}_1,\mathfrak{y}_1)\leq\alpha_1^\star\rho(\mathfrak{z}_1,\mathfrak{y}_1),
\end{aligned}$$

where

$$\alpha_1^\star=\frac{\mathtt{K}r_0^2}{(\mathrm{N}-2)^2}\left[\int_0^1|\mathrm{Y}(\zeta_{\mathfrak{d}})|^{\mathtt{q}}\mathtt{d}\zeta\right]^{\frac{1}{\mathtt{q}}}.$$

Similarly, for $0<\zeta_{\mathfrak{d}-2}<1$, we obtain

$$\begin{aligned}
&\left|\int_0^1\mathtt{Q}(\zeta_{\mathfrak{d}-2},\zeta_{\mathfrak{d}-1})\varrho(\zeta_{\mathfrak{d}-1})\mathtt{F}_{\mathfrak{d}-1}\left[\int_0^1\mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{z}_1(\zeta_{\mathfrak{d}})\big)\mathtt{d}\zeta_{\mathfrak{d}}\right]\mathtt{d}\zeta_{\mathfrak{d}-1}\right.\\
&\qquad\left.-\int_0^1\mathtt{Q}(\zeta_{\mathfrak{d}-2},\zeta_{\mathfrak{d}-1})\varrho(\zeta_{\mathfrak{d}-1})\mathtt{F}_{\mathfrak{d}-1}\left[\int_0^1\mathtt{Q}(\zeta_{\mathfrak{d}-1},\zeta_{\mathfrak{d}})\varrho(\zeta_{\mathfrak{d}})\mathtt{F}_{\mathfrak{d}}\big(\mathfrak{y}_1(\zeta_{\mathfrak{d}})\big)\mathtt{d}\zeta_{\mathfrak{d}}\right]\mathtt{d}\zeta_{\mathfrak{d}-1}\right|\\
&\leq\frac{\mathtt{K}r_0^2}{(\mathrm{N}-2)^2}\int_0^1|\mathrm{Y}(\zeta_{\mathfrak{d}-1})|\alpha_1\rho(\mathfrak{z}_1,\mathfrak{y}_1)\mathtt{d}\zeta_{\mathfrak{d}-1}\leq\widehat{\alpha}_1\alpha_1^\star\rho(\mathfrak{z}_1,\mathfrak{y}_1),
\end{aligned}$$

where

$$\widehat{\alpha}_1=\frac{\mathtt{K}r_0^2}{(\mathrm{N}-2)^2}\int_0^1|\mathrm{Y}(\zeta)|\mathtt{d}\zeta.$$

Thus, we have

$$|\mathtt{F}\mathfrak{z}_1(\zeta)-\mathtt{F}\mathfrak{y}_1(\zeta)|\leq\widehat{\alpha}_1^{\mathfrak{d}}\alpha_1^\star\rho(\mathfrak{z}_1,\mathfrak{y}_1);$$

that is,

$$\mathtt{d}(\mathtt{F}\mathfrak{z}_1,\mathtt{F}\mathfrak{y}_1)\leq\alpha_1\rho(\mathfrak{z}_1,\mathfrak{y}_1), \tag{17}$$

for some $\alpha_1=\widehat{\alpha}_1^{\mathfrak{d}}\alpha_1^{\star}>0$ for all $\mathfrak{z}_1,\mathfrak{y}_1\in\mathtt{X}$, this proves (14). Next, let $\mathfrak{z}_1,\mathfrak{y}_1\in\mathtt{X}$, and from (13) and (17), we obtain

$$\mathtt{d}(\mathtt{F}\mathfrak{z}_1,\mathtt{F}\mathfrak{y}_1)\leq\alpha_1\rho(\mathfrak{z}_1,\mathfrak{y}_1)\leq\alpha_1\mathtt{d}(\mathfrak{z}_1,\mathfrak{y}_1).$$

Thus, for $\varepsilon>0$, select $\eta=\varepsilon/\alpha_1$, we obtain $\mathtt{d}(\mathtt{F}\mathfrak{z}_1,\mathtt{F}\mathfrak{y}_1)<\varepsilon$, whenever $\mathtt{d}(\mathfrak{z}_1,\mathfrak{y}_1)<\eta$, which shows that $\mathtt{F}$ is continuous on $\mathtt{X}$ with metric $\mathtt{d}$. It remains to be shown that $\mathtt{F}$ is contractive on $\mathtt{X}$ with metric ρ. For each $\mathfrak{z}_1,\mathfrak{y}_1\in\mathtt{X}$, and from (17), we have

$$\begin{aligned}\left[\int_0^1|(\mathtt{F}\mathfrak{z}_1)(\zeta)-(\mathtt{F}\mathfrak{y}_1)(\zeta)|^{\mathtt{p}}\mathtt{d}\zeta\right]^{\frac{1}{\mathtt{p}}}&\leq\left[\int_0^1|\widehat{\alpha}_1^{\mathfrak{d}}\alpha_1^{\star}\rho(\mathfrak{z}_1,\mathfrak{y}_1)|^{\mathtt{p}}\mathtt{d}\zeta\right]^{\frac{1}{\mathtt{p}}}\\&\leq\left[\frac{\mathtt{K}r_0^2}{(\mathtt{N}-2)^2}\right]^{\mathfrak{d}+1}\left[\int_0^1|\Upsilon(\zeta)|\mathtt{d}\zeta\right]^{\mathfrak{d}}\left[\int_0^1|\Upsilon(\zeta)|^{\mathtt{q}}\mathtt{d}\zeta\right]^{\frac{1}{q}}\rho(\mathfrak{z}_1,\mathfrak{y}_1);\end{aligned}$$

that is

$$\rho(\mathtt{F}\mathfrak{z}_1,\mathtt{F}\mathfrak{y}_1)\leq\left[\frac{\mathtt{K}r_0^2}{(\mathtt{N}-2)^2}\right]^{\mathfrak{d}+1}\left[\int_0^1|\Upsilon(\zeta)|\mathtt{d}\zeta\right]^{\mathfrak{d}}\left[\int_0^1|\Upsilon(\zeta)|^{\mathtt{q}}\mathtt{d}\zeta\right]^{\frac{1}{q}}\rho(\mathfrak{z}_1,\mathfrak{y}_1).$$

From assumption (16), we have

$$\rho(\mathtt{F}\mathfrak{z}_1,\mathtt{F}\mathfrak{y}_1)\leq\alpha_2\rho(\mathfrak{z}_1,\mathfrak{y}_1)$$

for some $\alpha_2<1$ and all $\mathfrak{z}_1,\mathfrak{y}_1\in\mathtt{X}$. It follows from Theorem 10 that $\mathtt{F}$ has a unique fixed point in $\mathtt{X}$. Moreover, from Lemma 3, $\mathtt{F}$ is positive. Hence, the BVP (1) has a unique positive solution. □

Example 3. *Consider the problem,*

$$\triangle\mathfrak{z}_\beta-\frac{(\mathtt{N}-2)^2r_0^{2\mathtt{N}-2}}{|\nu|^{2\mathtt{N}-2}}\mathfrak{z}_\beta+\varrho(|\nu|)\mathtt{F}_\beta(\mathfrak{z}_{\beta+1})=0,\ 1<|\nu|<2, \tag{18}$$

$$\mathfrak{z}_\beta(0)=\mathfrak{z}_\beta(1)=0, \tag{19}$$

where $r_0=1$, $\mathtt{N}=3$, $\beta\in\{1,2\}$, $\mathfrak{z}_3=\mathfrak{z}_1$, $\varrho(\hat{\mathtt{r}})=\frac{1}{\hat{\mathtt{r}}^4}\prod_{i=1}^2\varrho_i(\hat{\mathtt{r}})$, $\varrho_i(\hat{\mathtt{r}})=\varrho_i\left(\frac{1}{\hat{\mathtt{r}}}\right)$, *in which* $\varrho_1(\hat{\mathtt{r}})=\varrho_2(\hat{\mathtt{r}})=\frac{\hat{\mathtt{r}}^3}{\sqrt{\hat{\mathtt{r}}+1}}$. *Let* $\mathtt{F}_1(\mathfrak{z})=\frac{3}{2}\sin(\mathfrak{z})$ *and* $\mathtt{F}_2(\mathfrak{z})=\frac{3}{2(\mathfrak{z}+1)}$*; then,*

$$|\mathtt{F}_1(\mathfrak{z})-\mathtt{F}_1(\mathtt{y})|=\frac{|\sin(\mathfrak{z})-\sin(\mathtt{y})|}{10^3}\leq\frac{3}{2}|\mathfrak{z}-\mathtt{y}|$$

and

$$|\mathtt{F}_2(\mathfrak{z})-\mathtt{F}_2(\mathtt{y})|=\frac{3}{2}\left|\frac{1}{\mathfrak{z}+1}-\frac{1}{\mathtt{y}+1}\right|\leq\frac{3}{2}|\mathfrak{z}-\mathtt{y}|.$$

Thus, $\mathtt{K}=\frac{3}{2}$. *Let* $\mathfrak{d}=2$ *and* $\mathtt{p}=\mathtt{q}=2$*; then,*

$$\left[\frac{\mathtt{K}r_0^2}{(\mathtt{N}-2)^2}\right]^{\mathfrak{d}+1}\left[\int_0^1|\Upsilon(\zeta)|\mathtt{d}\zeta\right]^{\mathfrak{d}}\left[\int_0^1|\Upsilon(\zeta)|^{\mathtt{q}}\mathtt{d}\zeta\right]^{\frac{1}{q}}\approx0.1508078067<1.$$

Hence, as an application of Theorem 11, the BVP (18) and (19) has a unique positive radial solution.

6. Conclusions

In this paper, we developed a theory to study the existence of single and coupled positive radial solutions for a certain type of iterative system of nonlinear elliptic equations, by applying Krasnoselskii's and Avery–Henderson's fixed-point theorems in a Banach space. In the future, we will study the existence of positive radial solutions for an iterative system of elliptic equations with a logarithmic nonlinear term. In addition, we will study global existence and ground-state solutions to the addressed problem.

Author Contributions: Conceptualization, X.W. and J.A.; methodology, M.K.; software, validation, M.F.; writing—original draft preparation, M.K.; writing—review and editing, J.A. and M.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research is supported by the National Natural Science Foundation of China (Grant No. 11861053).

Institutional Review Board Statement: This article does not contain any studies, performed by any of the authors, involving human participants or animals.

Data Availability Statement: Data sharing not applicable to this paper, as no data sets were generated or analyzed during the current study.

Acknowledgments: The authors would like to thank the referees for their valuable suggestions and comments for the improvement of the paper. Xiaoming Wang is thankful to the National Natural Science Foundation of China (Grant No. 11861053). J. Alzabut is thankful to Prince Sultan University and OSTİM Technical University, and Mahammad Khuddush is thankful to Lankapalli Bullayya College of Engineering for their tireless support during work on this paper. M. Fečkan is thankful to the Slovak Research and Development Agency, under contract No. APVV-18-0308, and to the Slovak Grant Agency VEGA No. 1/0084/23 and No. 2/0127/20.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Alsaedi, R. Positive solutions for some nonlinear elliptic systems in exterior domains of $\mathbb{R}^2$. *Abstr. Appl. Anal.* **2012**, *2012*, 273017. [CrossRef]
2. Boulbrachene, M. L^∞ error estimate for a class of semilinear elliptic systems of quasi-variational inequalities. *J. Nonlinear Var. Anal.* **2021**, *5*, 429–440.
3. Chen, H. A new two-grid P_0^2–P_1 mixed finite element algorithm for general elliptic optimal control problems. *J. Nonlinear Funct. Anal.* **2022**, *2022*, 41.
4. Ni, W.M. *Some Aspects of Semilinear Elliptic Equations on $\mathbb{R}^n$, in Nonlinear Diffusion Equations and Their Equilibrium States II*; Ni, W.M., Peletier, L.A., Serrin, J., Eds.; Springer: New York, NY, USA, 1988; pp. 171–205.
5. Yanagida, E. Uniqueness of positive radial solutions of $\triangle u + \mathrm{g}(|\mathrm{x}|)u + \mathrm{g}(|\mathrm{x}|)u^p = 0$ in $\mathbb{R}^n$. *Arch. Rational Mech. Anal.* **1991**, *115*, 257–274. [CrossRef]
6. Ali, J.; Brown, K.; Shivaji, R. Positive solutions for $n \times n$ elliptic systems with combined nonlinear effects. *Differ. Integral Equ.* **2011**, *24*, 307–324. [CrossRef]
7. Ali, J.; Shivaji, R.; Ramaswamy, M. Multiple positive solutions for classes of elliptic systems with combined nonlinear effects. *Differ. Integral Equ.* **2006**, *19*, 669–680. [CrossRef]
8. Dalmasso, R. Existence and uniqueness of positive solutions of semilinear elliptic systems. *Nonlinear Anal.* **2000**, *39*, 559–568. [CrossRef]
9. Hai, D.D.; Shivaji, R. An existence result on positive solutions for a class of pLaplacian systems. *Nonlinear Anal. Theory Methods Appl.* **2004**, *56*, 1007–1010. [CrossRef]
10. Ali, J.; Shivaji, R. Positive solutions for a class of p-Laplacian systems with multiple parameters. *J. Math. Anal. Appl.* **2007**, *335*, 1013–1019. [CrossRef]
11. Ali, J.; Padhi, S. Existence of multiple positive radial solutions to elliptic equations in an annulus. *Commun. Appl. Anal.* **2018**, *22*, 695–710. [CrossRef]
12. Beldzinski, M.; Galewski, M. On solvability of elliptic boundary value problems via global invertibility. *Opusc. Math.* **2020**, *40*, 37–47. [CrossRef]
13. Gala, S.; Galakhov, E.; Ragusa, M.A.; Salieva, O. Beale-Kato-Majda regularity criterion of smooth solutions for the Hall-MHD equations with zero viscosity. *Bull. Braz. Math. Soc.* **2022**, *53*, 229–241. [CrossRef]
14. Hai, D.D.; Shivaji, R. An existence result on positive solutions for a class of semilinear elliptic systems. *Proc. R. Soc. Edinb. Sect. A* **2004**, *134*, 137–141. [CrossRef]

15. Hai, D.D.; Shivaji, R. Uniqueness of positive solutions for a class of semipositone elliptic systems. *Nonlinear Anal.* **2007**, *66*, 396–402. [CrossRef]
16. Khuddush, M.; Prasad, K.R. Positive Solutions for an Iterative System of Nonlinear Elliptic Equations. *Bull. Malays. Math. Sci. Soc.* **2022**, *45*, 245–272. [CrossRef]
17. Khuddush, M.; Prasad, K.R. Existence of infinitely many positive radial solutions for an iterative system of nonlinear elliptic equations on an exterior domain. *Afr. Mat.* **2022**, *33*, 93. [CrossRef]
18. Khuddush, M.; Prasad, K.R.; Bharathi, B. Denumerably many positive radial solutions to iterative system of nonlinear elliptic equations on the exterior of a ball. *Nonlinear Dyn. Syst. Theory* **2023**, *1*, 95–106.
19. Li, Y. Positive radial solutions for elliptic equations with nonlinear gradient terms in an annulus. *Complex Var. Elliptic Equ.* **2018**, *63*, 171–187. [CrossRef]
20. Mei, L. Structure of positive radial solutions of a quasilinear elliptic problem with singular nonlinearity. *Complex Var. Elliptic Equ.* **2018**, *63*, 1595–1603. [CrossRef]
21. Padhi, S.; Graef, J.R.; Kanaujiya, A. Positive solutions to nonlinear elliptic equations depending on a parameter with Dirichlet boundary conditions. *Differ. Equ. Dyn. Syst.* **2023**, *31*, 319–336. [CrossRef]
22. Chrouda, M.B.; Hassine, K. Uniqueness of positive radial solutions for elliptic equations in an annulus. *Proc. Am. Math. Soc.* **2020**, *149*, 649–660. [CrossRef]
23. Dong, X.; Wei, Y. Existence of radial solutions for nonlinear elliptic equations with gradient terms in annular domains. *Nonlinear Anal.* **2019**, *187*, 93–109. [CrossRef]
24. Kajikiya, R.; Ko, E. Existence of positive radial solutions for a semipositone elliptic equation. *J. Math. Anal. Appl.* **2020**, *484*, 123735. [CrossRef]
25. Son, B.; Wang, P. Positive radial solutions to classes of nonlinear elliptic systems on the exterior of a ball. *J. Math. Anal. Appl.* **2020**, *488*, 124069. [CrossRef]
26. Lan, K.; Webb, J.R.L. Positive solutions of semilinear differential equations with singularities. *J. Differ. Equ.* **1998**, *148*, 407–421. [CrossRef]
27. Lee, Y.H. Multiplicity of positive radial solutions for multiparameter semilinear elliptic systems on an annulus. *J. Differ. Equ.* **2001**, *174*, 420–441. [CrossRef]
28. Cheung, W.S. Generalizations of Hölder's inequality. *Int. Math. Math. Sci.* **2001**, *26*, 7–10. [CrossRef]
29. Guo, D.; Lakshmikantham, V. *Nonlinear Problems in Abstract Cones*; Academic Press: San Diego, CA, USA, 1988.
30. Avery, R.I.; Henderson, J. Two positive fixed points of nonlinear operators on ordered Banach spaces. *Comm. Appl. Nonlinear Anal.* **2001**, *8*, 27–36.
31. Prasad, K.R.; Khuddush, M.; Leela, D. Existence, uniqueness and Hyers–Ulam stability of a fractional order iterative two-point boundary value problems. *Afr. Mat.* **2021**, *32*, 1227–1237. [CrossRef]
32. Rus, I.A. On a fixed point theorem of Maia. *Stud. Univ.-Babes-Bolyai Math.* **1977**, *1*, 40–42.